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# Boundary asymptotic analysis for an incompressible viscous flow: Navier wall laws

M. El Jarroudi, A. Brillard
Université Abdelmalek Essaâdi, FST Tanger
Département de Mathématiques, B.P. 416, Tanger, Morocco
Université de Haute-Alsace
Laboratoire de Gestion des Risques et Environnement
25 rue de Chemnitz, F-68200 Mulhouse, France

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**Abstract**

We consider a new way of establishing Navier wall laws. Considering a bounded domain $\Omega$ of $\mathbf{R}^N$, $N = 2, 3$, surrounded by a thin layer $\Sigma_\varepsilon$, along a part $\Gamma_2$ of its boundary $\partial\Omega$, we consider a Navier-Stokes flow in $\Omega \cup \partial\Omega \cup \Sigma_\varepsilon$ with Reynolds' number of order $1/\varepsilon$ in $\Sigma_\varepsilon$. Using $\Gamma$-convergence arguments, we describe the asymptotic behaviour of the solution of this problem and get a general Navier law involving a matrix of Borel measures having the same support contained in the interface $\Gamma_2$. We then consider two special cases where we characterize this matrix of measures. As a further application, we consider an optimal control problem within this context.

Navier law, Navier-Stokes flow, $\Gamma$-convergence, asymptotic behaviour, optimal control problem.
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## 1 Introduction

A common hypothesis used in fluid mechanics is that, at the interface between a solid and a fluid, the velocity $u$ of the fluid is equal to that of the solid. If the solid is at rest, the velocity of the fluid must thus vanish: $u = 0$, on the boundary of the solid. These are the so-called rigid boundary conditions. When writing this condition, one assumes that the fluid perfectly adheres to the solid.

This hypothesis has not always been accepted for a viscous fluid, although some verifications have been made through experiments. G. Taylor indeed verified in 1923 the correctness of this hypothesis, when studying the stability of the motion of a fluid flowing between two cylinders in rotation (Taylor-Couette's problem).

Another approach has then been suggested. A thin layer adhering to the solid exists with a tangential velocity different from 0 on the surface of the solid. Navier suggested that this tangential velocity is proportional to the shearing strains and thus is given through

$$\begin{cases} (Id - n \otimes n)\, \nu \dfrac{\partial u}{\partial n} = \kappa u, \\ u \cdot n = 0, \end{cases}$$

where $Id$ is the identity matrix, $n$ is the unit outer normal vector to the surface of the solid, $\nu$ is the viscosity of the fluid and $\kappa$ is a proportionality coefficient.

Many works have already been devoted to the derivation of Navier boundary conditions, see for example [2], [3], [13] and [14]. In [2] and [3], the authors considered a viscous and incompressible fluid, whose Reynolds number is of order $1/\varepsilon$, flowing in a domain with rugosities of thinness $\varepsilon$ and $\varepsilon$-periodically distributed on its boundary surface, and assuming an homogeneous Dirichlet boundary condition on the boundary of these rugosities. Using the asymptotic expansion method, they deduced, at the first-order level, a kind of Navier wall law

$$\left\{ \begin{array}{rcl} \varepsilon\left(Id - n\otimes n\right)\nu\dfrac{\partial u}{\partial n} & = & \kappa u, \\ u\cdot n & = & 0. \end{array} \right.$$

In [13], the authors considered the laminar flow in a pipe with rough pieces $\varepsilon$-periodically distributed on the surface of the pipe, and imposing an homogeneous Dirichlet boundary condition on the boundary of these rough pieces. They used an homogenization process and obtained a Navier wall law, computing a corrector term. In [14], the author considered an $\varepsilon$-periodic geometry built with rough pieces of thinness $\varepsilon^m$ and imposed there a boundary condition of the type

$$\left\{ \begin{array}{rcl} \left(Id - n\otimes n\right)\nu\dfrac{\partial u^\varepsilon}{\partial n} & = & \varepsilon^k\left(g^\varepsilon - \kappa u^\varepsilon\right), \\ u^\varepsilon\cdot n & = & 0. \end{array} \right.$$

The following limit law was obtained, depending on $k$ and $m$

$$\left\{ \begin{array}{rcl} \left(Id - n\otimes n\right)\nu\dfrac{\partial u}{\partial n} & = & \lambda\left(g - \kappa u\right), \\ u\cdot n & = & 0. \end{array} \right.$$

Throughout the present work, we consider a bounded domain $\Omega \subset \mathbf{R}^N$, $N = 2, 3$, whose boundary $\partial\Omega$ is Lipschitz continuous. We suppose that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, with $|\Gamma_1|$, $|\Gamma_2| > 0$, where $|\Gamma_i|$ denotes the Lebesgue measure of $\Gamma_i$. We suppose that near $\Gamma_2$ there exists a thin layer $\Sigma_\varepsilon$ of thinness $\varepsilon > 0$, which extends $\Omega$ into $\Omega_\varepsilon = \Omega \cup \Gamma_2 \cup \Sigma_\varepsilon$.

Figure 1: The domain under consideration.

We consider the steady-state, viscous and incompressible Navier-Stokes flow in $\Omega_\varepsilon$

$$\left\{ \begin{array}{rcll} -\nu\Delta u^\varepsilon + \left(u^\varepsilon\cdot\nabla\right)u^\varepsilon + \nabla p^\varepsilon & = & f & \text{in } \Omega, \\ -\nu\varepsilon\Delta u^\varepsilon + \left(u^\varepsilon\cdot\nabla\right)u^\varepsilon + \nabla p^\varepsilon & = & f & \text{in } \Sigma_\varepsilon, \\ \operatorname{div}\left(u^\varepsilon\right) & = & 0 & \text{in } \Omega_\varepsilon, \\ \left(u^\varepsilon\right)^+ & = & \left(u^\varepsilon\right)^- & \text{on } \Gamma_2, \\ \nu\left(\dfrac{\partial u^\varepsilon}{\partial n}\right)^+ & = & \nu\varepsilon\left(\dfrac{\partial u^\varepsilon}{\partial n}\right)^- & \text{on } \Gamma_2, \\ u^\varepsilon & = & 0 & \text{on } \partial\Omega_\varepsilon, \end{array} \right. \tag{1}$$

where the superscript $+$ (resp. $-$) denotes the trace seen from $\Omega$ (resp. from $\Sigma_\varepsilon$) on $\Gamma_2$. The thin layer $\Sigma_\varepsilon$ is here considered as an unstable thin boundary layer whose Reynolds' number $R_\varepsilon$ is of order $1/\varepsilon$ (see [12, pages 239-240], where Reynolds' number is allowed to depend on the thinness of the layer). In the problem (1), we suppose that the density $f$ of volumic forces belongs to $\mathbf{L}^\infty\left(\mathbf{R}^N, \mathbf{R}^N\right)$.

