

# Robin inequality for 7-free integers

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## Abstract

Recall that an integer is  $t$ -free iff it is not divisible by  $p^t$  for some prime  $p$ . We give a method to check Robin inequality  $\sigma(n) < e^\gamma n \log \log n$ , for  $t$ -free integers  $n$  and apply it for  $t = 6, 7$ . We introduce  $\Psi_t$ , a generalization of Dedekind  $\Psi$  function defined for any integer  $t \geq 2$  by

$$\Psi_t(n) := n \prod_{p|n} (1 + 1/p + \dots + 1/p^{t-1}).$$

If  $n$  is  $t$ -free then the sum of divisor function  $\sigma(n)$  is  $\leq \Psi_t(n)$ . We characterize the champions for  $x \mapsto \Psi_t(x)/x$ , as primorial numbers. Define the ratio  $R_t(n) := \frac{\Psi_t(n)}{n \log \log n}$ . We prove that, for all  $t$ , there exists an integer  $n_1(t)$ , such that we have  $R_t(N_n) < e^\gamma$  for  $n \geq n_1$ , where  $N_n = \prod_{k=1}^n p_k$ . Further, by combinatorial arguments, this can be extended to  $R_t(N) \leq e^\gamma$  for all  $N \geq N_n$ , such that  $n \geq n_1(t)$ . This yields Robin inequality for  $t = 6, 7$ . For  $t$  varying slowly with  $N$ , we also derive  $R_t(N) < e^\gamma$ .

**Keywords:** Dedekind  $\Psi$  function, Robin inequality, Riemann Hypothesis, Primorial numbers

## I. INTRODUCTION

The Riemann Hypothesis (RH), which describes the non trivial zeroes of Riemann  $\zeta$  function has been deemed the Holy Grail of Mathematics by several authors [1], [7]. There exist many equivalent formulations in the literature [5]. The one of concern here is that of Robin [12], which is given in terms of  $\sigma(n)$  the sum of divisor function

$$\sigma(n) < e^\gamma n \log \log n,$$

for  $n \geq 5041$ . Recall that an integer is  $t$ -free iff it is not divisible by  $p^t$  for some prime  $p$ . The above inequality was checked for many infinite families of integers in [3], for instance 5-free integers. In the present work we introduce a method to check the inequality for  $t$ -free integers for larger values of  $t$  and apply it to  $t = 6, 7$ . The idea of our method is to introduce the generalized Dedekind  $\Psi$  function defined for any integer  $t \geq 2$  by

$$\Psi_t(n) := n \prod_{p|n} (1 + 1/p + \dots + 1/p^{t-1}).$$

If  $t = 2$  this is just the classical Dedekind function which occurs in the theory of modular forms [4], in physics [10], and in analytic number theory [9]. By construction, if  $n$  is  $t$ -free then the sum of divisors function  $\sigma(n)$  is  $\leq \Psi_t(n)$ . To see this note that the multiplicative function  $\sigma$  satisfies for any integer  $a$  in the range  $t > a \geq 2$

$$\sigma(p^a) = 1 + p + \dots + p^a,$$

when the multiplicative function  $\Psi_t$  satisfies

$$\Psi_t(p^a) = p^a + \dots + 1 + \dots + 1/p^{t-1-a}.$$

It turns out that the structure of champion numbers for the arithmetic function  $x \mapsto \Psi_t(x)/x$  is much easier to understand than that of  $x \mapsto \sigma(x)/x$ , the super abundant numbers. They are exactly the so-called primorial numbers (product of first consecutive primes). We prove that, in order to maximize the ratio  $R_t$  it is enough to consider its value at primorial integers. Once this reduction is made, bounding above unconditionally  $R_t$  is easy by using classical lemmas on partial eulerian products. We conclude the article by some results on  $t$ -free integers  $N \geq N_n$ , valid for  $t$  varying slowly with  $N$ .

## II. REDUCTION TO PRIMORIAL NUMBERS

Define the primorial number  $N_n$  of index  $n$  as the product of the first  $n$  primes

$$N_n = \prod_{k=1}^n p_k,$$

so that  $N_0 = 1, N_1 = 2, N_2 = 6, \dots$  and so on. The primorial numbers (OEIS sequence A002110 [11]) play the role here of superabundant numbers in [12] or primorials in [8]. They are champion numbers (ie left to right maxima) of the function  $x \mapsto \Psi_t(x)/x$ :

$$\frac{\Psi_t(m)}{m} < \frac{\Psi_t(n)}{n} \text{ for any } m < n. \quad (1)$$

We give a rigorous proof of this fact.

