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HYPERSURFACES WITH SMALL EXTRINSIC RADIUS OR LARGE λ_1 IN EUCLIDEAN SPACES

ERWANN AUBRY, JEAN-FRANÇOIS GROSJEAN, JULIEN ROTH

Abstract. We prove that hypersurfaces of $\mathbb{R}^{n+1}$ which are almost extremal for the Reilly inequality on λ_1 and have L^p-bounded mean curvature ($p > n$) are Hausdorff close to a sphere, have almost constant mean curvature and have a spectrum which asymptotically contains the spectrum of the sphere. We prove the same result for the Hasanis-Koutroufiotis inequality on extrinsic radius. We also prove that when a supplementary L^q bound on the second fundamental is assumed, the almost extremal manifolds are Lipschitz close to a sphere when $q > n$, but not necessarily diffeomorphic to a sphere when $q \leqslant n$.

1. Introduction

Sphere theorems in positive Ricci curvature are now a classical matter of study. The canonical sphere $(\mathbb{S}^n, can)$ is the only manifold with $\mathrm{Ric} \geqslant n-1$ which is extremal for the volume, the radius, the first non zero eigenvalue λ_1 on functions or the diameter. Moreover, it was proved in [6, 7, 4] that manifolds with $\mathrm{Ric} \geqslant n-1$ and volume close to $\mathrm{Vol}\,(\mathbb{S}^n, can)$ are diffeomorphic and Gromov-Hausdorff close to the sphere. This stability result was extended in [14, 1], where it is proved that manifolds with $\mathrm{Ric} \geqslant n-1$ have almost extremal volume if and only if they have almost extremal radius, if and only if they have almost extremal λ_n. Almost extremal diameter and almost extremal λ_1 are also equivalent when $\mathrm{Ric} \geqslant n-1$ ([9, 11]), but, as shown in [2, 13], it does not force the manifold to be diffeomorphic nor Gromov-Hausdorff close to $(\mathbb{S}^n, can)$. In this paper, we study the stability of three optimal geometric inequalities involving the mean curvature of Euclidean hypersurfaces, and whose equality case characterizes the Euclidean spheres (see Inequalities (1.1), (1.2) and 1.3 below). More precisely we study the metric and spectral properties of the hypersurfaces which almost realize the equality case. It completes the results of [5, 16].

Let $X : (M^n, g) \to \mathbb{R}^{n+1}$ be a closed, connected, isometrically immersed n-manifold ($n \geqslant 2$). The first geometric inequality we are interested in is the following

$$\|H\|_2 \|X - \overline{X}\|_2 \geqslant 1 \tag{1.1}$$

where $\overline{X} := \frac{1}{\mathrm{Vol}\, M} \int_M X dv$, $\mathrm{Vol}\, M$ is the volume of (M^n, g), H is the mean curvature of the immersion X and $\|\cdot\|_p$ is the renormalized L^p-norm on $C^\infty(M)$ defined by $\|f\|_p^p = \frac{1}{\mathrm{Vol}\, M} \int_M |f|^p dv$. Equality holds in (1.1) if and only if $X(M)$ is a sphere of

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radius $\frac{1}{\|H\|_2}$ and center $\overline{X}$ (see section 2). From (1.1) we easily infer the Hasanis-Koutroufiotis inequality on extrinsic radius (i.e. the least radius of the balls of $\mathbb{R}^{n+1}$ which contains $X(M)$)

$$\|H\|_2 R_{ext} \geqslant 1 \tag{1.2}$$

whose equality case also characterizes the sphere of radius $\frac{1}{\|H\|_2}$ and center $\overline{X}$. The last inequality is the well-known Reilly inequality

$$\lambda_1 \leqslant n\|H\|_2^2 \tag{1.3}$$

Here also, the extremal hypersurfaces are the spheres of radius $\frac{1}{\|H\|_2} = \sqrt{\frac{n}{\lambda_1}}$. Let $p > 2$ and $\varepsilon \in (0,1)$ be some reals. We will say that M is almost extremal for Inequality (1.1) when it satisfies the pinching

$$\|H\|_p \big\|X - \overline{X}\big\|_2 \leqslant 1 + \varepsilon, \tag{$P_{p,\varepsilon}$}$$

We will say that M is almost extremal for Inequality (1.2) when it satisfies the pinching

$$\|H\|_p R_{ext} \leqslant 1 + \varepsilon \tag{$R_{p,\varepsilon}$}$$

We will say that M is almost extremal for Inequality (1.3) when it satisfies the pinching

$$(1+\varepsilon)\lambda_1 \geqslant n\|H\|_p^2 \tag{$\Lambda_{p,\varepsilon}$}$$

Remark 1.1. *It derives from the proof of the three above geometric inequalities, given in section 2, that Pinching* $(R_{p,\varepsilon})$ *or Pinching* $(\Lambda_{p,\varepsilon})$ *imply Pinching* $(P_{p,\varepsilon})$. *For that reason, Theorems 1.2, 1.7, 1.13 below are stated for hypersuraces satisfying Pinching* $(P_{p,\varepsilon})$ *but are obviously valid for Pinching* $(R_{p,\varepsilon})$ *or Pinching* $(\Lambda_{p,\varepsilon})$.

Our first result is that, when $\|H\|_q$ is bounded, almost extremal manifolds for one of the three Inequalities (1.1), (1.2) or (1.3) are Hausdorff close to an Euclidean sphere of radius $\frac{1}{\|H\|_2}$ and have almost constant mean curvature.

Theorem 1.2. *Let* $q > n$, $p > 2$ *and* $A > 0$ *be some reals. There exist some positive functions* $C = C(p,q,n,A)$ *and* $\alpha = \alpha(q,n)$ *such that if* M *satisfies* $(P_{p,\varepsilon})$ *and* $\operatorname{Vol} M\|H\|_q^n \leqslant A$, *then we have*

$$\Big\| |X - \overline{X}| - \frac{1}{\|H\|_2} \Big\|_\infty \leqslant C\varepsilon^\alpha \frac{1}{\|H\|_2}, \tag{1.4}$$

and there exist some positive functions $C = C(p,q,r,n,A)$ *and* $\beta = \frac{\alpha(q-r)}{r(q-1)}$ *so that*

$$\big\| |H| - \|H\|_2 \big\|_r \leqslant C\varepsilon^\beta \|H\|_2 \qquad \textit{for any } r \in [1,q). \tag{1.5}$$

We assume moreover that $q > \max(4,n)$. *For any* $r > 0$ *and* $\eta > 0$, *there exists* $\varepsilon_0 = \varepsilon_0(p,q,n,A,r,\eta) > 0$ *such that if* M *satisfies* $(P_{p,\varepsilon})$ *(for* $\varepsilon \leqslant \varepsilon_0$*) and* $\operatorname{Vol} M\|H\|_q^n \leqslant A$ *then for any* $x \in S = \overline{X} + \frac{1}{\|H\|_2}\cdot \mathbb{S}^n$, *we have*

$$\Big|\frac{\operatorname{Vol}\big(B(x,\frac{r}{\|H\|_2}) \cap X(M)\big)}{\operatorname{Vol} M} - \frac{\operatorname{Vol}\big(B(x,\frac{r}{\|H\|_2}) \cap S\big)}{\operatorname{Vol} S}\Big| \leqslant \eta \frac{\operatorname{Vol}\big(B(x,\frac{r}{\|H\|_2}) \cap S\big)}{\operatorname{Vol} S} \tag{1.6}$$

where $B(x,r)$ *is the Euclidean ball with center* x *and radius* r.

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Theorem 1.2 generalizes and improves the main results of [5] and [16], where only the pinchings $(R_{p,\varepsilon})$ and $(\Lambda_{p,\varepsilon})$ for $p \geqslant 4$ and $q = \infty$ were considered. The control on the mean curvature (Inequality (1.5)) and Inequality (1.6) are new, even under a L^∞ bound on the mean curvature. Note that (1.6) says not only that M goes near any point of the sphere S (as was proven in [5, 16]) but also that the density of M near each point of S is close to $\mathrm{Vol}\, M/\mathrm{Vol}\, S$.

Remark 1.3. *From Inequalities* (1.4) *and* (1.6) *we infer that almost extremal hypersurfaces for one of the three geometric inequalities* (1.1), (1.2) *or* (1.3) *converge in Hausdorff distance to a metric sphere of* $\mathbb{R}^{n+1}$. *As shown in Theorem 1.9, there is no Gromov-Hausdorff convergence if we do not assume a good enough bound on the second fundamental form.*

Remark 1.4. *By Theorem 1.2, when* $\mathrm{Vol}\, M \|H\|_q^n \leqslant A$ *(*$q > n$*), Pinching* $(P_{p,\varepsilon})$ *implies Pinching* $(R_{p,\varepsilon'})$ *for a constant* $\varepsilon' = \varepsilon'(\varepsilon|A, p, q, n)$*. In other words, Pinchings* $(P_{p,\varepsilon})$ *and* $(R_{p,\varepsilon})$ *are equivalent (in bounded mean curvature) and are both implied by Pinching* $(\Lambda_{p,\varepsilon})$*. However, we will see in Theorem 1.9 that Pinching* $(P_{p,\varepsilon})$ *(or* $(R_{p,\varepsilon})$*) does not imply Pinching* $(\Lambda_{p,\varepsilon})$.

Remark 1.5. *The constant* $C(p, q, n, A)$ *tends to* ∞ *when* $p \to 2$ *or* $q \to n$*, but the same result can be proved with* $\mathrm{Vol}\, M \|H\|_q^n \leqslant A$ *replaced by* $\mathrm{Vol}\, M \|H - \|H\|_2\|_n^n \leqslant A(n)$*, where* $A(n)$ *is a universal constant depending only on the dimension* n.

Inequality 1.4 follows from the following new pinching result on momenta.

Theorem 1.6. *Let* $q > n$ *be a real. There exists a constant* $C = C(q, n)$ *such that for any isometrically immersed hypersurface* M *of* $\mathbb{R}^{n+1}$*, we have*

$$\sup_M \Big| |X - \overline{X}| - \|X - \overline{X}\|_2 \Big| \leqslant C \big(\mathrm{Vol}\, M \|H\|_q^n\big)^\gamma \|X - \overline{X}\|_2 \Big(1 - \frac{\|X - \overline{X}\|_1}{\|X - \overline{X}\|_2}\Big)^{\frac{1}{2(1+n\gamma)}}$$

where $\gamma = \frac{q}{2(q-n)}$.

In particular, this gives

$$\|X - \overline{X}\|_\infty \leqslant C \big(\mathrm{Vol}\, M \|H\|_q^n\big)^\gamma \|X - \overline{X}\|_2$$

Our next result shows that almost extremal hypersurfaces must satisfy strong spectral constraints. We denote $0 = \mu_0 < \mu_1 < \cdots < \mu_i < \cdots$ the eigenvalues of the canonical sphere $\mathbb{S}^n$, m_i the multiplicity of μ_i and $\sigma_k = \sum_{0 \leqslant i \leqslant k} m_i$ (note that we have $\sigma_k = O(n^k)$ and $m_k = O(n^k)$). We also denote $0 = \lambda_0(M) < \lambda_1(M) \leqslant \cdots \leqslant \lambda_i(M) \leqslant \cdots$ the eigenvalues of M counted with multiplicities.

Theorem 1.7. *Let* $q > \max(n, 4)$*,* $p > 2$ *and* $A > 0$ *be some reals. There exist some positive functions* $C = C(p, q, n, A)$ *and* $\alpha = \alpha(q, n)$ *such that if* M *satisfies* $(P_{p,\varepsilon})$ *and* $\mathrm{Vol}\, M \|H\|_q^n \leqslant A$ *then for any* k *such that* $2\sigma_k C^{2k} \leqslant \varepsilon^{-\alpha}$*, the interval*

$$\big[(1 - \varepsilon^\alpha \sqrt{m_k} C^k)\|H\|_2^2 \mu_k, (1 + \varepsilon^\alpha \sqrt{m_k} C^k)\|H\|_2^2 \mu_k\big]$$

contains at least m_k *eigenvalues of* M *counted with multiplicities.*

Moreover, the previous intervals are disjoints and we get

$$\lambda_i(M) \leqslant \big(1 + \varepsilon^\alpha \sqrt{m_k} C^k\big)\|H\|_2^2 \lambda_i(\mathbb{S}^n) \qquad \text{for any } i \leqslant \sigma_k - 1,$$

and if $\lambda_{\sigma_k}(M) \geqslant \big(1+\varepsilon^\alpha\sqrt{m_k}C^k\big)\|H\|_2^2\mu_k$ *then*

$$\big|\lambda_i(M) - \|H\|_2^2\lambda_i(\mathbb{S}^n)\big| \leqslant \varepsilon^\alpha\sqrt{m_k}C^k\|H\|_2^2\lambda_i(\mathbb{S}^n) \qquad \text{for any } i \leqslant \sigma_k - 1.$$

Remark 1.8. *In the particular case of extremal hypersurfaces for Pinching* $(\Lambda_{p,\varepsilon})$*, Theorem 1.7 implies that*

$$\frac{n\|H\|_p^2}{1+\varepsilon} \leqslant \lambda_1(M) \leqslant \cdots \leqslant \lambda_{n+1}(M) \leqslant \big(1+C(n)\varepsilon^\alpha\big)n\|H\|_2^2$$

and so we must have the $n+1$*-first eigenvalues close to each other. Compare to positive Ricci curvature where* λ_n *close to* n *implies* λ_{n+1} *close to* n*, but we can have only* k *eigenvalues close to* n *for any* $k \leqslant n-1$ *(see* [1]*).*

Note that Theorem 1.7 does not say that the spectrum of almost extremal hypersurfaces for Inequality (1.1) is close to the spectrum of an Euclidean sphere, but only that the spectrum of the sphere $S = \overline{X} + \frac{1}{\|X\|_2}\cdot\mathbb{S}^n$ asymptotically appears in the spectrum of M. Our next two results show that this inclusion is strict in general (we have normalized the mean curvature by $\|H\|_2 = 1$ for sake of simplicity and $E(x)$ stands for the integral part of x).