Our purpose is to describe the asymptotic behavior of the solution $u^\varepsilon$ of (1) when $\varepsilon$ goes to 0, in order to derive the Navier wall law. We use $\Gamma$-convergence arguments (see [5] for the definition and the properties of the $\Gamma$-convergence) in order to characterize the limit problem. Our approach is based on the tools developed in [1], [4], [7], [8] and [9]. On $\Gamma_2$, we will get a general Navier law of the kind

$$\begin{cases} (Id - n \otimes n)\, \nu \dfrac{\partial u}{\partial n} + \mu^{\bullet} u &= 0, \\ u \cdot n &= 0, \end{cases}$$

where $\mu^{\bullet}$ is a symmetric matrix $\left(\mu_{ij}\right)_{i,j=1,\dots,N}$ of Borel measures having their support contained in $\Gamma_2$, which do not charge the polar subsets of $\mathbf{R}^N$ and which satisfy $\mu_{ij}\,(B)\,\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbf{R}^N$, $\forall B \in \mathcal{B}\left(\mathbf{R}^N\right)$, where $\mathcal{B}\left(\mathbf{R}^N\right)$ denotes the set of all Borel subsets of $\mathbf{R}^N$ and where we have used the summation convention with respect to repeated indices.

As a first special case, we prove that when $\Omega \subset \{x_3 > 0\}$, $\Gamma_2 = \partial\Omega \cap \{x_3 = 0\}$ and

$$\Sigma_\varepsilon = \left\{ x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2,\ -\varepsilon h\left(\frac{x'}{\varepsilon}\right) < x_3 < 0 \right\},$$

where $h$ is a periodic function, we get on $\Gamma_2$ the Robin type boundary conditions

$$\begin{cases} \dfrac{\partial u_1}{\partial x_3}(x', 0) &= -c_1 u_1 (x', 0), \\ \dfrac{\partial u_2}{\partial x_3}(x', 0) &= -c_2 u_2 (x', 0), \\ u_3 (x', 0) &= 0, \end{cases}$$

where $c_m$, $m = 1, 2$, are constants which will be computed in terms of the solution of appropriate local thin layer problems (21). This situation can be generalized to the case of a general open and bounded set $\Omega$, surrounded on a part of its boundary by such a rough thin layer.

As a second example, we will consider the case where

$$\Sigma_\varepsilon = \{ s + tn\,(s) \mid s \in \Gamma_2,\ -\varepsilon h\,(s) < x_3 < 0 \},$$

where $h$ is a Lipschitz continuous and positive function on $\Gamma_2$. We here prove that Navier's law takes the following expression on $\Gamma_2$

$$\begin{cases} (Id - n \otimes n) \dfrac{\partial u}{\partial n} + \dfrac{1}{h} u &= 0, \\ u \cdot n &= 0. \end{cases}$$

In the last part of this work, we consider an optimal control problem. Choosing $m > 0$, we consider the set $\Xi_m$ of all the matrices $\mathbf{h} = Diag\,(h_i)_{i=1,..,N}$ of functions $h_i : \Gamma_2 \to [0, +\infty]$, which are $d\Gamma_2$-measurable and satisfy $\int_{\Gamma_2} h_i d\Gamma_2 = m$, $\forall i = 1, \dots, N$. We suppose that $\Omega$ is smooth enough and consider the following problem with Navier conditions on $\Gamma_2$

$$\begin{cases} -\nu \Delta u^h + \left(u^h \cdot \nabla\right) u^h + \nabla p^h &= f \quad \text{in } \Omega, \\ \operatorname{div}\left(u^h\right) &= 0 \quad \text{in } \Omega, \\ \mathbf{h}\,(Id - n \otimes n) \dfrac{\partial u^h}{\partial n} + u^h &= 0 \quad \text{on } \Gamma_2, \\ u^h \cdot n &= 0 \quad \text{on } \Gamma_2. \end{cases} \tag{2}$$

Let $\left(u^h, p^h\right)$ be the solution of (2) and define the functional $\mathbf{F}$ through

$$\mathbf{F}(\mathbf{h}, u) = \begin{cases} \dfrac{\nu}{2} \displaystyle\int_\Omega |\nabla u|^2 dx + \frac{1}{2} \sum_{i=1}^{N} \int_{\Gamma_2} \frac{(u_i)^2}{h_i} d\Gamma_2 & \\ \qquad + \displaystyle\int_\Omega \left(u^h \cdot \nabla\right) u^h \cdot v dx - \int_\Omega f \cdot u dx & \text{if } u \in \mathbf{V}_{0,\Gamma_1}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathbf{V}_{0,\Gamma_1}(\Omega)$ is the functional space defined in (7). We consider the optimal control problem

$$\min_{\mathbf{h}\in\Xi_m} \min_{u\in\mathbf{V}_{0,\Gamma_1}(\Omega)} \mathbf{F}(\mathbf{h},u). \tag{3}$$

In the last section of this work, we describe the asymptotic behavior of the solution of (3), when $m$ goes to 0, and characterize the zones where some thin boundary layer appears. A problem of this kind has been considered in [11], but for a linear diffusion problem.

# 2 Functional framework

We define the $(H^1(\mathbf{R}^N))$ capacity of any compact subset $K$ of $\mathbf{R}^N$ as

$$\begin{aligned}&Cap(K)\\&= \inf\left\{\int_{\mathbf{R}^N} |\nabla\varphi|^2 dx + \int_{\mathbf{R}^N} |\varphi|^2 dx \mid \varphi\in\mathbf{C}_c^\infty(\mathbf{R}^N),\, \varphi\geq 1 \text{ on } K\right\}.\end{aligned}$$

