hal-00541563, version 1 - 30 Nov 2010

Abstract settings for stabilization of nonlinear parabolic system with a Riccati-based strategy. Application to Navier-Stokes and Boussinesq equations with Neumann or Dirichlet control

Mehdi Badra[1]

Abstract. Let $-A : \mathcal{D}(A) \to H$ be the generator of an analytic semigroup and $B : U \to [\mathcal{D}(A^*)]'$ a relatively bounded control operator such that $(A-\sigma, B)$ is stabilizable for some $\sigma > 0$. In this paper, we consider the stabilization of the nonlinear system $y' + Ay + G(y,u) = Bu$ by means of a feedback or a dynamical control u. The control is obtained from the solution to a Riccati equation which is related to a low-gain optimal quadratic minimization problem. We provide a general abstract framework to define exponentially stable solutions which is based on the contruction of Lyapunov functions. We apply such a theory to stabilize, around an unstable stationary solution, the 2D or 3D Navier-Stokes equations with a Neumann control and the 2D or 3D Boussinesq equations with a Dirichlet control.

Key words. Feedback stabilization, Riccati equation, Lyapunov function, Navier-Stokes, Boussinesq, Neumann control, Dirichlet control.

AMS subject classifications. 93D15, 93C20, 76D05, 76D55, 35Q30, 35Q35.

1. **Introduction.** The present paper is dedicated to the question of feedback stabilization of a nonlinear controlled system of the form:

$$y' + Ay + G(y,u) = Bu. \tag{1.1}$$

In the above setting, A is a closed linear operator defined on a Hilbert space H, B is a linear and possibly unbounded input operator defined on a Hilbert space of control U and such that $(A-\sigma, B)$ is stabilizable for some $\sigma > 0$, and G is a nonlinear mapping obeying $G'(0) = 0$. It is also assumed that $\widehat{A} \overset{\text{def}}{=} \lambda_0 + A$ has bounded imaginary powers for sufficiently large $\lambda_0 > 0$ and that $-A$ generates an analytic semigroup on H. Here, $t \mapsto y(t)$ is the state trajectory to be stabilized by means of a control function $t \mapsto u(t)$ that we want to express in a feedback form.

More precisely, we want to find a linear mapping $\mathfrak{F} : H \to U$ such that the solution to (1.1) with $u(t) = \mathfrak{F}y(t)$ obeys:

$$\|y(t)\| \leq Ce^{-\sigma t}\|y(0)\|,$$

for some norm $\|\cdot\|$ which has to be determined. Let us recall that a well-known strategy to construct $\mathfrak{F}$ consists in solving an auxiliary optimal quadratic cost problem stated over an infinite time horizon on the linear system:

$$y' + Ay - \sigma y = Bu.$$

The stabilizability of $(A-\sigma, B)$ guarantees the well-posedness of such problem and the resulting optimal control is then given by $u = -B^*\Pi y$ where Π is a linear mapping which is the unique solution to an algebraic Riccati equation, see [24, Chap. 2]. In the present paper we consider a feedback control related to a Riccati operator Π obtained from the minimization of a cost function of the form:

$$\int_0^\infty \|Ry(t)\|_Z^2 dt + \int_0^\infty \|u(t)\|_U^2 dt,$$

where R is a bounded, and boundedly invertible, linear mapping from H into a Hilbert space Z. We also make the additional assumption that R is bounded from $\mathcal{D}(\widehat{A}^{1/2})$ into $\mathcal{D}(\widehat{A}^{*1/2})$ to guarantee that Π maps H onto $\mathcal{D}(A^*)$. It then ensures that Π is the solution to a Riccati equation that can be written in the following strong formulation:

$$A^*\Pi + \Pi A + \Pi BB^*\Pi = R^*R + 2\sigma\Pi.$$

Thus, if we set $F(y) \overset{\text{def}}{=} G(y, -B^*\Pi y)$ then the nonlinear system (1.1) for such a feedback control has the following form:

$$y' + Ay + B(B^*\Pi)y + F(y) = 0. \tag{1.2}$$

Then our goal is to define stable solutions to the above equation and to provide a related Lyapunov function. Of course, some assumptions on F should be made, and these will be induced by the regularity theory of the non homogeneous closed-loop linear system:

$$y' + A_\Pi y = f \quad \text{where} \quad A_\Pi \overset{\text{def}}{=} A + B(B^*\Pi). \tag{1.3}$$

In the first part of the present paper (subsection 2.2), we prove that the natural spaces in which a trajectory $t \mapsto y(t)$ of (1.3) is continuous are $H_\Pi^r \overset{\text{def}}{=} \mathcal{D}(A_\Pi^r)$ and $H_\Pi^{-r} \overset{\text{def}}{=} [\mathcal{D}(A_\Pi^{*r})]'$, $r \geq 0$, and that the regularity theory for the linear closed-loop system can be enounced as follows: if $y(0) \in H_\Pi^r$ and $f \in L^2(H_\Pi^{r-1/2})$ then $t \mapsto y(t) \in H_\Pi^r$ is continuous. Then starting from this observation, we also prove that if F is a continuous mapping from $H_\Pi^{r-1/2}$ into $H_\Pi^{r+1/2}$ which satisfies some related Lipschitz and boundedness assumptions, then for a prescribed initial datum $y(0) \in H_\Pi^r$ in a neighborhood

[1]Laboratoire LMA, UMR CNRS 5142, Université de Pau et des Pays de l'Adour, 64013 Pau Cedex, France (mehdi.badra@univ-pau.fr).

of the origin there exists a unique continuous trajectory $t \mapsto y(t) \in H^r_\Pi$ of system (1.2), see Theorem 4 below. Moreover, we prove that the norm of H^r_Π defined by

$$\|\xi\|_r^2 \stackrel{\text{def}}{=} \langle A_\Pi^{*r+1/2}\,\Pi\, A_\Pi^{r+1/2}\xi|\xi\rangle_{[H^r_\Pi]',H^r_\Pi}$$

is a Lyapunov function of (1.2): for $\|y(0)\|_r$ in a neighborhood of the origin the mapping $t \mapsto \|y(t)\|_r$ is decreasing and

$$\|y(t)\|_r \le \|y(0)\|_r e^{-\sigma t}.$$

A direct consequence of the above result is that, when dealing with a particular controlled PDE system which can be rewritten in the form (1.2), the crucial point is to characterize the corresponding spaces of initial data H^r_Π for which the stabilization result is valid. Indeed, since the closed-loop dynamic is contained in the definition of $\mathcal{D}(A_\Pi)$, the elements of $H^r_\Pi \stackrel{\text{def}}{=} \mathcal{D}(A^r_\Pi)$ may verify a closed-loop compatibility condition for large r and the stabilization result may be irrelevant for such r. For instance, when considering the heat equation or Stokes like systems with Dirichlet feedback boundary control, a trace compatibility condition appears in the definition of H^r_Π when $r \ge 1/4$, see [4]. Then in such situation the relevant space of initial data is H^r_Π for $r < 1/4$. It means that to obtain a satisfactory stabilization result for the nonlinear system (1.2), the nonlinearity should not be "too strong": to define solutions which are continuous in H^r_Π for $r < 1/4$ the nonlinear mapping F should be continuous from $H^{r-1/2}_\Pi$ into $H^{r+1/2}_\Pi$ for $r < 1/4$. That is the reason why Dirichlet boundary feedback control obtained from a low gain Riccati operator fails to stabilize the 3D Navier-Stokes equations, and it explains why other strategies such that time dependent feedback control [28] or dynamical control [3] have been investigated. That is why the end of the first part of the paper is dedicated to a general framework to contruct stabilizing dynamical control obtained from an extended Riccati equation (see Subsection 2.3).

Let us underline that the results which are presented in the first part of the present paper deeply rely on the general theory of optimal quadratic cost problem of [24], and that, according to the terminology of [8], we are in the particular situation of "low-gain" feedback law (R is a bounded operator). Notice also that the last quoted work gives an abstract setting for general nonlinear closed-loop system (not necessarily obtained from a Riccati equation) but without providing a Lyapunov function.

In the second part of the paper (sections 3 and 4), we give two exemples of applications of the above abstract framework. We obtain new stabilization results for the Navier-Stokes equations with Neumann feedback control, and for the Boussinesq equations with Dirichlet feedback or dynamical control, see Theorem 8, Theorem 9 and Theorem 10 below. Unlike the Dirichlet case treated in [4], while considering the Navier-Stokes equation with Neumann control we obtain a 3D feedback stabilization result with no specific restriction on the initial datum. Indeed, since the spaces H^r_Π are closed subspaces of $(H^{2r}(\Omega))^3$ the 3D Navier-Stokes nonlinearity imposes to define a continuous trajectory $t \mapsto y(t) \in H^r_\Pi$ of (1.2) for and index r greater than 1/4, which is precisely the value above which a compatibility trace condition appears in the definition of H^r_Π in the case of Dirichlet control, see [4, Cor. 6 and Rem. 13]. In the case of Neumann control a compatibility trace condition also appears in the definition of H^r_Π but only for $r \ge 3/4$, and it allows to define solutions of the 3D Navier-Stokes equations for an initial datum in $(H^{2r}(\Omega))^3$ for $r \in [1/4, 3/4)$. Notice that analogous comments apply for Boussinesq equations, and that the last section dealing with Dirichlet control extends the results of [4, 3] to the Boussinesq equations. More generally, the abstract framework of the present paper can be applied to many other parabolic system with a nonlinear term of bilinear type, or even of multilinear type, such as power function for instance, see Remark 2 below.

We shall underline that the use of a Riccati based-strategy to stabilize the Navier-Stokes equations around a stationary state has been the object of numerous works. For internal control, let us mention [6, 12] or the recent book [7] for finite or infinite dimensional feedback control obtained from an infinite dimensional Riccati equation. For Dirichlet control, let us mention [27, 28, 4, 9, 8, 10] and [7, Chap 3, par. 4] for finite or infinite dimensional feedback control obtained from an infinite dimensional Riccati equation, [3] for dynamical control or [30, 5] for finite dimensional feedback control obtained from a finite dimensional Riccati equation. About the stabilizability questions related to [9, 8, 10] for tangential control, see [32]. Notice that [5] proposes a general abstract theory as well as a Lyapunov approach adapted to the case of finite dimensional control. Finally, we shall also mention the recent work [11] where an optimal quadratic cost problem is used to stabilize around an instationary state by means of an internal control.

The rest of the paper is organized as follows. Section 2 is dedicated to the construction of a feedback or a dynamical control in a general abstract setting: notations and general definitions are stated in subsection 2.1, subsection 2.2 is devoted to feedback control and subsection 2.3 is devoted to dynamical control. Thus, we apply the abstract framework in the case of Navier-Stokes equations with Neumann control in section 3, and in the case of Boussinesq equations with Dirichlet control in section 4.

2. **Abstract Closed-loop Linear and Nonlinear System.**

2.1. **Technical background and notations.** For a Hilbert space X, we denote by $\|\cdot\|_X$ its norm, we denote by $[X]'$ its dual space and by $\langle\cdot|\cdot\rangle_{[X]',X}$ the $[X]'$-X duality pairing. For two Hilbert spaces X_1 and X_2, we use the notation $X_1 \hookrightarrow X_2$ to say that X_1 is continuously embedded into X_2, for $\alpha \in (0,1)$ we denote by $[X_1, X_2]_\alpha$ the interpolation space obtained from X_1 and X_2 with the complex interpolation method [31, p.55], we denote by $\mathcal{L}(X_1, X_2)$ the space of

hal-00541563, version 1 - 30 Nov 2010

all bounded linear operators from X_1 into X_2 and we use the shorter expression $\mathcal{L}(X) \stackrel{\text{def}}{=} \mathcal{L}(X,X)$. If L is a closed linear mapping in X, we denote its domain by $\mathcal{D}(L)$, and we denote by L^* the adjoint of L.

Let us consider now a closed linear mapping L on X with bounded imaginary powers, and such that $-L$ is the infinitesimal generator of an analytic semigroup on X of negative type [13, Part. II, Chap 1, Par. 2.2. p.91]. Notice that in such case the powers L^r for $r \in \mathbb{C}$ are well-defined, see [31, Par. 1.15 p. 98] or [13, Part. II, Chap 1, Par. 5. p.167] for the particular case $r \in \mathbb{R}$. Thus, we can set $X^r = \mathcal{D}(L^r)$ when $r \geq 0$ and $X^r = [\mathcal{D}(L^{*-r})]'$ when $r < 0$, we recall that X^r can be equipped with:

$$\|\cdot\|_{X^r} \stackrel{\text{def}}{=} \|L^r \cdot\|_X, \tag{2.1}$$

and that we have the following interpolation equalities:

$$[X^{r_1}, X^{r_2}]_{1-\alpha} = X^{(1-\alpha)r_1+\alpha r_2}, \quad \forall \alpha \in (0,1), \quad r_2 < r_1. \tag{2.2}$$

Notice that when $r < 0$, since we have $L^r = (L^{-r})^{-1}$ the norm definition (2.1) can be justified by invoking [33, Chap. 2, Prop. 2.10.2, p. 60]. Moreover, equality (2.2) is obtained as follows. Since L^{-r_2} is an isomorphism from X onto X^{r_2} as well as from $X^{r_1-r_2}$ onto X^{r_2} [31, Thm.1.15.2, p. 101] we first obtain that $[X^{r_1}, X^{r_2}]_{1-\alpha} = L^{-r_2}[X^{r_1-r_2}, X]_{1-\alpha}$. Thus, since the fact that L has bounded imaginary powers implies $[X^{r_1-r_2}, X]_{1-\alpha} = X^{(1-\alpha)(r_1-r_2)}$ [31, Thm.1.15.3, p. 103], then (2.2) follows from $L^{-r_2}X^{(1-\alpha)(r_1-r_2)} = X^{(1-\alpha)r_1+\alpha r_2}$ because L^{-r_2} is an isomorphism from $X^{(1-\alpha)(r_1-r_2)}$ onto $X^{(1-\alpha)r_1+\alpha r_2}$. Next, for $(r, r_1, r_2) \in \mathbb{R}^3$, $r_2 < r_1$ and $0 < T \leq \infty$, we denote by $L^2(0,T;X^r)$ the usual vector-valued Lebesgue space equipped with the norm $\|z\|^2_{L^2(0,T;X^r)} \stackrel{\text{def}}{=} \int_0^T \|L^r z(t)\|^2_X dt$, and we define the space

$$W(0,T;X^{r_1},X^{r_2}) \stackrel{\text{def}}{=} \left\{ z \in L^2(0,T;X^{r_1}) \mid \frac{dz}{dt} \in L^2(0,T;X^{r_2}) \right\},$$

equipped with the norm

$$\|z\|^2_{W(0,T;X^{r_1},X^{r_2})} \stackrel{\text{def}}{=} \int_0^T \left(\|L^{r_1}z(t)\|^2_X + \left\| L^{r_2}\left(\frac{dz}{dt}(t)\right)\right\|^2_X \right) dt.$$

When $T = +\infty$ we use the shorter expressions $L^2(X^r) \stackrel{\text{def}}{=} L^2(0,+\infty;X^r)$ and $W(X^{r_1},X^{r_2}) \stackrel{\text{def}}{=} W(0,+\infty;X^{r_1},X^{r_2})$. Moreover, we denote by $L^\infty(X^r)$ (resp. $C_b(X^r)$) the space of bounded (resp. continuous and bounded) functions of $t \in [0,\infty[$ with values in X^r, we denote by $L^2_{\text{loc}}(X^r)$ the space of functions belonging to $L^2(0,T;X)$ for all $T > 0$, and we define $L^\infty_{\text{loc}}(X^r)$, $W_{\text{loc}}(X^{r_1},X^{r_2})$ analogously. Since it is well-known that $W(X^{r_1},X^{r_2}) \hookrightarrow C_b([X^{r_1},X^{r_2}]_{1/2})$ [31, 1.8 (2), p.44 and Rem.3 p.143] then from (2.2) with $\alpha = 1/2$ we have:

$$W(X^{r_1},X^{r_2}) \hookrightarrow C_b(X^{(r_1+r_2)/2}).$$

Finally, since $-L$ is the infinitesimal generator of an analytic semigroup on X of negative type, the mapping $z \mapsto (z' + Lz, z(0))$ is an isomorphism from $W(\mathcal{D}(L),X)$ onto $L^2(X) \times X^{1/2}$ [13, Part. II, Chap 1, Thm. 3.1. p.143 and Par. 6 eq. (6.4)], and from a change of variable $y = L^{-r}z$ one easily obtain that the mapping

$$y \mapsto (y' + Ly, y(0)) : W(X^{r+1/2},X^{r-1/2}) \to L^2(X^{r-1/2}) \times X^r \quad \text{is an isomorphism for all } r \in \mathbb{R}. \tag{2.3}$$

In the sequel, the letter C denotes a generic positive constant that may change from line to line.

2.2. **Linear and nonlinear systems with feedback control.** Let us now consider a closed linear operator A in a Hilbert space H, with domain $\mathcal{D}(A)$, and such that $\widehat{A} \stackrel{\text{def}}{=} A + \lambda_0$ fits the above framework for some $\lambda_0 > 0$: $-\widehat{A}$ is the infinitesimal generator of a stable analytic semigroup $(e^{-\widehat{A}t})_{t\geq 0}$ on H and $\widehat{A}$ has bounded imaginary powers. Notice that A^* and $\widehat{A}^* \stackrel{\text{def}}{=} A^* + \lambda_0$ also obey those properties. Then by applying the theoretical background of subsection 2.1 for $X := H$, $L := \widehat{A}$ and $L := \widehat{A}^*$ respectively, we have that the spaces

$$H^r \stackrel{\text{def}}{=} \mathcal{D}(\widehat{A}^r) \text{ if } r \geq 0 \quad \text{and} \quad H^r \stackrel{\text{def}}{=} [\mathcal{D}(\widehat{A}^{*-r})]' \text{ if } r < 0,$$

and

$$H^r_* \stackrel{\text{def}}{=} \mathcal{D}(\widehat{A}^{*r}) \text{ if } r \geq 0 \quad \text{and} \quad H^r_* \stackrel{\text{def}}{=} [\mathcal{D}(\widehat{A}^{-r})]' \text{ if } r < 0,$$

obey the following interpolation equalities:

$$[H^{r_1}, H^{r_2}]_{1-\alpha} = H^{(1-\alpha)r_1+\alpha r_2} \quad \text{and} \quad [H^{r_1}_*, H^{r_2}_*]_{1-\alpha} = H_*^{(1-\alpha)r_1+\alpha r_2}, \quad \forall \alpha \in (0,1), \quad r_2 < r_1.$$

Moreover, since $(e^{-\widehat{A}t})_{t\geq 0}$ is analytic on H, then obviously, $(e^{-At})_{t\geq 0}$ is also analytic on H, but not necessarily stable: the solutions to the dynamical system

$$y' + Ay = 0 \tag{2.4}$$

do not necessarily verify

$$\lim_{t\to\infty} y(t) = 0. \tag{2.5}$$

In order to obtain the above limit one choose to act through a control function $t \mapsto u(t)$ in the following way:

$$y' + Ay = Bu, \tag{2.6}$$

hal-00541563, version 1 - 30 Nov 2010

where B is a linear input operator defined on a Hilbert space of control U and with values in H^{-1}. In the whole following we assume that B is stictly relatively bounded with respect to A:

$$\widehat{A}^{-\gamma}B \in \mathcal{L}(U,H) \quad \text{for} \quad 0 \le \gamma < 1, \tag{2.7}$$

and that there is $\sigma > 0$ such that

$$(A-\sigma, B) \text{ is stabilizable.} \tag{2.8}$$

Thus, in order to construct a control in a feedback form $u(t) = \mathfrak{F}y(t)$, for $\mathfrak{F} \in \mathcal{L}(H,U)$, ensuring the exponential decrease:

$$\|y(t)\|_H \le Ce^{-\sigma t}\|y(0)\|_H \quad t \ge 0,$$

let us introduce an auxiliary optimal control problem. For an initial datum $\xi \in H$ and a control function $u \in L^2(U)$ we consider the solution $y \in W_{\mathrm{loc}}(H,H^{-1})$ to the following linear controlled system:

$$y' + (A-\sigma)y = Bu \ \in H^{-1}, \quad y(0) = \xi \in H. \tag{2.9}$$

We are then interested in the following minimization problem:

$$\inf \left\{ \mathcal{J}(y,u) \ \mid \ (y,u) \in W_{\mathrm{loc}}(H,H^{-1}) \times L^2(U) \ \text{ satisfies } \ (2.9) \right\}, \tag{2.10}$$

where the cost functional $\mathcal{J}$ is defined by

$$\mathcal{J}(y,u) \overset{\text{def}}{=} \int_0^\infty \|Ry(t)\|_Z^2 dt + \int_0^\infty \|u(t)\|_U^2 dt. \tag{2.11}$$

In the above setting, Z is a Hilbert space and $R \in \mathcal{L}(H,Z)$ is boundedly invertible (i.e. $R^{-1} \in \mathcal{L}(Z,H)$). Notice that the stabilizability of $(A-\sigma, B)$ guarantees the well-posedness of (2.10). Indeed, it implies the following condition:

$$\left\{ \begin{array}{l} \text{for all } \xi \in H \text{ there exists } u \in L^2(U) \\ \text{such that the corresponding solution} \\ y \in W_{\mathrm{loc}}(H,H^{-1}) \text{ to (2.9) belongs to } L^2(H) \end{array} \right. \tag{2.12}$$

which guarantees that for all $\xi \in H$ there is a pair (y,u) for which $\mathcal{J}(y,u)$ is finite, i.e. the set that we are looking the infimum (2.10) is not empty. According to [24, Chap. 2 Thm. 2.2.1] the solution of (2.10) is characterized as follows.

Theorem 1. *If* (2.12) *is satisfied then for all $\xi \in H$ problem (2.10) admits a unique solution (y_ξ, u_ξ). The optimal control obeys $u_\xi = -B^*\Phi_\xi$, where $(y_\xi, \Phi_\xi) \in W(H,H^{-1}) \times W(H^1_*, H)$ is the unique solution to:*

$$(\mathcal{S}_\xi) \left\{ \begin{array}{rll} y' + (A-\sigma)y & = -BB^*\Phi, & y(0) = \xi \in H, \\ -\Phi' + (A^*-\sigma)\Phi & = R^*Ry, & \Phi(\infty) = 0, \\ \Phi(t) & = \Pi\, y(t) & \forall t \ge 0, \end{array} \right.$$

where Π is the unique nonnegative and self-adjoint operator of $\mathcal{L}(H)$, which belongs to $\mathcal{L}(H, H^{1-\epsilon}_*)$ for all $\epsilon > 0$, solution to the following Riccati equation:

$$(\Pi\xi|A\zeta)_H + (A\xi|\Pi\zeta)_H + (B^*\Pi\xi|B^*\Pi\zeta)_U = (R\xi|R\zeta)_H + 2\sigma(\Pi\xi|\zeta)_H \quad \forall(\xi,\zeta) \in H^1 \times H^1. \tag{2.13}$$

Moreover, we also have $(y_\xi, u_\xi) \in C_b(H) \times C_b(U)$ and the estimate

$$\|y_\xi(t)\|_H + \|u_\xi(t)\|_U \le C\|\xi\|_H, \quad \forall t \ge 0, \tag{2.14}$$

and Π satisfies:

$$(\Pi\xi|\xi) = \mathcal{J}(y_\xi, u_\xi) = \inf \left\{ \mathcal{J}(y,u) \ \mid \ (y,u) \ \text{ satisfies } \ (2.9) \right\}. \tag{2.15}$$

Remark 1. *Since* (2.7) *implies $B \in \mathcal{L}(U, H^{-\gamma})$ and then $B^* \in \mathcal{L}(H^\gamma_*, U)$, a first immediate consequence of the above theorem with $\epsilon = 1-\gamma$ is that $B^*\Pi$ is bounded from H into U (which hopefully guarantees the well-posedness of the nonlinear term in* (2.13)*). Then it ensures that the linear map $A + B(B^*\Pi)$ is well defined as a bounded linear operator from H into H^{-1}:*

$$\langle A\xi + B(B^*\Pi)\xi|\zeta\rangle_{H^{-1},H^1_*} = (\xi|A^*\zeta + (B^*\Pi)^*B^*\zeta)_H \quad \forall(\xi,\zeta) \in H \times H^1_*.$$

Next, if we make the following additional assumption:

$$R^*R \in \mathcal{L}(H^{1/2}, H^{1/2}_*), \tag{2.16}$$

then arguing as in [9, Appendix B, Prop. B.4.1] we prove that Theorem 1 is also true with $\epsilon = 0$.

