Dynamic vs static scaling: an existence result

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Abstract: The relation between static and dynamic control Lyapunov functions scaling is discussed. It is shown that, under some technical assumptions, stabilizability by means of static scaling implies stabilizability by means of dynamic scaling. A motivating example and a worked out design example complement the theoretical part. *Copyright © 2010 IFAC*

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1. INTRODUCTION

Lyapunov function scaling is a well-established analysis and design tool in nonlinear control design. It has been used, for example, to establish a Lyapunov proof of the reduction principle arising in center manifold theory, see Carr (1981); Khalil (2002); in the study of stability properties of interconnected systems, see Jiang et al. (1996, 1994); Sontag and Teel (1995); Angeli and Astolfi (2007); Ito (2006); Ito and Jiang (2009); in the design of stabilizing control laws for cascaded or feedback interconnected systems, see Mazenc and Praly (1996); Jankovic et al. (1996), and in adaptive control systems, Krstic et al. (1995); Jiang (1999); Astolfi et al. (2008). Informally, the idea of Lyapunov function scaling can be described as follows. Consider a (nonlinear) system, and two functions V_1 and V_2 such that the time derivatives of each of these functions, along the solutions of the system, are non-positive on some sets of the state space, the union of which coincides with the whole state space. Lyapunov function scaling allows to determine, if possible, scaling functions l_1 and l_2 such that the function

$$l_1(V_1) + l_2(V_2)$$

is positive definite (and radially unbounded) and its time derivative is non-positive in the whole state space.

A second well-established design tool is dynamic scaling. Dynamic scaling essentially consists in adding a state component, the dynamics of which depend upon the system input and output signals, and using this component as a scaling factor. This scaling factor could play the role of a state norm observer, see Sontag and Wang (1997). As such it has been exploited in adaptive control, to render the boundedness property robust (see for instance Ioannou and Sun (1996) for linear adaptive control and Jiang and Praly (1992) for nonlinear adaptive control), in nonlinear stabilization, to cope with input disturbances (see Praly and Wang (1996)) and in nonlinear observers, to deal with non-Lipschitz nonlinearities (see Astolfi and Praly (2006)). Alternatively, it could be used to estimate the local incremental rate of a dynamical system. As such it is helpful in output feedback stabilization (see, for instance, Praly (2003) or Andrieu et al. (2009)).

By merging the above two tools Lyapunov-like functions, defined as sums of dynamically scaled partial Lyapunov functions, can be constructed. Preliminary results using this idea have been reported in Karagianis et al. (2009); Ortner and Astolfi (2009), for the case of observer design and adaptive control and in Carnevale and Astolfi (2009), for the stabilization of simple cascades.

2. AN INTRODUCTORY EXAMPLE

To illustrate the underlying ideas of static and dynamic Lyapunov function scaling we consider the problem of studying the stability properties of a simple cascade. Consider the nonlinear system

$$\begin{aligned} \dot{z} &= -z + zy \ , \\ \dot{y} &= -y \ . \end{aligned} \tag{1}$$

A simple analysis allows to conclude that the origin is a globally asymptotically stable equilibrium.

To establish this stability result by means of a Lyapunov function, following Sontag and Teel (1995), for instance, consider the two functions

$$V_1(y) \ = \ y^2 \ , \qquad V_2(z) \ = \ z^2 \ ,$$

two weighting functions ℓ_1 and ℓ_2, and the Lyapunov function candidate

$$V(y,z) \ = \ l_1(V_1(y)) + l_2(V_2(z)) \ .$$

Since

$$\frac{1}{2}\dot{V} \ = \ -\ell_1'(V_1(y))\, y^2 \ - \ \ell_2'(V_2(z))\, z^2 \ + \ \ell_2'(V_2(z))\, z^2\, y \ ,$$

$\dot{V}$ is negative definite if the functions ℓ_1 and ℓ_2 are chosen to satisfy the condition

$$\frac{\ell_1'(y^2)}{\ell_2'(z^2)} \; > \; \frac{z^2 \max\{0, |y|-1\}}{y^2} \qquad \forall (y,z) \tag{2}$$

or, alternatively, the conditions

$$\ell_1'(y^2) \; \geq \; 1 \; \geq \; \frac{\max\{0,|y|-1\}}{y^2}$$
$$\ell_2'(z^2) \; \leq \; \frac{1}{1+z^2} \; \leq \; \frac{1}{z^2} \; .$$

The above conditions yield the Lyapunov function

$$V(y,z) \; = \; y^2 \; + \; \log(1+z^2) \; .$$

which is such that $\dot{V} < 0$ for all nonzero (y,z).