If $U$ is an open subset of $\mathbf{R}^N$, then we define

$$Cap(U) = \sup\{Cap(K) \mid K\subset U,\, K \text{ compact}\}.$$

If $B\subset\mathbf{R}^N$ is a Borel subset of $\mathbf{R}^N$, then we define

$$Cap(B) = \inf\{Cap(U) \mid B\subset U,\, U \text{ open}\}.$$

**Definition 1** *Let $\mathcal{B}(\mathbf{R}^N)$ be the $\sigma$-algebra of all Borel subsets of $\mathbf{R}^N$.*

1. *A property is said to be true quasi-everywhere (q.e.) on $B\in\mathcal{B}(\mathbf{R}^N)$ if it is true except on a subset of $B$ of capacity $Cap$ equal to 0.*
2. *A function $u: B\to\overline{\mathbf{R}}$, with $B\in\mathcal{B}(\mathbf{R}^N)$, is quasi-continuous on $B$ if, for every $\varepsilon>0$, there exists an open subset $U\subset B$ with $Cap(U)<\varepsilon$ and such that the restriction of $u$ on $B\setminus U$ is continuous.*
3. *Every function $u\in\mathbf{H}^1(\mathbf{R}^N)$ has a quasi-continuous representative $\widetilde{u}$, which is unique for the equality quasi-everywhere in $\mathbf{R}^N$, (see [17], for example). $\widetilde{u}$ is given through*

$$\widetilde{u}(x) = \lim_{r\to 0^+} \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,dy,$$

*for q.e. $x\in\mathbf{R}^N$, where $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ of $\mathbf{R}^N$ of radius $r>0$ and centered at $x$.*

We define some notions concerning families of subsets of $\mathbf{R}^N$.

**Definition 2** 1. *A subset $\mathcal{D}\subset\mathcal{B}(\mathbf{R}^N)$ is a dense family in $\mathcal{B}(\mathbf{R}^N)$ if, for every $A,B\in\mathcal{B}(\mathbf{R}^N)$ with $\overline{A}\subset\overset{o}{B}$, there exists $D\in\mathcal{D}$ such that: $\overline{A}\subset\overset{o}{D}\subset\overline{D}\subset\overset{o}{B}$, where $\overset{o}{A}$ (resp. $\overline{A}$) denotes the interior (resp. the closure) of $A$.*

2. *A subset $\mathcal{R}\subset\mathcal{B}(\mathbf{R}^N)$ is a rich family in $\mathcal{B}(\mathbf{R}^N)$ if, for every family $(A_t)_{t\in]0,1[}\subset\mathcal{B}(\mathbf{R}^N)$ such that $\overline{A}_s\subset\overset{o}{A}_t$, for every $s<t$, the set $\{t\in]0,1[\mid A_t\notin\mathcal{R}\}$ is at most countable.*

Let $\mathcal{O}(\mathbf{R}^N)$ be the set of all open subsets of $\mathbf{R}^N$. We consider the class $\mathbb{F}$ of functionals $F$ from $\mathbf{H}^1(\mathbf{R}^N,\mathbf{R}^N)\times\mathcal{O}(\mathbf{R}^N)$ to $[0,+\infty]$ satisfying:

i) (*Lower semi-continuity*): for every open subset $\omega \in \mathcal{O}\left(\mathbf{R}^N\right)$, the functional $u \mapsto F(u,\omega)$ is lower semi-continuous with respect to the strong topology of $\mathbf{H}^1\left(\mathbf{R}^N, \mathbf{R}^N\right)$;

ii) *(Measure property)*: for every $u \in \mathbf{H}^1\left(\mathbf{R}^N, \mathbf{R}^N\right)$, $\omega \mapsto F(u,\omega)$ is the restriction to $\mathcal{O}\left(\mathbf{R}^N\right)$ of some Borel measure still denoted $F(u,\omega)$;

iii) (*Localization)*: for every $\omega \in \mathcal{O}\left(\mathbf{R}^N\right)$ and every $u, v \in \mathbf{H}^1\left(\mathbf{R}^N, \mathbf{R}^N\right)$:

$$u_{|\omega} = v_{|\omega} \Rightarrow F(u,\omega) = F(v,\omega);$$

iv) ($\mathbf{C}^1$-*convexity*): for every $\omega \in \mathcal{O}\left(\mathbf{R}^N\right)$, the functional $u \mapsto F(u,\omega)$ is convex on $\mathbf{H}^1\left(\mathbf{R}^N, \mathbf{R}^N\right)$ and moreover

$$\forall \varphi \in \mathbf{C}^1\left(\mathbf{R}^N\right),\ 0 \leq \varphi \leq 1 : F(\varphi u + (1-\varphi) v, \omega) \leq F(u,\omega) + F(v,\omega).$$

**Example 3** *Let us define $\Gamma_{2,\varepsilon} = \partial\Omega_\varepsilon \cap \overline{\Sigma_\varepsilon}$, for some thin layer $\Sigma_\varepsilon$, as defined above. We consider the functional $F^\varepsilon$ defined on the space $\mathbf{H}^1\left(\mathbf{R}^N, \mathbf{R}^N\right) \times \mathcal{O}\left(\mathbf{R}^N\right)$ through*

$$F^\varepsilon(u,\omega) = \begin{cases} 0 & \text{if } \widetilde{u} = 0, \text{ q.e. on } \Gamma_{2,\varepsilon} \cap \omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

*One can prove that $F^\varepsilon$ belongs to $\mathbb{F}$, for every $\varepsilon > 0$.*

Let us set the following definitions.

**Definition 4** *Let Cap be the above-defined capacity.*

1. *A Borel measure $\lambda$ is absolutely continuous with respect to the capacity Cap if*

$$\forall B \in \mathcal{B}\left(\mathbf{R}^N\right) : Cap(B) = 0 \Rightarrow \lambda(B) = 0.$$

2. *$\mathcal{M}_0$ is the set of nonnegative Borel measures $\mathbf{R}^N$ which are absolutely continuous with respect to the capacity Cap.*

We have the following example of measure in $\mathcal{M}_0$.

**Example 5** *For every $E \subset \mathbf{R}^N$ such that $Cap(E) > 0$, we define the measure $\infty_E$ through*

$$\infty_E(B) = \begin{cases} 0 & \text{if } Cap(B \cap E) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

*Then $\infty_E \in \mathcal{M}_0$.*

*Notice that, for every $u \in \mathbf{H}^1\left(\mathbf{R}^N, \mathbf{R}^N\right)$ and every $\omega \in \mathcal{O}\left(\mathbf{R}^N\right)$, the functional $F^\varepsilon$ defined in (4) can be written as*

$$F^\varepsilon(u,\omega) = \int_\omega |\widetilde{u}|^2 \, d\infty_{\Gamma_{2,\varepsilon}} = \int_\omega |u|^2 \, d\infty_{\Gamma_{2,\varepsilon}}.$$

One has the following representation theorem for the functionals of $\mathbb{F}$.