Theorem 2. *If* (2.16) *is satisfied then the solution Π of* (2.13) *belongs to $\mathcal{L}(H, H^1_*)$ and satisfies:*

$$(A^*\Pi\xi|\zeta)_H + (\xi|A^*\Pi\zeta)_H + (B^*\Pi\xi|B^*\Pi\zeta)_U = (R\xi|R\zeta)_H + 2\sigma(\Pi\xi|\zeta)_H \quad \forall(\xi,\zeta) \in H \times H. \tag{2.17}$$

hal-00541563, version 1 - 30 Nov 2010

Proof. Since (2.17) is an immediate consequence of (2.13) with $\Pi \in \mathcal{L}(H, H^1_*)$, let us focus on the proof of this last statement. First, for $\xi \in H$ we set $\widehat{u}_\xi = e^{-(\lambda_0+\sigma)(\cdot)}u_\xi$, $\widehat{y}_\xi = e^{-(\lambda_0+\sigma)(\cdot)}y_\xi$ and $\widehat{\Phi}_\xi = e^{-(\lambda_0+\sigma)(\cdot)}\Phi_\xi$ and we verify that

$$\widehat{\Phi}_\xi(t) = \int_t^\infty e^{-\widehat{A}^*(\tau-t)}(2(\lambda_0+\sigma)\Pi + R^*R)\widehat{y}_\xi(\tau)d\tau \quad \text{and} \quad \widehat{y}_\xi(t) = e^{-\widehat{A}t}\xi + \int_0^t e^{-\widehat{A}(t-\tau)}B\widehat{u}_\xi(\tau)d\tau.$$

Thus, by substituing the above expression of $\widehat{y}_\xi$ in the first above equality we obtain:

$$\Pi\xi = \widehat{\Phi}_\xi(0) = I_1\xi + I_2\xi + I_3\xi,$$

where

$$I_1\xi = \int_0^\infty e^{-\widehat{A}^*t}R^*Re^{-\widehat{A}t}\xi dt, \quad I_2\xi = 2(\lambda_0+\sigma)\int_0^\infty e^{-\widehat{A}^*t}\Pi e^{-\widehat{A}t}\xi dt$$

and

$$I_3\xi = \int_0^\infty e^{-\widehat{A}^*t}(2(\lambda_0+\sigma)\Pi + R^*R)\mathcal{L}(\widehat{u}_\xi)(t)dt \quad \text{where} \quad \mathcal{L}(\widehat{u}_\xi)(t) = \int_0^t e^{-\widehat{A}(t-\tau)}B\widehat{u}_\xi(\tau)d\tau.$$

To prove $\Pi \in \mathcal{L}(H, H^1_*)$, let us show $\|\widehat{A}^*I_i\xi\|_H \le C\|\xi\|_H$, $i = 1, 2, 3$. First, an obvious calculation give

$$(\widehat{A}^*I_1\xi|\zeta)_H = \int_0^\infty([\widehat{A}^{*1/2}R^*R\widehat{A}^{-1/2}]\widehat{A}^{1/2}e^{\widehat{A}t}\xi|\widehat{A}^{1/2}e^{\widehat{A}t}\zeta)_H dt$$

and since from (2.16) we have $[\widehat{A}^{*1/2}R^*R\widehat{A}^{-1/2}] \in \mathcal{L}(H)$, the continuity of $\xi \in H \mapsto \widehat{A}^{1/2}e^{\widehat{A}t}\xi \in L^2(H)$ (obtained from (2.3) with $r = 0$) combined with Cauchy-Schwarz inequality yields $\|\widehat{A}^*I_1\xi\|_H \le C\|\xi\|_H$. Moreover, estimate $\|\widehat{A}^*I_2\xi\|_H \le C\|\xi\|_H$ follows analogously from $[\widehat{A}^{*1/2}\Pi\widehat{A}^{-1/2}] \in \mathcal{L}(H)$ which is a consequence of $\Pi \in \mathcal{L}(H, H^{1/2}_*)$. Next, for $0 < \eta < \min(1-\gamma, 1/2)$, $R^*R \in \mathcal{L}(H^{1/2}, H^{1/2}_*) \cap \mathcal{L}(H)$ with and interpolation argument gives $R^*R \in \mathcal{L}(H^\eta, H^\eta_*)$ and since we also have $\Pi \in \mathcal{L}(H, H^\eta_*)$ we deduce that $\widehat{A}^{*\eta}(2(\lambda_0+\sigma)\Pi + R^*R)\widehat{A}^{-\eta} \in \mathcal{L}(H)$. Thus, by writing

$$\widehat{A}^{*\eta}(2(\lambda_0+\sigma)\Pi + R^*R)\mathcal{L}(\widehat{u}_\xi)(t) = [\widehat{A}^{*\eta}(2(\lambda_0+\sigma)\Pi + R^*R)\widehat{A}^{-\eta}]\int_0^t \widehat{A}^{\gamma+\eta}e^{-\widehat{A}(t-\tau)}\widehat{A}^{-\gamma}B\widehat{u}_\xi(\tau)d\tau,$$

then the Young inequality combined with the analyticity estimate $\|A^{\gamma+\eta}e^{-\widehat{A}(t-\tau)}\|_H \le C(t-\tau)^{-\gamma-\eta}$, $\widehat{A}^{-\gamma}B \in \mathcal{L}(U, H)$ and the bound of $\|u_\xi(t)\|_U$ given in (2.14), ensures that $\xi \in H \mapsto \widehat{A}^\eta(2(\lambda_0+\sigma)\Pi + R^*R)\mathcal{L}(\widehat{u}_\xi) \in C_b(H)$ is continuous. Finally, by writing

$$\widehat{A}^*I_3\xi = \int_0^\infty \widehat{A}^{*1-\eta}e^{-\widehat{A}^*t}\widehat{A}^{*\eta}(2(\lambda_0+\sigma)\Pi + R^*R)\mathcal{L}(\widehat{u}_\xi)(t)dt$$

Young inequality with the analyticity estimate $\|\widehat{A}^{*1-\eta}e^{-\widehat{A}^*t}\|_H \le Ct^{\eta-1}$ yields $\|\widehat{A}^*I_3\xi\|_H \le C\|\xi\|_H$. □

Notice that if (2.16) is true then the above theorem ensures that $A^*\Pi \in \mathcal{L}(H)$. Moreover, the self-adjointness of Π combined with $\Pi \in \mathcal{L}(H, H^1_*)$ and a duality argument guarantee $\Pi \in \mathcal{L}(H^{-1}, H)$. Then ΠA and ΠB also belong to $\mathcal{L}(H)$ and (2.17) can be rewritten as an equation stated in $\mathcal{L}(H)$.

Corollary 1. *If (2.16) is satisfied then the solution Π of (2.13) belongs to $\mathcal{L}(H, H^1_*) \cap \mathcal{L}(H^{-1}, H)$ and satisfies:*

$$A^*\Pi + \Pi A + \Pi BB^*\Pi = R^*R + 2\sigma\Pi. \tag{2.18}$$

In the sequel we will assume that (2.16) is satisfied. Next, according to Remark 1 we are allowed to introduce the linear operator $(\mathcal{D}(A_\Pi), A_\Pi))$ as follows:

$$\mathcal{D}(A_\Pi) = \{\xi \in H \mid A\xi + B(B^*\Pi)\xi \in H\}, \tag{2.19}$$
$$A_\Pi\xi = A\xi + B(B^*\Pi)\xi. \tag{2.20}$$

Let us state the first main theorem of this subsection.

Theorem 3. *The following results hold.*

1. *The adjoint of $(\mathcal{D}(A_\Pi), A_\Pi)$ is given by*

$$\mathcal{D}(A^*_\Pi) = H^1_* \quad \text{and} \quad A^*_\Pi = A^* + (B^*\Pi)^*B^*, \tag{2.21}$$

   and the following equality holds:

$$\mathcal{D}(A^{*r}_\Pi) = H^r_* \quad \forall r \in [0,1]. \tag{2.22}$$

2. *$(\mathcal{D}(A_\Pi), -A_\Pi)$ is the infinitesimal generator of an analytic and exponentially stable semigroup on H with an exponential rate of decrease greater than $\sigma > 0$: there exists $\epsilon > 0$ such that $\|e^{-A_\Pi t}\|_{\mathcal{L}(H)} \le Ce^{-(\sigma+\epsilon)t}$.*
3. *For $\xi \in H$ the optimal trajectory y_ξ satisfies $y_\xi(t) = e^{(\sigma-A_\Pi)t}\xi$ for all $t \ge 0$.*

hal-00541563, version 1 - 30 Nov 2010

4. *The spaces defined by*

$$H^r_\Pi \overset{def}{=} \mathcal{D}(A^r_\Pi) \ \text{ if } r \geq 0 \qquad \text{and} \qquad H^r_\Pi \overset{def}{=} [\mathcal{D}(\widehat{A}^{*-r}_\Pi)]' \ \text{ if } \ r < 0,$$

obey the interpolation equalities:

$$[H^{r_1}_\Pi, H^{r_2}_\Pi]_{1-\alpha} = H^{(1-\alpha)r_1+\alpha r_2}_\Pi \quad \forall \alpha \in (0,1), \quad r_2 < r_1. \tag{2.23}$$

Moreover, we have:

$$H^r_\Pi = H^r \quad \forall r \in [-1, 0]. \tag{2.24}$$

5. *The operator A_Π satisfies $\Pi A_\Pi \in \mathcal{L}(H)$ and $A^*_\Pi \Pi \in \mathcal{L}(H)$ and*

$$A^*_\Pi \Pi + \Pi A_\Pi = 2\sigma \Pi + R^* R + \Pi B B^* \Pi. \tag{2.25}$$

Proof. First, (2.21) follows by noticing that A_Π is exactly defined as the adjoint of $(\mathcal{D}(A^*), A^* + (B^*\Pi)^* B^*)$. Thus, since $(B^*\Pi)^* B^*$ belongs to $\mathcal{L}(H^\gamma_*, H)$, a perturbation argument ensures that $(\mathcal{D}(A^*), A^* + (B^*\Pi)^* B^*)$ is the infinitesimal generator of an analytic semigroup on H [26, Chap.3, Cor.2.4], and the analyticity of A_Π follows from a duality argument. As a consequence, for $\xi \in H$, the trajectory $t \mapsto y(t) = e^{(\sigma - A_\Pi)t}\xi$ is the (unique) weak solution of

$$y' = (\sigma - A_\Pi) y \in [\mathcal{D}(A^*_\Pi)]' = H^{-1}, \quad y(0) = \xi. \tag{2.26}$$

Moreover, since $B(B^*\Pi)$ is well-defined as a bounded operator from H into H^{-1} and since $y_\xi \in W(H, H^{-1})$, we deduce that $B(B^*\Pi)y_\xi \in L^2(H^{-1})$ and we are allowed to replace Φ_ξ by Πy_ξ in the first equality of $(\mathcal{S}_\xi)$: y_ξ is solution of

$$y'_\xi = \sigma y_\xi - A y_\xi - B(B^*\Pi) y_\xi \in H^{-1}, \quad y_\xi(0) = \xi,$$

which exactly means that y_ξ is solution of (2.26), and then that $y_\xi(t) = e^{(\sigma - A_\Pi)t}\xi$ for all $t \geq 0$. As a consequence, since by Theorem 1 we know that $e^{(\sigma - A_\Pi)(\cdot)}\xi$ belongs to $L^2(H)$ for all $\xi \in H$, the exponential stability of $(e^{(\sigma - A_\Pi)t})_{t\geq 0}$ follows from a well-known result due to Datko (see [26, Chap. 4, Th. 4.1] or [13, Th. 2.2 p. 93]) and then $\|e^{-A_\Pi t}\|_{\mathcal{L}(H)} \leq C e^{-(\sigma+\epsilon)t}$ for some $\epsilon > 0$. Moreover, since $-A_\Pi$ generates an exponentially stable semigroup on H, it is boundedly invertible [13, Part. II, Chap. 1, Prop. 2.9, p. 120], the fractional powers of A_Π are well-defined [13, Part II, Chap. 1, Par. 5 p. 167] and the fact that $\widehat{A}$ has bounded imaginary powers combined with a perturbation argument [16, Prop. 2.7] ensures that A_Π (and A^*_Π also) has bounded imaginary powers which ensures (2.23). Next, since (2.24) is a direct consequence of (2.22), it remains to shows (2.22). Since from (2.21) we have $\mathcal{D}(A^*_\Pi) = \mathcal{D}(\widehat{A}^*)$, the fact that A^*_Π has bounded imaginary powers implies $\mathcal{D}(A^{r*}_\Pi) = [H^1_*, H]_r = H^r_*$. Finally, from $\mathcal{D}(A^*_\Pi) = H^1_*$ and $\Pi \in \mathcal{L}(H, H^1_*) \cap \mathcal{L}(H^{-1}, H)$ we deduce that $\Pi A_\Pi \in \mathcal{L}(H)$ and $A^*_\Pi \Pi \in \mathcal{L}(H)$, and (2.25) follows from (2.18). □

As a first consequence of Theorem 3 we have that (2.3) with $L := A_\Pi$ is true, and then the following regularity result for system (1.3) holds.

Corollary 2. *Let $r \in \mathbb{R}$, $y_0 \in H^r_\Pi$ and $f \in L^2(H^{r-1/2}_\Pi)$. The solution to*

$$y' + A_\Pi y = f \qquad \text{and} \qquad y(0) = y_0, \tag{2.27}$$

belongs to $W(H^{r+1/2}_\Pi, H^{r-1/2}_\Pi)$ and obeys the following estimate:

$$\|y\|_{W(H^{r+1/2}_\Pi, H^{r-1/2}_\Pi)} \leq C(\|f\|_{L^2(H^{r-1/2}_\Pi)} + \|y_0\|_{H^r_\Pi}). \tag{2.28}$$

The second consequence of Theorems 2 and 3 is the possibility to construct a Lyapunov function for system (1.3) with $f = 0$. It is the subject of the following corollaries.

Corollary 3. *The linear operator Π obeys:*

$$A^{*r+1/2}_\Pi \, \Pi \, A^{r+1/2}_\Pi \in \mathcal{L}(H^r_\Pi, [H^r_\Pi]') \quad \forall r \in [0, 1]. \tag{2.29}$$

Proof. Since from Corollary 1 we know that $\Pi \in \mathcal{L}(H, H^1_*) \cap \mathcal{L}(H^{-1}, H)$, the fact that $\Pi \in \mathcal{L}(H^{-1/2}, H^{1/2}_*)$ follows by interpolation. Moreover, from (2.24) and (2.22) with $r = 1/2$ we deduce that $A^{*1/2}_\Pi \in \mathcal{L}(H^{1/2}_*, H)$ and $A^{1/2}_\Pi \in \mathcal{L}(H, H^{-1/2})$. Then we have $A^{*1/2}_\Pi \, \Pi \, A^{1/2}_\Pi \in \mathcal{L}(H)$ from which (8) is a direct consequence. □

Corollary 4. *For $r \in \mathbb{R}$, the linear operator*

$$\Pi_r \overset{def}{=} A^{*r+1/2}_\Pi \, \Pi \, A^{r+1/2}_\Pi$$

is bounded from H^r_Π onto $[H^r_\Pi]'$ and the bilinear form

$$(\xi|\zeta)_r \overset{def}{=} \langle \Pi_r \xi | \zeta \rangle_{[H^r_\Pi]', H^r_\Pi} \qquad \text{for all } \ (\xi, \zeta) \in H^r_\Pi \times H^r_\Pi, \tag{2.30}$$

defines an inner-product in H^r_Π: the norm defined by

$$\|\xi\|_r \overset{def}{=} \sqrt{(\xi|\xi)_r} \tag{2.31}$$

is equivalent to $\|\cdot\|_{H^r_\Pi}$.

hal-00541563, version 1 - 30 Nov 2010

Proof. The first part of the corollary is a direct consequence of (8), and since $\|\xi\|_r = \|A_\Pi^r \xi\|_0$ for all $\xi \in H_\Pi^r$, it suffices to prove the second part of the corollary for $r = 0$. First, $\|\cdot\|_0 \le C\|\cdot\|_H$ is a straightforward consequence of $\Pi^{(0)} \in \mathcal{L}(H)$. To prove the converse inequality, let us first pick $\xi \in H$ and set $\zeta = A_\Pi^{1/2}\xi \in H^{-1/2}$ and from (2.3) we have

$$\|\xi\|_H = \|A_\Pi^{-1/2}\zeta\|_H \le C\|e^{-A_\Pi t}\zeta\|_{W(H,H_\Pi^{-1})} = C(\|e^{-A_\Pi t}\zeta\|_{L^2(H)} + \left\|A_\Pi^{-1}\left(\frac{d}{dt}e^{-A_\Pi t}\zeta\right)\right\|_{L^2(H)}),$$

and $\frac{d}{dt}e^{-A_\Pi t}\zeta = -A_\Pi e^{-A_\Pi t}\zeta$ yields

$$\|\xi\|_H \le C\|e^{-A_\Pi t}\zeta\|_{L^2(H)} \le C\|R^{-1}\|_{\mathcal{L}(Z,X)}\|Re^{(\sigma - A_\Pi)t}\zeta\|_{L^2(H)}.$$

Finally, we conclude by observing that:

$$\|Re^{(\sigma-A_\Pi)t}\zeta\|^2_{L^2(H)} + \|(B^*\Pi)e^{(\sigma-A_\Pi)t}\zeta\|^2_{L^2(H)} = (\Pi\zeta|\zeta) = (\Pi_0\xi|\xi) = \|\xi\|_0^2.$$

□

Corollary 5. *The mappings*

$$\lceil|\xi|\rceil_{r+1/2} \overset{def}{=} \frac{1}{\sqrt{2}}\left(\|RA_\Pi^{r+1/2}\xi\|_H^2 + \|(B^*\Pi)A_\Pi^{r+1/2}\xi\|_U^2\right)^{1/2} \quad \text{and} \quad \lfloor|\xi|\rfloor_{r-1/2} \overset{def}{=} \sup_{\zeta \in H_\Pi^{r+1/2}} \frac{(\xi|\zeta)_r}{\lceil|\zeta|\rceil_{r+1/2}}, \tag{2.32}$$

define equivalent norms in $H_\Pi^{r+1/2}$ and in $H_\Pi^{r-1/2}$ respectively. Moreover, we have

$$(A_\Pi\xi|\xi)_r = \sigma\|\xi\|_r^2 + \lceil|\xi|\rceil_{r+1/2}^2 \tag{2.33}$$

Proof. First, from (2.25) we deduce that for all $\xi \in H$:

$$(\Pi A_\Pi \xi|\xi)_H = \sigma(\xi|\Pi\xi)_H + \frac{1}{2}\|R\xi\|_H^2 + \frac{1}{2}\|(B^*\Pi)\xi\|_U^2$$

and by replacing ξ by $A_\Pi^{r+1/2}\xi$ for $\xi \in H_\Pi^{r+1/2}$ in the above equation we obtain:

$$(A_\Pi\xi|\xi)_r = \sigma(\Pi_r\xi|\xi)_H + \frac{1}{2}\|RA_\Pi^{r+1/2}\xi\|_H^2 + \frac{1}{2}\|(B^*\Pi)A_\Pi^{r+1/2}\xi\|_U^2 \tag{2.34}$$

Then (2.33) is proved, and from $B^*\Pi \in \mathcal{L}(H,U)$ and the fact that $R : H \to Z$ is an isomorphism we deduce that that $\lceil|\cdot|\rceil_{r+1/2} \sim \|\cdot\|_{H_\Pi^{r+1/2}}$. Next, from $\Pi A_\Pi \in \mathcal{L}(H)$ and $\|A_\Pi^{r+1/2}\cdot\|_H \sim \lceil|\cdot|\rceil_{r+1/2}$ we deduce that

$$(\xi|\zeta)_r = (\Pi A_\Pi A_\Pi^{r-1/2}\xi|A_\Pi^{r+1/2}\zeta) \le C\|\xi\|_{H_\Pi^{r-1/2}}\lceil|\zeta|\rceil_{r+1/2}, \quad \forall(\xi,\zeta) \in H_\Pi^{r-1/2} \times H_\Pi^{r+1/2},$$

which first gives $\lfloor|\cdot|\rfloor_{r-1/2} \le C\|\cdot\|_{H_\Pi^{r-1/2}}$. To obtain the converse inequality we start by noticing that $\lceil|\cdot|\rceil_{r+1/2} \sim \|A_\Pi^{r+1/2}\cdot\|_H$ implies $\lceil|A_\Pi^{-1}\cdot|\rceil_{r+1/2} \sim \|A_\Pi^{r-1/2}\cdot\|_H$, and by replacing ξ in (2.34) by $A_\Pi^{-1}\xi$ for $\xi \in H_\Pi^{r-1/2}$ we obtain:

$$(\xi|A_\Pi^{-1}\xi)_r \ge \frac{1}{2}\|RA_\Pi^{r-1/2}\xi\|_H^2 \ge C\|A_\Pi^{r-1/2}\xi\|_H\lceil|A_\Pi^{-1}\xi|\rceil_{r+1/2}$$

and we conclude with:

$$\lfloor|\xi|\rfloor_{r-1/2} \ge \frac{(\xi|A_\Pi^{-1}\xi)_r}{\lceil|A_\Pi^{-1}\xi|\rceil_{r+1/2}} \ge C\|A_\Pi^{r-1/2}\xi\|_H.$$

□

Corollary 6. *For all $r \in \mathbb{R}$ and $\xi \in H_\Pi^r$, the mapping $t \longmapsto \|e^{-A_\Pi t}\xi\|_r$ decreases to 0 and obeys:*

$$\|e^{-A_\Pi t}\xi\|_r \le e^{-(\sigma+\beta_r)t}\|\xi\|_r \quad \text{where} \quad \beta_r = \inf_{0 \ne \xi \in H_\Pi^{r+1/2}} \frac{\lceil|\xi|\rceil_{r+1/2}^2}{\|\xi\|_r^2},$$

Proof. From $(y' + A_\Pi y|y)_r = 0$ we deduce that $\frac{d}{dt}\|y(t)\|_r^2 + 2\sigma\|y(t)\|_r^2 + 2\lceil|y(t)|\rceil_{r+1/2}^2 = 0$, and the concusion follows from $\beta_r\|\cdot\|_r^2 \le \lceil|\cdot|\rceil_{r+1/2}^2$. □

Finally, let us give an abstract characterization of the spaces H_Π^r in the case where $r \in [0,1]$.