An alternative way to study the properties of the solutions of system (1) is by means of dynamic Lyapunov function scaling. Following the arguments in Karagianis et al. (2009), consider the Lyapunov-like function [1]

$$V_r(z,y) = V_1(y) + \frac{1}{r} V_2(z),$$

where $r \geq r_\star > 0$ is the scaling variable. The time derivative of the Lyapunov-like functions along the trajectories of the system is

$$\frac{1}{2}\dot{V}_r = -y^2 - \frac{z^2}{r} + \frac{z^2 y}{r} - \frac{z^2}{r}\frac{\dot{r}}{2r},$$

hence selecting

$$\frac{\dot{r}}{r} = \frac{1}{2} + 2y^2 - \frac{r - r_\star}{r}, \tag{3}$$

with $r(0) \geq r_\star$, yields $r(t) \geq r_\star$ and

$$\frac{1}{2}\dot{V}_r \leq -y^2 - \frac{z^2}{r} + \frac{z^2}{2r}\frac{r-r_\star}{r} \leq -y^2 - \frac{z^2}{2r}.$$

As a result, $y(t) \in L_\infty$ and $z(t)/\sqrt{r(t)} \in L_\infty$. Note, however, that we cannot draw any conclusion on the properties of the zero equilibrium of the system, since no property of the behaviour of r has been established. One way to complete the analysis is via the (true) Lyapunov function

$$\mathcal{U}(y,z,r) \; = \; V_r \; + \; \frac{1}{2}\int_{2r_\star}^{r} \frac{\mathrm{sat}(s - 2r_\star)}{s} ds,$$

defined on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{+*}$ [2], the time derivative of which, along the trajectories of the system, satisfies the inequality

$$\dot{\mathcal{U}} \leq -2\left(y^2 + \frac{z^2}{2r}\right) + \frac{1}{2}\,\mathrm{sat}(r - 2r_\star)\left(\frac{2r_\star - r}{2r} + y^2\right)$$
$$\leq -\left(y^2 + \frac{z^2}{r} + \frac{(r-2r_\star)\mathrm{sat}(r-2r_\star)}{4r}\right) .$$

As a result, the point $(0, 0, 2r_\star)$ is asymptotically stable, with domain of attraction $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{+*}$, and locally exponentially stable.

The analysis by means of the dynamically scaled Lyapunov function presents a few advantages and disadvantages.

(1) The dynamically scaled Lyapunov function is (trivially) constructed as a linear combination of the two functions V_1 and V_2 with a coefficient which depends upon the scaling variable r. On the other hand, the dynamic of the scaling variable may be hard to select.

(2) Boundedness of the scaling variable r is established a-posteriori.

(3) There is no clear relation between the statically scaled Lyapunov function and the dynamically scaled one, *i.e.* between the constraint (2) on the ratio $\frac{\ell_1'(V_1)}{\ell_2'(V_2)}$ and the expression of $\dot{r}$ in (3). In particular, as far as we know today, existence of one does not imply, in general, existence of the other.

We conclude the section noting that in the simple, motivating, example discussed above we have focused on stability analysis, while in the rest of the paper we deal with a feedback design problem in a general context.

3. GOAL OF THE PAPER

Aim of this paper is to partly address the issues raised at the end of Section 2. In particular, a technical result, establishing a link between statically scaled control Lyapunov functions and dynamically scaled control Lyapunov functions is presented. This result gives conditions under which, with an additional technical assumption for each case, both scaled control-Lyapunov function and dynamically scaled Lypunov function exist.

4. A TECHNICAL RESULT

Consider a nonlinear system described by equations of the form

$$\dot{x} = f(x) + g(x)u, \tag{4}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and, without loss of generality, $f(0) = 0$.

