

SPECTRAL TRIPLES FOR FINITELY PRESENTED GROUPS, INDEX 1

SÉBASTIEN PALCOUX

ABSTRACT. Using a Cayley complex (generalizing the Cayley graph) and Clifford algebras, we are able to give, for a large class of finitely presented groups, a uniform construction of spectral triples with D_+ of index 1.

CONTENTS

1. Introduction	1
2. Basic definitions	2
3. Geometric construction	2
4. Clifford algebra	4
5. Dirac operator	5
References	6

Warning 0.1. *This paper is just a first draft, it contains very few proofs. It is possible that some propositions are false, or that some proofs are incomplete or trivially false.*

1. INTRODUCTION

In this paper, we define even θ -summable spectral triples for a large class of finitely presented groups such that D_+ is index 1. We just generalize the unbounded version of the construction of the Fredholm module for the free group given by Connes [1] and M. Pimsner-Voiculescu [5]. For so, we use the Clifford algebra in the same spirit that Julg-Valette do in [4]. We also use topics in geometric group theory as a Cayley complex (generalizing the Cayley graph).

2000 *Mathematics Subject Classification.* Primary 46L87. Secondary 20F65.

Key words and phrases. non-commutative geometry; spectral triple; geometric group theory; Clifford algebra; Cayley graph; Dirac operator; finitely presented groups.

2. BASIC DEFINITIONS

Definition 2.1. A spectral triple (\mathcal{A}, H, D) is given by a unital \star -algebra \mathcal{A} represented on the Hilbert space H , and an unbounded operator D , called the Dirac operator, such that:

- (1) D is self-adjoint.
- (2) $(D^2 + I)^{-1}$ is compact.
- (3) $\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$ is dense in \mathcal{A} .

See the article [6] of G. Skandalis, dedicated to A. Connes and spectral triple.

Definition 2.2. A group Γ is finitely presented if it exists a finite generating set S and a finite set of relations R such that $\Gamma = \langle S \mid R \rangle$. We always take S equals to S^{-1} and the identity element $e \notin S$ (see [3] for more details).

3. GEOMETRIC CONSTRUCTION

Definition 3.1. Let Γ_n be the set of irreducible n -blocks, defined by induction:

- $\Gamma_0 = \Gamma$.
- $\Gamma_1 := \{\{g, gs\} \mid g \in \Gamma, s \in S\}$

An $(n + 2)$ -block is a finite set a of $(n + 1)$ -blocks such that:

$$\forall b \in a, \forall c \in b, \exists ! b' \in a \text{ such that } b \cap b' = \{c\}.$$

Let a, a' be n -blocks then the commutative and associative composition:

$$a.a' := a \Delta a' = (a \cup a') \setminus a \cap a'$$

gives also an n -block if it's non empty (we take $n \neq 0$).

Let $n > 1$, an n -block a'' is called **irreducible** if $\forall a, a'$ n -blocks:

- (1) $a'' = a.a' \Rightarrow \text{card}(a)$ or $\text{card}(a') \geq \text{card}(a'')$
- (2) $\forall b \in a'', b$ is an irreducible $(n - 1)$ -block.

- Γ_{n+2} is the set of irreducible $(n + 2)$ -blocks.

Note that if $b \in \Gamma_n$, we call n the **dimension** of b .

Definition 3.2. An n -block is called **admissible** if it decomposes into irreducibles.

Example 3.3. Let $\mathbb{Z} = \langle s^{\pm 1} \mid \rangle$ then $a = \{e, s^{10}\}$ is an admissible 1-block because $a = \{e, s\}.\{s, s^2\} \dots \{s^9, s^{10}\}$; but, $b = \{\{e, s\}, \{e, s^{-1}\}, \{s^{-1}, s\}\}$ is a non-admissible 2-block, because there is no irreducible 2-block in this case.

Remark 3.4. The graph with vertices Γ_0 and edges Γ_1 is the Cayley graph \mathcal{G} .

Remark 3.5. Let a be an n -block then $a.a = \emptyset$ and if $a = \{b_1, \dots, b_r\}$ then $b_i = b_1.b_2 \dots b_{i-1}.b_{i+1} \dots b_r$ and $b_1.b_2 \dots b_r = \emptyset$.

Remark 3.6. $\Gamma_{n+1} \neq \emptyset$ iff $\exists r > 1; a_1, \dots, a_r \in \Gamma_n$ all distincts with $a_1 \dots a_r = \emptyset$.