Theorem 1.9. *For any integers* l, p *there exists sequence of embedded hypersurfaces* (M_j) *of* $\mathbb{R}^{n+1}$ *diffeomorphic to* p *spheres* $\mathbb{S}^n$ *glued by connected sum along* l *points, such that* $\|H_j\|_\infty \leqslant C(n)$*,* $\|B_j\|_n \leqslant C(n)$*,* $\big\||X_j|-1\big\|_\infty \to 0$*,* $\big\||H_j|-1\big\|_1 \to 0$*, and for any* $\sigma \in \mathbb{N}$ *we have*

$$\lambda_\sigma(M_j) \to \lambda_{E(\frac{\sigma}{p})}(\mathbb{S}^n).$$

In particular, the M_j *have at least* p *eigenvalues close to* 0 *whereas its extrinsic radius is close to* 1*.*

Theorem 1.10. *There exists sequence of immersed hypersurfaces* (M_j) *of* $\mathbb{R}^{n+1}$ *diffeomorphic to* 2 *spheres* $\mathbb{S}^n$ *glued by connected sum along* 1 *great subsphere* S^{n-2}*, such that* $\|H_j\|_\infty \leqslant C(n)$*,* $\|B_j\|_2 \leqslant C(n)$*,* $\big\||X_j|-1\big\|_\infty \to 0$*,* $\big\||H_j|-1\big\|_1 \to 0$*, and for any* $\sigma \in \mathbb{N}$ *we have*

$$\lambda_\sigma(M_j) \to \lambda_{E(\frac{\sigma}{2})}(\mathbb{S}^{n,d}),$$

where $\mathbb{S}^{n,d}$ *is the sphere* $\mathbb{S}^n$ *endowed with the singular metric, pulled-back of the canonical metric of* $\mathbb{S}^n$ *by the map* $\pi : (y,z,r) \in \mathbb{S}^1\times\mathbb{S}^{n-2}\times[0,\frac{\pi}{2}] \mapsto (y^d,z,r) \in \mathbb{S}^1\times\mathbb{S}^{n-2}\times[0,\frac{\pi}{2}]$*, where* $\mathbb{S}^1\times\mathbb{S}^{n-2}\times[0,\frac{\pi}{2}]$ *is identified with* $\mathbb{S}^n \subset \mathbb{R}^2\times\mathbb{R}^{n-1}$ *via the map* $\Phi(y,z,r) = \big((\sin r)y, (\cos r)z\big)$*. Note that* $\mathbb{S}^{n,d}$ *has infinitely many eigenvalues that are not eigenvalues of* $\mathbb{S}^n$*.*

Remark 1.11. *Theorem 1.9 shows that Pinching* $(\Lambda_{p,\varepsilon})$ *is not implied by Pinching* $(P_{p,\varepsilon})$ *nor Pinching* $(R_{p,\varepsilon})$*, even under an upper bound on* $\|B\|_n$*.*

Remark 1.12. *It also shows that almost extremal manifolds are not necessarily diffeomorphic nor Gromov-Hausdorff close to a sphere. We actually prove that the* (M_j) *can be constructed by gluing spheres along great subspheres* S^{k_i} *with* $k_i \leqslant k \leqslant n-2$ *and with* $\|B_j\|_{n-k} \leqslant C(k,n)$ *(see the last section of this article).*

In [5] and [16] it has been proved that when the L^∞-norm of the second fundamental form is bounded above, then almost extremal hypersurfaces are Lipschitz close to a sphere S of radius $\frac{1}{\|H\|_2}$ (which implies closeness of the spectra). In view of Theorem 1.9, we can wonder what stands when $\|B\|_q$ is bounded with $q > n$.

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Theorem 1.13. *Let $q > n$, $p > 2$ and $A > 0$ be some reals. There exist some positive functions $C = C(p, q, n, A)$ and $\alpha = \alpha(q, n)$ such that if M satisfies $(P_{p,\varepsilon})$ and $\mathrm{Vol}\, M \|B\|_q^n \leqslant A$, then the map*

$$\begin{array}{rccc} F & : & M & \longrightarrow & \frac{1}{\|H\|_2}\mathbb{S}^n \\ & & x & \longmapsto & \frac{1}{\|H\|_2}\frac{X_x}{|X_x|} \end{array}$$

is a diffeomorphism and satisfies $\left||dF(u)|^2 - |u|^2\right| \leqslant C\varepsilon^\alpha |u|^2$ for any vector $u \in TM$.

The structure of the paper is as follows: after preliminaries on the geometric inequalities for hypersurfaces in Section 2, we prove in Section 3 a general bound on extrinsic radius that depends on integral norms of the mean curvature (see Theorem 1.6). We prove Inequality (1.4) in Section 4 and Inequality (1.5) in Section 5. Theorem 1.13 is proven in Section 6. Section 7 is devoted to estimates on the trace on hypersurfaces of the homogeneous, harmonic polynomials of $\mathbb{R}^{n+1}$. These estimates are used in Section 8 to prove Theorem 1.7 and in section 9 to prove Inequality (1.6). We end the paper in section 10 by the constructions of Theorems 1.9 and 1.10.

Throughout the paper we adopt the notation that $C(p, q, n, A)$ is function greater than 1 which depends on p, q, n, A. These functions will always be of the form $C = D(p, q, n)A^{\beta(q,n)}$. But it eases the exposition to disregard the explicit nature of these functions. The convenience of this notation is that even though C might change from line to line in a calculation it still maintains these basic features.

2. Preliminaries

Let $X : (M^n, g) \to \mathbb{R}^{n+1}$ be a closed, connected, isometrically immersed n-manifold ($n \geqslant 2$). If ν denotes a local normal vector field of M in $\mathbb{R}^{n+1}$, the second fundamental form of (M^n, g) associated to ν is $B(\cdot\,,\cdot)=\left\langle \nabla^0_\cdot \nu, \cdot\right\rangle$ and the mean curvature is $H=(1/n)\mathrm{tr}\,(B)$, where ∇^0 and $\langle\cdot\,,\cdot\rangle$ are the Euclidean connection and inner product on $\mathbb{R}^{n+1}$.

Any function F on $\mathbb{R}^{n+1}$ gives rise to a function $F \circ X$ on M which, for more convenience, will be also denoted F subsequently. An easy computation gives the formula

$$\Delta F = nHdF(\nu) + \Delta^0 F + \nabla^0 dF(\nu, \nu), \tag{2.1}$$

where Δ denotes the Laplace-Beltrami operator of (M, g) and Δ^0 is the Laplace-Beltrami operator of $\mathbb{R}^{n+1}$. Applied to $F(x) = x_i$ or $F(x) = \langle x, x\rangle$, Formula 2.1 gives the following

$$\Delta X_i = nH\nu_i, \qquad \langle \Delta X, X\rangle = nH\langle \nu, X\rangle \tag{2.2}$$

$$\frac{1}{2}\Delta|X|^2 = nH\,\langle \nu, X\rangle - n, \qquad \int_M H\langle \nu, X\rangle dv = \mathrm{Vol}\, M \tag{2.3}$$

These formulas are fundamental to control the geometry of hypersurfaces by their mean curvature.

2.1. **A rough bound on geometry.** The integrated Hsiung formula (2.3) and the Cauchy-Schwarz inequality give the following

$$\int_M \frac{H\langle\nu, X\rangle dv}{\mathrm{Vol}\, M} = 1 \leqslant \|H\|_2 \big\|X - \overline{X}\big\|_2 = \|H\|_2 \inf_{u\in\mathbb{R}^{n+1}} \|X-u\|_2 \tag{2.4}$$

This inequality $\|H\|_2\|X - \overline{X}\|_2 \geqslant 1$ is optimal since M satisfies

$$\|H\|_2 \big\|X - \overline{X}\big\|_2 = 1$$

if and only if M is a sphere of radius $\frac{1}{\|H\|_2}$ and center $\overline{X}$. Indeed, in this case $X - \overline{X}$ and ν are everywhere colinear, hence the differential of the function $|X - \overline{X}|^2$ is zero on M. Equality (2.3) then implies that H is constant on M equal to $|X - \overline{X}|^{-1}$.

2.2. **Hasanis-Koutroufiotis inequality on extrinsic radius.** We set R the extrinsic Radius of M, i.e. the least radius of the balls of $\mathbb{R}^{n+1}$ which contain M. Then Inequality (2.4) gives

$$\|H\|_2 R_{ext} \geqslant \|H\|_2 \|X - \overline{X}\|_2 = \|H\|_2 \inf_{u\in\mathbb{R}^{n+1}} \|X-u\|_2$$

$$\|H\|_2 R_{ext} \geqslant 1 \tag{2.5}$$

and when $R_{ext} = \frac{1}{\|H\|_2}$, we have equality in (2.4), i.e. M is a sphere of radius $\frac{1}{\|H\|_2}$.

2.3. **Reilly inequality on λ_1.** We translate M so that $\overline{X} = 0$. By the min-max principle and Equality (2.2), we have

$$\frac{\lambda_1}{n}\|X\|_2^2 \leqslant \frac{\int_M \langle X, \Delta X\rangle dv}{n\mathrm{Vol}\, M} = \int_M \frac{H\langle\nu, X\rangle dv}{\mathrm{Vol}\, M} = 1 = \Big(\int_M \frac{H\langle\nu, X\rangle dv}{\mathrm{Vol}\, M}\Big)^2$$

where λ_1 is the first nonzero eigenvalue of M. Combined with Inequality (2.4), we get the Reilly inequality

$$\lambda_1 \leqslant \frac{n}{\mathrm{Vol}\, M}\int_M H^2 dv. \tag{2.6}$$

Here also, equality in the Reilly inequality gives equality in 2.4 and so it characterizes the sphere of radius $\frac{1}{\|H\|_2} = \|X\|_2 = \sqrt{\frac{n}{\lambda_1}}$.

3. Upper bound on the extrinsic radius

In this section we prove Theorem 1.6.

Proof. We translate M such that $\overline{X} = 0$. We set $\varphi = \big||X| - \|X\|_2\big|$. We have $|d\varphi^{2\alpha}| \leqslant 2\alpha\varphi^{2\alpha-1}$, hence, using the Sobolev inequality (see [12])

$$\|f\|_{\frac{n}{n-1}} \leqslant K(n)(\mathrm{Vol}\, M)^{\frac{1}{n}}\big(\|df\|_1 + \|Hf\|_1\big) \tag{3.1}$$

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we get for any $\alpha \geqslant 1$

$$\begin{aligned}\|\varphi\|_{\frac{2\alpha n}{n-1}}^{2\alpha} &\leqslant K(n)(\mathrm{Vol}\,M)^{\frac{1}{n}}\big(2\alpha\|\varphi\|_{2\alpha-1}^{2\alpha-1}+\|H\varphi^{2\alpha}\|_1\big)\\ &\leqslant K(n)(\mathrm{Vol}\,M)^{\frac{1}{n}}\big(2\alpha\|\varphi\|_{2\alpha-1}^{2\alpha-1}+\|H\|_q\|\varphi\|_{\frac{2\alpha q}{q-1}}^{2\alpha}\big)\\ &\leqslant K(n)(\mathrm{Vol}\,M)^{\frac{1}{n}}\big(2\alpha\|\varphi\|_{\frac{(2\alpha-1)q}{q-1}}^{2\alpha-1}+\|H\|_q\|\varphi\|_\infty\|\varphi\|_{\frac{(2\alpha-1)q}{q-1}}^{2\alpha-1}\big)\\ &\leqslant K(n)(\mathrm{Vol}\,M)^{\frac{1}{n}}\big(2\alpha+\|H\|_q\|\varphi\|_\infty\big)\|\varphi\|_{\frac{(2\alpha-1)q}{q-1}}^{2\alpha-1}\end{aligned}$$

We set $\nu=\frac{n(q-1)}{(n-1)q}$ and $\alpha=a_p\frac{q-1}{2q}+\frac{1}{2}$, where $a_{p+1}=\nu a_p+\frac{n}{n-1}$ and $a_0=\frac{2q}{q-1}$ (i.e. $a_p=a_0\nu^p+\frac{\nu^p-1}{\nu-1}\frac{n}{n-1}$). The previous inequality gives

$$\Big(\frac{\|\varphi\|_{a_{p+1}}}{\|\varphi\|_\infty}\Big)^{\frac{a_{p+1}}{\nu^{p+1}}}\leqslant\Big(K(n)(\mathrm{Vol}\,M)^{\frac{1}{n}}\big(\frac{a_p\frac{q-1}{q}+1}{\|\varphi\|_\infty}+\|H\|_q\big)\Big)^{\frac{n}{\nu^{p+1}(n-1)}}\Big(\frac{\|\varphi\|_{a_p}}{\|\varphi\|_\infty}\Big)^{\frac{a_p}{\nu^p}}$$

Since $q>n$ then $\nu>1$ and $\frac{a_p}{\nu^p}$ converges to $a_0+\frac{qn}{q-n}$ and we have

$$\begin{aligned}1 &\leqslant \Big(\frac{\|\varphi\|_{a_0}}{\|\varphi\|_\infty}\Big)^2\prod_{i=0}^{\infty}\Big(2K(n)(\mathrm{Vol}\,M)^{\frac{1}{n}}a_i\big(\frac{1}{\|\varphi\|_\infty}+\|H\|_q\big)\Big)^{\frac{1}{\nu^i}}\\ &= \Big(\frac{\|\varphi\|_{a_0}}{\|\varphi\|_\infty}\Big)^2\Big(\prod_{i=0}^{\infty}a_i^{\frac{1}{\nu^i}}\Big)\Big(2K(n)(\mathrm{Vol}\,M)^{\frac{1}{n}}\big(\frac{1}{\|\varphi\|_\infty}+\|H\|_q\big)\Big)^{\frac{\nu}{\nu-1}}\\ &= C(q,n)\Big(\frac{\|\varphi\|_{a_0}}{\|\varphi\|_\infty}\Big)^2\Big((\mathrm{Vol}\,M)^{\frac{1}{n}}\big(\frac{1}{\|\varphi\|_\infty}+\|H\|_q\big)\Big)^{\frac{n(q-1)}{q-n}}\\ &\leqslant C(q,n)\Big(\frac{\|\varphi\|_2}{\|\varphi\|_\infty}\Big)^{\frac{2(q-1)}{q}}\Big((\mathrm{Vol}\,M)^{\frac{1}{n}}\big(\frac{1}{\|\varphi\|_\infty}+\|H\|_q\big)\Big)^{\frac{n(q-1)}{q-n}}\end{aligned}$$

hence we have

$$\|\varphi\|_\infty\leqslant C(q,n)\Big((\mathrm{Vol}\,M)^{\frac{1}{n}}\big(\frac{1}{\|\varphi\|_\infty}+\|H\|_q\big)\Big)^{\frac{nq}{2(q-n)}}\|\varphi\|_2$$