**Theorem 6** *(see [9]) For every $F \in \mathbb{F}$, there exist a finite measure $\lambda \in \mathcal{M}_0$, a nonnegative Borel measure $\nu$ and a Borel function $g : \mathbf{R}^N \times \mathbf{R}^N \to [0, +\infty]$, with $\zeta \mapsto g(x,\zeta)$ convex and lower semi-continuous on $\mathbf{R}^N$, such that*

$$\forall u \in \mathbf{H}^1\left(\mathbf{R}^N, \mathbf{R}^N\right),\ \forall \omega \in \mathcal{O}\left(\mathbf{R}^N\right) : F(u,\omega) = \int_\omega g(x, \widetilde{u}(x))\, d\lambda + \nu(\omega).$$

Throughout the paper, we will need the following Corollary (see [9, Corollary 8.4]).

**Corollary 7** *Let $F \in \mathbb{F}$. If $F(.,\omega)$ is quadratic for every $\omega \in \mathcal{O}(\mathbf{R}^N)$, there exist $\lambda \in \mathcal{M}_0$ finite, a symmetric matrix $(a_{ij})_{i,j=1,..,N}$, of Borel functions from $\mathbf{R}^N$ to $\mathbf{R}$ satisfying $a_{ij}(x)\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbf{R}^N$ and for q.e. $x \in \mathbf{R}^N$, for every $x \in \mathbf{R}^N$ a subspace $\mathbf{V}(x)$ of $\mathbf{R}^N$, such that, for every $u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ and every $\omega \in \mathcal{O}(\mathbf{R}^N)$:*

*a) if $F(u,\omega) < +\infty$, then $u(x) \in \mathbf{V}(x)$, for q.e. $x \in \omega$,*

*b) if $u(x) \in \mathbf{V}(x)$, for q.e. $x \in \omega$*

$$F(u,\omega) = \int_\omega a_{ij} u_i u_j d\lambda. \tag{5}$$

**Remark 8** *Let $F \in \mathbb{F}$, $\lambda \in \mathcal{M}_0$ be the associated measure and $\Lambda$ be the set defined as $\Lambda = \cup_{\omega \in A(F)} \omega$, where*

$$A(F) = \left\{\omega \in \mathcal{O}(\mathbf{R}^N) \mid F(.,\omega) < +\infty, \text{ for q.e. } x \in \omega\right\}.$$

*We define the matrix $\mu^\bullet = (\mu_{ij}) = (a_{ij}\lambda)_{i,j=1,..,N} + \infty_{\mathbf{R}^N\setminus\Lambda} Id$ of measures, and, for every $x \in \mathbf{R}^N$, the subspace $\mathbf{V}(x)$ through*

$$\mathbf{V}(x) = \begin{cases} \mathbf{R}^N & \text{if } x \in \Lambda, \\ \{0\} & \text{if } x \in \mathbf{R}^N\setminus\Lambda. \end{cases} \tag{6}$$

*For every $u \in \mathbf{H}^1(\mathbf{R}^N, \mathbf{R}^N)$ and every $\omega \in \mathcal{O}(\mathbf{R}^N)$, one has, using the preceding definition of $\mu^\bullet$*

$$\int_\omega u_i u_j d\mu_{ij} = \begin{cases} \displaystyle\int_\omega a_{ij} u_i u_j d\lambda & \text{if } \omega \subset \Lambda, \\ \displaystyle\int_{\omega\cap\Lambda} a_{ij} u_i u_j d\lambda & \text{if } \begin{cases} u(x) = 0, \forall x \in \omega \cap \mathbf{R}^N\setminus\Lambda \\ \quad \text{and } Cap(\omega \cap \mathbf{R}^N\setminus\Lambda) > 0, \end{cases} \\ +\infty & \text{otherwise.} \end{cases}$$

*Thanks to (6), this expression can be written as*

$$\int_\omega u_i u_j d\mu_{ij} = \begin{cases} \displaystyle\int_\omega a_{ij} u_i u_j d\lambda & \text{if } u(x) \in \mathbf{V}(x), \text{ for q.e. } x \in \omega, \\ +\infty & \text{otherwise.} \end{cases}$$

*We can thus write the functional $F$ defined in (5) as*

$$F(u,\omega) = \int_\omega u_i u_j d\mu_{ij}.$$

# 3 Study of the problem (1)

We here suppose that the "outer" boundary $\Gamma_{2,\varepsilon}$ of $\Sigma_\varepsilon$ can be defined as

$$\Gamma_{2,\varepsilon} = \{(s,t) \mid s \in \Gamma_2, \, t = -\varepsilon h_\varepsilon(s)\},$$

where $h_\varepsilon$ is a locally Lipschitz continuous function satisfying

$$\|h_\varepsilon\|_{\mathbf{L}^\infty(\Gamma_2)} \leq C, \, \forall \varepsilon > 0,$$

for some constant $C$ independent of $\varepsilon$. The Lipschitz continuity of $h_\varepsilon$ ensures the almost everywhere existence of a unit outer normal vector to $\Gamma_{2,\varepsilon}$, thanks to Rademacher's Theorem, and ensures the existence of an extension of every function of $\mathbf{H}^1\left(\Omega_\varepsilon,\mathbf{R}^N\right)$ in a function of $\mathbf{H}^1\left(\mathbf{R}^N,\mathbf{R}^N\right)$. Let us define the functional spaces

$$\begin{array}{rcl}\mathbf{L}^2\left(\mathbf{R}^N,\mathrm{div}\right) & = & \left\{u\in\mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right)\mid \mathrm{div}\left(u\right)=0\text{ in }\mathbf{R}^N\right\},\\ \mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^N,\mathrm{div}\right) & = & \left\{\begin{array}{r}u\in\mathbf{H}^1\left(\mathbf{R}^N,\mathbf{R}^N\right)\mid \mathrm{div}\left(u\right)=0\text{ in }\mathbf{R}^N,\\ u=0\text{ on }\Gamma_1\end{array}\right\},\\ \mathbf{H}^1_{\Gamma_1}\left(\Omega,\mathrm{div}\right) & = & \left\{u\in\mathbf{H}^1\left(\Omega,\mathbf{R}^N\right)\mid \mathrm{div}\left(u\right)=0\text{ in }\Omega,\ u=0\text{ on }\Gamma_1\right\},\\ \mathbf{V}_{\Gamma_1}\left(\Omega\right) & = & \mathbf{L}^2\left(\mathbf{R}^N,\mathrm{div}\right)\cap\mathbf{H}^1_{\Gamma_1}\left(\Omega,\mathrm{div}\right),\\ \mathbf{V}_{0,\Gamma_1}\left(\Omega\right) & = & \mathbf{H}^1_{\Gamma_1}\left(\Omega,\mathrm{div}\right)\cap\left\{u\in\mathbf{H}^1\left(\Omega,\mathbf{R}^N\right)\mid u\cdot n=0\text{ on }\Gamma_2\right\}.\end{array} \tag{7}$$