Remark 3.7. Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then $\exists N$ such that $\Gamma_N \neq \emptyset$ and $\forall n > N, \Gamma_n = \emptyset$. In fact $N \leq \text{card}(S)$

Examples 3.8. For $\mathbb{F}_r = \langle s_1^{\pm 1}, \dots, s_r^{\pm 1} \mid \rangle$, we have $N = 1$. For $\mathbb{Z}^r = \langle s_1^{\pm 1}, \dots, s_r^{\pm 1} \mid s_i s_j s_i^{-1} s_j^{-1}, i, j = 1, \dots, r \rangle$, we have $N = r$. Here an n -block ($n \leq r$) is just an n -dimensional hypercube.

Definition 3.9. We define the action of Γ on Γ_n recursively:

- Γ acts on $\Gamma_0 = \Gamma$ as: $u_g : h \rightarrow g.h$ with $g, h \in \Gamma$.
- Action on Γ_{n+1} : $u_g : a \rightarrow g.a = \{g.b \mid b \in a\}$ with $g \in \Gamma, a \in \Gamma_{n+1}$.

Note that the action is well-defined: $g.\Gamma_n = \Gamma_n, \forall g \in \Gamma$.

Definition 3.10. Let a and b be blocks, then we say that $b \in a$ if $b = a$ or if $b \in a$ or if $\exists c \in a$ such that $b \in c$ (recursive definition).

Definition 3.11. Let $n > 1$ then an n -block c is **connected** if $\forall b \subset c$: ' b is an n -block' $\Rightarrow b = c$.

Definition 3.12. An n -block b is called **maximal** if there is no $(n + 1)$ -block c with $b \in c$. We note Γ_{\max} the set of maximal irreducible blocks.

Example 3.13. Let $\Gamma = \mathbb{Z}^2 \star \mathbb{Z} = \langle s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1} \mid s_1 s_2 s_1^{-1} s_2^{-1} \rangle$, then $\{e, s_3\}$ is a maximal 1-block, $\{\{e, s_1\}, \{s_1, s_1 s_2\}, \{s_1 s_2, s_2\}, \{s_2, e\}\}$ is a maximal 2-block.

Definition 3.14. We define the **block length** $\ell(\cdot)$ as follows: let b be a block, then $\ell(b)$ is the minimal number of irreducible blocks decomposing a connected admissible block c with $e \in c$ and, $b \in c$ or $b \cap c \neq \emptyset$.

Definition 3.15. Let b be a block, then a sequence $(c_1, \dots, c_{\ell(b)})$ with $b \in c_1, e \in c_{\ell(b)}, c_i$ irreducible and $c_i \cap c_{i+1} \neq \emptyset$ is called a **geodesic block-path**, from b to e beginning with c_1 .

Definition 3.16. Let Υ_b be the set of irreducible blocks of minimal dimension beginning a geodesic block-path from b to e .

Remark 3.17. In general, Υ_b is not of cardinal one. It is for $CAT(0)$ groups, but not for the Baumslag-Solitar group $BS(1, 2) = \langle a^{\pm 1}, b^{\pm 1} \mid bab^{-1} = a^2 \rangle$.

Remark 3.18. Consider the group Γ and its finite presentation $\langle S \mid R \rangle$, then we can complete the presentation as follows: let T be a finite subset of Γ with $T \cap S = \emptyset, T = T^{-1}$ and $e \notin T$, let $S' = T \cup S$ an amplified generating set and $R' = R \cup \{t = \bar{t} \mid t \in T\}$ where \bar{t} is t considered as a generator.

Then $\Gamma = \langle S' \mid R' \rangle$.

Lemma 3.19. We can choose T such that if we build the blocks with the completed presentation $\langle S' \mid R' \rangle$, then every irreducible blocks are triangular, i.e. $\forall b \in \Gamma_n, \text{card}(b) = n + 1$. We call $\langle S' \mid R' \rangle$ a **triangularized presentation**.

Example 3.20. *The complete triangularization: let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, then Γ acts on Γ_{max} (def. 3.9, 3.12); there are only finitely many orbits O_1, \dots, O_r ; choose $b_i \in O_i$; let $E_i = \{g \in \Gamma \mid g \in b_i\}$; let $T_i = \{gh^{-1} \mid g, h \in E_i, gh^{-1} \notin S \cup \{e\}\}$. Then amplifying the generating set with $T = \bigcup T_i$, we obtain obviously a triangularization called the complete triangularization. Note that this process increases the maximal dimension of the blocks. Note that $\text{card}(T)$ is finite because the group is finitely presented.*

4. CLIFFORD ALGEBRA

We first quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [7].