Assume that there exist three functions $V_1 : \mathbb{R}^n \to \mathbb{R}_+$, $V_2 : \mathbb{R}^n \to \mathbb{R}_+$ and $R : \mathbb{R}^n \to [r_0, +\infty)$, with $r_0 > 0$, such that the following holds.

(P1) The function $V_1 + V_2$ is positive definite and radially unbounded.

(P2) For each pair (x, r) satisfying
$$x \neq 0 \; , \quad r \geq R(x) \; , \quad rL_gV_1(x) + L_gV_2(x) \; = \; 0$$
the inequality
$$rL_fV_1(x) + L_fV_2(x) \; < \; 0$$
holds.

(P3) For each strictly positive real number ε there exists a strictly positive real number δ such that, for each pair (x, r) satisfying
$$|x| \leq \delta \; , \quad r \geq r_0 \; , \quad rL_gV_1(x) + L_gV_2(x) \neq 0 \; ,$$
the condition
$$\frac{rL_fV_1(x) + L_fV_2(x)}{rL_gV_1(x) + L_gV_2(x)} \; < \; \varepsilon$$
holds.

Since V_1 and V_2 take only non-negative values, (P1) means that to have a positive definite and radially unbounded function in x it is sufficient to combine in some appropriate way $V_1(x)$ and $V_2(x)$. Under assumption (P1), (P2) and (P3) are stating that, for r fixed to a sufficiently large positive value, the function $x \mapsto V_1(x) + \frac{1}{r}V_2(x)$ is a Control Lyapunov Function (CLF) satisfying the Small Control Property (SCP), see Sontag (1989).

[1] This is not a Lyapunov function *per se*, since it is not positive definite and radially unbounded in (y, z, r).

[2] $\mathbb{R}_{+*}$ denotes the set of strictly positive real numbers.

Remark. In the sequel we shall see that R in (P2) is the key ingredient to design the weights of the statically scaled control-Lyapunov function and to design the update law of the scaling factor of the dynamically scaled one. Specifically the weights ℓ_1 and ℓ_2 should be such that $\frac{\ell_1'(V_1(x))}{\ell_2'(V_2(x))} \geq R(x)$ and, similarly, $R(x(t))$ is what $r(t)$ should be. □

Remark. Without the knowledge of R one could try to define $\dot{r}$ indirectly, that is not from what it should be, but from the properties that it allows to achieve. For example, $\dot{r}$ may be such that, when r is large enough, a function h of the state is integrable along closed-loop solutions. This selection yields, for r large, the update law $\dot{r} = h(x)$, which however may lead to severe non-robustness problems. □

We are now ready to establish a preliminary result.

Lemma 1. Consider system (4). Assume conditions (P1) to (P3) hold. Then there exists a function ϕ defined and continuous in the set $\{(x,r) : r \geq R(x)\}$ satisfying

$$\begin{aligned}\overline{W}(x,r) = L_f V_1(x) + \frac{1}{r} L_f V_2(x) + \\ \left(L_g V_1(x) + \frac{1}{r} L_g V_2(x)\right) \phi(x,r) < 0\end{aligned} \tag{5}$$

for all (x,r) such that $x \neq 0$ and $r \geq R(x)$.

Proof. The result is a direct consequence of what is known on universal formulae for the design of state feedback laws exploiting CLFs satisfying the SCP, see Sontag (1989); Bacciotti (1991); Freeman and Kokotovic (1996). For instance, following Freeman and Kokotovic (1996), we can pick ϕ as [3]

$$\phi(x,r) = \begin{cases} -\dfrac{\max\{A(x,r) + |B(x,r)|^2, 0\}}{|B(x,r)|^2} B(x,r)^T, & \text{if } B \neq 0, \\ 0, & \text{if } B = 0, \end{cases}$$

with

$$A(x,r) = L_f V_1(x) + \frac{1}{r} L_f V_2(x),$$

$$B(x,r) = L_g V_1(x) + \frac{1}{r} L_g V_2(x).$$

◁

Remark. The reader should not be misled by the result in Lemma 1: the lemma does not establish that ϕ is a stabilizing state feedback. Indeed, the expression on the l.h.s. of inequality (5) is the time derivative of the *scaled* Lyapunov function $V_1 + \frac{1}{r} V_2$ for r constant, whereas inequality (5) holds only provided r is larger than $R(x)$. Hence, if R is a bounded function, a stabilizer from ϕ is obtained selecting $r \geq \sup_x R(x))$ whereas, if R is unbounded, either we consider only compact sets and obtain semi-global asymptotic stability, or we allow r to follow the variations of $R(x)$. This latter case has to be dealt with with care. In fact the function $x \mapsto \phi(x, R(x))$ is, in general, not a stabilizer since $x \mapsto V_1(x) + \dfrac{V_2(x)}{R(x)}$ may not be a CLF. □