We set $\gamma=\frac{nq}{2(q-n)}$. If $\|\varphi\|_\infty\geqslant\|H\|_q^{-\frac{\gamma}{1+\gamma}}\|\varphi\|_2^{\frac{1}{1+\gamma}}$ then we get the result since we have

$$\begin{aligned}\|\varphi\|_\infty &\leqslant C(q,n)\Big((\mathrm{Vol}\,M)^{\frac{1}{n}}\big(\|H\|_q^{\frac{\gamma}{1+\gamma}}\|\varphi\|_2^{\frac{-1}{1+\gamma}}+\|H\|_q\big)\Big)^\gamma\|\varphi\|_2\\ &\leqslant C(q,n)\big((\mathrm{Vol}\,M)^{\frac{1}{n}}\|H\|_q\big)^\gamma\Big(\|X\|_2^{\frac{1}{1+\gamma}}+\|\varphi\|_2^{\frac{1}{1+\gamma}}\Big)^\gamma\|\varphi\|_2^{\frac{1}{1+\gamma}}\end{aligned}$$

where we have used that $\|H\|_q\|X\|_2\geqslant 1$. We infer the result from the equality $\|\varphi\|_2=\sqrt{2}\|X\|_2\big(1-\frac{\|X\|_1}{\|X\|_2}\big)^{1/2}$. If $\|\varphi\|_\infty\leqslant\|H\|_q^{-\frac{\gamma}{1+\gamma}}\|\varphi\|_2^{\frac{1}{1+\gamma}}$, we get immediately the desired inequality of the Theorem from the above expression of $\|\varphi\|_2$ and the fact that $\|H\|_q\|X\|_2\geqslant 1$. □

4. Proof of Inequality (1.4)

Let M be an isometrically immersed hypersurface of $\mathbb{R}^{n+1}$. We can, up to translation, assume that $\int_M X dv = 0$. By the Hölder inequality and Pinching $(P_{p,\varepsilon})$, we have

$\|H\|_p\|X\|_2 \leqslant (1+\varepsilon) \leqslant (1+\varepsilon)\|H\|_p\|X\|_{\frac{p}{p-1}} \leqslant (1+\varepsilon)\|H\|_p\|X\|_1^{1-\frac{2}{p}}\|X\|_2^{\frac{2}{p}}$, hence

$$1-\frac{\|X\|_1}{\|X\|_2} \leqslant \big((1+\varepsilon)^{\frac{p}{p-2}}-1\big) \leqslant \frac{p}{p-2}2^{\frac{2}{p-2}}\varepsilon$$

On the other hand applying Inequality (3.1) to $f=1$ we get

$$1 \leqslant K(n)(\operatorname{Vol} M)^{\frac{1}{n}}\|H\|_1 \tag{4.1}$$

And combining the two above inequalities with Theorem 1.6 and $1 \leqslant \|H\|_2\|X\|_2 \leqslant 1+\varepsilon$ we get (1.4). More precisely we have $\big\||X|-\frac{1}{\|H\|_2}\big\|_\infty \leqslant \frac{C(p,q,n)A^{\gamma/n}}{\|H\|_2}\varepsilon^{\alpha(q,n)}$.

Remark 4.1. *Combining* (4.1) *with Inequality* (1.4) *we get*

$$\|X\|_\infty \leqslant C\big(p,q,n\big)A^{\gamma/n}(\operatorname{Vol} M)^{1/n} \tag{4.2}$$

Lemma 4.2. *For any $0<\varepsilon<1$ if $(P_{p,\varepsilon})$ is satisfied, then there exist some positive functions $C(p,q,n)$, $\alpha(q,n)$ and $\beta(q,n)$ so that the vector field $Z=\nu-HX$ satisfies*

$$\|Z\|_r \leqslant C(p,q,n)(1+A)^\beta\varepsilon^{\frac{\alpha(q-r)}{r(q-2)}} \qquad \textit{for any } r\in[2,q). \tag{4.3}$$

Proof. By the Hölder inequality we have for any $r\in[2,q)$

$$\begin{aligned}\|Z\|_r &\leqslant \|Z\|_q^{\frac{q(r-2)}{r(q-2)}}\|Z\|_2^{\frac{2(q-r)}{r(q-2)}} \leqslant (1+\|X\|_\infty\|H\|_q)^{\frac{q(r-2)}{r(q-2)}}\|Z\|_2^{\frac{2(q-r)}{r(q-2)}}\\ &\leqslant (1+\|X\|_\infty\|H\|_q)^{\frac{q}{q-2}}\|Z\|_2^{\frac{2(q-r)}{r(q-2)}}\end{aligned}$$

By remark 4.1, we have $\|H\|_q\|X\|_\infty \leqslant C(p,q,n)A^{\frac{\gamma+1}{n}}$. Then

$$\|Z\|_r \leqslant C(p,q,n)A^\beta\|Z\|_2^{\frac{2(q-r)}{r(q-2)}}$$

Moreover by integrating the Hsiung-Minkowsky formula (2.3) we have

$$\|Z\|_2^2 = 1-\frac{2}{\operatorname{Vol} M}\int_M H\langle\nu,X\rangle\,dv+\|HX\|_2^2 \leqslant -1+\|H\|_2^2\|X\|_\infty^2.$$

which, by Inequality (1.4), gives $\|Z\|_2^2 \leqslant C(p,q,n)A^{\beta(q,n)}\varepsilon^{\alpha(q,n)}$. □

5. Proof of Inequality (1.5)

Since we have $1=\frac{1}{\operatorname{Vol} M}\int_M H\langle X,\nu\rangle dv \leqslant \|H\|_2\|\langle X,\nu\rangle\|_2$, Inequality $(P_{p,\varepsilon})$ gives us

$$\|X\|_2 \leqslant (1+\varepsilon)\|\langle X,\nu\rangle\|_2, \qquad 1 \leqslant \|H\|_2\|X\|_2 \leqslant 1+\varepsilon,$$

and so

$$\|X-\langle X,\nu\rangle\nu\|_2 \leqslant \sqrt{3\varepsilon}\,\|X\|_2, \quad \|X-\frac{H\nu}{\|H\|_2^2}\|_2 = \sqrt{\|X\|_2^2-\|H\|_2^{-2}} \leqslant \sqrt{3\varepsilon}\,\|X\|_2.$$

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By Inequalities (1.4), this gives

$$\begin{aligned}\big\|H^2-\|H\|_2^2\big\|_1 &\leqslant \big\|H^2-|X|^2\|H\|_2^4\big\|_1+\big\||X|^2\|H\|_2^4-\|H\|_2^2\big\|_1\\ &=\|H\|_2^4\Big(\big\|\frac{H^2}{\|H\|_2^4}-|X|^2\big\|_1+\big\||X|^2-\frac{1}{\|H\|_2^2}\big\|_1\Big)\\ &\leqslant \|H\|_2^4\Big(\big\|\frac{H\nu}{\|H\|_2^2}+X\big\|_2\big\|X-\frac{H\nu}{\|H\|_2^2}\big\|_2+\frac{CA^{\gamma/n}\varepsilon^\alpha}{\|H\|_2}\Big(\|X\|_2+\frac{1}{\|H\|_2}\Big)\Big)\\ &\leqslant \|H\|_2^4\Big(\sqrt{3\varepsilon}\|X\|_2(\|\frac{H\nu}{\|H\|_2^2}\|_2+\|X\|_2)+CA^{\gamma/n}\varepsilon^\alpha\frac{1}{\|H\|_2^2}\Big)\\ &\leqslant CA^{\gamma/n}\varepsilon^\alpha\|H\|_2^2\end{aligned}$$

Hence we have $\big\||H|-\|H\|_2\big\|_1\leqslant\frac{\|H^2-\|H\|_2^2\|_1}{\|H\|_2}\leqslant CA^{\gamma/n}\varepsilon^\alpha\|H\|_2$. Moreover we have $\big\||H|-\|H\|_2\big\|_q\leqslant 2\|H\|_q\leqslant 2K(n)\|H\|_2(\operatorname{Vol}M)^{\frac{1}{n}}\|H\|_q$. Hence by the Hölder inequality, for any $r\in[1,q)$ we have

$$\begin{aligned}\big\||H|-\|H\|_2\big\|_r &\leqslant \big(\big\||H|-\|H\|_2\big\|_1\big)^{\frac{q-r}{r(q-1)}}\big(\big\||H|-\|H\|_2\big\|_q\big)^{\frac{q(r-1)}{r(q-1)}}\\ &\leqslant C(p,q,r,n)A^{\beta(q,r,n)}\varepsilon^{\frac{\alpha(q-r)}{r(q-1)}}\|H\|_2\end{aligned}$$

6. Proof of the theorem 1.13

Let $u\in TM$ be a unit vector and put $\psi=|X^\top|$ where $X^\top$ is the tangential projection of X on TM. For ε small enough we have from (1.4) $|X|\geqslant\frac{1}{2\|H\|_2}$ and then the application F is well defined. We have $dF(u)=\frac{1}{\|H\|_2|X|}\big(u-\frac{\langle X,u\rangle}{|X|^2}X\big)$ (see [5]), hence for any $\alpha\geqslant 1$

$$\begin{aligned}\big||dF_x(u)|^2-1\big| &\leqslant \frac{1}{|X|^2}\left|\frac{1}{\|H\|_2^2}-|X|^2\right|+\frac{1}{\|H\|_2^2}\frac{\langle u,X\rangle^2}{|X|^4}\\ &\leqslant \frac{C\varepsilon^\alpha}{|X|^2\|H\|_2^2}+\frac{\|\psi\|_\infty^2}{\|H\|_2^2|X|^4}\end{aligned}\tag{6.1}$$

Now an easy computation using 1.4 shows that $|d\psi|\leqslant|\langle X,\nu\rangle B-g|\leqslant\frac{CA^\beta}{\|H\|_2}|B|+n$. Now using the Sobolev inequality 3.1 and the fact that $\gamma_n\leqslant(\operatorname{Vol}M)^{1/n}\|H\|_2\leqslant(\operatorname{Vol}M)^{1/n}\|H\|_q\leqslant(\operatorname{Vol}M)^{1/n}\|B\|_q\leqslant A^{1/n}$ (see 4.1), we have

$$\begin{aligned}\|\psi\|_{\frac{2\alpha n}{n-1}}^{2\alpha} &\leqslant K(n)(\operatorname{Vol}M)^{1/n}\Big(2\alpha\frac{CA^\beta}{\|H\|_2}\|B\|_q+2\alpha n+\|H\|_q\|\psi\|_\infty\Big)\|\psi\|_{\frac{(2\alpha-1)q}{q-1}}^{2\alpha-1}\\ &\leqslant K(n)\Big(2\alpha CA^\beta(\operatorname{Vol}M)^{1/n}+A^{1/n}\|\psi\|_\infty\Big)\|\psi\|_{\frac{(2\alpha-1)q}{q-1}}^{2\alpha-1}\end{aligned}$$

And similarly to the proof of the theorem 1.6 we obtain

$$\|\psi\|_\infty\leqslant C(q,n)\big(\frac{(\operatorname{Vol}M)^{1/n}CA^\beta}{\|\psi\|_\infty}+A^{1/n}\big)^\gamma\|\psi\|_2$$

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And using the fact that $\|\psi\|_\infty \|H\|_2 \leqslant \|X\|_\infty \|H\|_2 \leqslant 1 + C$ and $A \geqslant \gamma_n$ we get $\|\psi\|_\infty \leqslant CA^\beta \left(\frac{1}{\|\psi\|_\infty \|H\|_2}\right)^\gamma \|\psi\|_2$ that is

$$\|\psi\|_\infty^{\gamma+1} \leqslant CA^\beta \frac{\|\psi\|_2}{\|H\|_2^\gamma}$$

Now since $\|\psi\|_2 = \|X - \langle X, \nu\rangle \nu\|_2 \leqslant \sqrt{3\varepsilon}\|X\|_2$ and $\|H\|_2\|X\|_2 \leqslant 1+\varepsilon$ we deduce that $\|\psi\|_\infty \leqslant \frac{CA^\beta}{\|H\|_2}\varepsilon^{\alpha(q,n)}$. And reporting this in (6.1) and using (1.4) with the fact that $|X| \geqslant \frac{1}{2\|H\|_2}$ we get $||dF_x(u)|^2 - 1| \leqslant CA^\beta \varepsilon^{\alpha(q,n)}$.

7. Homogeneous, harmonic polynomials of degree k

Let $\mathcal{H}^k(\mathbb{R}^{n+1})$ be the space of homogeneous, harmonic polynomials of degree k on $\mathbb{R}^{n+1}$. Note that $\mathcal{H}^k(\mathbb{R}^{n+1})$ induces on $\mathbb{S}^n$ the spaces of eigenfunctions of $\Delta^{\mathbb{S}^n}$ associated to the eigenvalues $\mu_k := k(n+k-1)$ with multiplicity $m_k := \binom{n+k-1}{k}\frac{n+2k-1}{n+k-1}$ (see [3]).

On the space $\mathcal{H}^k(\mathbb{R}^{n+1})$, we define the following inner product

$$(P,Q)_{\mathbb{S}^n} := \frac{1}{\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n} PQ dv_{\text{can}} ,$$

where dv_{can} denotes the element volume of the sphere with its standard metric. On the other hand the inner product on M will be defined by

$$(f,g) = \int_M \frac{fgdv}{\operatorname{Vol} M} \quad \text{ for } f, g \in C^\infty(M).$$

In this section we give some estimates on harmonic homogeneous polynomials needed subsequently. We set $(P_1, \cdots, P_{m_k})$ an arbitrary orthonormal basis of $\mathcal{H}^k(\mathbb{R}^{n+1})$. Remind that for any $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$ and any $Y \in \mathbb{R}^{n+1}$, we have $dP(X) = kP(X)$ and $\nabla^0 dP(X,Y) = (k-1)dP(Y)$.