Definition 4.1. *For V a n -dimensional Hilbert space, define the exterior algebra $\Lambda(V)$ equals to $\bigoplus_{k=0}^n \Lambda^k(V)$ with $\Lambda^0(V) = \mathbb{C}\Omega$. We called Ω the vacuum vector. Recall that $v_1 \wedge v_2 = -v_2 \wedge v_1$ so that $v \wedge v = 0$. Note that $\dim(\Lambda^k(V)) = C_n^k$ and $\dim(\Lambda(V)) = 2^n$.*

Definition 4.2. *Let α_v be the creation operator on $\Lambda(V)$ defined by:*

$$\alpha_v(v_1 \wedge \dots \wedge v_r) = v \wedge v_1 \wedge \dots \wedge v_r \text{ and } \alpha_v(\Omega) = v$$

Reminder 4.3. *The dual α_v^* is called the annihilation operator, then:*

$$\alpha_v^*(v_1 \wedge \dots \wedge v_r) = \sum_{i=0}^r (-1)^{i+1} (v, v_i) v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_r \text{ and } \alpha_v^*(\Omega) = 0$$

Reminder 4.4. *Let $\gamma_v = \alpha_v + \alpha_v^*$, then $\gamma_v = \gamma_v^*$ and $\gamma_v \gamma_w + \gamma_w \gamma_v = 2(v, w)I$.*

Definition 4.5. *The operators γ_v generate the Clifford algebra $\text{Cliff}(V)$. Note that the operators γ_v are bounded and that $\text{Cliff}(V).\Omega = \Lambda(V)$.*

Remark 4.6. *V admits the orthonormal basis $(v_a)_{a \in I}$.*

We will write γ_a instead of γ_{v_a} , so that $[\gamma_a, \gamma_{a'}]_+ = 2\delta_{a,a'}I$.

Let Γ be a finitely presented group, with a triangularized presentation $\langle S \mid R \rangle$.

Definition 4.7. *For any irreducible block c , let $\Delta_c = \{b \in \bigcup \Gamma_n \mid c \in \Upsilon_b\}$, with Υ_b defined on definition 3.16.*

Remark 4.8. *If $\Delta_c \neq \emptyset$ then $c \in \Delta_c$*

$\bigcup \Gamma_n = \bigcup \Delta_c$ (it's not a partition in general)

If $\Delta_c \cap \Delta_{c'} \neq \emptyset$ with $c \neq c'$ then $\dim(c) = \dim(c')$ and $c.c'$ is connected.

Definition 4.9. *Let $\bigsqcup_{\alpha \in \mathcal{J}} \mathcal{P}_\alpha$ be the **minimal partition** generated by $\bigcup \Delta_c$, ie $\bigsqcup_{\alpha \in \mathcal{J}} \mathcal{P}_\alpha = \bigcup \Gamma_n$, $\forall \alpha \in \mathcal{J}$, $\mathcal{P}_\alpha \neq \emptyset$ and $\exists n > 0$, $\exists c_1, \dots, c_n$ irreducibles such that $\mathcal{P}_\alpha = \Delta_{c_1} \cap \dots \cap \Delta_{c_n}$.*

Lemma 4.10. For any $\alpha \in \mathcal{J}$, \mathcal{P}_α admits a unique element c_α (resp. c'_α) of minimal dimension m (resp. of maximal dimension M). Denote by I_α the set of blocks of dimension $m + 1$ in \mathcal{P}_α , then \mathcal{P}_α is in one-to-one correspondence with the power set $\mathcal{P}(I_\alpha)$; in particular, the cardinality of \mathcal{P}_α is 2^{M-m} .

Definition 4.11. We naturally identify $\ell^2(\mathcal{P}_\alpha)$ with the exterior algebra $\Lambda(\ell^2(I_\alpha))$ on which operates the Clifford algebra $\text{Cliff}(\ell^2(I_\alpha))$ generated by γ_a , $a \in I_\alpha$.

5. DIRAC OPERATOR

Definition 5.1. We define the **n -block length** $\ell_n(\cdot)$ as follows: let b be a block, then $\ell_n(b)$ is the minimal number of irreducible blocks decomposing a connected admissible n -dimensional block c with $e \in c$ and, $b \in c$ or $b \cap c \neq \emptyset$.

Definition 5.2. Let b be a block, then a sequence $(c_1, \dots, c_{\ell_n(b)})$ with $b \in c_1$, $e \in c_{\ell(b)}$, $c_i \in \Gamma_n$ and $c_i \cap c_{i+1} \neq \emptyset$ is called a **geodesic n -block-path**, from b to e beginning with c_1 .

Definition 5.3. For any $\alpha \in \mathcal{J}$, let $n = \dim(c_\alpha) + 1$; for any $a \in I_\alpha$ define $p_a(\alpha)$ the number of geodesic n -block path from c_α to e beginning with a ; let $p(\alpha) = \sum_{a \in I_\alpha} p_a(\alpha)$; let $\lambda_a = \frac{p_a(\alpha)}{p(\alpha)} \ell_n(c_\alpha)$.