*Proposition 1:* The primorial numbers and their multiples are exactly the champion numbers of the function  $x \mapsto \Psi_t(x)/x$ .

*Proof:* The proof is by induction on  $n$ . The induction hypothesis  $H_n$  is that the statement is true up to  $N_n$ . Sloane sequence A002110 begins 1, 2, 4, 6... so that  $H_2$  is true. Assume  $H_n$  true. Let  $N_n \leq m < N_{n+1}$  denote a generic integer. The prime divisors of  $m$  are  $\leq p_n$ . Therefore  $\Psi_t(m)/m \leq \Psi_t(N_n)/N_n$  with equality iff  $m$  is a multiple of  $N_n$ . Further  $\Psi_t(N_n)/N_n < \Psi_t(N_{n+1})/N_{n+1}$ . The proof of  $H_{n+1}$  follows. ■

In this section we reduce the maximization of  $R_t(n)$  over all integers  $n$  to the maximization over primorials.

*Proposition 2:* Let  $n$  be an integer  $\geq 2$ . For any  $m$  in the range  $N_n \leq m < N_{n+1}$  one has  $R_t(m) < R_t(N_n)$ .

*Proof:* Like in the preceding proof we have

$$\Psi_t(m)/m \leq \Psi_t(N_n)/N_n.$$

Since  $0 < \log \log N_n \leq \log \log m$ , the result follows. ■

## III. $\Psi_t$ AT PRIMORIAL NUMBERS

We begin by an easy application of Mertens formula.

*Proposition 3:* For  $n$  going to  $\infty$  we have

$$\lim R_t(N_n) = \frac{e^\gamma}{\zeta(t)}.$$

*Proof:* Writing  $1 + 1/p = (1 - 1/p^2)/(1 - 1/p)$  in the definition of  $\Psi(n)$  we can combine the Eulerian product for  $\zeta(t)$  with Mertens formula

$$\prod_{p \leq x} (1 - 1/p)^{-1} \sim e^\gamma \log(x)$$

to obtain

$$\Psi(N_n) \sim \frac{e^\gamma}{\zeta(t)} \log(p_n).$$

Now the Prime Number Theorem [6, Th. 6, Th. 420] shows that  $x \sim \theta(x)$  for  $x$  large, where  $\theta(x)$  stands for Chebyshev's first summatory function:

$$\theta(x) = \sum_{p \leq x} \log p.$$

This shows that, taking  $x = p_n$  we have

$$p_n \sim \theta(p_n) = \log(N_n).$$

The result follows. ■

This motivates the search for explicit upper bounds on  $R_t(N_n)$  of the form  $\frac{e^\gamma}{\zeta(t)}(1 + o(1))$ . In that direction we have the following bound.

*Proposition 4:* For  $n$  large enough to have  $p_n \geq 20000$ , we have

$$\frac{\Psi_t(N_n)}{N_n} \leq \frac{\exp(\gamma + 2/p_n)}{\zeta(t)} \left( \log \log N_n + \frac{1.1253}{\log p_n} \right).$$

We prepare for the proof of the preceding Proposition by some Lemmas. First an upper bound on a partial Eulerian product from [13, (3.30) p.70].

*Lemma 1:* For  $x \geq 2$ , we have

$$\prod_{p \leq x} (1 - 1/p)^{-1} \leq e^\gamma \left( \log x + \frac{1}{\log x} \right).$$

Next an upper bound on the tail of the Eulerian product for  $\zeta(t)$ .

*Lemma 2:* For  $n \geq 2$  we have

$$\prod_{p > p_n} (1 - 1/p^t)^{-1} \leq \exp(2/p_n).$$

*Proof:* Use Lemma 6.4 in [3] with  $x = p_n$ . Bound  $\frac{t}{t-1}x^{1-t}$  above by  $2/x$ . ■

*Lemma 3:* For  $n \geq 2263$ , we have

$$\log p_n < \log \log N_n + \frac{0.1253}{\log p_n}.$$

*Proof:* If  $n \geq 2263$ , then  $p_n \geq 20000$ . By [13], we know then that

$$\log N_n > p_n \left( 1 - \frac{1}{8p_n} \right).$$

On taking log's we obtain

$$\log \log N_n > \log p_n - \frac{0.1253}{p_n},$$

upon using

$$\log \left( 1 - \frac{x}{8} \right) > -0.1253x$$

for  $x$  small enough. In particular  $x < 1/20000$  is enough. ■

We are now ready for the proof of Proposition 4.