In (1), let us replace throughout this section the homogeneous Dirichlet boundary condition $u^\varepsilon=0$, on $\partial\Omega_\varepsilon$ by a combination between the homogeneous Dirichlet boundary condition $u^\varepsilon=0$, on $\Gamma_{2,\varepsilon}\cap\omega$, for a given $\omega\in\mathcal{O}\left(\mathbf{R}^N\right)$, and homogeneous Neumann boundary conditions on $\Gamma_{2,\varepsilon}\setminus\left(\Gamma_{2,\varepsilon}\cap\omega\right)$. We introduce the functional space adpated to (1), with these modified boundary conditions

$$\mathbf{V}_{0,\omega}\left(\Omega_\varepsilon\right)=\left\{\begin{array}{r}v\in\mathbf{H}^1\left(\Omega_\varepsilon,\mathbf{R}^N\right)\mid \mathrm{div}\left(v\right)=0\text{ in }\Omega_\varepsilon,\\ v=0\text{ on }\Gamma_1\cup\left(\Gamma_{2,\varepsilon}\cap\omega\right)\end{array}\right\}.$$

The variational formulation of (1) can be written as

$$\begin{array}{c}\forall\varphi\in\mathbf{V}_{0,\omega}\left(\Omega_\varepsilon\right):\nu\displaystyle\int_\Omega\nabla u^\varepsilon\cdot\nabla\varphi dx+\nu\varepsilon\int_{\Sigma_\varepsilon}\nabla u^\varepsilon\cdot\nabla\varphi dx\\ +\displaystyle\int_{\Omega_\varepsilon}\left(u^\varepsilon\cdot\nabla\right)u^\varepsilon\cdot\varphi dx=\int_{\Omega_\varepsilon}f\cdot\varphi dx.\end{array} \tag{8}$$

Thanks to [15], for example, we deduce that (1) has a unique solution $\left(u^\varepsilon,p^\varepsilon\right)$ belonging to the space $\mathbf{V}_{0,\omega}\left(\Omega_\varepsilon\right)\times\mathbf{L}^2\left(\Omega_\varepsilon\right)/\mathbf{R}$.

**Proposition 9** *The solution $\left(u^\varepsilon,p^\varepsilon\right)$ of (1) satisfies the following estimates*

$$\begin{array}{rcl}\displaystyle\sup_\varepsilon\left(\int_\Omega\left|\nabla u^\varepsilon\right|^2dx+\varepsilon\int_{\Sigma_\varepsilon}\left|\nabla u^\varepsilon\right|^2dx\right) & < & +\infty,\\ \displaystyle\sup_\varepsilon\int_{\mathbf{R}^N}\left|u^\varepsilon\right|^2dx & < & +\infty,\\ \displaystyle\sup_\varepsilon\left\|p^\varepsilon\right\|_{\mathbf{L}^2(\Omega_\varepsilon)/\mathbf{R}} & < & +\infty.\end{array}$$

**Proof.** 1. Taking $u^\varepsilon$ as test-function in (8), we obtain

$$\begin{array}{rl}\multicolumn{2}{l}{\nu\displaystyle\int_\Omega\left|\nabla u^\varepsilon\right|^2dx+\nu\varepsilon\int_{\Sigma_\varepsilon}\left|\nabla u^\varepsilon\right|^2dx}\\ = & \displaystyle\int_\Omega f\cdot u^\varepsilon dx+\int_{\Sigma_\varepsilon}f\cdot u^\varepsilon dx\\ \le & \left\|f\right\|_{\mathbf{L}^\infty(\mathbf{R}^N,\mathbf{R}^N)}\left\|u^\varepsilon\right\|_{\mathbf{L}^1(\Omega_\varepsilon,\mathbf{R}^N)}\\ \le & \left\|f\right\|_{\mathbf{L}^\infty(\mathbf{R}^N,\mathbf{R}^N)}C\left(\Omega\right)\left\|\nabla u^\varepsilon\right\|_{\mathbf{L}^1(\Omega,\mathbf{R}^N)},\end{array}$$

using Poincaré's inequality. Cauchy-Schwarz' inequality implies

$$\begin{array}{l}\displaystyle\int_\Omega\left|\nabla u^\varepsilon\right|^2dx+\varepsilon\int_{\Sigma_\varepsilon}\left|\nabla u^\varepsilon\right|^2dx\\ \qquad\le C\left(f,\Omega\right)\left(\left(\displaystyle\int_\Omega\left|\nabla u^\varepsilon\right|^2dx\right)^{1/2}+\left(\varepsilon\int_{\Sigma_\varepsilon}\left|\nabla u^\varepsilon\right|^2dx\right)^{1/2}\right),\end{array}$$

whence, using the trivial inequality $\left(a+b\right)^2\le2\left(a^2+b^2\right)$

$$\int_\Omega\left|\nabla u^\varepsilon\right|^2dx+\varepsilon\int_{\Sigma_\varepsilon}\left|\nabla u^\varepsilon\right|^2dx\le C\Rightarrow\left\|\nabla u^\varepsilon\right\|_{\mathbf{L}^1(\Omega_\varepsilon,\mathbf{R}^N)}\le C.$$

The continuous embedding from $\mathbf{W}^{1,1}_{\Gamma_1}\left(\Omega_\varepsilon,\mathbf{R}^N\right)$ to $\mathbf{L}^2\left(\Omega_\varepsilon,\mathbf{R}^N\right)$ implies the existence of a constant $C$ independent of $\varepsilon$ such that

$$\int_{\Omega_\varepsilon}|u^\varepsilon|^2\,dx\leq C.$$

2. Let us define the zero mean value pressure $\overline{p^\varepsilon}=p^\varepsilon-\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}p^\varepsilon dx$, and let $\psi_\varepsilon$ be the solution of the following problem (see [15])

$$\left\{\begin{array}{rcll}\operatorname{div}(\psi_\varepsilon) & = & \overline{p^\varepsilon} & \text{in } \Omega_\varepsilon,\\ \psi_\varepsilon & = & 0 & \text{on } \Gamma_1\cup(\Gamma_2\cap\omega),\\ \|\nabla\psi_\varepsilon\|_{\mathbf{L}^2\left(\Omega_\varepsilon,\mathbf{R}^{N^2}\right)} & \leq & C(\Omega)\,\|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)}, & \end{array}\right. \tag{9}$$

for some constant $C(\Omega)$ independent of $\varepsilon$. Multiplying $(1)_{1,2}$ by $\psi_\varepsilon$ and using Green's formula, one obtains

$$\begin{aligned}&\nu\int_\Omega\nabla u^\varepsilon\cdot\nabla\psi_\varepsilon dx+\nu\varepsilon\int_{\Sigma_\varepsilon}\nabla u^\varepsilon\cdot\nabla\psi_\varepsilon dx+\int_{\Omega_\varepsilon}(u^\varepsilon\cdot\nabla)\,u^\varepsilon\cdot\psi_\varepsilon dx\\ &\quad=\int_{\Omega_\varepsilon}f\cdot\psi_\varepsilon dx+\int_{\Omega_\varepsilon}\left(\overline{p^\varepsilon}\right)^2dx.\end{aligned}$$