[3] Other selections are possible.

4.1 Static scaling

Consider system (4) and the problem of designing a static state feedback

$$u = \varphi(x) \tag{6}$$

such that the origin of the closed-loop system is asymptotically stable.

As expressed in the following statement this problem admits a solution if conditions (P1) to (P3) hold and provided an additional technical assumption is satisfied by the triple (V_1, V_2, R).

Proposition 1. Assume conditions (P1) to (P3) hold. If the triple (V_1, V_2, R) is such that there exists a pair (ℓ_1, ℓ_2) of C^1, class $\mathcal{K}^\infty$ functions, with nowhere zero derivative, satisfying

$$\ell_1'(V_1(x)) \geq R(x) \ell_2'(V_2(x)) \qquad \forall x \in \mathbb{R}^n \,, \tag{7}$$

then there exists a continuous functions φ such that the origin is an asymptotically stable equilibrium of the closed-loop system (4)-(6).

4.2 Dynamic scaling

Consider system (4) and the problem of designing a dynamic state feedback

$$\begin{aligned} \dot{r} &= \psi(x,r) \\ u &= \varphi(x,r) \end{aligned} \tag{8}$$

such that the closed-loop system (4)-(8) has the following properties:

- r remains in some compact subset of $[r_0, +\infty)$;
- there exists some *nominal* value $r_\star \geq r_0$ such that the point $(x,r) = (0, r_\star)$ is a globally stable equilibrium;
- the x component converges to zero as time goes to infinity.

As expressed in the following statement, this problem admits a solution if conditions (P1) to (P3) hold and provided an additional technical assumption is satisfied by the triple (V_1, V_2, R).

Proposition 2. Assume conditions (P1) to (P3) hold. If the triple (V_1, V_2, R) is such that the function $V_1 + \frac{1}{R} V_2$ is radially unbounded then there exist continuous functions φ and ψ and a constant $r_\star > R(0)$ such that the closed-loop system (4)-(8) has the following properties.

- The set $\mathbb{R}^n \times (R(0), +\infty)$ is forward invariant.
- The point $(x,r) = (0, r_\star)$ is a stable equilibrium.
- For each initial condition (x,r) in $\mathbb{R}^n \times (R(0), +\infty)$, the x component converges to zero as time goes to infinity.

Remark. The existence proof in Proposition 2 relies on the use of universal formulae. Note however that, in specific examples (see Section 5 and the introductory example), it is possible to design the feedback control and the dynamics of the scaling variable r directly, *i.e.* without the use of universal formulae. □

5. AN ILLUSTRATIVE EXAMPLE

To illustrate the theoretical result of Section 4 consider the system

$$\dot{x}_1 = -x_1 - x_2 + u\,,$$
$$\dot{x}_2 = x_1\,,$$
$$\dot{x}_3 = x_1^2 + u.$$

This is a feedforward system and a globally stabilizing state feedback can be designed, for instance, exploiting the results in Mazenc and Praly (1996) or Teel (1996).