Proposition 1. *For all $r \in [0,1]$ the linear operator $I + \widehat{A}^{-1}B(B^*\Pi)$ is an isomorphism from H_Π^r onto H^r and the following characterization hold:*

$$H_\Pi^r = \{\xi \in H \mid \xi + \widehat{A}^{-1}B(B^*\Pi)\xi \in H^r\}, \quad r \in [0,1]. \tag{2.35}$$

hal-00541563, version 1 - 30 Nov 2010

Proof. Let us set $T \stackrel{\rm def}{=} I + \widehat{A}^{-1}B(B^*\Pi)$ for readability convenience. First, from (2.19) we deduce that:

$$H_\Pi^1 = \{\xi \in H \mid \xi + \widehat{A}^{-1}B(B^*\Pi)\xi \in H^1\} = \{\xi \in H \mid T\xi \in H^1\}, \tag{2.36}$$

which also means that $T \in \mathcal{L}(H_\Pi^1, H^1)$. Then since T also belongs to $\mathcal{L}(H)$ and interpolation argument yields $T \in \mathcal{L}(H_\Pi^r, H^r)$ for $r \in [0,1]$. Next, to prove that T is injective in H we suppose that $\xi \in H$ obeys the equality $T\xi = \xi + \widehat{A}^{-1}B(B^*\Pi)\xi = 0$, and by multiplying by $\widehat{A}^*\Pi\xi$ and using (2.17) applied to (ξ,ξ) we obtain:

$$(\lambda_0 + \sigma)(\xi|\Pi\xi)_H + \frac{1}{2}\|R\xi\|_H^2 + \frac{1}{2}\|B^*\Pi\xi\|_U^2 = 0,$$

which ensures that $\xi = 0$. Thus, to prove that T is surjective, it suffices to remark that $T\xi = f \in H$ is equivalent to $\widehat{A}T\xi = \widehat{A}f \in H^{-1}$ or to $A_\Pi\xi = \widehat{A}f \in H^{-1}$. Then for $f \in H$ the element $\xi = A_\Pi^{-1}\widehat{A}f \in H$ obeys $T\xi = f \in H$. Then we have proved that T is an isomorphism from H onto H. Finally, since (2.36) exactly means that T^{-1} maps H^1 to H_Π^1, we obtain $T^{-1} \in \mathcal{L}(H^r, H_\Pi^r)$ by interpolation. □

We are now in position to state the existence and uniqueness of a stable solution to the following nonlinear system:

$$y' + A_\Pi y + F(y) = 0, \quad y(0) = y_0 \in H_\Pi^r, \tag{2.37}$$

where the nonlinear mapping $F(\cdot)$ satisfies the following assumptions.

$$\|F(\xi)\|_{H_\Pi^{r-1/2}} \leq C\|\xi\|_{H_\Pi^r}\|\xi\|_{H_\Pi^{r+1/2}}, \tag{2.38}$$

$$\|F(\xi) - F(\zeta)\|_{H_\Pi^{r-1/2}} \leq C(\|\xi-\zeta\|_{H_\Pi^r}\|\xi\|_{H_\Pi^{r+1/2}} + \|\zeta\|_{H_\Pi^r}\|\xi-\zeta\|_{H_\Pi^{r+1/2}}). \tag{2.39}$$

Theorem 4. *Assume* (2.38)-(2.39) *and* $y_0 \in H_\Pi^r$ *for* $r \in \mathbb{R}$. *There exist* $\rho > 0$ *and* $\mu > 0$ *such that, if* $\delta \in (0,\mu)$ *and* $\|y_0\|_r < \delta$, *then system (2.37) admits a solution* $y_{y_0} \in W(H_\Pi^{r+1/2}, H_\Pi^{r-1/2})$ *such that* $\|y_{y_0}\|_{W(H_\Pi^{r+1/2},H_\Pi^{r-1/2})} \leq \rho\delta$, *which is unique within the class of functions in* $L^\infty_{loc}(H_\Pi^r) \cap L^2_{loc}(H_\Pi^{r+1/2})$. *Moreover, every solution with an initial datum obeying:*

$$\|y_0\|_r < D_r \quad \text{where} \quad \frac{1}{D_r} = \sup_{0\neq\xi\in H_\Pi^{r+1/2}} \frac{\lfloor|F(\xi)|\rfloor_{r-1/2}}{\|\xi\|_r\lceil|\xi|\rceil_{r+1/2}},$$

is such that $t \longmapsto \|y_{y_0}(t)\|_r$ *is decreasing and we have:*

$$\|y_{y_0}(t)\|_r \leq \|y_0\|_r e^{-\sigma t - \beta_r(1-\|y_0\|_r/D_r)t}, \tag{2.40}$$

$$\int_0^\infty e^{\sigma t}\lceil|y_{y_0}(t)|\rceil_{r+1/2}^2 dt \leq \frac{D_r\|y_0\|_r^2}{2(D_r - \|y_0\|_r)}. \tag{2.41}$$

Proof. Let us use the notation $W_r \stackrel{\rm def}{=} W(H_\Pi^{r+1/2}, H_\Pi^{r-1/2})$ for readability convenience. In a first step, let us suppose that $\|y_0\|_r < D_r$ and that $y \in L^\infty_{\rm loc}(H_\Pi^r) \cap L^2_{\rm loc}(H_\Pi^{r+1/2})$ is a solution of (2.37) and let us prove that $y \in W_r$ as well as estimates (2.40) and (2.41). Since (2.38) ensures that $F(y) \in L^2_{\rm loc}(H_\Pi^{r-1/2})$, from (2.37) we obtain $y \in W_{\rm loc}(H_\Pi^{r+1/2}, H_\Pi^{r-1/2})$, and by $(\cdot|\cdot)_r$-multiplying the first equality in (2.37) by $y(t)$ and we obtain:

$$\frac{d}{dt}\|y(t)\|_r^2 + 2\sigma\|y(t)\|_r^2 + 2(1 - \|y(t)\|_r/D_r)\lceil|y(t)|\rceil_{r+1/2}^2 \leq 0.$$

Thus, because $\|y_0\|_r < D_r$, the mapping $t \longmapsto \|y(t)\|_r$ is a nonincreasing function lower than D_r and:

$$\frac{d}{dt}\|y(t)\|_r^2 + 2\sigma\|y(t)\|_r^2 + 2(1 - \|y_0\|_r/R_s)\lceil|y(t)|\rceil_{r+1/2}^2 \leq 0.$$

Then (2.40) follows from $\beta_r\|y(t)\|_r^2 \leq \lceil|y(t)|\rceil_{r+1/2}^2$, and multiplying the above equation by $e^{2\sigma t}$ and integrating over $(0,\infty)$ gives (2.41). Moreover, since the first equation in (2.37) with (2.38) yields:

$$\lfloor|y'(t)|\rfloor_{r-1/2} \leq K_r\lceil|y(t)|\rceil_{r+1/2} + \|y(t)\|_r\lceil|y(t)|\rceil_{r+1/2}/D_r \leq (K_r+1)\lceil|y(t)|\rceil_{r+1/2},$$

where K_r denotes the supremum of $\lfloor|A_\Pi\xi|\rfloor_{r-1/2}/\lceil|\xi|\rceil_{r+1/2}$ over $0 \neq \xi \in H_\Pi^{r+1/2}$, we also have:

$$\|y\|_{W_r}^2 \leq \frac{M_r}{1 - \|y_0\|_r/D_r}\|y_0\|_r^2. \tag{2.42}$$

for some $M_r > 0$. In a second step, in order to prove existence and uniqueness of a solution to (2.37), let us determine $\rho > 0$ and $\mu > 0$ such that for $\|y_0\|_r < \delta$ and $\delta \in (0,\mu)$ the mapping:

$$\Psi : z \in W_r \to y_z \in W_r \quad \text{where} \quad y_z' + A_\Pi y_z + F(z) = 0, \quad y_z(0) = y_0,$$

is a contraction of $B_\delta \stackrel{\rm def}{=} \{z \in W_r \mid \|z\|_{W_r} \leq \rho\delta\}$ into itself. First, by combining (2.28) and (2.38),(2.39) we obtain:

$$\|\Psi(z)\|_{W_r} \leq C_0(\|z\|_{W_r}^2 + \|y_0\|_r) \quad \text{and} \quad \|\Psi(z_1) - \Psi(z_2)\|_{W_r} \leq C_1(\|z_1\|_{W_r} + \|z_2\|_{W_r})\|z_1 - z_2\|_{W_r},$$

and for z, z_1, z_2 in B_δ and $\|y_0\|_r < \delta$, we deduce that:

$$\|\Psi(z)\|_{W_r} \leq C_0(\rho\mu + 1)\delta \quad \text{and} \quad \|\Psi(z_1) - \Psi(z_2)\|_{W_r} \leq 2C_1\rho\mu\|z_1 - z_2\|_{W_r}.$$

hal-00541563, version 1 - 30 Nov 2010

hal-00541563, version 1 - 30 Nov 2010

Then for any $\rho > 0$ and $\mu > 0$ obeying $\rho\mu < \frac{1}{2C_1}$ and $\rho(1 - C_0\mu) \geq C_0$ the mapping Ψ is a contraction of B_δ into itself and (2.37) admits a unique solution in B_δ. Moreover, if we also choose (ρ, μ) such that $\mu \in (0, D_r)$ and $\rho \geq \sqrt{\frac{M_r}{1-\mu/D_r}}$ then (2.42) ensures that every solution in $L^\infty_{\rm loc}(H^r_\Pi) \cap L^2_{\rm loc}(H^{r+1/2}_\Pi)$ belongs to B_δ. As a consequence, for such (ρ, μ) the fixed point solution of (2.37) is unique within the class of functions in $L^\infty_{\rm loc}(H^r_\Pi) \cap L^2_{\rm loc}(H^{r+1/2}_\Pi)$. □

Remark 2. *Notice that* (2.38)-(2.39) *suggests that the nonlinear term is of bilinear type:* $F(\xi) = \mathfrak{B}(\xi, \xi)$ *where* $\mathfrak{B}(\cdot, \cdot)$ *is bilinear. It is the main situation when considering Navier-Stokes type nonlinearity. In fact, Theorem 4 remains true if* (2.38)-(2.39) *hold only in a neighborhood of the origin in* H^{r+1}_Π*. It means that a nonlinearity obtained from a multilinear mapping can be considered, such as power functions for instance.*

2.3. **Linear and nonlinear systems with dynamical control.** Another way to obtain the limit (2.5) is to consider (2.6) with a function $t \mapsto u(t)$ itself solution to a dynamical system of the form:

$$u' + Eu = g \tag{2.43}$$

where $t \mapsto g(t)$ is now a control function for system (2.6)-(2.43). In the following, we suppose that E is a closed linear operator in U such that $-E$ generates an analytic semigroup on U, and that $\widehat{E} = \lambda_0 + E$ has bounded imaginary powers. Then we define the spaces $U^r = \mathcal{D}(\widehat{E}^r)$, $U^r_* = \mathcal{D}(\widehat{E}^{*r})$ and $U^{-r} = [\mathcal{D}(\widehat{E}^{*r})]'$, $U^{-r}_* = [\mathcal{D}(\widehat{E}^r)]'$ for $r \geq 0$ and we recall that the following interpolation equalities hold:

$$[U^{r_1}, U^{r_2}]_{1-\alpha} = U^{(1-\alpha)r_1+\alpha r_2} \quad \text{and} \quad [U^{r_1}_*, U^{r_2}_*]_{1-\alpha} = U^{(1-\alpha)r_1+\alpha r_2}_*, \quad \forall \alpha \in (0,1), \quad r_2 < r_1.$$

Thus, we introduce the extended state space $\mathbb{H} \stackrel{\text{def}}{=} H \times U$, we introduce the extended linear operator $\mathbb{A}$ defined in $\mathbb{H}$ by:

$$\mathcal{D}(\mathbb{A}) \stackrel{\text{def}}{=} \left\{ (y, u) \in \mathbb{H} \mid y - \widehat{A}^{-1}Bu \in H^1, \ u \in U^1 \right\}, \quad \mathbb{A}(y, u) \stackrel{\text{def}}{=} (Ay - Bu, Eu),$$

and we set $\widehat{\mathbb{A}} \stackrel{\text{def}}{=} \lambda_0 + \widehat{\mathbb{A}}$. Notice that $\mathcal{D}(\mathbb{A})$ is equipped with the norm $\|\widehat{\mathbb{A}}(y, u)\|_{\mathbb{H}} = \|\widehat{A}(y - \widehat{A}^{-1}Bu)\|_H + \|\widehat{E}u\|_U$. Next, if we also introduce that canonical projection $\mathbb{B} \in \mathcal{L}(\mathbb{H})$:

$$\mathbb{B}(w, g) \stackrel{\text{def}}{=} (0, g),$$

as well as the new state $Y = (y, u)$ and the new control $V = (w, g)$, system (2.6)-(2.43) can be rewritten as follows:

$$Y' + \mathbb{A}Y = \mathbb{B}V. \tag{2.44}$$

The following Theorem states that $\mathbb{A}$ fits the framework of section 2.2.

Theorem 5. *The following results hold.*

1. *The linear operator* $(\mathcal{D}(\mathbb{A}), -\mathbb{A})$ *is the infinitesimal generator of an analytic semigroup on* $\mathbb{H}$*, and* $\widehat{\mathbb{A}} \stackrel{def}{=} \lambda_0 + \widehat{\mathbb{A}}$ *has bounded imaginary powers.*
2. *The adjoint of* $\mathbb{A}$ *is given by*
$$\mathbb{A}^*(y, u) = (A^*y, -B^*y + E^*u) \quad \text{and} \quad \mathcal{D}(\mathbb{A}^*) = H^1_* \times U^1_*,$$
and the following characterization of $\mathcal{D}(\widehat{\mathbb{A}}^{*r})$ *for* $r \geq 0$ *hold*
$$\mathcal{D}(\widehat{\mathbb{A}}^{*r}) = H^r_* \times U^r_* \quad \forall r \geq 0. \tag{2.45}$$
3. *The following characterization of* $\mathcal{D}(\widehat{\mathbb{A}}^r)$ *for* $r \in [0, 1]$ *holds:*
$$\mathcal{D}(\widehat{\mathbb{A}}^r) \quad = \quad \{ (y, u) \in \mathbb{H} \mid y - \widehat{A}^{-1}Bu \in H^r, \ u \in U^r \} \quad \forall r \in [0, 1]. \tag{2.46}$$
Moreover, $\mathcal{D}(\widehat{\mathbb{A}}^r)$ *for* $r \in [0, 1]$ *can be equipped with the equivalent norm* $\|y - \widehat{A}^{-1}Bu\|_{H^r} + \|u\|_{U^r}$.

Proof. First, by remarking that for $\lambda \in \mathbb{C}$ the equality $(\lambda + \mathbb{A})(y, u) = (f, h)$ is equivalent to

$$(\lambda + A)y = Bu + f \quad \text{and} \quad (\lambda + E)u = h, \tag{2.47}$$

we deduce that the resolvent set of $\mathbb{A}$ is exactly the union of the resolvent sets of A and of E, and that the positive halfaxis $\mathbb{R}^+$ is contained in the resolvent set of $\widehat{\mathbb{A}} \stackrel{\text{def}}{=} \lambda_0 + \mathbb{A}$. Moreover, since we also notice that for λ in the resolvent set of $\mathbb{A}$, (2.47) is equivalent to

$$y = (\lambda + A)^{-1}\widehat{A}(\widehat{A}^{-1}B)(\lambda + E)^{-1}h + (\lambda + A)^{-1}f \quad \text{and} \quad u = (\lambda + E)^{-1}h,$$

by using the boundedness of $\widehat{A}^{-1}B$ as well as resolvent estimates related to the analyticity of $(e^{-At})_{t\geq 0}$ and of $(e^{-Et})_{t\geq 0}$, we deduce that there exists $M > 0$ such that for all $F = (f, h) \in \mathbb{H}$ and for all λ in an open sector of the complex plane, symmetric with respect to the real line and with an opening angle greater than π [13, Chap. II-1, Thm. 2.10], we have:

$$\|(\lambda + \mathbb{A})^{-1}F\|_{\mathbb{H}} \leq \frac{M}{|\lambda|}\|F\|_{\mathbb{H}}.$$

The above estimate proves that $-\mathbb{A}$ generates an analytic semigroup. Next, to characterize the adjoint of $\mathbb{A}$ let us show the inclusion $\mathcal{D}(\mathbb{A}^*) \subset \mathcal{D}(A^*) \times \mathcal{D}(E^*)$ which is the only non obvious fact to prove. If $Y = (y, u) \in \mathcal{D}(\mathbb{A}^*)$ then

hal-00541563, version 1 - 30 Nov 2010

$Z \in \mathcal{D}(\mathbb{A}) \mapsto (Y|\mathbb{A}Z)_{\mathbb{H}}$ is continuous for the topology of $\mathbb{H}$, and by successively remarking that $\mathcal{D}(A) \times \{0\} \subset \mathcal{D}(\mathbb{A})$ and that $\widehat{A}^{-1}B(U) \times U \subset \mathcal{D}(\mathbb{A})$ we deduce that $z \in \mathcal{D}(A) \mapsto (y|Az)_H$ is continuous for the topology of H and that $v \in \mathcal{D}(E) \mapsto (y|Ev)_U$ is continuous for the topology of U. Then it means that $y \in \mathcal{D}(A^*)$ and $u \in \mathcal{D}(E^*)$ and the desired inclusion is proved. Next, let us recall that $\widehat{\mathbb{A}}$ has bounded imaginary powers if and only if the operator defined for $z \in \mathbb{C}$ such that $\Re z > 0$ by

$$\widehat{\mathbb{A}}^{-z} = \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z}(t + \widehat{\mathbb{A}})^{-1} dt,$$

can be extended to strongly continuous functions from $\{z \in \mathbb{C} \mid \Re z \geq 0\}$ to $\mathcal{L}(\mathbb{H})$. Since an easy calculation gives

$$\widehat{\mathbb{A}}^{-z} = \begin{pmatrix} \widehat{A}^{-z} & -\beta(z) \\ 0 & \widehat{E}^{-z} \end{pmatrix} \quad \text{where} \quad \beta(z) = \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z}(t + \widehat{A})^{-1} B (t + \widehat{E})^{-1} dt,$$

then to prove that $\widehat{\mathbb{A}}$ has bounded imaginary powers it remains to prove that we can extend $\beta(z)$ to a strongly continuous function from $\{z \in \mathbb{C} \mid \Re(z) \geq 0\}$ in $\mathcal{L}(U, H)$. From $\widehat{A}(t + \widehat{A})^{-1} = I - t(t + \widehat{A})^{-1}$ and $\|(t + \widehat{A})^{-1}\|_{\mathcal{L}(H)} \leq C_0(1 + t)^{-1}$ we deduce that $\|(t+\widehat{A})^{-1}\|_{\mathcal{L}(H,\mathcal{D}(\widehat{A}))} = \|\widehat{A}(t+\widehat{A})^{-1}\|_{\mathcal{L}(H)} \leq 1+C_0$ and an interpolation argument with $[\mathcal{D}(\widehat{A}), H]_{1-\gamma} = \mathcal{D}(\widehat{A}^{\gamma})$ gives $\|(t + \widehat{A})^{-1}\|_{\mathcal{L}(H,\mathcal{D}(\widehat{A}^{\gamma}))} = \|\widehat{A}^{\gamma}(t + \widehat{A})^{-1}\|_{\mathcal{L}(H)} \leq (1 + C_0)^{\gamma}(C_0(1 + t)^{-1})^{1-\gamma}$. Then $(1 + t)^{1-\gamma}\widehat{A}^{\gamma}(t + \widehat{A})^{-1}$ is bounded independently of t and with $\widehat{A}^{-\gamma}B \in \mathcal{L}(H)$ we can bound the term under the integral and obtain that $\beta(z)$ is bounded independently on $z \in \{z \in \mathbb{C} \mid \Re(z) > 0\}$ in a neighborhood of 0. Then by [23, Ch. 17, Thm. 17.9.1] one can extend $\beta(z)$ to a strongly continuous function from $\{z \in \mathbb{C} \mid \Re(z) \geq 0\}$ in $\mathcal{L}(U, H)$. Finally, since the fact that $\widehat{\mathbb{A}}$, $\widehat{A}$ and $\widehat{E}$ have bounded imaginary powers means that for $r \in (0, 1)$ we the interpolation equalities $[\mathcal{D}(\widehat{\mathbb{A}}), H]_{1-r} = \mathcal{D}(\widehat{\mathbb{A}}^r)$, $[\mathcal{D}(\widehat{\mathbb{A}}^*), H]_{1-r} = \mathcal{D}(\widehat{\mathbb{A}}^{*r})$, $[\mathcal{D}(\widehat{A}), H]_{1-r} = \mathcal{D}(\widehat{A}^r)$, $[\mathcal{D}(\widehat{A}^*), H]_{1-r} = \mathcal{D}(\widehat{A}^{*r})$, $[\mathcal{D}(\widehat{E}), U]_{1-r} = \mathcal{D}(\widehat{E}^r)$ and $[\mathcal{D}(\widehat{E}^*), U]_{1-r} = \mathcal{D}(\widehat{E}^{*r})$ hold, then equalities (2.45) and (2.46) follow with an interpolation argument. Indeed, it suffices to remark that the mapping $(y, u) \mapsto (y - \widehat{A}^{-1}Bu, u)$ is an isomorphism from $\mathbb{H}$ onto $H \times U$ as well as from $\mathcal{D}(\mathbb{A})$ onto $\mathcal{D}(A) \times \mathcal{D}(E)$, and that $(y, u) \mapsto (y, u)$ is an isomorphism from $\mathbb{H}$ onto $H \times U$ as well as from $\mathcal{D}(\mathbb{A}^*)$ onto $\mathcal{D}(A^*) \times \mathcal{D}(E^*)$. □

Next, let us introduce the spaces:

$$\mathbb{H}^r \stackrel{\text{def}}{=} \mathcal{D}(\widehat{\mathbb{A}}^r) \quad \text{and} \quad \mathbb{H}^r_* \stackrel{\text{def}}{=} \mathcal{D}(\widehat{\mathbb{A}}^{*r}) \quad r \geq 0, \tag{2.48}$$

respectively equipped with norms

$$\|(y, u)\|_{\mathbb{H}^r} \stackrel{\text{def}}{=} \|y - \widehat{A}^{-1}Bu\|_{H^r} + \|u\|_{U^r} \quad \text{and} \quad \|(y, u)\|_{\mathbb{H}^r_*} \stackrel{\text{def}}{=} \|y\|_{H^r_*} + \|u\|_{U^r_*}$$

and let us we set $\mathbb{H}^{-r} \stackrel{\text{def}}{=} [\mathcal{D}(\widehat{\mathbb{A}}^{*r})]'$ and $\mathbb{H}^r_* \stackrel{\text{def}}{=} [\mathcal{D}(\widehat{\mathbb{A}}^{-r})]'$ for $r > 0$. According to Theorem 5, we have the following interpolation equalities:

$$[\mathbb{H}^{r_1}, \mathbb{H}^{r_2}]_{1-\alpha} = \mathbb{H}^{(1-\alpha)r_1+\alpha r_2} \quad \text{and} \quad [\mathbb{H}^{r_1}_*, \mathbb{H}^{r_2}_*]_{1-\alpha} = \mathbb{H}^{(1-\alpha)r_1+\alpha r_2}_*, \quad \forall \alpha \in (0, 1), \quad r_2 < r_1.$$

Thus, for a prescribed rate $\sigma > 0$ we consider

$$Y' + (\mathbb{A} - \sigma)Y = \mathbb{B}V. \tag{2.49}$$

Suppose now that there is a Hilbert space $\mathbb{Z}$ and a bounded operator $\mathbb{R} \in \mathcal{L}(\mathbb{H}, \mathbb{Z})$, boundedly invertible and such that $\mathbb{R}^*\mathbb{R} \in \mathcal{L}(\mathbb{H}^{1/2}, \mathbb{H}^{*1/2})$ (see Remark 4 below), and consider the following minimization problem:

$$\inf \Big\{ \mathcal{J}(Y, V) \ \mid \ (Y, V) \in W_{\text{loc}}(\mathbb{H}, \mathbb{H}^{-1}) \times L^2(\mathbb{H}) \ \text{ satisfies } \ (2.49) \Big\}, \tag{2.50}$$

where the cost functional $\mathcal{J}$ is defined by

$$\mathcal{J}(Y, V) \stackrel{\text{def}}{=} \int_0^{\infty} \|\mathbb{R}Y(t)\|^2_{\mathbb{Z}} dt + \int_0^{\infty} \|V(t)\|^2_{\mathbb{H}} dt. \tag{2.51}$$