Because

$$\begin{array}{rcl}\left|\int_{\Omega_\varepsilon}f\cdot\psi_\varepsilon dx\right| & \leq & \|f\|_{\mathbf{L}^2(\mathbf{R}^N,\mathbf{R}^N)}\,\|\psi_\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon,\mathbf{R}^N)}\\ & \leq & C\,\|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)}\\ \left|\int_{\Omega_\varepsilon}(u^\varepsilon\cdot\nabla)\,u^\varepsilon\cdot\psi_\varepsilon dx\right| & \leq & C\,\|\psi_\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon,\mathbf{R}^N)}\,\|\nabla u^\varepsilon\|^2_{\mathbf{L}^2(\Omega_\varepsilon,\mathbf{R}^N)}\\ & \leq & C\,\|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)}\,\|\nabla u^\varepsilon\|^2_{\mathbf{L}^2(\Omega_\varepsilon,\mathbf{R}^N)}\\ \left|\int_\Omega\nabla u^\varepsilon\cdot\nabla\psi_\varepsilon dx\right| & \leq & C\,\|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)}\,\|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon,\mathbf{R}^N)},\\ \left|\int_{\Sigma_\varepsilon}\nabla u^\varepsilon\cdot\nabla\psi_\varepsilon dx\right| & \leq & C\,\|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)}\,\|\nabla u^\varepsilon\|_{\mathbf{L}^2(\Omega_\varepsilon,\mathbf{R}^N)},\end{array}$$

thanks to $(9)_3$ and using Poincaré's inequality, we obtain

$$\|\overline{p^\varepsilon}\|^2_{\mathbf{L}^2(\Omega_\varepsilon)}\leq C\left(\|\nabla u^\varepsilon\|^2_{\mathbf{L}^2(\Omega_\varepsilon,\mathbf{R}^N)}+1\right)\|\overline{p^\varepsilon}\|_{\mathbf{L}^2(\Omega_\varepsilon)},$$

which proves the third estimate. ■

**Remark 10** *We can observe that, when we impose an homogeneous Dirichlet boundary condition on the whole $\Gamma_{2,\varepsilon}$, for example when $\omega=\mathbf{R}^N$, the above estimates can be obtained in a simpler way, assuming only that $f\in\mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right)$.*

# 4 Convergence

Every function $u\in\mathbf{H}^1_{\Gamma_1}(\Omega_\varepsilon,\operatorname{div})$ can be extended in a function of the space $\mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^N,\operatorname{div}\right)$, still denoted $u$ (see [16, Theorem 4.3.3], for example). We define the functional $\Phi^\varepsilon$ on $\mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right)$ associated to (1), with the above-described modified boundary conditions on $\Gamma_{2,\varepsilon}$ through

$$\Phi^\varepsilon(u)=\left\{\begin{array}{ll}\nu\int_\Omega|\nabla u|^2\,dx+\nu\varepsilon\int_{\mathbf{R}^N\setminus\boldsymbol{\Omega}}|\nabla u|^2\,dx & \text{if } u\in\mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^N,\operatorname{div}\right),\\ +\infty & \text{otherwise}\end{array}\right. \tag{10}$$

and the functional $\Phi^0$ defined on $\mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right)$ through

$$\Phi^0(u)=\left\{\begin{array}{ll}\nu\int_\Omega|\nabla u|^2\,dx & \text{if } u\in\mathbf{V}_{\Gamma_1}(\Omega),\\ +\infty & \text{otherwise.}\end{array}\right.$$

From the estimates given in Proposition 9, we can deduce that the asymptotic behaviour of the problem (1) is obtained when studying the Γ-limit of the associated energy functional for the following topology.

**Definition 11** *A sequence $(u_\varepsilon)_\varepsilon$ $\tau$-converges to $u$, if it converges to $u$ in the strong topology of* $\mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right)$ *and if* $\sup_\varepsilon \Phi^\varepsilon(u_\varepsilon) < +\infty$.

We first present the Γ-convergence result for $(\Phi^\varepsilon)_\varepsilon$.

**Proposition 12** *When $\varepsilon$ goes to 0, the sequence $(\Phi^\varepsilon)_\varepsilon$ Γ-converges to $\Phi^0$, in the topology $\tau$.*

**Proof.** Step 1: verification of the Γ-lim sup. Take any $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and consider the set $\Omega^{0,\varepsilon} = \Omega\cup\partial\Omega\cup\Sigma^{0,\varepsilon}$, with

$$\Sigma^{0,\varepsilon} = \left\{x\in\mathbf{R}^N \mid 0 < d(x,\partial\Omega) < \sqrt{\varepsilon}\right\},$$

where $d(x,\partial\Omega)$ denotes the euclidean distance between $x$ and the boundary $\partial\Omega$. Let $u^{1,\varepsilon}$ be such that $\operatorname{div}\left(u^{1,\varepsilon}\right) = 0$ in $\mathbf{R}^N$ and

$$\left\|u - u^{1,\varepsilon}\right\|_{\mathbf{L}^2(\mathbf{R}^N\setminus\Omega^{0,\varepsilon},\mathbf{R}^N)} < \varepsilon.$$

We define $\overline{u}^{1,\varepsilon}$ through

$$\overline{u}^{1,\varepsilon} = \begin{cases} u^{1,\varepsilon} & \text{in } \mathbf{R}^N\setminus\Omega^{0,\varepsilon},\\ 0 & \text{on } \partial\Omega^{0,\varepsilon}.\end{cases}$$