To pose the problem in the framework discussed above, let

$$V_1(x_1,x_2) = x_1^2 + x_1x_2 + x_2^2, \qquad V_2(x_3) = x_3^2,$$

and note that, for $u = 0$,

$$\overbrace{V_1(x_1,x_2)}^{\cdot} = -V_1(x_1,x_2),$$

while, for $u = x_1 = 0$,

$$\overbrace{V_2(x_3)}^{\cdot} = 0\,.$$

Note also that condition (P1) is satisfied. With this at hand, let

$$V_r(x_1,x_2,x_3) = \left[x_1^2 + x_1x_2 + x_2^2\right] + \frac{1}{r}x_3^2\,,$$

and note that, for r constant,

$$\overbrace{V_r(x_1,x_2,x_3)}^{\cdot} = -V_1(x_1,x_2) + \frac{2}{r}x_3x_1^2 + \left[(2x_1+x_2) + \frac{2}{r}x_3\right]u\,.$$

As a result

$$u = -\left[(2x_1+x_2) + \frac{2}{r}x_3\right]$$

is such that

$$\overbrace{V_r(x_1,x_2,x_3)}^{\cdot} < 0$$

for $(x_1,x_2,x_3) \neq 0$ and

$$r > \frac{2|x_3|x_1^2}{V_1(x_1,x_2)}.$$

This establishes that conditions (P2) and (P3) are satisfied if, given any strictly positive real number r_0, the function $R : \mathbb{R}^3 \mapsto [r_0,+\infty)$ is chosen as any continuous function satisfying

$$R(x_1,x_2,x_3) > \frac{2|x_3|\,x_1^2}{x_1^2 + x_1x_2 + x_2^2} \qquad \forall (x_1,x_2,x_3)\,.$$

To be more explicit, pick

$$R(x_1,x_2,x_3) = 4\sqrt{1+x_3^2}\,. \qquad (9)$$

and look for a pair (ℓ_1,ℓ_2) of C^1, class $\mathcal{K}^\infty$ functions, with nowhere zero derivative, satisfying

$$\ell_1'(V_1(x_1,x_2)) \geq R(x_1,x_2,x_3)\ell_2'(V_2(x_3)), \quad \forall (x_1,x_2,x_3)\,.$$

For example, let

$$\ell_1'(v_1) = 2 \quad , \qquad \ell_2'(v_2) = \frac{1}{2\sqrt{1+v_2}}$$

i.e.

$$\ell_1(v_1) = 2\,v_1 \quad , \qquad \ell_2(v_2) = \sqrt{1+v_2} - 1$$

The above selection yields the function

$$V_\ell(x_1,x_2,x_3) = \ell_1(V_1(x_1,x_2)) + \ell_2(V_2(x_3))$$
$$= 2\left[x_1^2 + x_1x_2 + x_2^2\right] + \sqrt{1+x_3^2} - 1,$$

which, consistently with the results in Mazenc and Praly (1996), is positive definite, radially unbounded, and it is a weak CLF satisfying the SCP. In addition, since

$$\dot{V}_\ell = -2V_1(x_1,x_2) + \frac{x_3}{\sqrt{1+x_3^2}}x_1^2 + 2\left[(2x_1+x_2) + \frac{1}{2}\frac{x_3}{\sqrt{1+x_3^2}}\right]u\,,$$

a globally stabilizing static state feedback is

$$u = -\left[(2x_1+x_2) + \frac{1}{2}\frac{x_3}{\sqrt{1+x_3^2}}\right]\,.$$

On the other hand, by equation (9), the function

$$V_1(x_1,x_2) + \frac{V_2(x_3)}{R(x_1,x_2,x_3} = \left[x_1^2 + x_1x_2 + x_2^3\right] + \frac{x_3^2}{4\sqrt{1+x_3^2}}$$

is radially unbounded. It follows that Proposition 2 applies. However, instead of following the (too) general design given in the proof of Proposition 2, we proceed with an ad-hoc design. To this end, let

$$V_r(x_1,x_2,x_3) = x_1^2 + x_1x_2 + x_2^2 + \frac{x_3^2}{r}$$

and note that

$$\dot{V}_r = -\left[x_1^2 + x_1x_2 + x_2^2\right] + \left[(2x_1+x_2) + \frac{2x_3}{r}\right]u + \frac{2x_3x_1^2}{r} - \frac{x_3^2}{r^2}\dot{r}\,.$$

Exploiting the inequality

$$\frac{2x_3x_1^2}{r} \leq \frac{x_1^2}{2} + \frac{2x_3^2x_1^2}{r^2}\,,$$

one has

$$\dot{V}_r \leq -\left[\frac{x_1^2}{2} + x_1x_2 + x_2^2\right] + \frac{x_3^2}{r^2}\left[2x_1^2 - \dot{r}\right] + \left[(2x_1+x_2) + \frac{2x_3}{r}\right]u\,,$$

which motivates the selection

$$u = -\left[(2x_1+x_2) + \frac{2x_3}{r}\right]\,, \qquad (10)$$
$$\dot{r} = 2x_1^2\,.$$