Lemma 7.1. *For any $x \in \mathbb{R}^{n+1}$, we have* $\sum_{i=1}^{m_k} P_i^2(x) = m_k |x|^{2k}$.

Proof. For any $x \in \mathbb{S}^n$, $Q_x(P) = P^2(x)$ is a quadratic form on $\mathcal{H}^k(\mathbb{R}^{n+1})$ whose trace is given by $\sum_{i=1}^{m_k} P_i^2(x)$. Since for any $x' \in \mathbb{S}^n$ and any $O \in O_{n+1}$ such that $x' = Ox$ we have $Q_{x'}(P) = Q_x(P \circ O)$ and since $P \mapsto P \circ O$ is an isometry of $\mathcal{H}^k(\mathbb{R}^{n+1})$, we have $\sum_{i=1}^{m_k} P_i^2(x) = \operatorname{tr}(Q_x) = \sum_{i=1}^{m_k} P_i^2(x') = \operatorname{tr}(Q_{x'})$. Now

$$\sum_{i=1}^{m_k} \frac{1}{\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n} P_i^2(x) dv = m_k = \frac{1}{\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n}\left(\sum_{i=1}^{m_k} P_i^2(x)\right) dv$$

and so $\sum_{i=1}^{m_k} P_i^2(x) = m_k$. We conclude by homogeneity of the P_i. □

As an immediate consequence, we have the following lemma.

Lemma 7.2. *For any $x, u \in \mathbb{R}^{n+1}$, we have*

$$\sum_{i=1}^{m_k}\big(d_x P_i(u)\big)^2 = m_k\left(\frac{\mu_k}{n}|x|^{2(k-1)}|u|^2 + \left(k^2 - \frac{\mu_k}{n}\right)\langle u, x\rangle^2 |x|^{2(k-2)}\right).$$

Proof. Let $x \in \mathbb{S}^n$ and $u \in \mathbb{S}^n$ so that $\langle u, x\rangle = 0$. Once again the quadratic forms $Q_{x,u}(P) = \big(d_x P(u)\big)^2$ are conjugate (since O_{n+1} acts transitively on orthonormal couples) and so $\sum_{i=1}^{m_k}\big(d_x P_i(u)\big)^2$ does not depend on $u \in x^\perp$ nor on $x \in \mathbb{S}^n$. By choosing an orthonormal basis $(u_j)_{1\leqslant j\leqslant n}$ of $x^\perp$, we obtain that

$$\sum_{i=1}^{m_k}\big(d_x P_i(u)\big)^2 = \frac{1}{n}\sum_{i=1}^{m_k}\sum_{j=1}^{n}\big(d_x P_i(u_j)\big)^2 = \frac{1}{n\mathrm{Vol}\,\mathbb{S}^n}\int_{\mathbb{S}^n}\sum_{i=1}^{m_k}|\nabla^{\mathbb{S}^n}P_i|^2$$
$$= \frac{1}{n\mathrm{Vol}\,\mathbb{S}^n}\int_{\mathbb{S}^n}\sum_{i=1}^{m_k}P_i\Delta^{\mathbb{S}^n}P_i = \frac{m_k\mu_k}{n}$$

Now suppose that $u \in \mathbb{R}^{n+1}$. Then $u = v + \langle u, x\rangle x$, where $v = u - \langle u, x\rangle x$, and we have

$$\begin{aligned}\sum_{i=1}^{m_k}\big(d_x P_i(u)\big)^2 &= \sum_{i=1}^{m_k}\big(d_x P_i(v) + k\langle u, x\rangle P_i(x)\big)^2\\ &= \sum_{i=1}^{m_k}\big(d_x P_i(v)\big)^2 + 2k\langle u, x\rangle\sum_{i=1}^{m_k}d_x P_i(v)P_i(x) + m_k\langle u, x\rangle^2 k^2\\ &= \frac{m_k\mu_k}{n}|v|^2 + m_k\langle u, x\rangle^2 k^2 = m_k\left(\frac{\mu_k}{n}|u|^2 + \left(k^2 - \frac{\mu_k}{n}\right)\langle u, x\rangle^2\right),\end{aligned}$$

where we derived the equality in Lemma 7.1 to make $\sum_{i=1}^{m_k}d_x P_i(v)P_i(x)$ disappear. We conclude by homogeneity of P_i. □

Lemma 7.3. *For any $x \in \mathbb{R}^{n+1}$, we have $\sum_{i=1}^{m_k}|\nabla^0 dP_i(x)|^2 = m_k\alpha_{n,k}|x|^{2(k-2)}$, where $\alpha_{n,k} = (k-1)(k^2+\mu_k)(n+2k-3) \leqslant C(n)k^4$.*

Proof. The Bochner equality gives

$$\begin{aligned}\sum_{i=1}^{m_k}|\nabla^0 dP_i(x)|^2 &= \sum_{i=1}^{m_k}\left(\langle d\Delta^0 P_i, dP_i\rangle - \frac{1}{2}\Delta^0|dP_i|^2\right)\\ &= -\frac{1}{2}m_k\big(k^2+\mu_k\big)\Delta^0|X|^{2k-2} = m_k\alpha_{n,k}|X|^{2k-4}\end{aligned}$$

□

Let $\mathcal{H}^k(M) = \{P\circ X\ ,\ P \in \mathcal{H}^k(\mathbb{R}^{n+1})\}$ be the space of functions induced on M by $\mathcal{H}^k(\mathbb{R}^{n+1})$. We will identify P and $P\circ X$ subsequently. There is no ambiguity since we have

Lemma 7.4. *Let M^n be a compact manifold immersed by X in $\mathbb{R}^{n+1}$ and let $(P_1, \ldots, P_m)$ be a linearly independent set of homogeneous polynoms of degree k on $\mathbb{R}^{n+1}$. Then the set $(P_1\circ X, \ldots, P_m\circ X)$ is also linearly independent.*

Proof. Any homogeneous polynomial P which is zero on M is zero on the cone $\mathbb{R}^+\cdot M$. Since M is compact there exists a point $x \in M$ so that $X_x \notin T_xM$ and so $\mathbb{R}^+\cdot M$ has non empty interior. Hence $P\circ X = 0$ implies $P = 0$. □

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Formula (2.1) implies

$$\Delta P=\mu_k H^2 P+(n+2k-2)HdP(Z)+\nabla^0 dP(Z,Z) \tag{7.1}$$

In order to estimate ΔP, we define two linear maps

$$\begin{array}{rcl} V_k^\star:\mathcal{H}^k(\mathbb{R}^{n+1}) & \longrightarrow & C^\infty(M)\\ P & \longmapsto & dP(V)\end{array}$$

and

$$\begin{array}{rcl} (V,W)_k^\star:\mathcal{H}^k(\mathbb{R}^{n+1}) & \longrightarrow & C^\infty(M)\\ P & \longmapsto & \nabla^0 dP(V,W)\end{array}$$

where $V,W\in\Gamma(M)$ are vector fields.

If $L:\mathcal{H}^k(\mathbb{R}^{n+1})\longrightarrow C^\infty(M)$ is a linear map, we set

$$|||L|||^2=\sum_{i=1}^{m_k}\|L(P_i)\|_2^2,$$

where $(P_1,\cdots,P_{m_k})$ is an orthonormal basis of $(\mathcal{H}^k(\mathbb{R}^{n+1}),\|\,.\,\|_{\mathbb{S}^n})$.

Remark 7.5. *For any $P\in\mathcal{H}^k(\mathbb{R}^{n+1})$, we have $\|L(P)\|_2^2\leqslant |||L\,|||^2\,\|P\|_{\mathbb{S}^n}^2$.*

We now give some estimates on $Z_k^\star$, $(HZ)_k^\star$ and $(Z,Z)_k^\star$.

Lemma 7.6. *We have*

$$|||Z_k^\star|||^2\leqslant\frac{m_k k^2}{\operatorname{Vol}M}\int_M |X|^{2(k-1)}|Z|^2dv \tag{7.2}$$

$$|||(HZ)_k^\star|||^2\leqslant\frac{m_k k^2}{\operatorname{Vol}M}\int_M |X|^{2(k-1)}H^2|Z|^2dv \tag{7.3}$$

$$|||(Z,Z)_k^\star|||^2\leqslant\frac{m_k\alpha_{k,n}}{\operatorname{Vol}M}\int_M |X|^{2(k-2)}|Z|^4dv \tag{7.4}$$

Proof. Let $(P_1,\cdots,P_{m_k})$ be an orthonormal basis of $\mathcal{H}^k(\mathbb{R}^{n+1})$. By Lemma 7.2 we have

$$|||Z_k^\star|||^2=\sum_{i=1}^{m_k}\|dP_i(Z)\|_2^2\leqslant\frac{m_k k^2}{\operatorname{Vol}M}\int_M |X|^{2(k-1)}|Z|^2dv$$

and

$$|||(HZ)_k^\star|||^2=\sum_{i=1}^{m_k}\|HdP_i(Z)\|_2^2\leqslant\frac{m_k k^2}{\operatorname{Vol}M}\int_M |X|^{2(k-1)}H^2|Z|^2dv$$

By Lemma 7.3, we have

$$|||(Z,Z)_k^\star|||^2=\sum_{i=1}^{m_k}\|\nabla^0 dP_i(Z,Z)\|_2^2\leqslant\frac{m_k\alpha_{k,n}}{\operatorname{Vol}M}\int_M |X|^{2(k-2)}|Z|^4dv,$$

which ends the proof. □

Lemma 7.7. *Let $q > n$ and $A > 0$ be some reals. There exist a constants $C = C(q,n)$ and $\beta(q,n)$ such that for any isometrically immersed hypersurface M of $\mathbb{R}^{n+1}$ which satisfies $\operatorname{Vol} M\|H\|_q^n \leqslant A$ and any $P \in \mathcal{H}^k(M)$, we have*

$$\left|\|H\|_2^{2k}\|P\|_2^2 - \|P\|_{\mathbb{S}^n}^2\right| \leqslant D\sigma_k(CA^\beta)^{2k-1}\,\|P\|_{\mathbb{S}^n}^2$$

where $D = \left\|H^2 - \|H\|_2^2\right\|_1\|X\|_\infty^2 + \left\|HZ\right\|_2\|X\|_\infty + \|Z\|_2^2 + \left\|Z\right\|_4^2$.

Proof. For any $P \in \mathcal{H}^k(M)$ we have

$$\begin{aligned}\|\nabla^0 P\|_2^2 &= \|dP(\nu)\|_2^2 + \|dP\|_2^2\\ &= \|dP(Z)\|_2^2 + k^2\|HP\|_2^2 + \frac{1}{\operatorname{Vol} M}\int_M \big(2kHdP(Z)P + P\Delta P\big)dv\end{aligned}$$

and from (7.1) we get

$$\begin{aligned}\|\nabla^0 P\|_2^2 =&\|dP(Z)\|_2^2 + \frac{1}{\operatorname{Vol} M}\int_M \big(P\nabla^0 dP(Z,Z) + (n+4k-2)HdP(Z)P\big)dv\\ &+ (\mu_k + k^2)\|HP\|_2^2\\ =&\frac{1}{\operatorname{Vol} M}\int_M \Big((\mu_k+k^2)\big(H^2 - \|H\|_2^2\big)P^2 + (n+4k-2)HdP(Z)P\Big)dv\\ &+ \frac{1}{\operatorname{Vol} M}\int_M P\nabla^0 dP(Z,Z)dv + (\mu_k+k^2)\|H\|_2^2\|P\|_2^2 + \|dP(Z)\|_2^2\end{aligned}$$

Now we have

$$\left\|\nabla^0 P\right\|_{\mathbb{S}^n}^2 = \left\|\nabla^{\mathbb{S}^n} P\right\|_{\mathbb{S}^n}^2 + k^2\,\|P\|_{\mathbb{S}^n}^2 = (\mu_k + k^2)\,\|P\|_{\mathbb{S}^n}^2 \tag{7.5}$$

Hence

$$\begin{aligned}&\|H\|_2^{2k-2}\|\nabla^0 P\|_2^2 - \left\|\nabla^0 P\right\|_{\mathbb{S}^n}^2 = (\mu_k+k^2)\big(\|H\|_2^{2k}\|P\|_2^2 - \|P\|_{\mathbb{S}^n}^2\big) + \|H\|_2^{2k-2}\|dP(Z)\|_2^2\\ &+\frac{\|H\|_2^{2k-2}}{\operatorname{Vol} M}\int_M P\Big((\mu_k+k^2)\big(H^2 - \|H\|_2^2\big)P + H(n+4k-2)dP(Z) + \nabla^0 dP(Z,Z)\Big)dv\end{aligned}$$

Which gives

(7.6)

$$\begin{aligned}\left|\|H\|_2^{2k}\|P\|_2^2 - \|P\|_{\mathbb{S}^n}^2\right| \leqslant& \frac{1}{\mu_k + k^2}\left|\|H\|_2^{2k-2}\|\nabla^0 P\|_2^2 - \left\|\nabla^0 P\right\|_{\mathbb{S}^n}^2\right|\\ &+ \frac{\|H\|_2^{2k-2}}{\mu_k+k^2}\Big((n+4k-2)\big|((HZ)_k^\star P,P)\big| + \|Z_k^\star(P)\|_2^2 + \big|((Z,Z)_k^\star P,P)\big|\Big)\\ &+ \frac{\|H\|_2^{2k-2}}{\operatorname{Vol} M}\int_M\big|H^2 - \|H\|_2^2\big|P^2 dv\end{aligned}$$

Note that, by Lemma 7.1 and Remark 7.5, we have

$$\begin{aligned}\frac{1}{\operatorname{Vol} M}\int_M\big|H^2 - \|H\|_2^2\big|P^2 dv \leqslant& \frac{\|P\|_{\mathbb{S}^n}^2}{\operatorname{Vol} M}\int_M\big|H^2 - \|H\|_2^2\big|\sum_{i=1}^{m_k}P_i^2 dv\\ =& \frac{m_k\,\|P\|_{\mathbb{S}^n}^2}{\operatorname{Vol} M}\int_M\big|H^2 - \|H\|_2^2\big||X|^{2k}dv\\ \leqslant& m_k\|X\|_\infty^{2k}\|H^2 - \|H\|_2^2\|_1\,\|P\|_{\mathbb{S}^n}^2\end{aligned}$$