We then take a nonnegative and smooth function $\rho_\varepsilon \in \mathbf{C}_c^\infty\left(\mathbf{R}^N\right)$ with support in $B(0,\varepsilon)$ and satisfying $\int_{\mathbf{R}^N}\rho_\varepsilon(x)\,dx = 1$. We define the function $\overline{u}^{0,\varepsilon}$ through $\overline{u}^{0,\varepsilon} = \left(\rho_\varepsilon * \overline{u}^{1,\varepsilon}\right)_{|\mathbf{R}^N\setminus\overline{\Omega^{0,\varepsilon}}}$. There exists $\widehat{u}\in \mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right)$ such that $\operatorname{curl}(\widehat{u}) = u$ in $\mathbf{R}^N$ (see [15], for example). We finally define the function $u^{0,\varepsilon}$ through

$$u^{0,\varepsilon} = \begin{cases} \overline{u}^{0,\varepsilon} & \text{in } \mathbf{R}^N\setminus\overline{\Omega^{0,\varepsilon}},\\ \operatorname{curl}\left(\widehat{u}\dfrac{\sqrt{\varepsilon}-d(x,\partial\Omega)}{\sqrt{\varepsilon}}\right) & \text{in } \Sigma^{0,\varepsilon},\\ u & \text{in } \Omega.\end{cases}$$

We immediately satisfy that $u^{0,\varepsilon}\in\mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^N,\operatorname{div}\right)$, that the sequence $\left(u^{0,\varepsilon}\right)_\varepsilon$ converges to $u$ in the topology $\tau$ and that

$$\limsup_{\varepsilon\to0}\Phi^\varepsilon\left(u^{0,\varepsilon}\right) \le \nu\int_\Omega |\nabla u|^2\,dx = \Phi^0(u).$$

Step 2: verification of the Γ-lim inf. We take any sequence $(u_\varepsilon)_\varepsilon$ contained in $\mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^N,\operatorname{div}\right)$ which converges to $u$ in the topology $\tau$. We trivially have

$$\Phi^0(u) \le \liminf_{\varepsilon\to0}\Phi^0(u_\varepsilon) \le \liminf_{\varepsilon\to0}\Phi^\varepsilon(u_\varepsilon),$$

thanks to the lower semi-continuity property of $\Phi^0$ for the weak topology of $\mathbf{H}^1\left(\mathbf{R}^N,\mathbf{R}^N\right)$. ■

We define the functional $G^\varepsilon$ on $\mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right)\times\mathcal{O}\left(\mathbf{R}^N\right)$ through

$$G^\varepsilon(u,\omega) = \begin{cases}\Phi^\varepsilon(u) + F^\varepsilon(u,\omega) & \text{if } u\in\mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^N,\operatorname{div}\right),\\ +\infty & \text{otherwise,}\end{cases}$$

where $F^\varepsilon$ is defined in (4). Our main result is the following.

**Theorem 13** *There exist a rich family $\mathcal{R}\subset\mathcal{B}\left(\mathbf{R}^N\right)$ and a symmetric matrix $\mu^\bullet = \left(\mu_{ij}\right)_{i,j=1,\ldots,N}$ of Borel measures having their support contained in $\Gamma_2$, which are absolutely continuous with respect to the above-defined capacity Cap, and satisfying $\mu_{ij}(B)\zeta_i\zeta_j \ge 0$, $\forall\zeta\in\mathbf{R}^N$, $\forall B\in\mathcal{B}\left(\mathbf{R}^N\right)$, such that, for every $u\in\mathbf{V}_{\Gamma_1}(\Omega)$ and every $\omega\in\mathcal{R}\cap\mathcal{O}\left(\mathbf{R}^N\right)$*

$$\left(\Gamma\text{-}\lim_{\varepsilon\to0}G^\varepsilon\right)(u,\omega) = \nu\int_\Omega|\nabla u|^2 + \int_{\Gamma_2\cap\omega}u_iu_j\,d\mu_{ij} =: G^0(u,\omega),$$

*where the Γ-limit is taken with respect to the topology $\tau$.*

**Proof.** The upper and lower Γ-limits of the sequence $(G^\varepsilon)_\varepsilon$, with respect to the topology $\tau$, exist, which are respectively defined through

$$\forall u \in \mathbf{V}_{\Gamma_1}(\Omega), \forall B \in \mathcal{B}\left(\mathbf{R}^N\right) : \left\{ \begin{array}{lcl} G^s(u,B) & = & \inf\limits_{u_\varepsilon \xrightarrow{\tau} u} \limsup\limits_{\varepsilon\to 0} G^\varepsilon(u_\varepsilon, B), \\ G^i(u,B) & = & \inf\limits_{u_\varepsilon \xrightarrow{\tau} u} \liminf\limits_{\varepsilon\to 0} G^\varepsilon(u_\varepsilon, B). \end{array}\right. \tag{11}$$

Because $F^\varepsilon$ takes nonnegative values and thanks to Proposition 12, we observe that, for every $B \in \mathcal{B}\left(\mathbf{R}^N\right)$, one has

$$G^s(.,B) \geq \Phi^0(.) \ ; G^i(.,B) \geq \Phi^0(.).$$

Let us define the functionals $F^s$ and $F^i$ on $\mathbf{L}^2\left(\mathbf{R}^N, \mathbf{R}^N\right) \times \mathcal{B}\left(\mathbf{R}^N\right)$ through

$$\left(F^0\right)^\alpha(u,B) = \left\{ \begin{array}{ll} G^\alpha(u,B) - \Phi^0(u) & \text{if } u \in \mathbf{V}_{\Gamma_1}(\Omega), \\ +\infty & \text{otherwise,} \end{array}\right.$$

with $\alpha = s, i$. Let $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and $(u_\varepsilon)_\varepsilon \subset \mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^N, \mathrm{div}\right)$ be such that $(u_\varepsilon)_\varepsilon$ converges to $u$ in the topology $\tau$. We define $z_\varepsilon = u_\varepsilon - u$. Thus $(z_\varepsilon)_\varepsilon \subset \mathbf{H}^1\left(\mathbf{R}^N, \mathbf{R}^N\right)$ and $(z_\varepsilon)_\varepsilon$ converges to 0 in the topology $\tau$. Replacing $u_\varepsilon$ by $z_\varepsilon + u$ in (11), one obtains, using the quadratic property of $\Phi^\varepsilon$

$$\begin{array}{lcl} \left(F^0\right)^s(u,B) & = & \inf\limits_{z_\varepsilon \xrightarrow{\tau} 0} \limsup\limits_{\varepsilon\to 0} \left(\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(u + z_\varepsilon, B)\right), \\ \left(F^0\right)^i(u,B) & = & \inf\limits_{z_\varepsilon \xrightarrow{\tau} 0} \liminf\limits_{\varepsilon\to 0} \left(\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(u + z_\varepsilon, B)\right). \end{array}$$

The functionals $\left(F^0\right)^s$ and $\left(F^0\right)^i$ satisfy the following properties.