Such an expression for $\dot{r}$ is not satisfactory since it leads to a monotonic behavior of r along closed-loop solutions. We therefore modify the above by introducing a damping term, *i.e.* selecting

$$\dot{r} = 2\,x_1^2 - \mu(x,r)\,(r - r_*), \qquad (11)$$

with $\mu : \mathbb{R}^3 \times \mathbb{R}_{+*} \to \mathbb{R}_+$ a function to be defined and r_* a strictly positive real number. This selection renders the set $\{r \geq r_*\}$ positively invariant and yields

$$\dot{V}_r \leq -\left[\frac{x_1^2}{2} + x_1x_2 + x_2^2\right] - \left[(2x_1+x_2) + \frac{2x_3}{r}\right]^2 + \mu(x,r)\,|r - r_*|\,\frac{x_3^2}{r^2}\,.$$

Observe now that

$$\left[\frac{x_1^2}{2} + x_1x_2 + x_2^2\right] \geq \frac{3}{16}\left[2x_1 + x_2\right]^2$$

and that

$$\frac{3}{16}\,[2x_1+x_2]^2 \;+\; \left[(2x_1+x_2)+\frac{2x_3}{r}\right]^2 \;-\; \frac{12}{19}\frac{x_3^2}{r^2} = \frac{19}{16}\left((2x_1+x_2)-\frac{32}{19}\frac{x_3}{r}\right)^2 .$$

Hence,imposing the condition

$$\mu(x,r)\,|r-r_*| \;\leq\; \frac{6}{19} \tag{12}$$

yields

$$\begin{aligned}\dot{V}_r &\leq -W_r(x_1,x_2,\frac{x_3}{r}) \\ &= -\frac{1}{2}\left[\frac{x_1^2}{2}+x_1x_2+x_2^2\right] \;-\; \frac{1}{2}\left[(2x_1+x_2)+\frac{2x_3}{r}\right]^2 .\end{aligned}$$

Observe now that since W_r is a positive definite quadratic form in $(x_1, x_2, \frac{x_3}{2})$, there exists a strictly positive real number κ satisfying

$$\kappa\, W_r \;>\; 2\,x_1^2 .$$

To conclude the design of μ consider the (true) Lyapunov function

$$\mathcal{U}(x,r) = 2\kappa V_r \;+\; \left[\sqrt{1+(r-r_*)^2}-1\right] ,$$

yielding

$$\begin{aligned}\dot{\mathcal{U}} &\leq -2\kappa\, W_r \;+\; \frac{r-r_*}{\sqrt{1+(r-r_*)^2}}\left[2\,x_1^2 \;-\; \mu(x,r)\,(r-r_*)\right] ,\\ &\leq -\kappa\, W_r \;-\; \mu(x,r)\frac{(r-r_*)^2}{\sqrt{1+(r-r_*)^2}} .\end{aligned}$$

The only constraint on μ is given by equation (12), hence selecting, for instance,

$$\mu(x,r) \;=\; \frac{6}{19r} ,$$

proves that the state feedback (10) and the scaling factor update (11) render the point $(0, 0, 0, r_*)$ an asymptotically stable equilibrium with $\mathbb{R}^3 \times \mathbb{R}_{+*}$ as basin of attraction.

Note, finally, that r_* is a free parameter which can be chosen, for instance, to match a linear feedback designed from the first order approximation of the system at the origin.

6. CONCLUSIONS

The relation between static and dynamic Lyapunov functions scaling has been discussed. It has been shown that, under proper conditions, the two tools are *equivalent*. This theoretical, existence, result has been motivated by means of a simple example and has been illustrated on a worked out design problem. Applications of the proposed tool to the stabilization of general cascaded systems (see the preliminary results in Carnevale and Astolfi (2009)) and to output feedback stabilization of system with iISS inverse dynamics (in the spirit of the results in Jiang et al. (2004)) are under investigation.

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