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which, combined with (7.6), gives

$$\Big|\|H\|_2^{2k}\|P\|_2^2-\|P\|_{\mathbb{S}^n}^2\Big|\leqslant\frac{1}{\mu_k+k^2}\Big|\|H\|_2^{2k-2}\|\nabla^0P\|_2^2-\big\|\nabla^0P\big\|_{\mathbb{S}^n}^2\Big|$$
$$+\,m_k\|H\|_2^{2k-2}\|X\|_\infty^{2k}\|H^2-\|H\|_2^2\|_1\,\|P\|_{\mathbb{S}^n}^2$$
$$+\,\frac{\|H\|_2^{2k-2}\,\|P\|_{\mathbb{S}^n}}{\mu_k+k^2}\Big((n+4k-2)\,|||\,(HZ)_k^\star\,|||\,\|P\|_2+|||Z_k^\star\,|||^2\,\|P\|_{\mathbb{S}^n}+|||(Z,Z)_k^\star\,|||\,\|P\|_2\Big)$$

Now, as above, we have

$$\|P\|_2\leqslant\sqrt{\sum_i\|P_i\|_2^2}\,\|P\|_{\mathbb{S}^n}\leqslant\sqrt{\frac{m_k}{\mathrm{Vol}\,M}\int_M|X|^{2k}dv}\,\|P\|_{\mathbb{S}^n}\tag{7.7}$$

and from Lemma 7.6, we get

$$\frac{\|H\|_2^{2k-2}\,\|P\|_{\mathbb{S}^n}}{\mu_k+k^2}\Big((n+4k-2)\,|||\,(HZ)_k^\star\,|||\,\|P\|_2+|||Z_k^\star\,|||^2\,\|P\|_{\mathbb{S}^n}+|||(Z,Z)_k^\star\,|||\,\|P\|_2\Big)$$
$$\leqslant C(n)m_k(\|H\|_2\|X\|_\infty)^{2k-2}(\|X\|_\infty\|HZ\|_2+\|Z\|_2^2+\|Z\|_4^2)\|P\|_{\mathbb{S}^n}^2$$

and

$$\Big|\|H\|_2^{2k}\|P\|_2^2-\|P\|_{\mathbb{S}^n}^2\Big|\leqslant\frac{1}{\mu_k+k^2}\Big|\|H\|_2^{2k-2}\|\nabla^0P\|_2^2-\big\|\nabla^0P\big\|_{\mathbb{S}^n}^2\Big|$$
$$+\big(\|X\|_\infty\|H\|_2\big)^{2k-2}m_kC(n)D\,\|P\|_{\mathbb{S}^n}^2$$

with

$$D:=\Big(\big\|H^2-\|H\|_2^2\big\|_1\|X\|_\infty^2+\big\|HZ\big\|_2\|X\|_\infty+\|Z\|_2^2+\big\|Z\big\|_4^2\Big)$$

In particular for $k=1$, we have $|\nabla^0P|$ constant and so

$$\big|\|H\|_2^2\|P\|_2^2-\|P\|_{\mathbb{S}^n}^2\big|\leqslant m_1C(n)D\,\|P\|_{\mathbb{S}^n}^2$$

Let $B_k=\sup\Big\{\frac{|\|H\|_2^{2k}\|P\|_2^2-\|P\|_{\mathbb{S}^n}^2|}{\|P\|_{\mathbb{S}^n}^2}\;\;|\;\;P\in\mathcal{H}^k(\mathbb{R}^{n+1})\setminus\{0\}\Big\}$. Then using that $\nabla^0P\in\mathcal{H}^{k-1}(\mathbb{R}^{n+1})$ and (7.5), we get for $1\leqslant i\leqslant k$

$$B_k\leqslant B_{k-1}+m_k\big(\|X\|_\infty\|H\|_2\big)^{2k-2}C(n)D\leqslant C(n)D\sigma_k\big(\|X\|_\infty\|H\|_2\big)^{2k-1}$$

We conclude using Theorem 1.6. □

8. Proof of Theorem 1.7

Under the assumption of Theorem 1.7 we can use Lemma 4.2, Theorem 1.6 and Inequality (1.4) to improve the estimate in Lemma 7.7.

Lemma 8.1. *Let $q>\max(4,n)$, $p>2$ and $A>0$ be some reals. There exist some constants $C=C(p,q,n)$, $\alpha=\alpha(q,n)$ and $\beta=\beta(q,n)$ such that for any isometrically immersed hypersurface M of $\mathbb{R}^{n+1}$ satisfying $(P_{p,\varepsilon})$ and $\mathrm{Vol}\,M\|H\|_q^n\leqslant A$, and for any $P\in\mathcal{H}^k(M)$, we have*

$$\Big|\|H\|_2^{2k}\|P\|_2^2-\|P\|_{\mathbb{S}^n}^2\Big|\leqslant\varepsilon^\alpha\sigma_k(CA^\beta)^{2k}\,\|P\|_{\mathbb{S}^n}^2\,.$$

This allows to prove the following estimate on ΔP.

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Lemma 8.2. *Let k be an integer such that $\varepsilon^\alpha\sigma_k(CA^\beta)^{2k}\leqslant\frac12$ and $P\in\mathcal{H}^k(M)$, we have*

$$\big\|\Delta P-\mu_k\|H\|_2^2P\big\|_2\leqslant\sqrt{m_k}\mu_k(CA^\beta)^k\varepsilon^\alpha\|H\|_2^2\|P\|_2$$

Proof. From Formula (7.1), we have

$$\|\Delta P-\mu_k\|H\|_2^2P\|_2\leqslant\|\mu_k(H^2-\|H\|_2^2)P\|_2+(n+2k-2)\|HdP(Z)\|_2+\|\nabla^0dP(Z,Z)\|_2$$

If $\varepsilon^\alpha\sigma_k(CA^\beta)^{2k}\leqslant\frac12$ we deduce from Lemma 8.1 that $\|P\|_{\mathbb{S}^n}^2\leqslant2\|H\|_2^{2k}\|P\|_2^2$. And using Lemma 7.1 and Inequality (1.4), we have

$$\|\mu_k(H^2-\|H\|_2^2)P\|_2^2\leqslant\frac{\mu_k^2m_k}{\operatorname{Vol}M}\|P\|_{\mathbb{S}^n}^2\int_M|H^2-\|H\|_2^2|^2|X|^{2k}dv$$
$$\leqslant\frac{2\mu_k^2m_k}{\operatorname{Vol}M}\|P\|_2^2\big(\|H\|_2\|X\|_\infty\big)^{2k}\int_M(H^2-\|H\|_2^2)^2dv\leqslant(CA^\beta)^{2k}\varepsilon^\alpha\mu_k^2m_k\|H\|_2^4\|P\|_2^2$$

where the last inequality comes from Inequality 1.5 and the Hölder Inequality. By technical Lemma of Section 7, we have

$$\|HdP(Z)\|_2^2\leqslant\|P\|_{\mathbb{S}^n}^2\,|||\,(HZ)_k^*|||^2\leqslant\|P\|_{\mathbb{S}^n}^2\frac{m_kk^2}{\operatorname{Vol}M}\|X\|_\infty^{2k-2}\int_MH^2|Z|^2dv$$
$$\leqslant\varepsilon^\alpha(CA^\beta)^{2k}k^2m_k\|H\|_2^4\|P\|_2^2$$
$$\|\nabla^0dP(Z,Z)\|_2^2\leqslant|||(Z,Z)_k^\star\,|||^2\,\|P\|_{\mathbb{S}^n}^2\leqslant m_k\alpha_{k,n}\|X\|_\infty^{2k-4}\|Z\|_4^4\|P\|_{\mathbb{S}^n}^2$$
$$\leqslant(CA^\beta)^{2k}\varepsilon^\alpha m_k\alpha_{k,n}\|H\|_2^4\|P\|_2^2$$

which gives the result. □

Let $\nu>0$ and E_k^ν be the space spanned by the eigenfunctions of M associated to an eigenvalue in the interval $\big[(1-\varepsilon^\alpha\sqrt{m_k}C^k-\nu)\|H\|_2^2\mu_k,(1+\varepsilon^\alpha\sqrt{m_k}C^k+\nu)\|H\|_2^2\mu_k\big]$. If $\dim E_k^\nu<m_k$, then there exists $P\in\mathcal{H}^k(M)\setminus\{0\}$ which is L^2-orthogonal to E_k^ν. Let $P=\sum_i f_i$ be the decomposition of P in the Hilbert basis given by the eigenfunctions f_i of M associated respectively to λ_i. Putting $N:=\{i\mid f_i\notin E_k^\nu\}$, by assumption on P we have

$$(C^k\sqrt{m_k}\varepsilon^\alpha+\nu)^2\|H\|_2^4\mu_k^2\|P\|_2^2\leqslant\sum_{i\in N}\big(\lambda_i-\|H\|_2^2\mu_k\big)^2\|f_i\|_2^2=\|\Delta P-\mu_k\|H\|_2^2P\|_2^2$$
$$\leqslant\mu_k^2C^{2k}m_k\|H\|_2^4\varepsilon^{2\alpha}\|P\|_2^2$$

which gives a contradiction. We then have $\dim E_k^\nu\geqslant m_k$. We get the result by letting ν tends to 0.

9. Proof of Inequality 1.6

We can assume $\eta\leqslant1$ and $\|H\|_2=1$ by a homogeneity argument. Let $x\in\mathbb{S}^n$ and set $V^n(s)=\operatorname{Vol}(B(x,s)\cap\mathbb{S}^n)$. Let $\beta>0$ small enough so that $(1+\eta/2)V^n\big((1+2\beta)r\big)\leqslant(1+\eta)V^n(r)$ and $(1-\eta/2)V^n\big((1-2\beta)r\big)\geqslant(1-\eta)V^n(r)$. Let $f_1:\mathbb{S}^n\to[0,1]$ (resp. $f_2:\mathbb{S}^n\to[0,1]$) be a smooth function such that $f_1=1$ on $B\big(x,(1+\beta)r\big)\cap\mathbb{S}^n$ (resp. $f_2=1$ on $B\big(x,(1-2\beta)r\big)\cap\mathbb{S}^n$) and $f_1=0$ outside $B\big(x,(1+2\beta)r\big)\cap\mathbb{S}^n$ (resp. $f_2=0$ outside $B\big(x,(1-\beta)r\big)\cap\mathbb{S}^n$). There exists a family $(P_k^i)_{k\leqslant N}$ such that

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$P_k^i \in \mathcal{H}^k(\mathbb{R}^{n+1})$ and $A = \sup_{\mathbb{S}^n}\big|f_i - \sum_{k\leqslant N} P_k^i\big| \leqslant \|f_i\|^2_{\mathbb{S}^n}\eta/18$. We extend f_i to $\mathbb{R}^{n+1}\setminus\{0\}$ by $f_i(X) = f_i\big(\frac{X}{|X|}\big)$. Then we have

$$\Big|\|f_i\|_2^2 - \frac{1}{\mathrm{Vol}\,\mathbb{S}^n}\int_{\mathbb{S}^n}|f_i|^2\Big| \leqslant I_1 + I_2 + I_3$$

where

$$I_1 := \Big|\frac{1}{\mathrm{Vol}\,M}\int_M\Big(|f_i|^2 - (\sum_{k\leqslant N}|X|^{-k}P_k^i)^2\Big)dv\Big|$$

$$I_2 := \Big|\frac{1}{\mathrm{Vol}\,M}\int_M(\sum_{k\leqslant N}|X|^{-k}P_k^i)^2dv - \sum_{k\leqslant N}\|P_k^i\|^2_{\mathbb{S}^n}\Big|$$

and

$$I_3 := \Big|\frac{1}{\mathrm{Vol}\,\mathbb{S}^n}\int_{\mathbb{S}^n}\Big((\sum_{k\leqslant N}P_k^i)^2 - f_i^2\Big)\Big|.$$

On $\mathbb{S}^n$ we have $\big|f_i^2 - (\sum_{k\leqslant N}P_k^i)^2\big| \leqslant A\big(2\sup_{\mathbb{S}^n}|f_i| + A\big) \leqslant \|f_i\|^2_{\mathbb{S}^n}\eta/6$ and on M we have

$$\Big|f_i^2(X) - (\sum_{k\leqslant N}|X|^{-k}P_k^i(X))^2\Big| = \Big|f_i^2(\frac{X}{|X|}) - \Big(\sum_{k\leqslant N}P_k^i(\frac{X}{|X|})\Big)^2\Big| \leqslant \|f_i\|^2_{\mathbb{S}^n}\eta/6$$

.