Moreover, we assume the following finite cost condition:

$$\left\{ \begin{array}{l} \text{for all } Y_0 \in \mathbb{H} \text{ there exists } V \in L^2(\mathbb{H}) \text{ such} \\ \text{that the corresponding solution to (2.49)} \\ \text{with } Y(0) = Y_0 \text{ satisfies } Y \in L^2(\mathbb{H}). \end{array} \right. \tag{2.52}$$

Then the results of section 2.2 apply: for a prescribed rate $\sigma > 0$ there exists a self-adjoint operator $\mathbf{\Pi} \in \mathcal{L}(\mathbb{H}, \mathbb{H}^1_*)$ which is the unique solution to the Riccati equation

$$\mathbb{A}^*\mathbf{\Pi} + \mathbf{\Pi}\mathbb{A} + \mathbf{\Pi}\mathbb{B}\mathbb{B}^*\mathbf{\Pi} = \mathbb{R}^*\mathbb{R} + 2\sigma\mathbf{\Pi}, \tag{2.53}$$

the closed-loop operator $\mathbb{A}_{\mathbf{\Pi}} \stackrel{\text{def}}{=} \mathbb{A} + \mathbb{B}\mathbb{B}^*\mathbf{\Pi}$ is such that $-\mathbb{A}_{\mathbf{\Pi}}$ generates an analytic and exponentially stable semigroup on $\mathbb{H}$, and the norm of $\mathbb{H}^r_{\mathbf{\Pi}} \stackrel{\text{def}}{=} \mathcal{D}(\mathbb{A}^r_{\mathbf{\Pi}})$ if $r \geq 0$, or $\mathbb{H}^r_{\mathbf{\Pi}} \stackrel{\text{def}}{=} [\mathcal{D}(\mathbb{A}^{*-r}_{\mathbf{\Pi}})]'$ if $r < 0$, which is defined by

$$|||\cdot|||^2_r \stackrel{\text{def}}{=} \langle \mathbb{A}^{*r+1/2}_{\mathbf{\Pi}} \, \mathbf{\Pi} \, \mathbb{A}^{r+1/2}_{\mathbf{\Pi}} \cdot | \cdot \rangle_{[\mathbb{H}^r_{\Pi}]', \mathbb{H}^r_{\Pi}}$$

is such that for $Y_0 \in \mathbb{H}^r_{\mathbf{\Pi}}$ the mapping $t \longmapsto |||e^{-\mathbb{A}_{\mathbf{\Pi}}t}Y_0|||_r$ decreases to 0 and obeys:

$$|||e^{-\mathbb{A}_{\mathbf{\Pi}}t}Y_0|||_r \le e^{-\sigma t}|||Y_0|||_r.$$

Moreover, for an extended nonlinear mapping $\mathbb{F}(\cdot)$ satisfying the analogue extended version of (2.38)-(2.39), which is to say with $\mathbb{F}(\cdot)$ instead of $F(\cdot)$ and by replacing the norms of $H_\Pi^{r-1/2}$, H_Π^r and $H_\Pi^{r+1/2}$ by the norms of $\mathbb{H}_{\mathbf{\Pi}}^{r-1/2}$, $\mathbb{H}^r_{\mathbf{\Pi}}$ and $\mathbb{H}_{\mathbf{\Pi}}^{r+1/2}$, then the analogue extended version of Theorem 8 applies and guarantees existence and uniqueness of a stable solution to the nonlinear system:

$$Y' + \mathbb{A}_{\mathbf{\Pi}}Y + \mathbb{F}(Y) = 0, \quad Y(0) = Y_0, \tag{2.54}$$

provided that Y_0 is in a neighborhood of the origin of $\mathbb{H}^r_{\mathbf{\Pi}}$. Thus, since $\mathbb{B}$ is a bounded operator in $\mathbb{H}$, the closed-loop operator $\mathbb{A}_{\mathbf{\Pi}}$ is a bounded perturbation of $\mathbb{A}$ and we have $\mathcal{D}(\mathbb{A}_{\mathbf{\Pi}}) = \mathcal{D}(\mathbb{A})$ as well as the following equalities:

$$\mathbb{H}^r_{\mathbf{\Pi}} = \mathbb{H}^r \quad \forall r \in [0,1].$$

Then it means that when $r \in [0,1]$ the stabilization result for system (2.54) hold for an initial datum $Y_0 \in \mathbb{H}^r$. In the following, we suppose that $r \in [0,1/2]$ and that the nonlinear mapping has the form $\mathbb{F}((y,u)) = (G(y,u),0)$ where $G(\cdot) : \mathbb{H}^{r+1/2} \to H^{r-1/2}$ is a nonlinear mapping satisfying

$$\|G(\xi,\theta)\|_{H^{r-1/2}} \le C\|(\xi,\theta)\|_{\mathbb{H}^r}\|(\xi,\theta)\|_{\mathbb{H}^{r+1/2}}, \tag{2.55}$$

$$\|G(\xi,\theta) - G(\zeta,\tau)\|_{H^{r-1/2}} \le C(\|(\xi-\zeta,\theta-\tau)\|_{\mathbb{H}^r}\|(\xi,\theta)\|_{\mathbb{H}^{r+1/2}} + \|(\zeta,\tau)\|_{\mathbb{H}^r}\|(\xi-\zeta,\theta-\tau)\|_{\mathbb{H}^{r+1/2}}). \tag{2.56}$$

Since we have $\mathbb{H}^r_{\mathbf{\Pi}} = \mathbb{H}^r$ and $\mathbb{H}^{-r}_{\mathbf{\Pi}} = \mathbb{H}^{-r} = H^{-r} \times U^{-r}$ it is easily seen that such $\mathbb{F}(\cdot)$ satisfy the extended version of (2.38)-(2.39). Moreover, if we introduce the components $\Pi_1 = \Pi_1^* \in \mathcal{L}(H, H^1_*)$, $\Pi_2 \in \mathcal{L}(H, U^1_*)$ and $\Pi_3 = \Pi_3^* \in \mathcal{L}(U, U^1_*)$ of $\mathbf{\Pi}$, i.e.

$$\mathbf{\Pi} = \begin{pmatrix} \Pi_1 & \Pi_2^* \\ \Pi_2 & \Pi_3 \end{pmatrix} \tag{2.57}$$

then (2.54) can be rewritten as

$$y' + Ay + G(y,u) = Bu, \quad y(0) = y_0 \tag{2.58}$$

$$u' + Eu + \Pi_2 y + \Pi_3 u = 0, \quad u(0) = u_0, \tag{2.59}$$

and the following corollary is a consequence of the extended version of Theorem 4.

Corollary 7. *Assume* (2.55)-(2.56) *and* $(y_0,u_0) \in H \times U$ *such that* $y_0 - \widehat{A}^{-1}Bu_0 \in H^r$ *and* $u \in U^r$ *for* $r \in [0,1/2]$. *There exist* $\rho > 0$ *and* $\mu > 0$ *such that, if* $\delta \in (0,\mu)$ *and* $|||(y_0,u_0)|||_r < \delta$, *then system* (2.58)-(2.59) *admits a solution* $(y_{y_0}, u_{y_0}) \in W(\mathbb{H}^{r+1/2}, \mathbb{H}^{r-1/2})$ *such that* $\|(y_{y_0},u_{y_0})\|_{W(\mathbb{H}^{r+1/2},\mathbb{H}^{r-1/2})} \le \rho\delta$, *which is unique within the class of functions in* $L^\infty_{loc}(\mathbb{H}^r) \cap L^2_{loc}(\mathbb{H}^{r+1/2})$. *Moreover, there is* $D_r > 0$ *such that every solution with an initial datum obeying:*

$$|||(y_0,u_0)|||_r < D_r,$$

is such that $t \longmapsto |||(y_{y_0}(t), u_0(t))|||_r$ *is decreasing and we have:*

$$|||(y_{y_0}(t),u_0(t))|||_r \le |||(y_0,u_0)|||_r e^{-\sigma t}. \tag{2.60}$$

Remark 3. *Notice that* (2.60) *implies the following estimate:*

$$\|y(t) - \widehat{A}^{-1}Bu(t)\|_{H^r} + \|u(t)\|_{U^r} \le Ce^{-\sigma t}(\|y_0 - \widehat{A}^{-1}Bu_0\|_{H^r} + \|u_0\|_{U^r}).$$

Remark 4. *Suppose that* $B^*\widehat{A}^{*-1} \in \mathcal{L}(H^{1/2}_*, U^{1/2}_*)$, *and that for two Hilbert spaces* Z_1 *and* Z_2 *we have two bounded linear operators* $R \in \mathcal{L}(H, Z_1)$ *and* $\Theta \in \mathcal{L}(U, Z_2)$, *both boundedly invertible and satisfying* $R \in \mathcal{L}(H^{1/2}, H^{1/2}_*)$ *and* $\Theta \in \mathcal{L}(U^{1/2}, U^{1/2}_*)$. *Then if we set* $\mathbb{Z} \overset{def}{=} Z_1 \times Z_2$ *an adequate bounded linear mapping* $\mathbb{R} \in \mathcal{L}(\mathbb{H}, \mathbb{Z})$ *can be defined as follows:*

$$\mathbb{R}(y,u) \overset{def}{=} (R(y - \widehat{A}^{-1}Bu), \Theta u).$$

Indeed, its bounded invertibility is a direct consequence of the bounded invertibility of R *and of* Θ, *and the fact that* $\mathbb{R} \in \mathcal{L}(\mathbb{H}^{1/2}, \mathbb{H}^{1/2}_*)$ *follows by remarking that:*

$$\mathbb{R}^*\mathbb{R}(y,u) = (R^*R(y - \widehat{A}^{-1}Bu), \Theta^*\Theta u - B^*\widehat{A}^{*-1}R^*R(y - \widehat{A}^{-1}Bu)).$$

Finally, let us give a sufficient condition for (2.52).

Theorem 6. *Assume that* (A,B) *is approximatively controllable and that* $B^*(\mathcal{D}(A^{*n})) \hookrightarrow U^1$ *for some* $n \in \mathbb{N}^*$. *Then the finite cost condition* (2.52) *is satisfied.*

Proof. In a first step, let us prove that a sufficient condition for (2.52) is:

$$\left\{ \begin{array}{l} \text{for all } \xi \in H^1 \text{ there exists } u \in W(U^1, U) \\ \text{such that the corresponding solution} \\ y \in W_{\text{loc}}(H, H^{-1}) \text{ to (2.9) belongs to } L^2(H). \end{array} \right. \tag{2.61}$$

Indeed, under the assumption (2.61) let us construct $V = (0,g)$ with $g \in L^2(U)$ such that $Y = (y,u)$ solution to (2.49) with $Y(0) = (y_0,u_0) \in H \times U$ belongs to $L^2(H) \times L^2(U)$. It will then implies (2.52). First, we fix $\epsilon > 0$ and we set

hal-00541563, version 1 - 30 Nov 2010

$g = 0$ on $(0, \epsilon/3)$ so that the analyticity of $(e^{-Et})_{t\geq 0}$ ensures that $u(\epsilon/3) \in U^{1/2}$. Thus, we set $g = u' + (E - \sigma)u$ where $u \in W(\epsilon/3, \epsilon, U^1, U)$ is chosen so that it is identically zero on $(2\epsilon/3, \epsilon)$. Then the control $g \in L^2(U)$ constructed on $(0, \epsilon)$ drives u_0 to 0 at $2\epsilon/3$ and fix u at zero on $(2\epsilon/3, \epsilon)$, y obeys (2.4) on $(2\epsilon/3, \epsilon)$ and the analyticity of $(e^{-At})_{t\geq 0}$ guarantees $y(\epsilon) \in H^1$. Finally, we choose $g = u' + (E - \sigma)u$ on $(\epsilon, +\infty)$ where $u \in W(\epsilon, +\infty; U^1, U)$ is given by (2.61).

In a second step, it remains to prove that (2.61) is true under the assumption of the theorem. According to [5] the approximate controllability of (A, B) guarantees the stabilizability of (2.9) by means of a finite dimensional control of the form

$$u(t) = \sum_{j=1}^{K} u_j(t) v_j$$

where $(u_1, \ldots, u_K) \in (H^1(\mathbb{R}))^K$ is solution to a differential equation and v_j, $j = 1, \ldots, K$ is a linear combination of real and imaginary parts of eigenvectors of A^*. Then we have $u \in H^1(U)$, and since each eigenvector of A^* belongs to $\mathcal{D}(A^{*n})$, the fact that $B^*(\mathcal{D}(A^{*n})) \hookrightarrow U^1$ guarantees $v_j \in U^1$, $j = 1, \ldots, K$ which also implies $u \in L^2(U^1)$. As a consequence, $u \in W(U^1, U)$ and the desired result is obtained. $\square$

3. **Stabilization of Navier-Stokes equations with Neumann feedback control.** We suppose here that Ω is a bounded and connected domain in $\mathbb{R}^d$ for $d = 2$ or $d = 3$, with a boundary $\Gamma = \partial\Omega$ of class $C^{2,1}$. By $L^2(\Omega)$, $L^2(\Gamma)$, $H^{2r}(\Omega)$, $H^{2r}(\Gamma)$, $H_0^{2r}(\Omega)$ and $H^{-2r}(\Omega) = (H_0^{2r}(\Omega))'$ for $r \geq 0$, we denote the usual Lebesgue and Sobolev spaces of scalar functions in Ω or in Γ, and we write in bold the spaces of vector-valued functions: $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{L}^2(\Gamma) = (L^2(\Gamma))^d$, etc. Moreover, $(\cdot|\cdot)$ is the usual inner product in $\mathbf{L}^2(\Omega)$. We also introduce the space of free divergence vector fields:

$$\mathbf{V}^{2r}(\Omega) = \Big\{ y \in \mathbf{H}^{2r}(\Omega) \mid \nabla \cdot y = 0 \text{ in } \Omega \Big\}, \qquad r \geq 0.$$

Let us underline that the following interpolation equalities hold:

$$[\mathbf{V}^{2r_1}(\Omega), \mathbf{V}^{2r_2}(\Omega)]_{1-\alpha} = \mathbf{V}^{2((1-\alpha)r_1 + \alpha r_2)}(\Omega), \quad \alpha \in (0,1), \quad 0 \leq r_2 \leq r_1 \leq 1. \tag{3.62}$$

To justify the above equalities it suffices to remark that the orthogonal projection operator $\mathbf{P} : \mathbf{L}^2(\Omega) \to \mathbf{V}^0(\Omega)$ is bounded from $\mathbf{H}^{2r_i}(\Omega)$ onto $\mathbf{V}^{2r_i}(\Omega)$, $i = 1, 2$ and then to apply [31, Thm. 1.17.1.1, p.118]. Indeed, since we have $\mathbf{P}f = f + \nabla p$ where $p \in H^1(\Omega)$ is the solution of $-\Delta p = \nabla \cdot f$ in Ω and $p = 0$ on Γ, then regularity results for the Laplace problem with a homogeneous Dirichlet condition guarantees the claimed boundedness properties of $\mathbf{P}$. Then [31, Thm. 1.17.1.1, p.118] yields $[\mathbf{H}^{2r_1}(\Omega) \cap \mathbf{V}^0(\Omega), \mathbf{H}^{2r_2}(\Omega) \cap \mathbf{V}^0(\Omega)]_{1-\alpha} = [\mathbf{H}^{2r_1}(\Omega), \mathbf{H}^{2r_2}(\Omega)]_{1-\alpha} \cap \mathbf{V}^0(\Omega)$ and the conclusion follows from $[\mathbf{V}^{2r_1}(\Omega), \mathbf{V}^{2r_2}(\Omega)]_{1-\alpha} = [\mathbf{H}^{2r_1}(\Omega) \cap \mathbf{V}^0(\Omega), \mathbf{H}^{2r_2}(\Omega) \cap \mathbf{V}^0(\Omega)]_{1-\alpha}$ and $[\mathbf{H}^{2r_1}(\Omega), \mathbf{H}^{2r_2}(\Omega)]_{1-\alpha} = \mathbf{H}^{2((1-\alpha)r_1+\alpha r_2)}(\Omega)$.

In the following, C denotes a positive constant which may change from line to line and which only depends on the geometry. We recall that $n = (n_1, \ldots, n_d)$ denotes the unit interior normal vector field defined near Γ, for a vector field y defined near the boundary we denote by $y_n n \stackrel{\text{def}}{=} (y \cdot n)n$ and $y_\tau \stackrel{\text{def}}{=} y - (y \cdot n)n$ the normal and the tangential component of y respectively, and for a scalar function or a vector field y its normal derivative is defined by $\frac{dy}{dn} = \sum_{i=1}^d n_i \frac{dy}{dx_i}$. We also underline that $(\frac{dy}{dn})_\tau = \frac{dy_\tau}{dn}$ and $(\frac{dy}{dn})_n = \frac{dy_n}{dn}$, see [21, App. A] for details.

Next, for $\nu > 0$ and $f \in \mathbf{L}^2(\Omega)$ we consider a pair $(z_e, r_e) \in \mathbf{V}^2(\Omega) \times H^1(\Omega)$ solution to the stationary Navier-Stokes equations:

$$-\nu\Delta z_e + (z_e \cdot \nabla)z_e + \nabla r_e = f, \quad \nabla \cdot z_e = 0 \ \text{ in } \ \Omega, \tag{3.63}$$

and we focus on the question of stabilizing around (z_e, r_e) the unstationary solution (z, r) of the Navier-Stokes equations

$$\partial_t z - \nu\Delta z + (z \cdot \nabla)z + \nabla r = f, \quad \nabla \cdot z = 0 \ \text{ in } \ \Omega \times (0, +\infty), \tag{3.64}$$

by means of a Neumann control. Our goal is to apply the Riccati approach presented in section 2 to construct a feedback law $\mathfrak{F} : \mathbf{V}^0(\Omega) \to \mathbf{L}^2(\Gamma)$ such that the solution to the above equations with

$$\nu \frac{d(z - z_e)}{dn} - (r - r_e)n = \mathfrak{F}(z - z_e) \quad \text{ on } \quad \Gamma \times (0, +\infty)$$

obeys:

$$\lim_{t \to +\infty} z(t) = z_e.$$

In the following we will use the notations:

$$\chi(y, p) \stackrel{\text{def}}{=} \nu \frac{dy}{dn} - pn \quad \text{ and } \quad \chi_e(y, p) \stackrel{\text{def}}{=} \nu \frac{dy}{dn} + (z_e \cdot n)y - pn.$$

We shall underline that most of the technical statements dealing with the well-posedness of Stokes-type system with Neumann boundary condition which are given in the following can be found in [14] or in [21]. The last quoted work is a complete study of stationary and instationary Stokes and Navier-Stokes system with Neumann boundary condition treated by pseudo-differential methods. However, in order to be complete an self-contained, we choose here to give all proofs exept the following two lemma's. The first one is a lifting theorem which is a direct consequence of a theorem due to Amrouche and Girault [1]. The second one is a regularity theorem for Stokes system with Neumann boundary condition which can be found in [21, Thm. 6.3], see also [14].

hal-00541563, version 1 - 30 Nov 2010

Lemma 1. *Let Ω be a bounded and connected open subset of $\mathbb{R}^d$ of class $C^{k+1,1}$ for $k \in \mathbb{N}$ and let $(b_0, b_1) \in \mathbf{H}^{k+3/2}(\Gamma) \times \mathbf{H}^{k+1/2}(\Gamma)$ such that $\int_\Gamma b_0 \cdot n = 0$. Then there exists $(u_b, p_b) \in \mathbf{H}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ satisfying:*

$$\nabla \cdot u_b = 0 \quad \text{in} \quad \Omega, \quad u_b = b_0 \quad \text{and} \quad \chi(u_b, p_b) = b_1 \quad \text{on} \quad \Gamma$$

and

$$\|u_b\|_{\mathbf{H}^{k+2}(\Omega)} + \|p_b\|_{H^{k+1}(\Omega)} \leq C(\|b_0\|_{\mathbf{H}^{k+3/2}(\Gamma)} + \|b_1\|_{\mathbf{H}^{k+1/2}(\Gamma)}),$$

Moreover, the result is still valid with χ_e instead of χ.

Proof. The lemma relies on [1, Thm. A.] which states that for all $(g_0, g_1) \in \mathbf{H}^{k+3/2}(\Gamma) \times \mathbf{H}^{k+1/2}(\Gamma)$ satisfying $\int_\Gamma g_0 \cdot n = 0$ and $g_1 \cdot n = \Psi(g_0) \stackrel{\text{def}}{=} 2\nu K g_0 \cdot n - \nu \nabla_\Gamma \cdot (g_0)_\tau$ there exists $u \in \mathbf{H}^{k+2}(\Omega)$ such that:

$$\nabla \cdot u = 0 \ \text{ in } \ \Omega, \quad u = g_0 \ \text{ and } \ \nu \frac{du}{dn} = g_1 \ \text{ on } \ \Gamma \ \text{ and } \ \|u\|_{\mathbf{H}^{k+2}(\Omega)} \leq C(\|g_0\|_{\mathbf{H}^{k+3/2}(\Gamma)} + \|g_1\|_{\mathbf{H}^{k+1/2}(\Gamma)}).$$

In the above setting K denotes the mean curvature of Γ and $\nabla_\Gamma \cdot$ denotes the surface divergence operator. Thus, it suffices to define $u_b \in \mathbf{H}^{k+2}(\Omega)$ as the vector field obtained for $g_0 = b_0$ and $g_1 = (b_1)_\tau + \Psi(b_0)n$, and to define $p_b \in H^{k+1}(\Omega)$ as a pressure function obtained from a continuous right inverse of the trace operator [20, Thm. 1.5.1.5] and such that $p_b = -b_1 \cdot n + \Psi(b_0)$ on Γ. The results whith χ_e instead of χ can be obtained analogously, with $g_1 = (b_1)_\tau - (z_e \cdot n)(b_0)_\tau + \Psi(b_0)n$ and $p_b = -b_1 \cdot n + (z_e \cdot n)b_0 \cdot n + \Psi(b_0)$ on Γ. □

Lemma 2. *Let Ω be a bounded and connected open subset of $\mathbb{R}^d$ of class $C^{k+1,1}$ for $k \in \mathbb{N}$ and let $f \in \mathbf{H}^k(\Omega)$. If $(u, p) \in \mathbf{V}^1(\Omega) \times L^2(\Omega)$ satisfies*

$$\nu \int_\Omega \nabla u : \nabla v - \int_\Omega p \nabla \cdot v = \int_\Omega f \cdot v \quad \forall v \in \mathbf{H}^1(\Omega), \tag{3.65}$$

then we have $(u, p) \in \mathbf{V}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ and the following estimate hold:

$$\|u\|_{\mathbf{H}^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \leq C\|f\|_{\mathbf{H}^k(\Omega)}.$$

Corollary 8. *Let the assumptions of Lemma 2 be satisfied and let $g \in \mathbf{H}^{k+1/2}(\Gamma)$. If $(u, p) \in \mathbf{V}^1(\Omega) \times L^2(\Omega)$ satisfies*

$$\nu \int_\Omega \nabla u : \nabla v - \int_\Omega p \nabla \cdot v = \int_\Omega f \cdot v + \int_\Gamma g \cdot v \quad \forall v \in \mathbf{H}^1(\Omega),$$

then we have $(u, p) \in \mathbf{V}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ and the following estimate hold:

$$\|u\|_{\mathbf{H}^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \leq C(\|f\|_{\mathbf{H}^k(\Omega)} + \|g\|_{\mathbf{H}^{k+1/2}(\Gamma)}).$$

Proof. The conclusion follows from Lemma 2 by remarking that if we write $(u, p) = (\widetilde{u}, \widetilde{p}) + (u_g, p_g)$ where $(u_g, p_g) \in \mathbf{V}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ is given by Lemma 1 with $b_0 = 0$ and $b_1 = g$, then an integration by parts shows that $(\widetilde{u}, \widetilde{p})$ obeys (3.65) with $f + \Delta u_g - \nabla p_g \in \mathbf{H}^k(\Omega)$ instead of f at the right side of the equality. □

Next, let $m \in C^2(\Gamma; \mathbb{R}^+)$ be a compactly supported function of Γ which is not identically equal to zero. Then our objective is to prove that for a prescribed rate of decrease $\sigma > 0$ there is a unique nonnegative and self-adjoint linear mapping $\Pi \in \mathcal{L}(\mathbf{V}^0(\Omega))$ belonging to $\mathcal{L}(\mathbf{V}^0(\Omega), \mathbf{V}^2(\Omega))$ and solution to the Riccati equation

$$\int_\Omega \nabla \Pi \xi : \nabla \zeta + \int_\Omega \nabla \xi : \nabla \Pi \zeta + \int_\Gamma m \Pi \xi \cdot m \Pi \zeta = \int_\Omega \xi \cdot \zeta + 2\sigma \int_\Omega \Pi \xi \cdot \zeta, \quad \forall (\xi, \zeta) \in \mathbf{V}^1(\Omega) \times \mathbf{V}^1(\Omega), \tag{3.66}$$

such that for z_0 close enough to z_e in $\mathbf{V}^{2r}(\Omega)$ for $r \in [0, 3/4)$, system

$$\partial_t z - \nu \Delta z + (z \cdot \nabla) z + \nabla r = f, \quad \nabla \cdot z = 0 \quad \text{in } \Omega \times (0, \infty), \tag{3.67}$$

$$\chi(z, r) = \chi(z_e, r_e) + m^2 \Pi(z_e - z) \quad \text{on } \Gamma \times (0, \infty), \tag{3.68}$$

with initial datum

$$z(0) = z_0 \tag{3.69}$$

admits a unique solution which satisfies

$$\|z(t) - z_e\|_{\mathbf{V}^{2r}(\Omega)} \leq C e^{-\sigma t} \|z_0 - z_e\|_{\mathbf{V}^{2r}(\Omega)}.$$

To achieve this goal we need: first to prove that system (3.67), (3.68) can be rewritten in the form (2.37), second to characterize the spaces H^r_Π and finally to apply Theorem 4.