*Proof:*

Write

$$\frac{\Psi_t(N_n)}{N_n} = \prod_{k=1}^n \frac{1 - 1/p_k^t}{1 - 1/p_k} = \frac{\prod_{p > p_n} (1 - 1/p^t)^{-1}}{\zeta(t)} \prod_{p \leq p_n} (1 - 1/p)^{-1}$$

and use both Lemmas to derive

$$\frac{\Psi_t(N_n)}{N_n} \leq \frac{\exp(\gamma + 2/p_n)}{\zeta(t)} \left( \log p_n + \frac{1}{\log p_n} \right).$$

Now we get rid of the first log in the RHS by Lemma 3.

The result follows. ■

So, armed with this powerful tool, we derive the following significant Corollaries. For convenience let

$$f(n) = \left(1 + \frac{1.1253}{\log p_n \log \log N_n}\right).$$

*Corollary 1:* Let  $n_0 = 2263$ . Let  $n_1(t)$  denote the least  $n \geq n_0$  such that  $e^{2/p_n} f(n) < \zeta(t)$ . For  $n \geq n_1(t)$  we have  $R_t(N_n) < e^\gamma$ .

*Proof:*

Let  $n \geq n_0$ . We need to check that

$$\exp(2/p_n) \left(1 + \frac{1.1253}{\log p_n \log \log N_n}\right) \leq \zeta(t).$$

which, for fixed  $t$  holds for  $n$  large enough. Indeed  $\zeta(t) > 1$  and the LHS goes monotonically to  $1^+$  for  $n$  large. ■

We give a numerical illustration of Corollary 1 in Table 1.

| $t$ | $n_1(t)$ | $N_{n_1(t)}$            |
|-----|----------|-------------------------|
| 3   | 10       | $6.5 \times 10^9$       |
| 4   | 24       | $2.4 \times 10^{34}$    |
| 5   | 79       | $4.1 \times 10^{163}$   |
| 6   | 509      | $5.8 \times 10^{1551}$  |
| 7   | 10 596   | $2.5 \times 10^{48337}$ |

TABLE I: The numbers in Corollary 1.

We can extend this Corollary to all integers  $\geq n_0$  by using the reduction of preceding section.

*Corollary 2:* For all  $N \geq N_n$  such that  $n \geq n_1(t)$  we have  $R_t(N) < e^\gamma$ .

*Proof:* Combine Corollary 1 with Proposition 2. ■

We are now in a position to derive the main result of this note.

*Theorem 1:* If  $N$  is a 7-free integer, then  $\sigma(N) < Ne^\gamma \log \log N$ .

*Proof:* If  $N$  is  $\geq N_n$  with  $n \geq n_1(7)$ , then the above upper bound holds for  $\Psi_7(N)$  by Corollary 2, hence for  $\sigma(N)$  by the remark in the Introduction. If not, we invoke the results of [2], who checked Robin inequality for  $5040 < N \leq 10^{10^{10}}$ , and observe that all 7-free integers are  $> 5040$ . ■

#### IV. VARYING $t$

We begin with an easy Lemma.

*Lemma 4:* Let  $t$  be a real variable. For  $t$  large, we have  $\zeta(t) = 1 + \frac{1}{2^t} + o(\frac{1}{2^t})$ .

*Proof:* By definition, for  $t > 1$  we may write

$$\zeta(t) = \sum_{n=1}^{\infty} \frac{1}{n^t}$$

so that

$$\zeta(t) \geq 1 + \frac{1}{2^t}.$$

In the other direction, we write

$$\zeta(t) = 1 + \frac{1}{2^t} + \frac{1}{3^t} + \sum_{n=4}^{\infty} \frac{1}{n^t},$$

and compare the remainder of the series expansion of the  $\zeta$  function with an integral:

$$\sum_{n=4}^{\infty} \frac{1}{n^t} < \int_3^{\infty} \frac{du}{u^t} = \frac{3}{(t-1)3^t} = O\left(\frac{1}{3^t}\right).$$

The result follows. ■

We can derive a result when  $t$  grows slowly with  $n$ .