Definition 5.4. On $\ell^2(\mathcal{P}_\alpha) = \Lambda(\ell^2(I_\alpha))$, define the Dirac operator D_α by:

$$D_\alpha = \sum_{a \in I_\alpha} \lambda_a \cdot \gamma_a$$

Remark 5.5. $\mathcal{P}_e := \Delta_e = \{e\}$, $\ell^2(\mathcal{P}_e) = \mathbb{C}e_1$, $I_e = \emptyset$ and $D_e = 0$.

Definition 5.6. Consider then the Hilbert space:

$$\mathcal{H} = \bigoplus_n \ell^2(\Gamma_n) = \sum_c \ell^2(\Delta_c) = \bigoplus_{\alpha \in \mathcal{J}} \ell^2(\Delta_\alpha) = \bigoplus_{\alpha \in \mathcal{J}} \Lambda(\ell^2(I_\alpha))$$

\mathbb{Z}_2 -graded by the decomposition into even and odd dimensional blocks:

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

Define the unbounded selfadjoint operator $\mathcal{D} = \sum_{\alpha \in \mathcal{J}} D_\alpha$.

Lemma 5.7. $\mathcal{D}^2 = \sum_\alpha \lambda_\alpha^2 \cdot p_\alpha$,
with $\lambda_\alpha^2 = \sum_{a \in I_\alpha} \lambda_a^2$ and p_α the projection on $\ell^2(\Delta_\alpha)$.

Proof. We use the orthonormal decomposition and the Clifford relations. □

Proposition 5.8. $\mathcal{D}_+ : \mathcal{H}^+ \rightarrow \mathcal{H}^-$ is a Fredholm operator of index 1.

Proposition 5.9. $(\mathcal{D}^2 + I)^{-1}$ is compact.
For $t > 0$, the operator $e^{-t\mathcal{D}^2}$ is trass-class.

Definition 5.10. For any $g \in \Gamma$ and for any $s \in S$ define $p_s(g)$ the number of geodesic 1-block path from g to e beginning with $\{g, gs\}$; let $p(g) = \sum_{s \in S} p_s(g)$.

Definition 5.11. Let \mathcal{C} be the class of finitely presented groups $\Gamma = \langle S \mid R \rangle$ such that $\forall g \in \Gamma, \exists K_g \in \mathbb{R}_+$ such that $\forall s \in S$ and $\forall h \in \Gamma$ (with $h, gh \neq e$):

$$\left| \frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)} \right| \leq \frac{K_g}{\ell_1(h)}$$

Examples 5.12. The class \mathcal{C} is stable by direct or free product, it contains $\mathbb{Z}^n, \mathbb{F}_n$, the finite groups, and probably every amenable or automatic groups (containing the hyperbolic groups, see [2]).

Proposition 5.13. Let Γ of class \mathcal{C} , $\mathcal{A} = C_r^*(\Gamma)$ and \mathcal{D} as previously then: $\{a \in \mathcal{A} \mid [\mathcal{D}, a] \in B(\mathcal{H})\}$ is dense in \mathcal{A} .

Theorem 5.14. $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even θ -summable spectral triple and \mathcal{D}_+ is index 1. It then gives a non-trivial element for the K -homology of \mathcal{A} .

REFERENCES

- [1] A. Connes, *Noncommutative differential geometry*. Inst. Hautes études Sci. Publ. Math. No. 62 (1985), 257360.
- [2] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.
- [3] P. de la Harpe, *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.
- [4] P. Julg, A. Valette, *Fredholm modules associated to Bruhat-Tits buildings*. Mini-conferences on harmonic analysis and operator algebras (Canberra, 1987), 143155, Proc. Centre Math. Anal. Austral. Nat. Univ., 16, Austral. Nat. Univ., Canberra, 1988.
- [5] M. Pimsner, D. Voiculescu, *KK-groups of reduced crossed products by free groups*. J. Operator Theory 8 (1982), no. 1, 131156.
- [6] G. Skandalis *Géométrie non commutative d'après Alain Connes: la notion de triplet spectral*. Gaz. Math. No. 94 (2002), 4451.
- [7] A. Wassermann, *Lecture notes on Atiyah-Singer index theorem*, Lent 2010 course, <http://www.dpmms.cam.ac.uk/~ajw/AS10.pdf>

INSTITUT DE MATHÉMATIQUES DE LUMINY, MARSEILLE, FRANCE.

E-mail address: palcoux@iml.univ-mrs.fr, <http://iml.univ-mrs.fr/~palcoux>