First, for $z_0 \in \mathbf{V}^0(\Omega)$ we should say that $(z, p) \in W_{\text{loc}}(\mathbf{V}^1(\Omega), [\mathbf{V}^1(\Omega)]') \times L^2_{\text{loc}}(L^2(\Omega))$ is a solution to (3.67), (3.68), (3.69), if and only if, it satisfies (3.69) and for all $v \in \mathbf{H}^1(\Omega)$ and $t \geq 0$:

$$\frac{d}{dt} \int_\Omega z(t) \cdot v + \int_\Omega (\nu \nabla z(t) : \nabla v + (z(t) \cdot \nabla z(t)) \cdot v - r(t) \nabla \cdot v) = \int_\Gamma (\chi(z_e, r_e) + m^2 \Pi(z_e - z)) \cdot v + \int_\Omega f \cdot v. \tag{3.70}$$

Obviously, if (z, p) is regular an integration by parts shows that a solution to (3.70) obeys (3.67), (3.68) in a classical sense. Thus, an easy calculation shows that $(y, p) = (z - z_e, r - r_e)$ is solution of the variational formulation:

$$\frac{d}{dt} \int_\Omega y(t) \cdot v + \int_\Omega (\nu \nabla y(t) : \nabla v + (y(t) \cdot \nabla z_e) \cdot v + (z_e \cdot \nabla y(t)) \cdot v) + \int_\Omega (y(t) \cdot \nabla) y(t) \cdot v + \int_\Gamma m^2 \Pi y(t) \cdot v = 0,$$

hal-00541563, version 1 - 30 Nov 2010

hal-00541563, version 1 - 30 Nov 2010

for all $v \in \mathbf{H}^1(\Omega)$ and $t \geq 0$. Then, if we introduce the linear operator:

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \{v \in \mathbf{V}^1(\Omega) \mid w \mapsto a(v,w) \text{ is } \mathbf{V}^0(\Omega) - \text{continuous}\} \quad (Av|w) \stackrel{\text{def}}{=} a(v,w), \tag{3.71}$$

defined from the continuous bilinear form on $\mathbf{H}^1(\Omega)$:

$$a(v,w) \stackrel{\text{def}}{=} \int_\Omega (\nu\nabla v : \nabla w + (v\cdot\nabla z_e)\cdot w + (z_e\cdot\nabla v)\cdot w) \quad \forall (v,w) \in \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega),$$

if we introduce the input linear operator $B \in \mathcal{L}(\mathbf{L}^2(\Gamma), [\mathbf{V}^1(\Omega)]')$ defined by

$$\langle Bu|v\rangle_{[\mathbf{V}^1(\Omega)]',\mathbf{V}^1(\Omega)} \stackrel{\text{def}}{=} \int_\Gamma mu\cdot v \quad \forall (u,v) \in \mathbf{L}^2(\Gamma)\times\mathbf{V}^1(\Omega), \tag{3.72}$$

with adjoint $B^* \in \mathcal{L}(\mathbf{V}^1(\Omega), \mathbf{L}^2(\Gamma))$ given by $B^*v = mv|_\Gamma$, and if we define the nonlinear mapping:

$$F : \mathbf{V}^1(\Omega) \to [\mathbf{V}^1(\Omega)]', \quad \langle F(y)|v\rangle_{[\mathbf{V}^1(\Omega)]',\mathbf{V}^1(\Omega)} = \int_\Omega (y\cdot\nabla)y\cdot w \quad \forall (y,v) \in \mathbf{V}^1(\Omega)\times\mathbf{V}^1(\Omega), \tag{3.73}$$

the above system can be rewritten as follows:

$$y' + Ay + B(B^*\Pi)y + F(y) = 0. \tag{3.74}$$

In the following, we are going to prove that A, B and F fit the framework of section 2 with $H = \mathbf{V}^0(\Omega)$ and $U = \mathbf{L}^2(\Gamma)$ and we are going to make more precise the equivalence between formulation (3.70) and formulation (3.74).

First, recall that the trilinear form $(v_1,v_2,v_3) \mapsto \int_\Omega (v_1\cdot\nabla)v_2\cdot v_3$ obeys:

$$\Big|\int_\Omega (v_1\cdot\nabla)v_2\cdot v_3\Big| \leq C\|v_1\|_{\mathbf{H}^{s_1}(\Omega)}\|v_2\|_{\mathbf{H}^{1+s_2}(\Omega)}\|v_3\|_{\mathbf{H}^{s_3}(\Omega)}, \tag{3.75}$$

for all $(v_1,v_2,v_3) \in \mathbf{H}^{s_1}(\Omega)\times\mathbf{H}^{1+s_2}(\Omega)\times\mathbf{H}^{s_3}(\Omega)$, where s_1, s_2 and s_3 are real nonnegative numbers such that $s_1+s_2+s_3 \geq \frac{d}{2}$ if $s_i \neq \frac{d}{2}$, $i = 1,2,3$ or $s_1+s_2+s_3 > \frac{d}{2}$ if $s_i = \frac{d}{2}$, for at least one i [15, Prop. 6.1, (6.10)]. Then it yields

$$\left|\int_\Omega (v\cdot\nabla z_e)\cdot v + \int_\Omega (z_e\cdot\nabla v)\cdot v\right| \quad \leq \quad C_0\|z_e\|_{\mathbf{H}^2(\Omega)}\|v\|_{\mathbf{L}^2(\Omega)}\|v\|_{\mathbf{H}^1(\Omega)},$$

and we deduce the existence of $\lambda_0 >$ such that:

$$a(v,v) + \lambda_0\|v\|^2_{\mathbf{L}^2(\Omega)} \geq \frac{\nu}{2}\|v\|^2_{\mathbf{H}^1(\Omega)} \quad \forall v \in \mathbf{H}^1(\Omega). \tag{3.76}$$

We set $\widehat{A} \stackrel{\text{def}}{=} \lambda_0 + A$ and we are going to prove that

1. $-A$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^0(\Omega)$.
2. A and A^* are characterized by

$$\begin{cases} \mathcal{D}(A) &= \left\{y \in \mathbf{V}^2(\Omega) \mid \dfrac{dy_\tau}{dn} = 0 \text{ on } \Gamma\right\} \\ Ay &= -\nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + \nabla Ry, \end{cases} \tag{3.77}$$

and:

$$\begin{cases} \mathcal{D}(A^*) &= \left\{y \in \mathbf{V}^2(\Omega) \mid \nu\dfrac{dy_\tau}{dn} + (z_e\cdot n)y_\tau = 0 \text{ on } \Gamma\right\} \\ A^*y &= -\nu\Delta y + {}^t(\nabla z_e)y - (z_e\cdot\nabla)y + \nabla Sy. \end{cases} \tag{3.78}$$

3. $\widehat{A}$ has bounded imaginary powers and the fractional powers of $\widehat{A}$ and of $\widehat{A}^*$ satisfy:

$$\mathcal{D}(\widehat{A}^r) = \mathcal{D}(\widehat{A}^{*r}) = \mathbf{V}^{2r}(\Omega), \quad \forall r \in [0,3/4), \tag{3.79}$$

$$\mathcal{D}(\widehat{A}^r) = \left\{y \in \mathbf{V}^{2r}(\Omega) \,\middle|\, \frac{dy_\tau}{dn} = 0 \text{ on } \Gamma\right\}, \quad \forall r \in (3/4,3/2], \tag{3.80}$$

$$\mathcal{D}(\widehat{A}^{*r}) = \left\{y \in \mathbf{V}^{2r}(\Omega) \,\middle|\, \nu\frac{dy_\tau}{dn} + (z_e\cdot n)y_\tau = 0 \quad \text{on } \Gamma\right\}, \quad \forall r \in (3/4,3/2]. \tag{3.81}$$

In the above setting, the mappings $R \in \mathcal{L}(\mathbf{V}^2(\Omega), H^1(\Omega))$ and $S \in \mathcal{L}(\mathbf{V}^2(\Omega), H^1(\Omega))$ are defined by $p = Ry$ and $q = Sy$ where p and q are the respective solutions to

$$-\Delta p = \nabla\cdot[(y\cdot\nabla)z_e + (z_e\cdot\nabla)y] \text{ in } \Omega \quad \text{and} \quad p = \nu\frac{dy_n}{dn} \text{ on } \Gamma, \tag{3.82}$$

and

$$-\Delta q = \nabla\cdot[{}^t(\nabla z_e)y - (z_e\cdot\nabla)y] \text{ in } \Omega \quad \text{and} \quad q = \nu\frac{dy_n}{dn} + (z_e\cdot n)y_n \text{ on } \Gamma. \tag{3.83}$$

Remark 5. *Notice that from* (3.75) *we deduce that* $\|(y\cdot\nabla)z_e + (z_e\cdot\nabla)y\|_{\mathbf{L}^2(\Omega)} \leq C\|z_e\|_{\mathbf{H}^2(\Omega)}\|y\|_{\mathbf{V}^1(\Omega)}$. *Then* $\nabla\cdot[(y\cdot\nabla)z_e + (z_e\cdot\nabla)y]$ *belongs to* $H^{-1}(\Omega)$ *and regularity results for Laplace problem with nonhomogeneous Dirichlet condition ensures that* R *belongs to* $\mathcal{L}(\mathbf{V}^{2r}(\Omega), H^{2r-1}(\Omega))$ *for* $r \in]3/4,1]$. *Analogously, we can prove that* $S \in \mathcal{L}(\mathbf{V}^{2r}(\Omega), H^{2r-1}(\Omega))$ *for* $r \in]3/4,1]$.

We first need the following Lemma which is consequence of De Rham's theorem.

Lemma 3. *If $f \in [\mathbf{H}^1(\Omega)]'$ obeys $\langle f \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = 0$ for all $v \in \mathbf{V}^1(\Omega)$ then there exists a unique $p \in L^2(\Omega)$ such that*

$$\langle f \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \int_\Omega p\nabla\cdot v \quad \forall v \in \mathbf{H}^1(\Omega).$$

Proof. First, under the above assumptions we have in particular $\langle f \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = 0$ for all $v \in \mathbf{V}^1(\Omega)\cap\mathbf{H}^1_0(\Omega)$ and De Rham's theorem [2, Thm 2.8] ensures that there exists $p \in L^2(\Omega)$, defined up to a constant, and such that $\langle f \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \int_\Omega p\nabla\cdot v$ for all $v \in \mathbf{H}^1_0(\Omega)$. Thus, we verify that each $v \in \mathbf{H}^1(\Omega)$ can be decomposed as $v = v_1 + v_2 + c(v)\varphi_d$ with $v_1 \in \mathbf{H}^1_0(\Omega)$, $v_2 \in \mathbf{V}^1(\Omega)$, $\varphi_d(x) = \frac{1}{d}\,{}^t(x_1, x_2, \ldots, x_d)$ and $c(v) = |\Omega|^{-1}\int_\Gamma v\cdot n$. Indeed, since we have $\nabla\cdot\varphi_d \equiv 1$ then $\int_\Omega \nabla\cdot(v - c(v)\varphi_d) = 0$ and we can choose $v_1 \in \mathbf{H}^1_0(\Omega)$ such that $\nabla\cdot v_1 = \nabla\cdot(v - c(v)\varphi_d)$, see [2, Cor. 3.1, ii)]. Finally, since $\langle f \mid v_1\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \int_\Omega p\nabla\cdot v_1$ and $\langle f \mid v_2\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = 0$ an easy computation gives:

$$\langle f \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} - \int_\Omega p\nabla\cdot v = (\langle f \mid \varphi_d\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} - \int_\Omega p)c(v),$$

and by choosing $p \in L^2(\Omega)$ such that $\int_\Omega p = \langle f \mid \varphi_d\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)}$ we obtain the desired result. □

Remark 6. *Notice that since the orthogonal projection operator $\mathbf{P} : \mathbf{L}^2(\Omega) \to \mathbf{V}^0(\Omega)$ is also bounded from $\mathbf{H}^1(\Omega)$ into $\mathbf{V}^1(\Omega)$ then its adjoint $\mathbf{P}^* : \mathbf{V}^0(\Omega) \to \mathbf{L}^2(\Omega)$, which is simply the injection operator, can be extended as a bounded operator from $[\mathbf{V}^1(\Omega)]'$ into $[\mathbf{H}^1(\Omega)]'$ as follows:*

$$\langle \mathbf{P}^* f|v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \langle f|\mathbf{P}v\rangle_{[\mathbf{V}^1(\Omega)]',\mathbf{V}^1(\Omega)} \quad \forall v \in \mathbf{H}^1(\Omega).$$

Then Lemma 3 has the following interpretation: if $f \in [\mathbf{H}^1(\Omega)]'$ and $g \in [\mathbf{V}^1(\Omega)]'$ coincide on $\mathbf{V}^1(\Omega)$ then

$$\langle f \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \langle \mathbf{P}^* g \mid v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} + \int_\Omega p\nabla\cdot v \quad \forall v \in \mathbf{H}^1(\Omega).$$

Theorem 7. *The operator $(\mathcal{D}(A), -A)$ [resp. $(\mathcal{D}(A^*), -A^*)$] is the infinitesimal generator of an analytic semigroup $(e^{-At})_{t\geq 0}$ [resp. $(e^{-A^*t})_{t\geq 0}$] on $\mathbf{V}^0(\Omega)$ and equalities (3.77) and (3.78) are satisfied. Moreover, $\widehat{A}$ has bounded imaginary powers and the fractional powers of $\widehat{A}$ and of $\widehat{A}^*$ satisfy (3.79),(3.80),(3.81).*

Proof. First, according to [13, Part.I, Chap. 1, Thm 2.12, p.115] the coercivity condition (3.76) ensures that $(\mathcal{D}(A), -A)$ and $(\mathcal{D}(A^*), -A^*)$ are infinitesimal generators of analytic semigroups on $\mathbf{V}^0(\Omega)$. Next, let us characterize $\mathcal{D}(A)$. For $y \in \mathcal{D}(A)$ we have $Ay \in \mathbf{V}^0(\Omega)$ and $a(y, v) = \int_\Omega Ay\cdot v$ for all $v \in \mathbf{V}^1(\Omega)$, and by Lemma 3 there is $p \in L^2(\Omega)$ obeying:

$$\nu\int_\Omega \nabla y : \nabla v - \int_\Omega p\nabla\cdot v = \int_\Omega (Ay - (y\cdot\nabla)z_e - (z_e\cdot\nabla)y)\cdot v \qquad \forall v \in \mathbf{H}^1(\Omega).$$

Thus, because $Ay - (y\cdot\nabla)z_e - (z_e\cdot\nabla)y \in \mathbf{L}^2(\Omega)$, Lemma 2 ensures that (y, p) belongs to $\mathbf{V}^2(\Omega)\times H^1(\Omega)$, and an integration by parts yields:

$$-\nu\Delta y + \nabla p = Ay - (y\cdot\nabla)z_e - (z_e\cdot\nabla)y \;\text{ in }\; \Omega, \quad \chi(y, p) = 0 \;\text{ on }\; \Gamma.$$

The above trace condition means that $(\frac{dy}{dn})_\tau = \frac{dy_\tau}{dn} = 0$, and the application of the divergence operator to the above first equation gives $p = Ry$. Then (3.77) is proved. Next, to characterize $\mathcal{D}(A^*)$, let us denote $V^\sharp = \{y \in \mathbf{V}^2(\Omega) \mid \nu\frac{dy_\tau}{dn} + (z_e\cdot n)y_\tau = 0 \text{ on } \Gamma\}$ and $A^\sharp y = -\nu\Delta y + {}^t(\nabla z_e)y - (z_e\cdot\nabla)y + \nabla Sy$ and let us prove that $\mathcal{D}(A^*) = V^\sharp$ and $A^* = A^\sharp$. First, for all $(y, v) \in \mathcal{D}(A)\times V^\sharp$ an integration by parts gives:

$$\int_\Omega (-\nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + \nabla Ry)\cdot v = \int_\Omega y\cdot(-\nu\Delta v + {}^t(\nabla z_e)v - (z_e\cdot\nabla)v + \nabla Sv),$$

which means that $(Ay|v) = (y|A^\sharp v)$. Then we have $V^\sharp \subset \mathcal{D}(A^*)$ and the operators $A^\sharp$ and A^* coincide on $V^\sharp$. Conversely, if $v \in \mathcal{D}(A^*)$ then for all $y \in \mathcal{D}(A)$ we have $(Ay|v) = (y|A^* v)$ and then $(\widehat{A}y|v) = (y|\widehat{A}^* v)$. Moreover, according to the Lax-Milgram theorem there is unique $\widetilde{v} \in \mathbf{V}^1(\Omega)$ obeying $\lambda_0(w|\widetilde{v}) + a(w, \widetilde{v}) = (w|\widehat{A}^* v)$ for all $w \in \mathbf{V}^1(\Omega)$. Then by choosing $w = y \in \mathcal{D}(A)$ we obtain that $\widetilde{v}$ obeys $(\widehat{A}y|\widetilde{v}) = (y|\widehat{A}^* v)$ and $(\widehat{A}y|v - \widetilde{v}) = 0$ for all $y \in \mathcal{D}(A)$, which means that $v = \widetilde{v} \in \mathbf{V}^1(\Omega)$ and that $\lambda_0(y|v) + a(y, v) = (y|\widehat{A}^* v)$ for all $y \in \mathcal{D}(A)$. A density argument ensures that this last equality remains valid for all $y \in \mathbf{V}^1(\Omega)$, and by Lemma 3 there is $q \in L^2(\Omega)$ such that $\lambda_0(y|v) + a(y, v) - \int_\Omega q\nabla\cdot y = (y|\widehat{A}^* v)$ for all $y \in \mathbf{H}^1(\Omega)$ and

$$\nu\int_\Omega \nabla y : \nabla v + (y\cdot\nabla)z_e\cdot v + (z_e\cdot\nabla y)\cdot v - \int_\Omega q\nabla\cdot y = \int_\Omega y\cdot A^* v \quad \forall y \in \mathbf{H}^1(\Omega). \tag{3.84}$$

Thus, an integration by parts gives

$$\nu\int_\Omega \nabla v : \nabla y + \int_\Gamma (z_e\cdot n)v\cdot y - \int_\Omega q\nabla\cdot y = \int_\Omega (A^* v - {}^t(\nabla z_e)v + (z_e\cdot\nabla v))\cdot y \quad \forall y \in \mathbf{H}^1(\Omega).$$

Moreover, since $(z_e\cdot n)v \in \mathbf{H}^{1/2}(\Gamma)$ and $A^* v - {}^t(\nabla z_e)v + (z_e\cdot\nabla v) \in \mathbf{L}^2(\Omega)$, then Corollary 8 ensures that (y, p) belongs to $\mathbf{V}^2(\Omega)\times H^1(\Omega)$, and integrating by parts in (3.84) yields

$$\int_\Omega (-\nu\Delta v + {}^t(\nabla z_e)v - (z_e\cdot\nabla)v + \nabla q)\cdot y + \int_\Gamma (\nu\frac{dv}{dn} - qn + (z_e\cdot n)v)\cdot y = \int_\Omega A^* v\cdot y \quad \forall y \in \mathbf{H}^1(\Omega),$$

hal-00541563, version 1 - 30 Nov 2010

which means:

$$-\nu\Delta v + {}^t(\nabla z_e)v - z_e\cdot\nabla v + \nabla q = A^* v \;\text{ in }\; \Omega \quad \text{ and } \quad \chi_e(v,q) = 0 \;\text{ on }\; \Gamma.$$

Finally, the application of the divergence operator to the first above equation gives $q = S(v)$ and the second above equation means that $\nu\frac{dv_\tau}{dn} + (z_e\cdot n)v_\tau = 0$. Then we have proved (3.78).