*Theorem 2:* Let  $S_n$  be a sequence of integers such that  $S_n \geq N_n$  for  $n$  large, and such that  $S_n$  is  $t$ -free with  $t = o(\log \log n)$ . For  $n$  large enough, Robin inequality holds for  $S_n$ .

*Proof:* For Corollary 2 to hold we need

$$e^{2/p_n} f(n) < \zeta(t)$$

to hold, or, taking logs, the exact bound

$$2/p_n + \log f(n) < \log \zeta(t),$$

or up to  $o(1)$  terms

$$2/p_n + \frac{1.1253}{\log p_n \log \log N_n} \leq \log \zeta(t).$$

In the LHS, the dominant term is of order  $1/(\log p_n)^2$ , since, like in the proof of Proposition 3, we may write  $p_n \sim \log N_n$ . Now  $p_n \sim n \log n$  by [6, Th. 8], entailing  $\log p_n \sim \log n$  and  $(\log p_n)^2 \sim (\log n)^2$ . In the RHS, with the hypothesis made on  $t$  we have, by Lemma 4, the estimate  $\log \zeta(t) \sim \frac{1}{2t}$ . The result follows after comparing logarithms of both sides. ■

## V. CONCLUSION

In this article we have proposed a technique to check Robin inequality for  $t$ -free integers for some values of  $t$ . The main idea has been to investigate the complex structure of the divisor function  $\sigma$  through the sequence of Dedekind psi functions  $\psi_t$ . The latter are simpler for the following reasons

- $\Psi_t(n)$  solely depends on the prime divisors of  $n$  and not on their multiplicity
- the champions of  $\Psi_t$  are the primorials instead of the colossally abundant numbers
- $\Psi_t$  is easier to bound for  $n$  large because of connections with Eulerian products

Further,  $\sigma(n) \leq \Psi_t(n)$  for  $t$ -free integers  $n$ . We checked Robin inequality for  $t$ -free integers for  $t = 6, 7$  and  $t = o(\log \log n)$ . It is an interesting and difficult open problem to apply Theorem 2 to superabundant numbers or colossally abundant numbers for instance. We do not believe it is possible. New ideas are required to prove Robin inequality in full generality.

## REFERENCES

- [1] Peter Borwein, Stephen Choi, Brendan Rooney and Andrea Weirathmueller *The Riemann hypothesis. A resource for the aficionado and virtuoso alike*. CMS Books in Mathematics. Springer, New York, 2008.
- [2] Keith Briggs, Abundant numbers and the Riemann hypothesis. *Experiment. Math.* 15 (2006), no. 2, 251–256.
- [3] YoungJu Choie, Nicolas Lichiardopol, Pieter Moree, Patrick Solé, On Robin's criterion for the Riemann hypothesis, *J. Théor. Nombres Bordeaux* 19 (2007), no. 2, 357–372.
- [4] J. A. Csirik, M. Zieve and J. Wetherell, On the genera of  $X_0(N)$ , unpublished manuscript (2001); available online at <http://www.csirik.net/papers.html>
- [5] Brian J. Conrey, The Riemann hypothesis. *Notices Amer. Math. Soc.* 50 (2003), no. 3, 341–353.
- [6] G.H. Hardy, E.M. Wright, *An introduction to the theory of numbers*, Oxford (1979).
- [7] Gilles Lachaud, L'hypothèse de Riemann : le Graal des mathématiciens. *La Recherche Hors-Série* no 20, August 2005.
- [8] Jean-Louis Nicolas, Petites valeurs de la fonction d'Euler. *J. Number Theory* 17 (1983), no. 3, 375–388.
- [9] Patrick Solé, Michel Planat, Extreme values of the Dedekind  $\Psi$  function, <http://fr.arxiv.org/abs/1011.1825>
- [10] Michel Planat, Riemann hypothesis from the Dedekind psi function, [hal.archives-ouvertes.fr/docs/00/52/64/54/PDF/RiemannHyp.pdf](http://hal.archives-ouvertes.fr/docs/00/52/64/54/PDF/RiemannHyp.pdf)
- [11] [www.research.att.com/njas/sequences/](http://www.research.att.com/njas/sequences/)
- [12] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *J. Math. Pures Appl.* (9) 63 (1984), 187–213.
- [13] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers. *Illinois J. Math.* 6 (1962), 64–94.