Hence $I_1 + I_3 \leqslant \|f_i\|^2_{\mathbb{S}^n}\eta/3$. Now

$$\begin{aligned}
I_2 \leqslant& \Big|\frac{1}{\mathrm{Vol}\,M}\int_M\sum_{k\leqslant N}\frac{(P_k^i)^2}{|X|^{2k}}dv - \sum_{k\leqslant N}\|P_k^i\|^2_{\mathbb{S}^n}\Big| + \frac{1}{\mathrm{Vol}\,M}\Big|\int_M\sum_{1\leqslant k\neq k'\leqslant N}\frac{P_k^iP_{k'}^i}{|X|^{k+k'}}dv\Big|\\
\leqslant& \frac{1}{\mathrm{Vol}\,M}\int_M\sum_{k\leqslant N}\Big|\frac{1}{|X|^{2k}} - \|H\|_2^{2k}\Big|(P_k^i)^2dv\\
&+ \frac{1}{\mathrm{Vol}\,M}\int_M\sum_{1\leqslant k\neq k'\leqslant N}\Big|\frac{1}{|X|^{k+k'}} - \|H\|_2^{k+k'}\Big||P_k^iP_{k'}^i|dv\\
&+ \sum_{k\leqslant N}\Big|\|H\|_2^{2k}\|P_k^i\|_2^2 - \big\|P_k^i\big\|^2_{\mathbb{S}^n}\Big| + \sum_{1\leqslant k\neq k'\leqslant N}\frac{\|H\|_2^{k+k'}}{\mathrm{Vol}\,M}\Big|\int_M P_k^iP_{k'}^idv\Big|
\end{aligned}$$

From (1.4) we have $\Big|\frac{1}{|X|^{k+k'}} - \|H\|_2^{k+k'}\Big| \leqslant NC^N\varepsilon^\alpha\|H\|_2^{k+k'}$. From this and Lemma 8.1, we have

$$I_2 \leqslant N^2C^N\varepsilon^\alpha\sum_{k\leqslant N}\|H\|_2^{2k}\|P_k^i\|_2^2 + \varepsilon^\alpha\sum_{k\leqslant N}\sigma_kC^{2k}\big\|P_k^i\big\|^2_{\mathbb{S}^n} + \sum_{1\leqslant k\neq k'\leqslant N}\frac{\|H\|_2^{k+k'}}{\mathrm{Vol}\,M}\Big|\int_M P_k^iP_{k'}^idv\Big|$$

and, by Lemma 8.2, we have

$$\begin{aligned}\Big|\frac{\|H\|_2^2(\mu_k-\mu_{k'})}{\operatorname{Vol}M}\int_M P_k^iP_{k'}^i dv\Big| &\leqslant \int_M \frac{|P_k^i(\Delta P_{k'}^i-\|H\|_2^2\mu_{k'}P_{k'}^i)|}{\operatorname{Vol}M}dv\\ &\quad+\int_M \frac{|P_{k'}^i(\Delta P_k^i-\|H\|_2^2\mu_k P_k^i)|}{\operatorname{Vol}M}dv\\ &\leqslant \|P_k^i\|_2\big\|\Delta P_{k'}^i-\|H\|_2^2\mu_{k'}P_{k'}^i\big\|_2+\|P_{k'}^i\|_2\big\|\Delta P_k^i-\|H\|_2^2\mu_kP_k^i\big\|_2\\ &\leqslant 2\sqrt{m_N}\mu_N C^N\varepsilon^\alpha\|H\|_2^2\|P_{k'}^i\|_2\|P_k^i\|_2\end{aligned}$$

under the condition $\varepsilon^\alpha\sigma_N C^{2N}\leqslant\frac12$. Since $\mu_k-\mu_{k'}\geqslant n$ when $k\neq k'$, we have

$$\Big|\frac{1}{\operatorname{Vol}M}\int_M P_k^iP_{k'}^i dv\Big|\leqslant\frac{2}{n}\sqrt{m_N}\mu_N C^N\varepsilon^\alpha\|P_{k'}^i\|_2\|P_k^i\|_2$$

hence

$$\begin{aligned}I_2 &\leqslant N^2C^N\varepsilon^\alpha\sum_{k\leqslant N}\|H\|_2^{2k}\|P_k^i\|_2^2+\varepsilon^\alpha\sum_{k\leqslant N}\sigma_kC^{2k}\left\|P_k^i\right\|_{\mathbb{S}^n}^2\\ &\quad+\frac{2}{n}\sqrt{m_N}\mu_N C^N\varepsilon^\alpha\sum_{1\leqslant k\neq k'\leqslant N}\|H\|_2^{k+k'}\|P_{k'}^i\|_2\|P_k^i\|_2\\ &\leqslant D_N\varepsilon^\alpha\sum_{k\leqslant N}\|H\|_2^{2k}\|P_k^i\|_2^2+\varepsilon^\alpha\sum_{k\leqslant N}\sigma_kC^{2k}\left\|P_k^i\right\|_{\mathbb{S}^n}^2\\ &\leqslant\varepsilon^\alpha\sum_{k\leqslant N}(D_N(1+\varepsilon^\alpha\sigma_kC^{2k})+\sigma_kC^{2k})\left\|P_k^i\right\|_{\mathbb{S}^n}^2\leqslant D'_N\varepsilon^\alpha\end{aligned}$$

Where we have used the fact that $\|\sum_{k\leqslant N}P_k^i\|_{\mathbb{S}^n}^2$ is bounded by a constant. We infer that if $\varepsilon^\alpha\leqslant\frac{V^n((1-2\beta)r)\eta}{6D'_N\operatorname{Vol}\mathbb{S}^n}\leqslant\frac{\|f_i\|_{\mathbb{S}^n}^2\eta}{6D'_N}$, then we have

$$\Big|\|f_i\|_2^2-\frac{1}{\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n}|f_i|^2\Big|\leqslant\eta\|f_i\|_{\mathbb{S}^n}^2/2$$

Note that N depends on r and β but not on x since $O(n+1)$ acts transitively on $\mathbb{S}^n$. Eventually, by assumption on f_1 and f_2 and by estimate (1.4), we have

$$\begin{aligned}\frac{\operatorname{Vol}\,(B(x,(1+\beta)r-C\varepsilon^\alpha)\cap X(M))}{\operatorname{Vol}M}&\leqslant\|f_1\|_2^2\leqslant(1+\eta/2)\|f_1\|_{\mathbb{S}^n}^2\\ &\leqslant(1+\eta/2)\frac{V^n((1+2\beta)r)}{\operatorname{Vol}\mathbb{S}^n}\leqslant(1+\eta)\frac{V^n(r)}{\operatorname{Vol}\mathbb{S}^n}\\ \frac{\operatorname{Vol}\,(B(x,(1-\beta)r+C\varepsilon^\alpha)\cap X(M))}{\operatorname{Vol}M}&\geqslant\|f_2\|_2^2\geqslant(1-\eta/2)\|f_2\|_{\mathbb{S}^n}^2\\ &\geqslant(1-\eta/2)\frac{V^n((1-2\beta)r)}{\operatorname{Vol}\mathbb{S}^n}\geqslant(1-\eta)\frac{V^n(r)}{\operatorname{Vol}\mathbb{S}^n}\end{aligned}$$

And by choosing $\varepsilon^\alpha=\min\left(\frac{\beta r}{C},\frac{V^n((1-2\beta)r)\eta}{6D'_N\operatorname{Vol}\mathbb{S}^n)}\right)$ we get

$$\Big|\frac{\operatorname{Vol}\,(B(x,r)\cap X(M))}{\operatorname{Vol}M}-\frac{V^n(r)}{\operatorname{Vol}\mathbb{S}^n}\Big|\leqslant\eta\frac{V^n(r)}{\operatorname{Vol}\mathbb{S}^n}$$

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10. Some examples

We set $I_\varepsilon = [\varepsilon, \frac{\pi}{2}]$ for $\varepsilon > 0$ and let $\varphi : I_\varepsilon \longrightarrow (-1, +\infty)$ be a function continuous on I_ε and smooth on $(\varepsilon, \frac{\pi}{2}]$. For any $0 \leqslant k \leqslant n-2$, we consider the map

$$\begin{array}{rcl}\Phi_\varphi : \mathbb{S}^{n-k-1} \times \mathbb{S}^k \times I_\varepsilon & \longrightarrow & \mathbb{R}^{n+1} = \mathbb{R}^{n-k} \oplus \mathbb{R}^{k+1} \\ x = (y, z, r) & \longmapsto & (1+\varphi(r))\big((\sin r) y + (\cos r) z\big)\end{array}$$

which is an embedding onto a manifold $X_\varphi \subset \mathbb{R}^{n+1}$. We denote respectively by $B(\varphi)$ and $H(\varphi)$ the second fundamental form and the mean curvature of X_φ. We have

Lemma 10.1. *Let $x = (y, z, r) \in \mathbb{S}^{n-k-1} \times \mathbb{S}^k \times I_\varepsilon$, $q = \Phi_\varphi(x)$ and $(u, v, h) \in T_x X_\varepsilon$. Then we have*

$$\begin{aligned} nH_q(\varphi) = &\big(\varphi'^2 + (1+\varphi)^2\big)^{-3/2}\Big[-(1+\varphi(r))\varphi''(r) + (1+\varphi(r))^2 + 2\varphi'^2(r)\Big] \\ &+ \frac{\big(\varphi'^2 + (1+\varphi)^2\big)^{-1/2}}{1+\varphi(r)}\Big[-(n-k-1)\varphi'(r)\cot r + (n-1)(1+\varphi(r)) + k\varphi'(r)\tan r\Big]\end{aligned}$$

$$|B_q(\varphi)| = \frac{(1+\varphi(r))^{-1}}{\big(1 + (\frac{\varphi'(r)}{1+\varphi(r)})^2\big)^{1/2}} \max\Big(\big|1 - \frac{\varphi'}{1+\varphi}\cot r\big|, \big|1 + \frac{\varphi'}{1+\varphi}\tan r\big|, \big|1 + \frac{(\varphi')^2 - (1+\varphi)\varphi''}{\varphi'^2 + (1+\varphi)^2}\big|\Big)$$

Proof. Let $(u, v, h) \in T_x S_\varepsilon$ and put $w = d(\Phi_\varphi)_x(u, v, h) \in T_q X_\varphi$ where $S_\varepsilon = \mathbb{S}^{n-k-1} \times \mathbb{S}^k \times I_\varepsilon$. An easy computation shows that

$$\begin{aligned} w = &(1+\varphi(r))((\sin r)u + (\cos r)v) \\ &+ \varphi'(r)((\sin r)y + (\cos r)z)h + (1+\varphi(r))((\cos r)y - (\sin r)z)h \end{aligned} \tag{10.1}$$

We set

$$\tilde{N}_q = -\varphi'(r)((\cos r)y - (\sin r)z) + (1+\varphi(r))((\sin r)y + (\cos r)z)$$

and $N_q = \dfrac{\tilde{N}_q}{(\varphi'^2 + (1+\varphi)^2)^{1/2}}$ is a unit normal vector field on X_φ. Then we have

$$\begin{aligned} B_q(\varphi)(w, w) &= \big\langle \nabla^0_w N, w\big\rangle = \big(\varphi'^2 + (1+\varphi)^2\big)^{-1/2}\big\langle \nabla^0_w \tilde{N}, w\big\rangle \\ &= \big(\varphi'^2 + (1+\varphi)^2\big)^{-1/2}\Big\langle \sum_{i=1}^{n+1} w(\tilde{N}^i)\partial_i, w\Big\rangle \end{aligned} \tag{10.2}$$

where $(\partial_i)_{1\leqslant i\leqslant n+1}$ is the canonical basis of $\mathbb{R}^{n+1}$. A straightforward computation shows that

$$\begin{aligned} \sum_{i=1}^{n+1} w(\tilde{N}^i)\partial_i = &-\varphi'(r)((\cos r)u - (\sin r)v) + (1+\varphi(r))((\sin r)u + (\cos r)v) \\ &- \varphi''(r)((\cos r)y - (\sin r)z)h + 2\varphi'(r)((\sin r)y + (\cos r)z)h \\ &+ (1+\varphi(r))((\cos r)y - (\sin r)z)h \end{aligned}$$

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Reporting this in (10.2) and using (10.1) we get

$$B_q(\varphi)((u,v,h),(u,v,h)) = \frac{1}{\sqrt{\varphi'^2+(1+\varphi)^2}}\Big[-\varphi'(r)\big(1+\varphi(r)\big)\sin r\cos r(|u|^2-|v|^2)$$
$$+(1+\varphi(r))^2(\sin^2 r|u|^2+\cos^2 r|v|^2)-\big(1+\varphi(r)\big)\varphi''(r)h^2+2\varphi'^2(r)h^2+(1+\varphi(r))^2h^2\Big]$$

Now let $(u_i)_{1\leqslant i\leqslant n-k-1}$ and $(v_i)_{1\leqslant i\leqslant k}$ be orthonormal bases of respectively $\mathbb{S}^{n-k-1}$ at y and $\mathbb{S}^k$ at z. We set $g=\Phi_\varphi^\star can$ and $\xi=(0,0,1)$, then we have

$$g(u_i,u_j)=(1+\varphi(r))^2\sin^2 r\delta_{ij},\quad g(v_i,v_j)=(1+\varphi(r))^2\cos^2 r\delta_{ij},\quad g(u_i,v_j)=0,$$
$$g(\xi,\xi)=\varphi'^2+(1+\varphi)^2,\qquad\qquad g(u_i,\xi)=g(v_j,\xi)=0.$$

Now setting $\tilde u_i=d(\Phi_\varphi)_x(u_i)$, $\tilde v_i=d(\Phi_\varphi)_x(u_i)$ and $\tilde\xi=d(\Phi_\varphi)_x(\xi)$, the relation above allows us to compute the trace and norm

$$|B_q(\varphi)|=\max\Big(\max_i\frac{|B_q(\varphi)(\tilde u_i,\tilde u_i)|}{g(u_i,u_i)},\max_j\frac{|B_q(\varphi)(\tilde v_j,\tilde v_j)|}{g(v_j,v_j)},\frac{|B_q(\varphi)(\tilde\xi,\tilde\xi)|}{g(\xi,\xi)}\Big)$$
$$=\frac{1}{\sqrt{\varphi'^2+(1+\varphi)^2}}\max\Big(\big|1-\frac{\varphi'}{1+\varphi}\cot r\big|,\big|1+\frac{\varphi'}{1+\varphi}\tan r\big|,\big|1+\frac{(\varphi')^2-(1+\varphi)\varphi''}{\varphi'^2+(1+\varphi)^2}\big|\Big)$$

of the second fundamental form. □

To prove Theorem 1.9, we set $a<\frac{\pi}{10}$ and define the function φ_ε on I_ε by

$$\varphi_\varepsilon(r)=\begin{cases} f_\varepsilon(r)=\varepsilon\displaystyle\int_1^{\frac{r}{\varepsilon}}\frac{dt}{\sqrt{t^{2(n-k-1)}-1}} & \text{if } \varepsilon\leqslant r\leqslant a+\varepsilon,\\ u_\varepsilon(r) & \text{if } r\geqslant a+\varepsilon,\\ b_\varepsilon & \text{if } r\geqslant 2a+\varepsilon,\end{cases}$$

where u_ε is chosen such that φ_ε is smooth on $(\varepsilon,\frac{\pi}{2}]$ and strictly concave on $(\varepsilon,2a+\varepsilon]$, and b_ε is a constant. We have $f_\varepsilon(a+\varepsilon)\to 0$, $f_\varepsilon'(a+\varepsilon)\to 0$, $f_\varepsilon''(a+\varepsilon)\to 0$ and so $b_\varepsilon\to 0$ as $\varepsilon\to 0$. Hence b_ε can be chosen less than $\frac{1}{2}$ and u_ε can be chosen such that φ_ε tends uniformly on I_ε and $\varphi_\varepsilon'\to 0$, $\varphi_\varepsilon''\to 0$ uniformly on any compact of $(\varepsilon,\frac{\pi}{2}]$.