Now, let us prove (3.79) and (3.80)-(3.81). First, since $\widehat{A}$ is closed maximal accretive and that $\widehat{A}^{-1}$ is bounded in $\mathbf{V}^0(\Omega)$, then according to [13, Part.I, Chap. 1, Thm 6.1 and Prop. 6.1, p.171] $\widehat{A}$ has bounded imaginary powers and the fractional powers of $\widehat{A}$ and of $\widehat{A}^*$ satisfy $\mathcal{D}(\widehat{A}^r) = [\mathcal{D}(A), \mathbf{V}^0(\Omega)]_{1-r}$ and $\mathcal{D}(\widehat{A}^{*r}) = [\mathcal{D}(A^*), \mathbf{V}^0(\Omega)]_{1-r}$ for all $r\in[0,1]$. We underline that the same argument applies for the following auxiliary selfadjoint linear operators A_1 and A_2 defined by:

$$\mathcal{D}(A_i) \overset{\text{def}}{=} \left\{v\in\mathbf{V}^1(\Omega) \mid w\mapsto a_i(v,w) \text{ is } \mathbf{V}^0(\Omega) - \text{continuous }\right\}, \quad (A_i v|w) \overset{\text{def}}{=} a_i(v,w), \quad i=1,2$$

and

$$a_1(v,w) \overset{\text{def}}{=} \int_\Omega (v\cdot w + \nu\nabla v:\nabla w) \quad \text{ and } \quad a_2(v,w) \overset{\text{def}}{=} \lambda_2 a_1(v,w) + \int_\Gamma (z_e\cdot n) w\cdot v,$$

where $\lambda_2>0$ is large enough so that $a_2(\cdot,\cdot)$ is coercive. Then we also have $\mathcal{D}(A_i^r) = [\mathcal{D}(A_i), \mathbf{V}^0(\Omega)]_{1-r}$, $i=1,2$, for all $r\in[0,1]$. Moreover, analogously as in the first part of the proof, by invoking Corollary 8 and integrating by parts, we verify that $\mathcal{D}(A_1) = \mathcal{D}(A)$ and $\mathcal{D}(A_2) = \mathcal{D}(A^*)$. As a consequence, we have $\mathcal{D}(\widehat{A}^r) = [\mathcal{D}(A_1), \mathbf{V}^0(\Omega)]_{1-r} = \mathcal{D}(A_1^r)$ and $\mathcal{D}(\widehat{A}^{*r}) = [\mathcal{D}(A_2), \mathbf{V}^0(\Omega)]_{1-r} = \mathcal{D}(A_2^r)$ and the proof of (3.79)-(3.80)-(3.81) is then reduced to the characterization of $\mathcal{D}(A_i^r)$, $i=1,2$. We then fix $i=1,2$ in the following and we first remark that the continuity and coercivity of $a_i(\cdot,\cdot)$ with the obvious calculation:

$$\|A_i^{1/2}y\|^2_{\mathbf{V}^0(\Omega)} = (A_i y|y) = a_i(y,y),$$

yield $\mathcal{D}(A_i^{1/2}) = \mathbf{V}^1(\Omega)$, which proves (3.79) for $r=1/2$. Moreover, since we know from Lemma 3 that there exists $p\in L^2(\Omega)$ satisfying:

$$a_i(v,w) - \int_\Omega p\nabla\cdot v = \int_\Omega A_i y\cdot v \quad \forall v\in\mathbf{H}^1(\Omega),$$

and since $y\in\mathcal{D}(A_i^{3/2})$ is equivalent to $A_i y\in\mathcal{D}(A_i^{1/2}) = \mathbf{V}^1(\Omega)$, then equalities (3.80) and (3.81) for $r=3/2$ are direct consequences of Corollary 8. Then it remains to conclude for $r\in(0,1/2)$ and for $r\in(1/2,3/2)$ with an interpolation argument. According to [31, Thm. 1.15.3.1, p. 103] the fact that A_i has bounded imaginary powers also yields the general interpolation equalities:

$$\mathcal{D}(A_i^{\alpha a+(1-\alpha)b}) = [\mathcal{D}(A_i^a), \mathcal{D}(A_i^b)]_{1-\alpha}, \quad \forall\alpha\in(0,1), \; a\geq b\geq 0. \tag{3.85}$$

Then using (3.85) for $(a,b) = (1/2,0)$ with $\mathcal{D}(A_i^{1/2}) = \mathbf{V}^1(\Omega)$ yield $\mathcal{D}(A_i^r) = [\mathbf{V}^1(\Omega), \mathbf{V}^0(\Omega)]_{1/2-r}$ for all $r\in(0,1/2)$, and (3.79) for $r\in(0,1/2)$ follows from (3.62) with $(r_1,r_2) = (1,0)$. Next, to prove (3.79)-(3.80)-(3.81) for $r\in(1/2,3/2)$ we introduce

$$\mathbf{H}_1^3(\Omega) \overset{\text{def}}{=} \left\{y\in\mathbf{H}^3(\Omega) \;\middle|\; \frac{dy_\tau}{dn} = 0 \text{ on } \Gamma\right\} \quad \text{ and } \quad \mathbf{H}_2^3(\Omega) \overset{\text{def}}{=} \left\{y\in\mathbf{H}^3(\Omega) \;\middle|\; \nu\frac{dy_\tau}{dn} + (z_e\cdot n)y_\tau = 0 \text{ on } \Gamma\right\},$$

and we write (3.85) with $(a,b) = (3/2,1/2)$ as follows:

$$\mathcal{D}(A_i^r) = [\mathbf{H}_i^3(\Omega)\cap\mathbf{V}^1(\Omega), \mathbf{V}^1(\Omega)]_{3/2-r}, \quad \forall r\in(1/2,3/2).$$

Thus, we define the projection operator $\mathbf{P}_i : \mathbf{H}^1(\Omega)\to\mathbf{V}^1(\Omega)$ by

$$\mathbf{P}_i f \overset{\text{def}}{=} y \quad \text{ where } \quad a_i(y,v) = a_i(f,v) \quad \forall v\in\mathbf{V}^1(\Omega),$$

and with Lemma 3 and Corollary 8 we can verify that $\mathbf{P}_i$ is also continuous from $\mathbf{H}_i^3(\Omega)$ onto $\mathbf{H}_i^3(\Omega)\cap\mathbf{V}^1(\Omega)$. Then by applying [31, Thm. 1.17.1.1, p.118] we deduce that $\mathcal{D}(A_i^r) = [\mathbf{H}_i^3(\Omega), \mathbf{H}^1(\Omega)]_{3/2-r}\cap\mathbf{V}^1(\Omega)$ for $r\in(1/2,3/2)$. Finally, since by [31, Thm. 4.3.3.1, p.321] or [19] we have $\mathbf{H}^1(\Omega) = [\mathbf{H}_i^3(\Omega), \mathbf{L}^2(\Omega)]_{2/3}$ then the reiteration Theorem [31, Thm. 1.10.3.2, p.66] yields $[\mathbf{H}_i^3(\Omega), \mathbf{H}^1(\Omega)]_{3/2-r} = [\mathbf{H}_i^3(\Omega), [\mathbf{H}_i^3(\Omega), \mathbf{L}^2(\Omega)]_{2/3}]_{3/2-r} = [\mathbf{H}_i^3(\Omega), \mathbf{L}^2(\Omega)]_{1-2r/3}$ for $r\in(1/2,3/2)$, which gives $\mathcal{D}(A_i^r) = [\mathbf{H}_i^3(\Omega), \mathbf{L}^2(\Omega)]_{1-2r/3}\cap\mathbf{V}^1(\Omega)$ for $r\in(1/2,3/2)$ and [31, Thm. 4.3.3.1, p.321] or [19] allows to conclude. □

Let us now give an expression of B defined in (3.72) in terms of the Neumann operator associated with $\lambda_0 + A$. For $u\in\mathbf{L}^2(\Gamma)$ set $Nu = w$ where w obeys:

$$\lambda_0 w - \nu\Delta w + (w\cdot\nabla)z_e + (z_e\cdot\nabla)w + \nabla(Rw + Tu) = 0, \quad \nabla\cdot w = 0 \;\text{ in }\; \Omega, \quad \chi(w, Rw+Tu) = u \;\text{ on }\Gamma. \tag{3.86}$$

In the above setting, T is the linear mapping defined by $p = Tu$ where p is the solution to

$$-\Delta p = 0 \;\text{ in }\; \Omega \quad \text{ and } \quad p = u_n \;\text{ on }\; \Gamma. \tag{3.87}$$

For rough data $u\in\mathbf{L}^2(\Gamma)$, defining a solution to (3.86) can be done with the transposition method. It consists in looking for a velocity $w\in\mathbf{V}^0(\Omega)$ obeying:

$$\int_\Gamma u\cdot\varphi = \int_\Omega w\cdot f \quad \forall f\in\mathbf{V}^0(\Omega), \tag{3.88}$$

hal-00541563, version 1 - 30 Nov 2010

where $\varphi \in \mathbf{V}^2(\Omega)$ is the unique solution to $\widehat{A}^*\varphi = f$: $\varphi \in \mathbf{H}^2(\Omega)$ and

$$\lambda_0\varphi - \nu\Delta\varphi + {}^t(\nabla z_e)\varphi - (z_e\cdot\nabla)\varphi + \nabla S\varphi = f, \quad \nabla\cdot\varphi = 0, \quad \text{in } \Omega, \quad \chi_e(\varphi, S\varphi) = 0 \text{ on } \Gamma. \tag{3.89}$$

The existence and uniqueness of $w \in \mathbf{V}^0(\Omega)$ solution to (3.88) is a consequence of the Riesz representation theorem, and an integration by parts allows to prove that a smooth velocity (say $w \in \mathbf{V}^2(\Omega)$ and $u \in \mathbf{H}^{1/2}(\Gamma)$) solution to (3.86) in a classical sense is also the solution to (3.88).

Proposition 2. *The operator N is bounded from $\mathbf{L}^2(\Gamma)$ into $\mathbf{V}^0(\Omega)$ and it satisfies:*

$$N \in \mathcal{L}(\mathbf{H}^{2r-3/2}(\Gamma), \mathbf{V}^{2r}(\Omega)) \quad \text{for all} \quad r \in [0, 3/2]. \tag{3.90}$$

Proof. For $u \in \mathbf{H}^{3/2}(\Gamma)$, the successive use of Lax-Milgram theorem and Lemma 3 first ensures the existence of a unique pair $(w, q) \in \mathbf{V}^1(\Omega) \times L^2(\Omega)$ satisfying:

$$\int_\Omega (\lambda_0 w\cdot v + \nu\nabla w : \nabla v + (w\cdot\nabla)z_e\cdot v + (z_e\cdot\nabla)w\cdot v) - \int_\Omega q\nabla\cdot v = \int_\Gamma u\cdot v \qquad \forall v \in \mathbf{H}^1(\Omega). \tag{3.91}$$

Thus, since by Corollary 8 we have $(w, q) \in \mathbf{V}^3(\Omega) \times H^1(\Omega)$, an integration by part in (3.91) shows that (w, q) obeys:

$$\lambda_0 w - \nu\Delta w + (w\cdot\nabla)z_e + (z_e\cdot\nabla)w + \nabla q = 0 \quad \text{in} \quad \Omega, \quad \chi(w, q) = u \text{ on } \Gamma,$$

and by taking the divergence of the first above equality one verifies that $q = Rw + Tu$ and $Nu = w \in \mathbf{V}^3(\Omega)$. Then (3.90) for $r = 3/2$ follows. Next, let us prove that N can be uniquely extended to a bounded operator from $\mathbf{H}^{-3/2}(\Gamma)$ into $\mathbf{V}^0(\Omega)$. For $f \in \mathbf{V}^0(\Omega)$ and φ solution to (3.89) we obtain $\int_\Omega Nu\cdot f = \int_\Gamma u\cdot\varphi$ by setting $v = \varphi$ in (3.91) and integrating by parts. Thus, by taking the sup over all $f \in \mathbf{V}^0(\Omega)$, with $\|\varphi\|_{\mathbf{H}^{3/2}(\Gamma)} \le C\|\varphi\|_{\mathbf{V}^2(\Omega)}$ and $\|\varphi\|_{\mathbf{V}^2(\Omega)} \le C\|f\|_{\mathbf{V}^0(\Omega)}$ we deduce that $\|Nu\|_{\mathbf{V}^0(\Omega)} \le C\|u\|_{\mathbf{H}^{-3/2}(\Gamma)}$, and the density of $\mathbf{H}^{3/2}(\Gamma)$ into $\mathbf{H}^{-3/2}(\Gamma)$ ensures that N can be extended to a bounded operator from $\mathbf{H}^{-3/2}(\Gamma)$ into $\mathbf{V}^0(\Omega)$ in a unique way. Finally, (3.90) follows with an interpolation argument. ☐

Remark 7. *(i) Notice that to define a solution $w \in \mathbf{V}^0(\Omega)$ of (3.86) with the transposition method it is sufficient to have a boundary value u in the dual space of $\mathbf{V}^{3/2}(\Gamma) \stackrel{def}{=} \{y \in \mathbf{H}^{3/2}(\Gamma) \mid \int_\Gamma y\cdot n = 0\}$, that we denote by $\mathbf{V}^{-3/2}(\Gamma)$. Then in (3.88) the sign $\int_\Gamma$ must be understood as a duality product between $\mathbf{V}^{-3/2}(\Gamma)$ and $\mathbf{V}^{3/2}(\Gamma)$.*
(ii) In fact, for all solution $w \in \mathbf{V}^0(\Omega)$ defined by transposition the trace condition:

$$\chi(w, Rw + Tu) = u \quad \text{on} \quad \Gamma,$$

is still valid. Indeed, from (3.88) one verifies that the transposition solution belongs to the space

$$\Xi(\Omega) \stackrel{def}{=} \{(y, p) \in \mathbf{V}^0(\Omega) \times \mathcal{D}'(\Omega) \mid -\nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + \nabla p \in \mathbf{L}^2(\Omega)\}$$

normed with

$$\|(y, p)\|_{\Xi(\Omega)} \stackrel{def}{=} \|y\|_{\mathbf{L}^2(\Omega)} + \| - \nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + \nabla p\|_{\mathbf{L}^2(\Omega)}.$$

Moreover, we can verify that $\Xi(\Omega)$ is a Hilbert space and that $\mathbf{V}^2(\Omega) \times H^1(\Omega)$ is dense in $\Xi(\Omega)$, and then arguing as in [25, Thm. 6.5, Chap. 2] *we can prove that χ can be extended in a unique way to a bounded operator from $\Xi(\Omega)$ onto $\mathbf{V}^{-3/2}(\Gamma)$. Here is the argument. According to Lemma 2, for all $b \in \mathbf{V}^{3/2}(\Gamma)$ we can choose $(v_b, p_b) \in \mathbf{V}^2(\Omega) \times H^1(\Omega)$, which continuously depends on $b \in \mathbf{V}^{3/2}(\Gamma)$ and such that $\chi_e(v_b, p_b) = 0$ and $v_b = b$ on Γ. Thus, since an integration by parts ensures that every $(y, p) \in \mathbf{V}^2(\Omega) \times H^1(\Omega)$ obeys:*

$$\int_\Gamma \chi(y, p)\cdot b = \int_\Omega (-\nu\Delta v_b + {}^t(\nabla z_e)v_b - (z_e\cdot\nabla)v_b + \nabla p_b)\cdot y - \int_\Omega v_b\cdot(-\nu\Delta y + (y\cdot\nabla)z_e + (z_e\cdot\nabla)y + \nabla p),$$

then by taking the supremum over all $b \in \mathbf{V}^{3/2}(\Gamma)$ we deduce that:

$$\|\chi(y, p)\|_{\mathbf{V}^{-3/2}(\Gamma)} \le C(\|y\|_{\mathbf{V}^0(\Omega)} + \|\nu\Delta y - (y\cdot\nabla)z_e - (z_e\cdot\nabla)y + \nabla p\|_{\mathbf{L}^2(\Omega)}),$$

and the conclusion follows from a density argument.

Next, we recall that $m \in C^2(\Gamma; \mathbb{R}^+)$ is a compactly supported function of Γ which is not identically equal to zero.

Proposition 3. *The following equality holds:*

1. *For all $u \in \mathbf{L}^2(\Omega)$ we have $Bu = \widehat{A}N(mu) \in [\mathbf{V}^1(\Omega)]'$.*
2. *For all $v \in \mathbf{V}^1(\Omega)$ we have $B^*v = mv|_\Gamma$.*
3. *For all $\varepsilon \in]0, 1/4[$ we have $\widehat{A}^{-1/4-\varepsilon}B \in \mathcal{L}(\mathbf{L}^2(\Gamma), \mathbf{V}^0(\Omega))$.*

Proof. The two first statements are straightforward consequences of (3.88). Thus, from $\mathcal{D}(\widehat{A}^{*1/4+\varepsilon}) = \mathbf{V}^{1/2+2\varepsilon}(\Omega)$ we deduce that $\widehat{A}^{*-1/4-\varepsilon} \in \mathcal{L}(\mathbf{V}^0(\Omega), \mathbf{V}^{1/2+2\varepsilon}(\Omega))$ and with $B^* \in \mathcal{L}(\mathbf{V}^{1/2+2\varepsilon}(\Omega), \mathbf{V}^0(\Omega))$ we obtain $B^*\widehat{A}^{*-1/4-\varepsilon} \in \mathcal{L}(\mathbf{V}^0(\Omega), \mathbf{L}^2(\Gamma))$. Then the third statement follows from a duality argument. ☐

The following proposition states precisely the equivalence between formulations (3.74) and (3.70).

hal-00541563, version 1 - 30 Nov 2010

hal-00541563, version 1 - 30 Nov 2010

Proposition 4. *Let $r \in [0,1]$ and set*

$$W_{loc}^{2r} \stackrel{def}{=} W_{loc}(\mathbf{V}^{1+2r}(\Omega), [\mathbf{V}^{1-2r}(\Omega)]') \ \text{ if } \ r \le 1/2 \quad \text{ and } \quad W_{loc}^{2r} \stackrel{def}{=} W_{loc}(\mathbf{V}^{1+2r}(\Omega), \mathbf{V}^{-1+2r}(\Omega)) \ \text{ if } \ r > 1/2.$$

Then $y \in W_{loc}^{2r}$ obeys (3.74), if and only if, there is a unique $p \in L^2_{loc}(H^{2r}(\Omega))$ such that $(z,r) = (z_e, r_e) + (y,p)$ satisfies (3.70) for all $v \in \mathbf{H}^1(\Omega)$.

Proof. Let us show that formulation (3.74) implies formulation (3.70), which is the only non obvious fact to prove. Suppose that $y \in W^{2r}_{\mathrm{loc}}$ obeys (3.74), which means that:

$$\langle y'|v\rangle_{[\mathcal{D}(A^*)]',\mathcal{D}(A^*)} + a(y,v) + \int_\Gamma m\Pi y \cdot v + \int_\Omega (y\cdot\nabla y)\cdot v = 0 \quad \forall v \in \mathcal{D}(A^*).$$

Since $y' \in L^2_{\mathrm{loc}}([\mathbf{V}^1(\Omega)]')$ and $(y\cdot\nabla y) \in L^2_{\mathrm{loc}}([\mathbf{H}^1(\Omega)]')$ the above equality can be extended to $v \in \mathbf{V}^1(\Omega)$ with a density argument and by Lemma 3 there exists a unique $p \in L^2_{\mathrm{loc}}(L^2(\Omega))$ such that:

$$\langle \mathbf{P}^* y'|v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} + a(y,v) - \int_\Omega p\nabla\cdot v + \int_\Gamma m\Pi y\cdot v + \int_\Omega (y\cdot\nabla y)\cdot v = 0 \quad \forall v \in \mathbf{H}^1(\Omega). \tag{3.92}$$

In the above setting, $\mathbf{P}^* : [\mathbf{V}^1(\Omega)]' \to [\mathbf{H}^1(\Omega)]'$ is the extension of the injection operator $\mathbf{P}^* : \mathbf{V}^0(\Omega) \to \mathbf{L}^2(\Omega)$ (see Remark 6) and it is obvious to see that $\mathbf{P}^*$ is also bounded from $[\mathbf{V}^{1-2r}(\Omega)]'$ into $[\mathbf{H}^{1-2r}(\Omega)]'$ if $r \le 1/2$, or from $\mathbf{V}^{-1+2r}(\Omega)$ into $\mathbf{H}^{-1+2r}(\Omega)$ if $r > 1/2$. Then $y \in W^{2r}_{\mathrm{loc}}$ implies $\mathbf{P}^*y' + (y\cdot\nabla)y \in L^2_{\mathrm{loc}}([\mathbf{H}^{1-2r}(\Omega)]')$ if $r \le 1/2$, or $\mathbf{P}^*y' + (y\cdot\nabla)y \in L^2_{\mathrm{loc}}(\mathbf{H}^{-1+2r}(\Omega))$ if $r > 1/2$, and with $m(\Pi y)|_\Gamma \in \mathbf{H}^{3/2}(\Gamma)$ we obtain $p \in L^2_{\mathrm{loc}}(H^{2r}(\Omega))$ (apply Corollary 8 with an interpolation argument). Finally, for all $\phi \in C_0^\infty((0,\infty))$ the following calculation

$$\begin{aligned}\int_0^\infty \langle \mathbf{P}^* y'(t)|v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)}\phi(t)dt &= \langle \int_0^\infty y'(t)\phi(t)dt|\mathbf{P}v\rangle_{[\mathbf{V}^1(\Omega)]',\mathbf{V}^1(\Omega)} = \int_\Omega \Big(-\int_0^\infty y(t)\phi'(t)dt\Big)\cdot \mathbf{P}v \\ &= -\int_0^\infty \Big(\int_\Omega y(t)\cdot\mathbf{P}v\Big)\phi'(t)dt = -\int_0^\infty \Big(\int_\Omega y(t)\cdot v\Big)\phi'(t)dt,\end{aligned}$$

ensures that $\langle \mathbf{P}^* y'(t)|v\rangle_{[\mathbf{H}^1(\Omega)]',\mathbf{H}^1(\Omega)} = \frac{d}{dt}\int_\Omega y(t)\cdot v$. Then (3.92) becomes

$$\frac{d}{dt}\int_\Omega y(t)\cdot v + a(y(t),v) - \int_\Omega p(t)\nabla\cdot v + \int_\Gamma m\Pi y(t)\cdot v + \int_\Omega (y(t)\cdot\nabla)y(t)\cdot v = 0 \quad \forall v \in \mathbf{H}^1(\Omega),\ \forall t \ge 0,$$

and $(z,r) = (y+z_e, p+r_e)$ obeys the desired equation. □

We are then in the framework of Section 2 with $H = \mathbf{V}^0(\Omega)$, A and $\widehat{A} = \lambda_0 + A$ defined by (3.71), (3.76), $U = \mathbf{L}^2(\Gamma)$ and B defined by (3.72). Indeed, as required, A is the infinitesimal generator of an analytic semigroup on H and has bounded imaginary powers (Theorem 7), the mapping B obeys (2.7) with $\gamma \in (1/4, 1/2)$ (Proposition 3). Then problem (2.9)-(2.10)-(2.11) with $Z = \mathbf{V}^0(\Omega)$ and R equal to the identity in $\mathbf{V}^0(\Omega)$ guarantees the existence of a self-adjoint operator $\Pi \in \mathcal{L}(\mathbf{V}^0(\Omega), \mathcal{D}(\widehat{A}^*))$ which is the unique solution to the Riccati equation (3.66). Notice that such a problem is well-posed since the finite cost condition (2.12) can be obtained from [17] with a classical extension of the domain procedure. Then in order to obtain a local feedback stabilization theorem for system (3.74) it suffices to apply Theorem 4. But for such stabilization result to be relevant, one first need to characterize the spaces H^r_Π introduced in Theorem 3.