Note that φ_ε satisfies

$$\varphi_\varepsilon''=-\frac{(n-k-1)(1+\varphi_\varepsilon'^2)}{r}\varphi_\varepsilon'\quad\text{on}\quad(\varepsilon,a+\varepsilon]. \tag{10.3}$$

Moreover we have $\varphi_\varepsilon(\varepsilon)=0$ and $\lim\limits_{t\to\varepsilon}\varphi_\varepsilon'(t)=+\infty=-\lim\limits_{t\to\varepsilon}\varphi_\varepsilon''(t)$. Moreover, we can define on $(-b_\varepsilon,b_\varepsilon)$ an application $\tilde\varphi_\varepsilon$ so that $\tilde\varphi_\varepsilon(t)=\varphi_\varepsilon^{-1}(t)$ on $[0,b_\varepsilon)$ and $\tilde\varphi_\varepsilon(-t)=\tilde\varphi_\varepsilon(t)$.

Now let us consider the two applications $\Phi_{\varphi_\varepsilon}$ and $\Phi_{-\varphi_\varepsilon}$ defined as above, and put $M_\varepsilon^+=X_{\varphi_\varepsilon}$ and $M_\varepsilon^-=X_{-\varphi_\varepsilon}$. Since $\tilde\varphi_\varepsilon$ satisfies the equation $yy''=(n-k-1)\big(1+(y')^2\big)$ with initial data $\tilde\varphi_\varepsilon(0)=\varepsilon$ and $\tilde\varphi_\varepsilon'(0)=0$, it is smooth at 0, hence on $(-b_\varepsilon,b_\varepsilon)$, and so $M_\varepsilon^k=M_\varepsilon^+\cup M_\varepsilon^-$ is a smooth submanifold of $\mathbb{R}^{n+1}$. Indeed, the function $F_\varepsilon(p_1,p_2)=|p_1|^2-|p|^2\sin^2\big(\tilde\varphi_\varepsilon(|p|-1)\big)$, defined on

$$U=\{p=(p_1,p_2)\in\mathbb{R}^{n-k}\oplus\mathbb{R}^{k+1}/\,p_1\neq 0,\,p_2\neq 0,\,-b_\varepsilon+1<|p|<b_\varepsilon+1\}$$

is a smooth, local equation of M_ε^k at the neighborhood of $M_\varepsilon^+\cap M_\varepsilon^-$ which satisfies

$$\nabla F_\varepsilon(p_1,p_2)=2p_1\cos^2\varepsilon-2p_2\sin^2\varepsilon\neq 0$$

on $M_\varepsilon^+ \cap M_\varepsilon^-$.

We denote respectively by H_ε and B_ε, the mean curvature and the second fundamental form of M_ε^k.

Theorem 10.2. *$\|H_\varepsilon\|_\infty$ and $\|B_\varepsilon\|_{n-k}$ remain bounded whereas $\|H_\varepsilon - 1\|_1 \to 0$ and $\big\||X|-1\big\|_\infty \to 0$ when $\varepsilon \to 0$.*

Remark 10.3. *We have $\|B_\varepsilon\|_q \to \infty$ when $\varepsilon \to 0$, for any $q > n-k$.*

Proof. From the lemma 10.1 and the definition of φ_ε, H_ε and $|B_\varepsilon|$ converge uniformly to 1 on any compact of $M_\varepsilon^k \setminus M_\varepsilon^+ \cap M_\varepsilon^-$. On the neighborhood of $M_\varepsilon^+ \cap M_\varepsilon^-$, we have $n(H_\varepsilon)_x = nh_\varepsilon^\pm(r)$ and $nh_\varepsilon^\pm \leqslant h_{1,\varepsilon}^\pm + h_{2,\varepsilon}^\pm + h_{3,\varepsilon}^\pm$, where

$$h_{2,\varepsilon}^\pm(r) = k\frac{(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-1/2}}{1\pm\varphi_\varepsilon}\varphi_\varepsilon' \tan(r) \leqslant \frac{k}{1-b_\varepsilon}\tan\frac{\pi}{5}$$
$$h_{3,\varepsilon}^\pm(r) = (n-1)(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-1/2} + (\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-3/2}((1\pm\varphi_\varepsilon)^2 + 2\varphi_\varepsilon'^2) \leqslant \frac{n+1}{1-b_\varepsilon}$$

and by differential Equation (10.3) we have

$$\begin{aligned}
h_{1,\varepsilon}^\pm(r) &= \Big|(n-k-1)\frac{(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-1/2}}{1\pm\varphi_\varepsilon}\varphi_\varepsilon'\cot(r) + (\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-3/2}(1\pm\varphi_\varepsilon)\varphi_\varepsilon''\Big| \\
&\leqslant (n-k-1)\frac{(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-1/2}}{1\pm\varphi_\varepsilon}\varphi_\varepsilon'\Big|\cot(r) - \frac{1}{r}\Big| \\
&\quad + \frac{n-k-1}{r}\Big|\frac{(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-1/2}}{1\pm\varphi_\varepsilon}\varphi_\varepsilon' - (\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-3/2}(1\pm\varphi_\varepsilon)(1+\varphi_\varepsilon'^2)\varphi_\varepsilon'\Big| \\
&\leqslant \frac{n}{1-b_\varepsilon}\Big(\frac{1}{r} - \cot(r)\Big) \\
&\quad + \frac{n\left(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2\right)^{-3/2}}{r(1\pm\varphi_\varepsilon)}\varphi_\varepsilon'\Big|\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2 - (1\pm\varphi_\varepsilon)^2(1+\varphi_\varepsilon'^2)\Big| \\
&\leqslant \frac{n}{1-b_\varepsilon}\Big(\frac{1}{r} - \cot(r)\Big) + \frac{n}{r}\varphi_\varepsilon\frac{2\pm\varphi_\varepsilon}{1\pm\varphi_\varepsilon}\frac{\varphi_\varepsilon'^3}{[\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2]^{3/2}} \\
&\leqslant \frac{n}{1-b_\varepsilon}\Big(\frac{1}{r} - \cot(r)\Big) + \frac{n}{r}\varphi_\varepsilon\frac{2+b_\varepsilon}{1-b\varepsilon}
\end{aligned}$$

Since $\dfrac{\varphi_\varepsilon}{r} = \dfrac{\varepsilon}{r}\displaystyle\int_1^{r/\varepsilon}\frac{dt}{\sqrt{t^{2(n-k-1)}-1}} \leqslant \frac{r}{\varepsilon}\int_1^{r/\varepsilon}\frac{dt}{\sqrt{t^2-1}}$ and $\frac{1}{x}\int_1^x \frac{dt}{\sqrt{t^2-1}} \sim_{+\infty} \frac{\ln x}{x}$, we get that $h_{1,\varepsilon}^\pm$ is bounded on M_ε^k, hence H_ε is bounded on M_ε. By the Lebesgue theorem we have $\|H_\varepsilon - 1\|_1 \to 0$.

We now bound $\|B_\varepsilon\|_q$ with $q = n-k$. The volume element at the neighbourhood of $M_\varepsilon^+ \cap M_\varepsilon^-$ is

$$dv_{g_\varepsilon} = (1\pm\varphi_\varepsilon)^n(1 + (\frac{\varphi_\varepsilon'}{1\pm\varphi_\varepsilon})^2)^{1/2}\sin^{n-k-1}(r)\cos^k(r)dv_{n-k-1}dv_k dr \tag{10.4}$$

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where dv_{n-k-1} and dv_k are the canonical volume element of $\mathbb{S}^{n-k-1}$ and $\mathbb{S}^k$ respectively. By Lemma 10.1 and Equation (10.3), we have

$$|B_\varepsilon|^q dv_{g_\varepsilon} = \frac{1}{(\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2)^{\frac{q}{2}}} \max\Big(\big|1 - \frac{\varphi_\varepsilon'}{1 \pm \varphi_\varepsilon} \cot r\big|, \big|1 + \frac{\varphi_\varepsilon'}{1 \pm \varphi_\varepsilon} \tan r\big|, \\ \big|1 + \frac{\varphi_\varepsilon'^2 + (n-k-1)(1 \pm \varphi_\varepsilon)(1 + \varphi_\varepsilon'^2)\varphi_\varepsilon'/r}{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}\big|\Big)\Big]^q dv_{g_\varepsilon}$$

Noting that $\frac{x}{\sqrt{1+x^2}} \leqslant \min(1, x)$, it is easy to see that, if we set $h_\varepsilon = \min(1, |\varphi_\varepsilon'|)$

$$\begin{aligned} \frac{\big|1 - \frac{\varphi_\varepsilon'}{1 \pm \varphi_\varepsilon} \cot r\big|}{\sqrt{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}} &\leqslant \frac{1}{\sqrt{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}} + \frac{\frac{\varphi_\varepsilon'}{1 \pm \varphi_\varepsilon}}{\sqrt{\frac{\varphi_\varepsilon'^2}{(1 \pm \varphi_\varepsilon)^2} + 1}} \frac{\cot r}{1 \pm \varphi_\varepsilon} \\ &\leqslant \frac{1}{1 - \varphi_\varepsilon} + \frac{h_\varepsilon \cot r}{(1 - \varphi_\varepsilon)^2} \\ &\leqslant 4\left(1 + \frac{h_\varepsilon}{r}\right) \end{aligned}$$

Similarly for $r \in [\varepsilon, \pi/5 + \varepsilon]$ and ε small enough, we have

$$\frac{\big|1 + \frac{\varphi_\varepsilon'}{1 \pm \varphi_\varepsilon} \tan r\big|}{\sqrt{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}} \leqslant 4(1 + h_\varepsilon \tan r) \leqslant 8(1 + h_\varepsilon r) \leqslant 8\big(1 + \frac{h_\varepsilon}{r}\big)$$

And since $\varphi_\varepsilon' = 0$ for $r \geqslant \pi/5 + \varepsilon$, this inequality is also true for $r \in (\varepsilon, \pi/2]$. Moreover

$$\begin{aligned} &\frac{1}{\sqrt{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}} \Big|1 + \frac{\varphi_\varepsilon'^2 + (n-k-1)(1 \pm \varphi_\varepsilon)(1 + \varphi_\varepsilon'^2)\varphi_\varepsilon'/r}{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}\Big| \\ &\leqslant \frac{1}{1 \pm \varphi_\varepsilon} + \frac{\varphi_\varepsilon'^2}{(\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2)^{3/2}} + \frac{n}{r} \frac{(1 \pm \varphi_\varepsilon)(1 + \varphi_\varepsilon'^2)}{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2} \frac{|\varphi_\varepsilon'|}{(\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2)^{1/2}} \\ &\leqslant \frac{2}{1 \pm \varphi_\varepsilon} + \frac{n h_\varepsilon}{r(1 - \varphi_\varepsilon)} \frac{(1 \pm \varphi_\varepsilon)(1 + \varphi_\varepsilon'^2)}{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2} \\ &\leqslant \frac{2}{1 \pm \varphi_\varepsilon} + 2\frac{n h_\varepsilon}{r} \frac{(1 + \varphi_\varepsilon)^2}{(1 - \varphi_\varepsilon)^2} \\ &\leqslant 2\big(2 + 9\frac{n h_\varepsilon}{r}\big) \end{aligned}$$

It follows that

$$\begin{aligned} |B_\varepsilon|^q dv_{g_\varepsilon} &\leqslant C(n,k)\big(1 + \frac{h_\varepsilon}{r}\big)^q dv_{g_\varepsilon} \\ &\leqslant C(n,k)(r + h_\varepsilon)^q r^{-1}\big(1 + \frac{\varphi_\varepsilon'}{1 \pm \varphi_\varepsilon}\big) dv_{n-k-1} dv_k dr \\ &\leqslant C(n,k) r^{-1} (r + h_\varepsilon)^q \Big(1 + \frac{1}{\sqrt{(r/\varepsilon)^{2(n-k-1)} - 1}}\Big) dv_{n-k-1} dv_k dr \end{aligned}$$

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Now

$$\int_{M_\varepsilon^k}|B_\varepsilon|^q dv_{g_\varepsilon}\leqslant C(n,k)\Big(\int_\varepsilon^{2^{\frac{1}{2(n-k-1)}}\varepsilon} r^{-1}\Big(1+\frac{1}{\sqrt{(r/\varepsilon)^{2(n-k-1)}-1}}\Big)dr$$

$$+\int_{2^{\frac{1}{2(n-k-1)}}\varepsilon}^{2a+\varepsilon} r^{n-k-1}\Big(1+\frac{1}{r\sqrt{(r/\varepsilon)^{2(n-k-1)}-1}}\Big)^q dr\Big)$$

$$\leqslant C(n,k)\Big(\int_1^{2^{\frac{1}{2(n-k-1)}}} s^{-1}\Big(1+\frac{1}{\sqrt{s^{2(n-k-1)}-1}}\Big)ds+\int_{2^{\frac{1}{2(n-k-1)}}}^{2a/\varepsilon+1} s^{n-k-1}\big(\varepsilon+\frac{1}{s^q}\big)^q ds\Big)$$

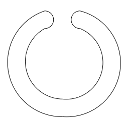
Since $\varepsilon^{\frac{-1}{q}}\leqslant\frac{2a}{\varepsilon}+1$ for ε small enough we have

$$\int_{M_\varepsilon^k}|B_\varepsilon|^q dv_{g_\varepsilon}\leqslant C(n,k)\Big(1+\int_{2^{\frac{1}{2(n-k-1)}}}^{\varepsilon^{\frac{-1}{q}}}\frac{2s^{n-k-1}}{s^{q^2}}ds+\int_{\varepsilon^{\frac{-1}{q}}}^{2a/\varepsilon+1}2s^{n-k-1}\varepsilon^q ds\Big)$$

$$\leqslant C(n,k)\big(1+\varepsilon^{n-k-1}\big)$$

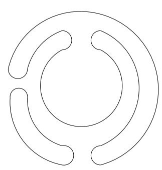
which remains bounded when $\varepsilon\to 0$. □

Since φ_ε is constant outside a neighborhood of $M_\varepsilon^+\cap M_\varepsilon^-$ (given by a), M_ε^k is a smooth submanifold diffeomorphic to the sum of two spheres $\mathbb{S}^n$ along a (great) subsphere $\mathbb{S}^k\subset\mathbb{S}^n$.