Proposition 5. *The following equalities holds:*

$$H^r_\Pi = \mathbf{V}^{2r}(\Omega) \quad \forall r \in [0,3/4), \tag{3.93}$$

$$H^r_\Pi = \{\xi \in \mathbf{V}^{2r}(\Omega) \mid \nu\frac{d\xi_\tau}{dn} + m(\Pi\xi)_\tau = 0 \text{ on } \Gamma\} \quad \forall r \in (3/4, 3/2]. \tag{3.94}$$

Proof. Let us first consider the case $r \in [0,1]$. From $B = \widehat{A}N$ and (2.35) we deduce that:

$$H^r_\Pi = \{\xi \in \mathbf{V}^0(\Omega) \mid \xi + N(m(\Pi\xi)|_\Gamma) \in \mathcal{D}(\widehat{A}^r)\}, \quad \forall r \in [0,1].$$

Thus, for $\xi \in \mathbf{V}^0(\Omega)$ the boundedness of Π from $\mathbf{V}^0(\Omega)$ into $\mathcal{D}(\widehat{A}^*) \hookrightarrow \mathbf{V}^2(\Omega)$ combined with and boundedness property of the trace operator yields $m(\Pi\xi)|_\Gamma \in \mathbf{H}^{3/2}(\Gamma)$. Then (3.90) yields $N(m(\Pi\xi)|_\Gamma) \in \mathbf{V}^3(\Omega) \hookrightarrow \mathbf{V}^{2r}(\Omega)$, and with $\mathcal{D}(\widehat{A}^r) \hookrightarrow \mathbf{V}^{2r}(\Omega)$ we deduce that H^r_Π is the closed subspace of $\mathbf{V}^{2r}(\Omega)$ defined by:

$$H^r_\Pi = \{\xi \in \mathbf{V}^{2r}(\Omega) \mid \xi + N(m(\Pi\xi)|_\Gamma) \in \mathcal{D}(\widehat{A}^r)\}. \tag{3.95}$$

If $r \in [0,3/4)$ then $\mathcal{D}(\widehat{A}^r) = \mathbf{V}^{2r}(\Omega)$ and (3.93) is an obvious consequence of (3.95). If $r \in (3/4,1]$, then $\xi + N(m(\Pi\xi)|_\Gamma) \in \mathcal{D}(\widehat{A}^r)$ means that $\xi + N(m(\Pi\xi)|_\Gamma) \in \mathbf{V}^r(\Omega)$ and $\nu\frac{d\xi_\tau}{dn} + \nu(\frac{dN(m(\Pi\xi)|_\Gamma)}{dn})_\tau = 0$. Then we obtain (3.94) by recalling that $\nu(\frac{dN(m(\Pi\xi)|_\Gamma)}{dn})_\tau = m(\Pi\xi)_\tau$ on Γ. Next, let us consider the case $r \in (1,3/2]$. Starting from $H^r_\Pi = \{\xi \in H^1_\Pi \mid A_\Pi\xi \in H^{r-1}_\Pi\}$ and using the fact that $H^{r-1}_\Pi = \mathbf{V}^{2r-2}(\Omega) = \mathcal{D}(\widehat{A}^{r-1})$ we first deduce that $H^r_\Pi = \{\xi \in H^1_\Pi \mid A_\Pi\xi \in \mathcal{D}(\widehat{A}^{r-1})\}$. Thus, since $A_\Pi\xi \in \mathcal{D}(\widehat{A}^{r-1})$ is equivalent to $\xi + N(m(\Pi\xi)|_\Gamma) \in \mathcal{D}(\widehat{A}^r)$, and according to the characterization of H^1_Π given by (3.95), we deduce that $H^r_\Pi = \{\xi \in \mathbf{V}^1(\Omega) \mid \xi + N(m(\Pi\xi)|_\Gamma) \in \mathcal{D}(\widehat{A}^r)\}$. Then with $N(m(\Pi\xi)|_\Gamma) \in \mathbf{V}^3(\Omega) \hookrightarrow \mathbf{V}^{2r}(\Omega)$ we obtain that (3.95) remains valid for $r \in (1,3/2]$ and the conclusion follows from (3.80) with $r \in (1,3/2]$, analogously as in the case $r \in [0,1]$. □

hal-00541563, version 1 - 30 Nov 2010

Finally, from (3.75) with $(s_1, s_2, s_3) = (2r, 2r, 1-2r)$ we verify that the nonlinear mapping F defined by (3.73) fits the assumptions (2.38)-(2.39) for $r \in (0,1]$ if $d=2$ and $r \in [\frac{1}{4}, 1]$ if $d=3$. Then Theorem 4 provides a stabilization result for the abstract system (3.74), and with Proposition 4 and Proposition 5 we obtain the following stabilization Theorem.

Theorem 8. *Let $r \in (0,1]\backslash\{\frac{3}{4}\}$ if $d=2$ or $r \in [\frac{1}{4},1]\backslash\{\frac{3}{4}\}$ if $d=3$, and set*

$$W^{2r} \stackrel{def}{=} W(\mathbf{V}^{1+2r}(\Omega), [\mathbf{V}^{1-2r}(\Omega)]') \;\; \text{if} \;\; r \leq 1/2 \qquad \text{and} \qquad W^{2r} \stackrel{def}{=} W(\mathbf{V}^{1+2r}(\Omega), \mathbf{V}^{-1+2r}(\Omega)) \;\; \text{if} \;\; r > 1/2.$$

Let $z_0 \in \{z_e\} + \mathbf{V}^{2r}(\Omega)$ and if $r \in (3/4, 1]$ we also assume that

$$\nu \frac{d(z_0)_\tau}{dn} + m(\Pi z_0)_\tau = \nu \frac{d(z_e)_\tau}{dn} + m(\Pi z_e)_\tau.$$

Then there exists $\mu > 0$ such that if $\|z_0 - z_e\|_{\mathbf{V}^{2r}(\Omega)} \leq \mu$, system (3.67), (3.68), admits a solution $(z,r) \in \{z_e, r_e\} + W^{2r} \times L^2(H^{2r}(\Omega))$ which is unique within the class of functions in $\{z_e, r_e\} + L^\infty_{loc}(\mathbf{V}^{2r}(\Omega)) \cap L^2_{loc}(\mathbf{V}^{1+2r}(\Omega)) \times L^2_{loc}(H^{2r}(\Omega))$. Moreover, for all $t \geq 0$ the following estimate holds:

$$\|z(t) - z_e\|_{\mathbf{H}^{2r}(\Omega)} \leq Ce^{-\sigma t}\|z_0 - z_e\|_{\mathbf{H}^{2r}(\Omega)}.$$

4. **Stabilization of Boussinesq equations with feedback or dynamical Dirichlet control.** In this section, we still consider an open subset Ω of $\mathbb{R}^d$ with $d=2$ or $d=3$ with a boundary Γ of class $C^{2,1}$ and we consider a trajectory (z, r, τ) of the Boussinesq equations:

$$\begin{aligned} \partial_t z - \Delta z + (z\cdot\nabla)z + \nabla r &= \tau e + f && \text{in } \Omega \times (0,+\infty), && (4.96)\\ \nabla\cdot z &= 0 && \text{in } \Omega \times (0,+\infty), && (4.97)\\ \partial\tau - \Delta\tau + z\cdot\nabla\tau &= h && \text{in } \Omega \times (0,+\infty). && (4.98) \end{aligned}$$

In the above setting, $z = z(x,t)$ represents the velocity of the particules of the fluid, $\tau = \tau(x,t)$ their temperature, $r = r(x,t)$ is the pressure function, e stands for the gravity vector field, and $f \in \mathbf{L}^2(\Omega)$ and $h \in L^2(\Omega)$. We consider here the question of stabilizing (z, r, τ) around a stationary state $(z_e, r_e, \tau_e) \in \mathbf{H}^2(\Omega) \times H^1(\Omega) \times H^2(\Omega)$ by means of boundary control. For $\bar{u} \stackrel{\text{def}}{=} (u_1, \dots, u_d) \in \mathbf{L}^2(\Gamma) \stackrel{\text{def}}{=} (L^2(\Gamma))^d$ and $u \stackrel{\text{def}}{=} (\bar{u}, u_{d+1}) \in (L^2(\Gamma))^{d+1}$, we consider the Dirichlet control

$$z = z_e + M(\bar{u}) \qquad \text{and} \qquad \tau = \tau_e + m u_{d+1} \qquad \text{on } \Gamma \times (0,+\infty), \tag{4.99}$$

where $m \in C^2(\Gamma; \mathbb{R}^+)$ is a compactly supported function of Γ which is not identically equal to zero and M is an operator used to localize the action of the control in the support of m, see (4.105) below. Then the triplet $(w, p, \theta) \stackrel{\text{def}}{=} (z - z_e, r - r_e, \tau - \tau_e)$ satisfies:

$$\begin{aligned} \partial_t w - \Delta w + (w\cdot\nabla)z_e + (z_e\cdot\nabla)w + (w\cdot\nabla)w + \nabla p &= \theta e && \text{in } \Omega \times (0,+\infty), && (4.100)\\ \nabla\cdot w &= 0 && \text{in } \Omega \times (0,+\infty), && (4.101)\\ \partial_t\theta - \Delta\theta + w\cdot\nabla\tau_e + z_e\cdot\nabla\theta + w\cdot\nabla\theta &= 0 && \text{in } \Omega \times (0,+\infty), && (4.102)\\ z &= M(\bar{u}) && \text{on } \Gamma \times (0,+\infty), && (4.103)\\ \theta &= m u_{d+1} && \text{on } \Gamma \times (0,+\infty), && (4.104) \end{aligned}$$

and the question of stabilizing (4.96), (4.97), (4.98) around (z_e, r_e, τ_e) is equivalent to the question of stabilizing (4.100), (4.101), (4.102) around zero.

In addition to notations of section 3, we need to define some other function spaces. Let $\mathcal{H}_0^{2r}(\Omega) \stackrel{\text{def}}{=} H^{2r}(\Omega)$ for $r \in [0, 1/4)$, let $\mathcal{H}_0^{\frac{1}{2}}(\Omega) \stackrel{\text{def}}{=} H^{\frac{1}{2}}(\Omega) \cap L^2_{-\frac{1}{2}}(\Omega)$, where $L^2_{-1/2}(\Omega)$ is the space of functions $y \in L^2(\Omega)$ such that $\int_\Omega \text{dist}(x,\Gamma)^{-1}|y|^2\, dx < +\infty$, and let $\mathcal{H}_0^{2r}(\Omega) \stackrel{\text{def}}{=} \{y \in H^{2r}(\Omega) \mid y = 0 \text{ on } \Gamma\}$ for $r \in (1/4, 1]$. Moreover, we set $\mathcal{H}_0^{2r}(\Omega) = [\mathcal{H}_0^{-2r}(\Omega)]'$ for $r \in [-1, 0]$. Let us also introduce:

$$\begin{aligned} \mathbf{V}_n^{2r}(\Omega) &\stackrel{\text{def}}{=} \Big\{ y \in \mathbf{H}^{2r}(\Omega) \; ; \; \nabla\cdot y = 0 \;\; \text{in } \Omega, \;\; y\cdot n = 0 \;\; \text{on } \Gamma \Big\}, \qquad r \geq 0,\\ \mathbf{V}_0^{2r}(\Omega) &\stackrel{\text{def}}{=} \Big\{ y \in \mathbf{H}^{2r}(\Omega) \; ; \; \nabla\cdot y = 0 \;\; \text{in } \Omega, \;\; y = 0 \;\; \text{on } \Gamma \Big\}, \qquad r > \frac{1}{4},\\ \mathbf{V}^{2r}(\Gamma) &\stackrel{\text{def}}{=} \Big\{ y \in \mathbf{H}^{2r}(\Gamma) \; ; \; \int_\Gamma y\cdot n = 0 \Big\}, \qquad r \in [0,1]. \end{aligned}$$

Moreover, we define $\mathbf{V}_0^{2r}(\Omega)$ for $r \in [0, \frac{1}{4})$ by $\mathbf{V}_0^{2r}(\Omega) \stackrel{\text{def}}{=} \mathbf{V}_n^{2r}(\Omega)$, for $r = 1/4$ by $\mathbf{V}_0^{\frac{1}{2}}(\Omega) \stackrel{\text{def}}{=} \{y \in \mathbf{V}_n^{\frac{1}{2}}(\Omega) \mid y \in (L^2_{-1/2}(\Omega))^d\}$, and for $r < 0$ by $\mathbf{V}_0^{2r}(\Omega) \stackrel{\text{def}}{=} [\mathbf{V}_0^{-2r}(\Omega)]'$. Notice that the subscript 0 in $\mathcal{H}_0^{2r}(\Omega)$ and in $\mathbf{V}_0^{2r}(\Omega)$ only means that one may have a vanishing Dirichlet boundary condition.

In order to rewrite the system in the form (1.1), we introduce:

1. the (Leray) orthogonal projection operator $P : \mathbf{L}^2(\Omega) \to \mathbf{V}_n^0(\Omega)$. Notice that from the Neumann problem related to P we can verify that $P \in \mathcal{L}(\mathbf{H}^{2r}(\Omega), \mathbf{V}_n^{2r}(\Omega))$ for $r \in [0,1]$, [18, Chap. I, Thm. 1.10].

hal-00541563, version 1 - 30 Nov 2010

2. the Oseen operator:

$$\mathcal{D}(A_1) \stackrel{\text{def}}{=} \mathbf{V}_0^2(\Omega) \quad \text{and} \quad A_1\varphi \stackrel{\text{def}}{=} P(-\Delta\varphi + (\varphi\cdot\nabla)z_e + (z_e\cdot\nabla)\varphi).$$

Notice that $-A_1$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}_n^0(\Omega)$, that $\widehat{A}_1 \stackrel{\text{def}}{=} \lambda_0 + A_1$ for $\lambda_0 > 0$ large enough has bounded imaginary powers, that the adjoint of A_1 is given by

$$\mathcal{D}(A_1^*) \stackrel{\text{def}}{=} \mathbf{V}_0^2(\Omega) \quad \text{and} \quad A_1^*\varphi \stackrel{\text{def}}{=} P(-\Delta\varphi + {}^t(\nabla z_e)\varphi - (z_e\cdot\nabla)\varphi),$$

and that $\mathcal{D}(\widehat{A}_1^\alpha) = \mathcal{D}(\widehat{A}_1^{*\alpha}) = \mathbf{V}_0^{2\alpha}(\Omega)$ for $\alpha \in [0,1]$, see for instance [5].

3. the Dirichlet operator $D_1 : \mathbf{V}^0(\Gamma) \longrightarrow \mathbf{V}^0(\Omega)$ associated with $\lambda_0 + A_1$: $D_1 v \stackrel{\text{def}}{=} \varphi$ where φ is the solution of

$$\lambda_0\varphi - \Delta\varphi + (\varphi\cdot\nabla)z_e + (z_e\cdot\nabla)\varphi + \nabla q = 0 \quad \text{and} \quad \nabla\cdot\varphi = 0 \ \text{ in } \ \Omega, \qquad \varphi = v \ \text{ on } \ \Gamma.$$

Notice that $D_1 \in \mathcal{L}(\mathbf{V}^{2r}(\Gamma), \mathbf{V}^{2r+\frac{1}{2}}(\Omega))$ for $r \in [0, 3/4]$, see [29].

4. the localization self-adjoint operator $M \in \mathcal{L}(\mathbf{L}^2(\Gamma); \mathbf{V}^0(\Gamma))$:

$$Mv \stackrel{\text{def}}{=} m\left(v - \left(\int_\Gamma m\right)^{-1}\left(\int_\Gamma mv\cdot n\right) n\right), \tag{4.105}$$

where $m \in C^2(\Gamma; \mathbb{R}^+)$ is a compactly supported function of Γ which is not identically equal to zero. Notice that $M \in \mathcal{L}(\mathbf{V}^{2r}(\Gamma), \mathbf{V}^{2r}(\Gamma))$ for $r \in [0,1]$ and that $\mathrm{Supp}(M(v)) \subset \mathrm{Supp}(m)$.

5. the input operator

$$B_1 u = \widehat{A}_1 P D_1 M(\bar{u}) : (L^2(\Gamma))^d \to [\mathcal{D}(A^*)]'.$$

Notice that B_1 obeys (2.7) with $\gamma \in (3/4, 1)$, see [4, Prop. 2], and according to [4, Prop. 3] its adjoint is given by

$$B_1^*\varphi = \begin{pmatrix} -m\dfrac{d\varphi}{dn} + m\phi(\varphi)n \\ 0 \end{pmatrix},$$

where $\phi(\varphi)$ is the solution to the Neumann problem:

$$\Delta\phi = \nabla\cdot(z_e\cdot\nabla - {}^t(\nabla z_e))\varphi \ \text{ in } \Omega, \quad \int_\Gamma m\phi = 0, \tag{4.106}$$

$$\frac{d\phi}{dn} = (\Delta - {}^t(\nabla z_e) + z_e\cdot\nabla)\varphi\cdot n \ \text{ on } \Gamma. \tag{4.107}$$

6. the heat type operator on $L^2(\Omega)$:

$$\mathcal{D}(A_2) \stackrel{\text{def}}{=} \mathcal{H}_0^2(\Omega) \quad \text{and} \quad A_2\varrho \stackrel{\text{def}}{=} -\Delta\varrho + z_e\cdot\nabla\varrho.$$

Notice that since $-\Delta$ is the infinitesimal generator of a stable analytic semigroup on $L^2(\Omega)$ and has bounded imaginary powers, a perturbation argument ensures that so does $-\widehat{A}_2$ where $\widehat{A}_2 \stackrel{\text{def}}{=} \lambda_0 + A_2$ for $\lambda_0 > 0$ large enough, see [26, Chap.3, Cor.2.4] and [16, Prop. 2.7]. Moreover, the adjoint of A_2 is given by

$$\mathcal{D}(A_2^*) = \mathcal{H}_0^2(\Omega) \quad \text{and} \quad A_2^*\varrho \stackrel{\text{def}}{=} -\Delta\varrho - z_e\cdot\nabla\varrho,$$

and we have $\mathcal{D}(\widehat{A}_2^\alpha) = \mathcal{D}(\widehat{A}_2^{*\alpha}) = \mathcal{H}_0^{2\alpha}(\Omega)$ for $\alpha \in [0,1]$.

7. the Dirichlet operator $D_2 : L^2(\Gamma) \longrightarrow L^2(\Omega)$ associated with $\lambda_0 + A_2$: $D_2 b = \varrho$ where ϱ is the solution of

$$\lambda_0\varrho - \Delta\varrho + z_e\cdot\nabla\varrho = 0 \ \text{ in } \ \Omega, \qquad \varrho = b \ \text{ on } \ \Gamma.$$

Notice that $D_2 \in \mathcal{L}(H^{2r}(\Gamma), H^{2r+\frac{1}{2}}(\Omega))$ for $r \in [0, 3/4]$.

8. the input operator

$$B_2 u = -(I-P)D_1 M(\bar{u})\cdot\nabla\tau_e + \widehat{A}_2 D_2(m u_{d+1}) : (L^2(\Gamma))^{d+1} \to [\mathcal{D}(A_2^*)]'.$$

Notice that from the regularizing property of P, D_1, M and D_2 one can verify that B_1 obeys (2.7) with $\gamma \in (3/4, 1)$. Moreover, from the expression of D_1^*, see [29], and from the Neumann problem related to P one verifies that the adjoint $B_2^* \in \mathcal{L}(\mathcal{D}(A_2^*), (L^2(\Gamma))^{d+1})$ is given by

$$B_2^*\varrho = \begin{pmatrix} -MD_1^*(I-P)(\nabla\tau_e\varrho) \\ mD_2^*\widehat{A}_2^*\varrho \end{pmatrix} = \begin{pmatrix} m\chi(\varrho)n \\ -m\dfrac{d\varrho}{dn} \end{pmatrix},$$

where $\chi(\varrho) = \chi$ is the unique solution to the Neumann problem:

$$-\Delta\chi = \nabla\cdot(\nabla\tau_e\varrho) \ \text{ in } \Omega, \quad \int_\Omega m\chi = 0, \tag{4.108}$$

$$\frac{d\chi}{dn} = 0 \ \text{ on } \Gamma. \tag{4.109}$$

According to [29] system (4.100), (4.101), (4.102), (4.103), (4.104) can be equivalently rewritten in the abstract form:

$$\begin{array}{rcl} Pw' + A_1Pw + P(w\cdot\nabla)w - P\theta e &=& \widehat{A}_1P_1D_1M(\bar{u}) \in [\mathcal{D}(A_1^*)]', \quad Py(0) = P(z(0)-z_e),\\ (I-P)w &=& (I-P)D_1M(\bar{u}),\\ \theta' + A_2\theta + w\cdot\nabla\tau_e + w\cdot\nabla\theta &=& \widehat{A}_2D_2(mu_{d+1}) \in [\mathcal{D}(A_2)]', \quad \theta(0) = \tau(0)-\tau_e. \end{array}$$

Then with $w = Pw + (I-P)D_1M(\bar{u})$, by setting

$$G_1(Pw,\bar{u}) = P((Pw + (I-P)D_1M(\bar{u}))\cdot\nabla)(Pw + (I-P)D_1M(\bar{u})),$$

and

$$G_2(Pw,\bar{u},\theta) = ((Pw + (I-P)D_1M(\bar{u}))\cdot\nabla\theta,$$

and by renaming $z(0)-z_e$ by w_0 and $\tau(0)-\tau_e$ by θ_0 for simplicity, system (4.100), (4.101), (4.102), (4.103), (4.104) can be equivalently rewritten in the following abstract form:

$$Pw' + A_1Pw - P\theta e + G_1(Pw,\bar{u}) = B_1u \in [\mathcal{D}(A_1^*)]', \quad Pw(0) = Pw_0, \tag{4.110}$$

$$\theta' + A_2\theta + Pw\cdot\nabla\tau_e + G_2(Pw,\bar{u},\theta) = B_2u \in [\mathcal{D}(A_2^*)]', \quad \theta(0) = \theta_0. \tag{4.111}$$

Thus, we introduce:

9. the closed and densely defined linear operator on $\mathbf{V}^0_n(\Omega)\times L^2(\Omega)$

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \mathcal{D}(A_1)\times\mathcal{D}(A_2) \quad\text{and}\quad A\begin{pmatrix}\varphi\\ \varrho\end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} A_1\varphi - P\varrho e\\ A_2\varrho + \varphi\cdot\nabla\tau_e\end{pmatrix}.$$

Notice that an easy verification shows that the adjoint of A is given by

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \mathcal{D}(A_1^*)\times\mathcal{D}(A_2^*) \quad\text{and}\quad A^*\begin{pmatrix}\varphi\\ \varrho\end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} A_1^* + \varrho\nabla\tau_e\\ A_2^*\varrho - \varphi\cdot e\end{pmatrix},$$

and that the known properties of A_1, A_2 combined with a perturbation argument ensures that $-A$ generates an analytic semigroup on $\mathbf{V}^0_n(\Omega)\times L^2(\Omega)$, that $\widehat{A} \stackrel{\text{def}}{=} \lambda_0 + A$ for $\lambda_0 > 0$ large enough has bounded imaginary powers and that $\mathcal{D}(\widehat{A}^\alpha) = \mathcal{D}(\widehat{A}^{*\alpha}) = \mathbf{V}^2_0(\Omega)\times\mathcal{H}^{2\alpha}_0(\Omega)$, for $\alpha\in[0,1]$.

10. the linear input operator

$$B \stackrel{\text{def}}{=} \begin{pmatrix} B_1\\ B_2\end{pmatrix} : (L^2(\Gamma))^{d+1} \to [\mathcal{D}(A^*)]'.$$

Notice that from the known properties of B_1 and B_2, we have that B obeys (2.7) with $\gamma\in(3/4,1)$ and that:

$$\widehat{A}^{-1}B \in \mathcal{L}((H^{2r}(\Gamma))^{d+1}), \mathbf{V}^{2r}_n(\Omega)\times H^{2r}(\Omega)), \quad r\in[0,1]. \tag{4.112}$$

Moreover, its adjoint is given by

$$B^*\begin{pmatrix}\varphi\\ \varrho\end{pmatrix} = \begin{pmatrix} -m\dfrac{d\varphi}{dn} + m(\phi(\varphi) + \chi(\varrho))n\\ -m\dfrac{d\varrho}{dn}\end{pmatrix}$$

where $\phi(\varphi)$ and $\chi(\varrho)$ are the respective solution to (4.106), (4.107) and to (4.108), (4.109).