If we denote $\tilde{M}_\varepsilon^k$ one connected component of the points of M_ε^k corresponding to $r\leqslant 3a$, we get some pieces of hypersurfaces

that can be glued together along pieces of spheres of constant curvature to get a smooth submanifold M_ε, diffeomorphic to p spheres $\mathbb{S}^n$ glued each other along l subspheres S_i, and with curvature satisfying the bounds of Theorem 1.9 (when all the subspheres have dimension 0) or of Remark 1.12.



Since the surgeries are performed along subsets of capacity zero, the manifold constructed have a spectrum close to the spectrum of p disjoints spheres of radius close to 1 (i.e. close to the spectrum of the standard $\mathbb{S}^n$ with all multiplicities multiplied by p). More precisely, we set $\eta\in[2\varepsilon,\frac{\pi}{20}]$, and for any subsphere S_i, we set $N_{i,\eta,\varepsilon}$ the tubular neighborhood of radius η of the submanifold $\tilde{S}_i = M_{\varepsilon,i}^+\cap M_{\varepsilon,i}^-$ in the local parametrization of M_ε given by the map $\Phi_{\varphi_{\varepsilon,i}}$ associated to the subsphere S_i. We have $M_\varepsilon=\Omega_{1,\eta,\varepsilon}\cup\cdots\cup\Omega_{p,\eta,\varepsilon}\cup N_{1,\eta,\varepsilon}\cup\cdots\cup N_{l,\eta,\varepsilon}$ where $\Omega_{i,\eta,\varepsilon}$ are the connected component of $M\setminus\cup_i N_{i,\eta,\varepsilon}$. The $\Omega_{i,\eta,\varepsilon}$ are diffeomorphic to some $S_{i,\eta}$ (which does not depend on ε

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and η) open set of $\mathbb{S}^n$ which are complements of neighborhoods of subspheres of dimension less than $n-2$ and radius η, endowed with metrics which converge in $\mathcal{C}^1$ topology to standard metrics of curvature 1 on $S_{i,\eta}$. Indeed, φ_ε converge to 0 in topology $\mathcal{C}^2$ on $[r^{i,\pm}_{\varepsilon,\eta},\frac{\pi}{2}]$, where $\int_\varepsilon^{r^{i,\pm}_{\varepsilon,\eta}}\sqrt{(1\pm\varphi_{\varepsilon,i})^2+(\varphi'_{\varepsilon,i})^2}=\eta$ since it converges in $\mathcal{C}^1$ topology on any compact of $[\varepsilon,\frac{\pi}{2}]$ and since we have

$$\eta\geqslant\int_\varepsilon^{r^{i,\pm}_{\varepsilon,\eta}}(1-b_{i,\varepsilon})\,dt=(r^{i,\pm}_{\varepsilon,\eta}-\varepsilon)(1-b_{i,\varepsilon})$$

$$\eta\leqslant\int_\varepsilon^{r^{i,\pm}_{\varepsilon,\eta}}(1+b_{i,\varepsilon})\,dt+\int_\varepsilon^{r^{i,\pm}_{\varepsilon,\eta}}\frac{dt}{\sqrt{(\frac{t}{\varepsilon})^{2(n-k-1)}-1}}=(r^{i,\pm}_{\varepsilon,\eta}-\varepsilon)(1+b_{i,\varepsilon})$$
$$+\varepsilon\int_1^{+\infty}\frac{dt}{\sqrt{t^{2(n-k-1)}-1}}$$

so $r^{\pm}_{\varepsilon,\eta}\to\eta$ when $\varepsilon\to 0$. So the spectrum of $\cup_i\Omega_{i,\eta,\varepsilon}\subset M_\varepsilon$ for the Dirichlet problem converges to the spectrum of $\amalg_i S_{i,\eta}\subset\amalg_i\mathbb{S}^n$ for the Dirichlet problem as ε tends to 0 (by the min-max principle). Since any subsphere of codimension at least 2 has zero capacity in $\mathbb{S}^n$, we have that the spectrum of $\amalg_i S_{i,\eta}\subset\amalg_i\mathbb{S}^n$ for the Dirichlet problem converges to the spectrum of $\amalg_i\mathbb{S}^n$ when η tends to 0 (see for instance [8] or adapt what follows). Since the spectrum of $\amalg_i\mathbb{S}^n$ is the spectrum of $\mathbb{S}^n$ with all multiplicities multiplied by p, by diagonal extraction we infer the existence of two sequences (ε_m) and (η_m) such that $\varepsilon_m\to 0$, $\eta_m\to 0$ and the spectrum of $\cup_i\Omega_{i,\eta_m,\varepsilon_m}\subset M_{\varepsilon_m}$ for the Dirichlet problem converges to the spectrum of $\mathbb{S}^n$ with all multiplicities multiplied by p.

Finally, note that $\lambda_\sigma(M_\varepsilon)\leqslant\lambda_\sigma(\cup_i\Omega_{i,2\eta,\varepsilon})$ for any σ by the Dirichlet principle. On the other hand, by using functions of the distance to the $\tilde{S}_i$ we can easily construct on M_ε a function ψ_ε with value in $[0,1]$, support in $\cup_i\Omega_{i,\eta,\varepsilon}$, equal to 1 on $\cup_i\Omega_{i,2\eta,\varepsilon}$ and whose gradient satisfies $|d\psi_\varepsilon|_{g_\varepsilon}\leqslant\frac{2}{\eta}$. It readily follows that

$$\|1-\psi_\varepsilon^2\|_1+\|d\psi_\varepsilon\|_2^2\leqslant(1+\frac{4}{\eta^2})\sum_i\frac{\operatorname{Vol}N_{i,2\eta,\varepsilon}}{\operatorname{Vol}M_\varepsilon}$$

To estimate $\sum_i\operatorname{Vol}N_{i,2\eta,\varepsilon}$, note that $N_{i,2\eta,\varepsilon}$ corresponds to the set of points with $r^{i,\pm}\leqslant r^{i,\pm}_{\varepsilon,2\eta}$ in the parametrization of M_ε given by $\Phi_{\varphi_{\varepsilon,i}}$ at the neighborhood of $\tilde{S}_i$, where, as above, $r^{i,\pm}_{\varepsilon,2\eta}$ is given by

$$\int_\varepsilon^{r^{i,\pm}_{\varepsilon,2\eta}}\sqrt{(1\pm\varphi_{\epsilon,i})^2+(\varphi'_{\epsilon,i})^2}=2\eta$$

hence satisfies $\frac{1}{2}(r^{i,\pm}_{\varepsilon,2\eta}-\varepsilon)\leqslant 2\eta$ (since we have $1-\varphi_{\varepsilon,i}\geqslant\frac{1}{2}$). By formula 10.4, we have

$$\operatorname{Vol}N_{i,2\eta,\varepsilon}\leqslant C(n)\int_\varepsilon^{r^-_\eta}(1-\varphi_{\varepsilon,i})^{n-1}\sqrt{(1-\varphi_{\varepsilon,i})^2+(\varphi'_{\varepsilon,i})^2}t^{n-k-1}dt$$
$$+C(n)\int_\varepsilon^{r^+_\eta}(1+\varphi_{\varepsilon,i})^{n-1}\sqrt{(1+\varphi_{\varepsilon,i})^2+(\varphi'_{\varepsilon,i})^2}t^{n-k-1}dt$$
$$\leqslant C(n)(4\eta+\varepsilon)^{n-k-1}\eta\leqslant C(n,k)\eta^{n-k}$$

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where we have used that $\varphi_{\varepsilon,i} \leqslant 2$ and $2\varepsilon \leqslant \eta$. We then have

$$\|1-\psi_\varepsilon^2\|_1 + \|d\psi_\varepsilon\|_2^2 \leqslant C(n,k,l,p)\eta^{n-k}$$

To end the proof of the fact that M_{ε_m} has a spectrum close to that of $\cup_i \Omega_{i,\eta_m,\varepsilon_m}$ we need the following proposition, whose proof is a classical Moser iteration (we use the Sobolev Inequality 3.1).

Proposition 10.4. *For any $q > n$ there exists a constant $C(q,n)$ so that if (M^n, g) is any Riemannian manifold isometrically immersed in $\mathbb{R}^{n+1}$ and $E_N = \langle f_0, \cdots, f_N\rangle$ is the space spanned by the eigenfunctions associated to $\lambda_0 \leqslant \cdots \leqslant \lambda_N$, then for any $f \in E_N$ we have*

$$\|f\|_\infty \leqslant C(q,n)\Big((\operatorname{Vol} M)^{1/n}(\lambda_N^{1/2} + \|H\|_q)\Big)^\gamma \|f\|_2$$

where $\gamma = \frac{1}{2}\frac{qn}{q-n}$.

Since we already know that $\lambda_\sigma(M_{\varepsilon_m}) \leqslant \lambda_\sigma(\cup_i \Omega_{i,\eta_m,\varepsilon_m}) \to \lambda_{E(\sigma/p)}(\mathbb{S}^n)$ for any σ when $m \to \infty$, we infer that for any N there exists $m = m(N)$ large enough such that on M_{ε_m} and for any $f \in E_N$, we have (with $q = 2n$ and since $\|H\|_\infty \leqslant C(n)$)

$$\|f\|_\infty \leqslant C(p,N,n)\|f\|_2$$

By the previous estimates, if we set

$$L_{\varepsilon_m} : f \in E_N \mapsto \psi_{\varepsilon_m} f \in H_0^1(\cup_i \Omega_{i,\eta_m,\varepsilon_m})$$

then we have

$$\|f\|_2^2 \geqslant \|L_{\varepsilon_m}(f)\|_2^2 \geqslant \|f\|_2^2 - \|f\|_\infty^2 \|1-\psi_{\varepsilon_m}^2\|_1 \geqslant \|f\|_2^2\big(1 - C(k,l,p,N,n)\eta_m^{n-k}\big)$$

and

$$\begin{aligned}
\|dL_{\varepsilon_m}(f)\|_2^2 &= \frac{1}{\operatorname{Vol} M_{\varepsilon_m}} \int_{M_{\varepsilon_m}} |f d\psi_{\varepsilon_m} + \psi_{\varepsilon_m} df|^2 \\
&\leqslant (1+h)\|df\|_2^2 + (1+\frac{1}{h})\frac{1}{\operatorname{Vol} M_{\varepsilon_m}}\int_{M_{\varepsilon_m}} f^2 |d\psi_{\varepsilon_m}|^2 \\
&\leqslant (1+h)\|df\|_2^2 + (1+\frac{1}{h})C(k,l,p,N,n)\|f\|_2^2 \eta_m^{n-k}
\end{aligned}$$

for any $h > 0$. We set $h = \eta_m^{\frac{n-k}{2}}$. For $m = m(k,l,p,N,n)$ large enough, $L_{\varepsilon_m} : E_N \to H_0^1(\cup_i \Omega_{i,\eta_m,\varepsilon_m})$ is injective and for any $f \in E_N$, we have

$$\frac{\|dL_{\varepsilon_m}(f)\|_2^2}{\|L_{\varepsilon_m}(f)\|_2^2} \leqslant (1 + C(k,l,p,N,n)\eta_m^{\frac{n-k}{2}})\frac{\|df\|_2^2}{\|f\|_2^2} + C(k,l,p,N,n)\eta_m^{\frac{n-k}{2}}$$

By the min-max principle, we infer that for any $\sigma \leqslant N$, we have

$$\lambda_\sigma(M_{\varepsilon_m}) \leqslant \lambda_\sigma(\cup_i \Omega_{i,\eta_m,\varepsilon_m}) \leqslant (1 + C(k,l,p,N,n)\eta_m^{\frac{n-k}{2}})\lambda_\sigma(M_{\varepsilon_m}) + C(k,l,p,N,n)\eta_m^{\frac{n-k}{2}}$$

Since $\lambda_\sigma(\cup_i \Omega_{i,\eta_M,\varepsilon_m}) \to \lambda_{E(\sigma/p)}(\mathbb{S}^n)$, this gives that $\lambda_\sigma(M_{\varepsilon_m}) \to \lambda_{E(\sigma/p)}(\mathbb{S}^n)$ for any $\sigma \leqslant N$. By diagonal extraction we get the sequence of manifolds (M_j) of Theorem 1.9.

To construct the sequence of Theorem 1.10, we consider the sequence of embedded submanifolds (M_j) of Theorem 1.9 for $p = 2$, $k = n-2$ and $l = 1$. Each element of the sequence admits a covering of degree d given by $y \mapsto y^d$ in the local charts associated to the maps Φ. We endow these covering with the pulled back metrics. Arguing as

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above, we get that the spectrum of the new sequence converge to the spectrum of two disjoint copies of

$$\big(\mathbb{S}^1\times\mathbb{S}^{n-2}\times[0,\frac{\pi}{2}],dr^2+d^2\sin^2 r g_{\mathbb{S}^1}+\cos^2 r g_{\mathbb{S}^{n-2}}\big).$$

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(E. Aubry) LJAD, Université de Nice Sophia-Antipolis, CNRS; 28 avenue Valrose, 06108 Nice, France
E-mail address: eaubry@unice.fr

(J.-F. Grosjean) Institut Élie Cartan (Mathématiques), Université Henri Poincaré Nancy I, B.P. 239, F-54506 Vandœuvre-les-Nancy cedex, France
E-mail address: grosjean@iecn.u-nancy.fr

(J. Roth) LAMA, Université Paris-Est - Marne-la-Vallée, 5 bd Descartes, Cité Descartes, Champs-sur-Marne, F-77454 Marne-la-Vallée
E-mail address: Julien.Roth@univ-mlv.fr

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