Next, by setting:

$$y \stackrel{\text{def}}{=} \begin{pmatrix} Pw\\ \theta\end{pmatrix}, \quad y_0 \stackrel{\text{def}}{=} \begin{pmatrix} Pw_0\\ \theta_0\end{pmatrix} \quad\text{and}\quad G(y,u) \stackrel{\text{def}}{=} \begin{pmatrix} G_1(Pw,u)\\ G_2(Pw,u,\theta)\end{pmatrix},$$

system (4.110), (4.111) can be rewritten as follows:

$$y' + Ay + G(y,u) = Bu \in [\mathcal{D}(A^*)]', \quad y(0) = y_0. \tag{4.113}$$

With the change of variable $y = {}^t(P(z-z_e), \tau-\tau_e)$ we have then transformed (4.96),(4.97),(4.98),(4.99) to the abstract system (4.113) with $y_0 = {}^t(P(z(0)-z_e), \tau(0)-\tau_e)$. Moreover, operators A and B fit the framework of section 2 with $H = \mathbf{V}^0_n(\Omega)\times L^2(\Omega)$ and $U = (L^2(\Gamma))^{d+1}$: $\widehat{A}$ has bounded imaginary powers, $-A$ generates an analytic semigroup on H and B satisfies (2.7) for $\gamma\in(3/4,1)$. Notice also that the required finite cost condition (2.12) can be obtained from the null controllability results with internal control stated in [22], by means of a usual geometrical extension procedure. Thus, since we have $\mathcal{D}(\widehat{A}^{1/2}) = \mathcal{D}(\widehat{A}^{*1/2}) = \mathbf{V}^1_0(\Omega)\times\mathcal{H}^1_0(\Omega)$ we can apply the abstract Theory of Subsection 2.2 with $Z = \mathbf{V}^0_n(\Omega)\times L^2(\Omega)$ and R equal to the identity in $\mathbf{V}^0_n(\Omega)\times L^2(\Omega)$: for a prescribed rate $\sigma > 0$ problem (2.10), (2.11) guarantees the existence of a self-adjoint operator $\Pi\in\mathcal{L}(\mathbf{V}^0_n(\Omega)\times L^2(\Omega), \mathbf{V}^2_0(\Omega)\times\mathcal{H}^2_0(\Omega))$ which is the unique solution to the Riccati equation (2.18). Notice that it will be convenient to write Π as follows:

$$\Pi = \begin{pmatrix} \pi_1 & \pi_2^*\\ \pi_2 & \pi_3\end{pmatrix} \quad\text{where}\quad \pi_1\in\mathcal{L}(\mathbf{V}^0_n(\Omega)), \quad \pi_2\in\mathcal{L}(\mathbf{V}^0_n(\Omega), L^2(\Omega)) \quad\text{and}\quad \pi_3\in\mathcal{L}(L^2(\Omega))$$

so that for $\xi = {}^t(\varphi,\varrho)\in\mathbf{V}^0_n(\Omega)\times L^2(\Omega)$ we have $\Pi\xi = {}^t(\pi_1\varphi + \pi_2^*\varrho, \pi_2\varphi + \pi_3\varrho)$. Moreover, the nonlinear abstract system subjected to the feedback control $u = -B^*\Pi y$ has the following form:

$$y' + A_\Pi y + F(y) = 0, \quad y(0) = y_0, \tag{4.114}$$

hal-00541563, version 1 - 30 Nov 2010

where $F(y) \stackrel{\text{def}}{=} G(y, -B^*\Pi y)$. In a way similar to what is done in [4, Cor. 6] one can prove that the space $H^r_\Pi \stackrel{\text{def}}{=} \mathcal{D}(A^r_\Pi)$ is a closed linear subspace of $\mathbf{V}^{2r}_n(\Omega) \times H^{2r}(\Omega)$ when $r \in [0,1]$. More precisely, as in [4] on can prove that when $r \neq 1/4$ we have $H^r_\Pi = \{(P\varphi, \varrho) \mid (\varphi, \varrho) \in \Xi^{2r}_\Pi(\Omega)\}$, where for $r \in [0, 1/4)$ we have $\Xi^{2r}_\Pi(\Omega) \stackrel{\text{def}}{=} \mathbf{V}^{2r}(\Omega) \times H^{2r}(\Omega)$, and for $r \in (1/4, 1]$ the space $\Xi^{2r}_\Pi(\Omega)$ is composed with elements $(\varphi, \varrho) \in \mathbf{V}^{2r}(\Omega) \times H^{2r}(\Omega)$ which satisfy the trace condition

$$\begin{aligned} \varphi &= m^2 \frac{d}{dn}(\pi_1 P\varphi + \pi_2^* \varrho) - m^2(\phi((\pi_1 P\varphi + \pi_2^*\varrho)) + \chi((\pi_2 P\varphi + \pi_3\varrho)))n \quad \text{on} \quad \Gamma, \\ \varrho &= m^2 \frac{d\varrho}{dn} \quad \text{on} \quad \Gamma. \end{aligned}$$

Then since the nonlinear mappings $w \mapsto (w \cdot \nabla)w$ and $(w, \theta) \mapsto w \cdot \nabla\theta$ satisfy the following estimates for $r \in (\frac{d-2}{4}, \frac{1}{2}]$:

$$\|(w \cdot \nabla)w\|_{\mathbf{H}^{2r-1}(\Omega)} \leq C\|w\|_{\mathbf{H}^{2r}(\Omega)}\|w\|_{\mathbf{H}^{1+2r}(\Omega)}, \tag{4.115}$$

$$\|w \cdot \nabla\theta\|_{H^{2r-1}(\Omega)} \leq C\|w\|_{\mathbf{H}^{2r}(\Omega)}\|\theta\|_{H^{1+2r}(\Omega)}, \tag{4.116}$$

we can deduce that $y \mapsto F(y)$ obeys (2.38)-(2.39) for $r \in (\frac{d-2}{4}, \frac{1}{2}]$, and Theorem 4 provides a local stabilization result for system (4.114) with $y_0 \in H^r_\Pi$ and $r \in (\frac{d-2}{4}, \frac{1}{2}]$. Then with an easy adaptation of [4, Thm. 12] one can obtain a stabilization theorem for system (4.96), (4.97), (4.98) with the boundary condition

$$\begin{aligned} z - z_e &= m^2 \frac{d}{dn}(\pi_1 P(z - z_e) + \pi_2^*(\tau - \tau_e)) \qquad (4.117)\\ &\quad -m^2(\phi((\pi_1 P(z - z_e) + \pi_2^*(\tau - \tau_e))) + \chi((\pi_2 P(z - z_e) + \pi_3(\tau - \tau_e))))n \quad \text{on} \quad \Gamma \times (0, +\infty), \\ \tau - \tau_e &= m^2 \frac{d(\tau - \tau_e)}{dn} \quad \text{on} \quad \Gamma \times (0, +\infty), \qquad (4.118) \end{aligned}$$

and with initial data

$$z(0) = z_0 \quad \text{and} \quad \tau(0) = \tau_0. \tag{4.119}$$

Theorem 9. *Let $r \in (\frac{d-2}{4}, \frac{1}{2}] \backslash \{\frac{1}{4}\}$ and $(z_0, \tau_0) \in \{(z_e, \tau_e)\} + \mathbf{V}^{2r}(\Omega) \times H^{2r}(\Omega)$. If $r > \frac{1}{4}$ we also assume that $(z_0 - z_e, \tau_0 - \tau_e) \in \Xi^{2r}_\Pi(\Omega)$, which is to say that the following initial compatibility conditions are satisfied:*

$$\begin{aligned} z_0 - z_e &= m^2 \frac{d}{dn}(\pi_1 P(z_0 - z_e) + \pi_2^*(\tau_0 - \tau_e)) \qquad (4.120)\\ &\quad -m^2(\phi((\pi_1 P(z_0 - z_e) + \pi_2^*(\tau - \tau_e))) + \chi((\pi_2 P(z_0 - z_e) + \pi_3(\tau_0 - \tau_e))))n \quad \textit{on} \quad \Gamma, \\ \tau_0 - \tau_e &= m^2 \frac{d(\tau_0 - \tau_e)}{dn} \quad \textit{on} \quad \Gamma. \qquad (4.121) \end{aligned}$$

Then there exist $\rho > 0$ and $\mu > 0$ such that, if $\delta \in (0, \mu)$ and $\|P(z_0 - z_e)\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau_0 - \tau_e\|_{H^{2r}(\Omega)} \leq \delta$, system (4.96), (4.97), (4.98), (4.117), (4.118), (4.119) admits a solution (z, r, τ) in

$$\{(z_e, r_e, \tau_e)\} + L^2(\mathbf{V}^{1+2r}(\Omega)) \cap H^{1/2+r}(\mathbf{L}^2(\Omega)) \times H^{-1/2+r}(H^{2r}(\Omega)/\mathbb{R}) \times W(H^{2r+1}(\Omega), H^{2r-1}(\Omega)),$$

which is unique within the class of function in

$$\{(z_e, r_e, \tau_e)\} + L^2_{loc}(\mathbf{V}^{1+2r}(\Omega)) \cap H^{1/2+r}_{loc}(\mathbf{L}^2(\Omega)) \times H^{-1/2+r}_{loc}(H^{2r}(\Omega)/\mathbb{R}) \times W_{loc}(H^{2r+1}(\Omega), H^{2r-1}(\Omega)).$$

Moreover, for all $t \geq 0$ the following estimate holds:

$$\|z(t) - z_e\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau(t) - \tau_e\|_{H^{2r}(\Omega)} \leq Ce^{-\sigma t}(\|z_0 - z_e\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau_0 - \tau_e\|_{H^{2r}(\Omega)}).$$

According to the above theorem, to define a solution to system (4.96), (4.97), (4.98), (4.117), (4.118), (4.119) when $d = 3$ one must impose the initial velocity to fit the feedback trace condition (4.120), (4.121). Since such a condition is very restrictive in practice, another strategy consists in looking for a control function $u \stackrel{\text{def}}{=} (\bar{u}, u_{d+1}) \stackrel{\text{def}}{=} (u_1, \ldots, u_{d+1})$ itself solution to an evolution equation:

$$\partial_t u_i - \Delta_\Gamma u_i = g_i \quad \text{in} \quad \Gamma \times (0, +\infty), \quad i = 1, \ldots, d+1, \tag{4.122}$$

where $g \stackrel{\text{def}}{=} (g_1, \ldots, g_{d+1})$ plays the role of a control function for the whole system (4.96), (4.97), (4.98), (4.122). In the above setting, Δ_Γ denotes the Laplace Beltrami operator. Then if we consider (4.122) with the initial condition $u(0) = 0$, every initial datum obeying $z_0 = z_e$ and $\tau_0 = \tau_e$ on Γ would fit the initial compatibility conditions $z_0 - z_e = M\bar{u}(0)$ and $\tau_0 - \tau_e = mu_{d+1}(0)$ on Γ. It then allows to define a fixed-point solution to the Boussinesq system, see the introduction of [3] for the particular case of Navier-Stokes equations. In the following, we apply the framework of section 2.3 to contruct a stabilizing control g in the feedback form:

$$g(t) = \mathfrak{F}(z(t) - z_e, \tau(t) - \tau_e, u) \quad \forall t \geq 0.$$

Let us denote by Δ_b the vectorial Laplace Beltrami operator, i.e. $(\Delta_b u)_i = \Delta_\Gamma u_i$, for all $i = 1, \ldots, d+1$, and let us introduce:

hal-00541563, version 1 - 30 Nov 2010

hal-00541563, version 1 - 30 Nov 2010

11. the unbounded operator E on $U \stackrel{\text{def}}{=} (L^2(\Gamma))^{d+1}$:

$$\mathcal{D}(E) = (H^2(\Gamma))^{d+1} \quad \text{and} \quad Eu = -\Delta_b u.$$

Notice that E is self-adjoint, that $-E$ is the infinitesimal generator of an analytic semigroup on $(L^2(\Gamma))^{d+1}$, that $\widehat{E} \stackrel{\text{def}}{=} \lambda_0 + E$ for $\lambda_0 > 0$ has bounded imaginary powers and that $\mathcal{D}(\widehat{E}^{\alpha}) = \mathcal{D}(\widehat{E}^{*\alpha}) = (H^{2\alpha}(\Gamma))^{d+1}$ for $\alpha \in [0,1]$.

Thus, (4.122) can be simply rewritten as

$$u' + Eu = g,$$

and Subsection 2.3 applies with $\mathbb{A}$ defined from the pair (A, B) which has been introduced above, and for $\mathbb{R}$ given in Remark 4 with R and Θ equal to the identity in $\mathbf{V}^0_n(\Omega) \times L^2(\Omega)$ and in $(L^2(\Gamma))^{d+1}$ respectively. Indeed, the assumption $B^* \widehat{A}^{*-1} \in \mathcal{L}(H^{1/2}_*, U^{1/2}_*) = \mathcal{L}(\mathbf{V}^1_0(\Omega) \times \mathcal{H}^1_0(\Omega), (H^1(\Gamma))^{d+1})$ is a consequence of regularity results for the Oseen and for the heat equation which guarantees that $\mathcal{D}(A^{*3/2}) \hookrightarrow \mathbf{V}^3_0(\Omega) \times H^3(\Omega)$ and then $B^*(\mathcal{D}(A^{*3/2})) \hookrightarrow U^{1/2} = (H^1(\Gamma))^{d+1}$. Notice that the finite cost condition (2.52) can be obtained from the null controllability result [22], by using Theorem 6, if we additionally assume that Γ is of class $C^{3,1}$. Indeed, with such an assumption regularity results for the Oseen and for the heat equation guarantee $\mathcal{D}(A^{*2}) \hookrightarrow \mathbf{V}^4_0(\Omega) \times H^4(\Omega)$ and that $B^*(\mathcal{D}(A^{*2})) \hookrightarrow U^1_* = (H^2(\Gamma))^{d+1}$. As a consequence, for a prescribed rate $\sigma > 0$ we have the existence of a self-adjoint operator $\mathbf{\Pi} \in \mathcal{L}(\mathbb{H}, \mathbb{H}^1_*)$ which is the unique solution to the Riccati equation (2.53). Notice that it will be convenient to write $\mathbf{\Pi}$ as follows:

$$\mathbf{\Pi} = \begin{pmatrix} \pi_1 & \pi_2^* & \pi_3^* \\ \pi_2 & \pi_4 & \pi_5^* \\ \pi_3 & \pi_5 & \pi_6 \end{pmatrix} \in \mathcal{L}(\mathbf{V}^0_n(\Omega) \times L^2(\Omega) \times (L^2(\Gamma))^{d+1}, \mathbf{V}^2_0(\Omega) \times \mathcal{H}^2_0(\Omega) \times (H^2(\Gamma))^{d+1}),$$

with $\pi_1 = \pi_1^* \in \mathcal{L}(\mathbf{V}^0_n(\Omega))$, $\pi_2 \in \mathcal{L}(\mathbf{V}^0_n(\Omega), L^2(\Omega))$, $\pi_3 \in \mathcal{L}(\mathbf{V}^0_n(\Omega), (L^2(\Gamma))^{d+1})$, $\pi_4 = \pi_4^* \in \mathcal{L}(L^2(\Omega))$, $\pi_5 \in \mathcal{L}(L^2(\Omega), (L^2(\Gamma))^{d+1})$ and $\pi_6 = \pi_6^* \in \mathcal{L}((L^2(\Gamma))^{d+1})$. In terms of the notations (2.57) we have the following correspondances:

$$\Pi_1 = \begin{pmatrix} \pi_1 & \pi_2^* \\ \pi_2 & \pi_4 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} \pi_3 & \pi_5 \end{pmatrix} \quad \text{and} \quad \Pi_3 = \pi_6.$$

Then (2.58), (2.59) with $u_0 = 0$ can be rewritten as (4.96), (4.97), (4.98) with the following boundary conditions

$$z = z_e + M(\bar{u}) \quad \text{and} \quad \tau = \tau_e + m u_{d+1} \quad \text{on} \quad \Gamma \times (0, +\infty), \tag{4.123}$$

and

$$\partial_t u - \Delta_b u + \pi_6 u + \pi_3 P(z - z_e) + \pi_5(\tau - \tau_e) = 0 \quad \text{in} \quad \Gamma \times (0, +\infty), \tag{4.124}$$

and with the following initial conditions

$$z(0) = z_0, \quad \tau(0) = \tau_0 \quad \text{and} \quad u(0) = 0. \tag{4.125}$$

Finally, from (4.112) we deduce that the space $\mathbb{H}^r$ for $r \in [0,1]$ (defined in (2.48), (2.46)) is the closed subspace of $\mathbf{V}^{2r}_n(\Omega) \times H^{2r}(\Omega) \times (H^{2r}(\Gamma))^{d+1}$ defined by:

$$\mathbb{H}^r = \{ (P\varphi, \varrho, u) \mid (\varphi, \varrho, u) \in \mathbf{V}^{2r}(\Omega) \times H^{2r}(\Omega) \times (H^{2r}(\Gamma))^{d+1} \text{ s. t. } \varphi = M(\bar{u}) \text{ and } \varrho = u_{d+1} \text{ on } \Gamma \}, \quad r \in [0,1],$$

and that $\mathbb{H}^{-r} = \mathbf{V}^{-2r}_0(\Omega) \times \mathcal{H}^{-2r}_0(\Omega) \times (H^{-2r}(\Gamma))^{d+1}$ for $r \in [0,1]$. Then it follows that (2.55), (2.56) can be obtained from (4.115), (4.116) and the following stabilization theorem follows from corollary 7.

Theorem 10. *Assume that Γ is of class $C^{3,1}$, let $r \in (\frac{1}{4}, \frac{1}{2}]$ and $(z_0, \tau_0) \in \{(z_e, \tau_e)\} + \mathbf{V}^{2r}_0(\Omega) \times \mathcal{H}^{2r}_0(\Omega)$. There exist $\rho > 0$ and $\mu > 0$ such that, if $\delta \in (0, \mu)$ and $\|z_0 - z_e\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau_0 - \tau_e\|_{H^{2r}(\Omega)} \leq \delta$, system* (4.96), (4.97), (4.98), (4.123), (4.124), (4.125) *admits a solution (z, r, τ, u) in*

$$\{(z_e, r_e, \tau_e, 0)\} + W(\mathbf{V}^{2r+1}(\Omega), \mathbf{V}^{2r-1}_0(\Omega)) \times H^{-1/2+r}(H^{2r}(\Omega)/\mathbb{R}) \\ \times W(H^{2r+1}(\Omega), H^{2r-1}(\Omega)) \times W((H^{2r+1}(\Gamma))^{d+1}, (H^{2r-1}(\Gamma))^{d+1}),$$

which is unique within the class of function in

$$\{(z_e, r_e, \tau_e, 0)\} + W_{loc}(\mathbf{V}^{2r+1}(\Omega), \mathbf{V}^{2r-1}_0(\Omega)) \times H^{-1/2+r}_{loc}(H^{2r}(\Omega)/\mathbb{R}) \\ \times W_{loc}(H^{2r+1}(\Omega), H^{2r-1}(\Omega)) \times W_{loc}((H^{2r+1}(\Gamma))^{d+1}, (H^{2r-1}(\Gamma))^{d+1}).$$

Moreover, for all $t \geq 0$ the following estimate holds:

$$\|z(t) - z_e\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau(t) - \tau_e\|_{H^{2r}(\Omega)} + \|u(t)\|_{(H^{2r}(\Gamma))^{d+1}} \leq C e^{-\sigma t}(\|z_0 - z_e\|_{\mathbf{H}^{2r}(\Omega)} + \|\tau_0 - \tau_e\|_{H^{2r}(\Omega)}).$$

hal-00541563, version 1 - 30 Nov 2010

REFERENCES

[1] C. Amrouche and V. Girault. Problèmes généralisés de Stokes. *Portugal. Math.*, 49(4):463–503, 1992.

[2] C. Amrouche and V. Girault. Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. *Czechoslovak Math. J.*, 44(119)(1):109–140, 1994.

[3] M. Badra. Feedback stabilization of the 2-D and 3-D Navier-Stokes equations based on an extended system. *ESAIM: Control, Optimisation and Calculus of Variations*, 15(4):934–968, 2009.

[4] M. Badra. Lyapunov function and local feedback boundary stabilization of the Navier-Stokes equations. *SIAM J. Control Optim.*, 48(3):1797–1830, 2009.

[5] M. Badra and T. Takahashi. Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers. application to the Navier-Stokes system.

[6] V. Barbu. Feedback stabilization of Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 9:197–206 (electronic), 2003.

[7] V. Barbu. *Stabilization of Navier-Stokes Flow.* To appear.

[8] V. Barbu, I. Lasiecka, and R. Triggiani. Abstract settings for tangential boundary stabilization of Navier-Stokes equations by high- and low-gain feedback controllers. *Nonlinear Anal.*, 64(12):2704–2746, 2006.

[9] V. Barbu, I. Lasiecka, and R. Triggiani. Tangential boundary stabilization of Navier-Stokes equations. *Mem. Amer. Math. Soc.*, 181(852):x+128, 2006.

[10] V. Barbu, I. Lasiecka, and R. Triggiani. Local exponential stabilization strategies of the Navier-Stokes equations, $d = 2, 3$, via feedback stabilization of its linearization. In *Control of coupled partial differential equations*, volume 155 of *Internat. Ser. Numer. Math.*, pages 13–46. Birkhäuser, Basel, 2007.

[11] V. Barbu, S.S. Rodrigues, and A. Shirikyan. Internal exponential stabilization for Navier-Stokes equations by means of finite-dimensional distributed controls. 2010. http://hal.archives-ouvertes.fr/hal-00455250/fr/.

[12] V. Barbu and R Triggiani. Internal stabilization of Navier-Stokes equations with finite-dimensional controllers. *Indiana Univ. Math. J.*, 53(5):1443–1494, 2004.

[13] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and control of infinite dimensional systems.* Systems & Control: Foundations & ApplicationsḂirkhäuser Boston Inc., Boston, MA, second edition, 2007.

[14] F. Boyer and P. Fabrie. *Éléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles*, volume 52 of *Mathématiques & Applications (Berlin) [Mathematics & Applications].* Springer-Verlag, Berlin, 2006.

[15] P. Constantin and C. Foias. *Navier-Stokes equations.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[16] R. Denk, M. Hieber, and J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788):viii+114, 2003.

[17] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, and J.-P. Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.

[18] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations. Theory and algorithms*, volume 5 of *Springer Series in Computational Mathematics.* Springer-Verlag, Berlin, 1986.

[19] P. Grisvard. Caractérisation de quelques espaces d'interpolation. *Arch. Rational Mech. Anal.*, 25:40–63, 1967.

[20] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1985.

[21] G. Grubb and V. A. Solonnikov. Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods. *Math. Scand.*, 69(2):217–290 (1992), 1991.

[22] S. Guerrero. Local exact controllability to the trajectories of the Boussinesq system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(1):29–61, 2006.

[23] E. Hille and R. S. Phillips. *Functional analysis and semi-groups.* American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957. rev. ed.

[24] I. Lasiecka and R. Triggiani. *Control theory for partial differential equations: continuous and approximation theories. I. Abstract parabolic systems*, volume 74 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2000.

[25] J.-L. Lions and E. Magenes. *Problemes aux limites non homognes et applications. Vol. I.* Dunod, Paris, 1968.

[26] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1983.

[27] J.-P. Raymond. Feedback boundary stabilization of the two-dimensional Navier-Stokes equations. *SIAM J. Control Optim.*, 45(3):790–828 (electronic), 2006.

[28] J.-P. Raymond. Feedback boundary stabilization of the three-dimensional incompressible Navier-Stokes equations. *J. Math. Pures Appl. (9)*, 87(6):627–669, 2007.

[29] J.-P. Raymond. Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24(6):921–951, 2007.

[30] J-P. Raymond and L. Thevenet. Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers. *Issue in Discrete and Continuous Dynamical Systems A*, 27(3):1159–1187, 2010.

[31] H. Triebel. *Interpolation theory, function spaces, differential operators.* Johann Ambrosius Barth, Heidelberg, second edition, 1995.

[32] Roberto Triggiani. Unique continuation of boundary over-determined Stokes and Oseen eigenproblems. *Discrete Contin. Dyn. Syst. Ser. S*, 2(3):645–677, 2009.

[33] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.