1. For every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$, $\left(F^0\right)^s(u,.)$ and $\left(F^0\right)^i(u,.)$ are nonnegative measures, because $F^\varepsilon(u + z_\varepsilon, .)$ is a measure for every $\varepsilon > 0$ and for every sequence $(z_\varepsilon)_\varepsilon \subset \mathbf{V}_{\Gamma_1}(\Omega)$ which converges to 0 in the topology $\tau$.

2. $\left(F^0\right)^s(.,B)$ and $\left(F^0\right)^i(.,B)$ are lower semi-continuous on $\mathbf{H}^1\left(\mathbf{R}^N, \mathbf{R}^N\right)$, when equipped with its strong topology, because $G^s(.,B)$, $G^i(.,B)$ and $\Phi^0$ are lower semi-continuous as upper, lower, or Γ-limits of functionals which are lower semi-continuous for this strong topology.

3. Let $\omega \in \mathcal{O}\left(\mathbf{R}^N\right)$ and $u, v \in \mathbf{V}_{\Gamma_1}(\Omega)$ be such that $u_{|\omega} = v_{|\omega}$. Then $\left(F^0\right)^s(u,\omega) = \left(F^0\right)^s(v,\omega)$ and $\left(F^0\right)^i(u,\omega) = \left(F^0\right)^i(v,\omega)$, because $F^\varepsilon(u + z_\varepsilon, \omega) = F^\varepsilon(v + z_\varepsilon, \omega)$, for every sequence $(z_\varepsilon)_\varepsilon$ such that $u + z_\varepsilon \in \mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^N, \mathrm{div}\right)$, for every $\varepsilon > 0$.

4. Take any $\varphi \in \mathbf{C}^1\left(\mathbf{R}^N\right)$ such that $0 \leq \varphi \leq 1$, $u, v \in \mathbf{V}_{\Gamma_1}(\Omega)$ and $B \in \mathcal{B}\left(\mathbf{R}^N\right)$. One has, for every sequence $(z_\varepsilon)_\varepsilon \subset \mathbf{V}_{\Gamma_1}(\Omega)$ converging to 0 in the topology $\tau$

$$\begin{array}{lcl} F^\varepsilon(z_\varepsilon + \varphi u + (1-\varphi)v, B) & = & F^\varepsilon((z_\varepsilon + u)\varphi + (1-\varphi)(z_\varepsilon + v), B) \\ & \leq & F^\varepsilon(z_\varepsilon + u, B) + F_\varepsilon(z_\varepsilon + v, B), \end{array}$$

because $F^\varepsilon$ is $\mathbf{C}^1$-convex. Because $\Phi^\varepsilon$ takes nonnegative values, for every $\varepsilon > 0$, one has

$$\begin{array}{l} \limsup\limits_{\varepsilon\to 0} \left(\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + \varphi u + (1-\varphi)v, B)\right) \\ \quad \leq \limsup\limits_{\varepsilon\to 0} \left(\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + u, B) + \Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + v, B)\right) \\ \quad \leq \limsup\limits_{\varepsilon\to 0} \left(\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + u, B)\right) \\ \qquad + \limsup\limits_{\varepsilon\to 0} \left(\Phi^\varepsilon(z_\varepsilon) + F^\varepsilon(z_\varepsilon + v, B)\right). \end{array}$$

Taking the infimum over all sequences $(z_\varepsilon)_\varepsilon \subset \mathbf{H}^1\left(\mathbf{R}^N, \mathbf{R}^N\right)$ which converge to 0 in the topology $\tau$, one obtains

$$\left(F^0\right)^s(\varphi u + (1-\varphi)v, B) \leq \left(F^0\right)^s(u,B) + \left(F^0\right)^s(v,B).$$

We prove in a similar way that $\left(F^0\right)^s$ is convex. Thus $\left(F^0\right)^s$ is $\mathbf{C}^1$-convex.

Thanks to the compacity theorem of [10], there exist a subsequence $(\varepsilon_k)_k$ and a dense and countable family $\mathcal{D} \subset \mathcal{B}\left(\mathbf{R}^N\right)$ such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $B \in \mathcal{D}$

$$\left(\Gamma\text{-}\lim_{k\to+\infty} G^{\varepsilon_k}\right)(u,B) = G^0(u,B),$$

where the $\Gamma$-limit is taken with respect to the topology $\tau$. We then define the functional $F^0$ on $\mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right) \times \mathcal{D}$ as

$$F^0(u,B) = \begin{cases} G^0(u,B) - \Phi^0(u) & \text{if } u \in \mathbf{V}_{\Gamma_1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{12}$$

We have $F^0 = \left(F^0\right)^s = \left(F^0\right)^i$ on $\mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right) \times \mathcal{D}$. We then extend $F^0$ on $\mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right) \times \mathcal{B}\left(\mathbf{R}^N\right)$ defining

$$F^0(u,B) = \sup_{D\in\mathcal{D},\overline{D}\subset \overset{o}{B}} \left(F^0\right)^s(u,D) = \sup_{D\in\mathcal{D},\overline{D}\subset \overset{o}{B}} \left(F^0\right)^i(u,D). \tag{13}$$

We define the family $\mathcal{R}(F)$ of Borel subsets of $\mathbf{R}^N$ through

$$\mathcal{R}(F) = \left\{ \begin{array}{c} B \in \mathcal{B}\left(\mathbf{R}^N\right) \mid \forall u \in \mathbf{L}^2\left(\mathbf{R}^N,\mathbf{R}^N\right) : \left(F^0\right)^s_+(u,B) = \\ \sup_{D\in\mathcal{D},\overline{D}\subset \overset{o}{B}} \left(F^0\right)^s(u,D) = \inf_{D\in\mathcal{D},\overline{B}\subset \overset{o}{D}} \left(F^0\right)^s(u,D) \\ = \left(F^0\right)^s_-(u,B) \end{array} \right\}.$$

Then we prove (see [5, Proposition 14.14]) that $\mathcal{R}\left(F^0\right)$ is a rich family in $\mathcal{B}\left(\mathbf{R}^N\right)$ and $F^0 = \left(F^0\right)^s = \left(F^0\right)^s_+ = \left(F^0\right)^s_- = \left(F^0\right)^i_+ = \left(F^0\right)^i_- = \left(F^0\right)^i$ on $\mathcal{R}\left(F^0\right)$. One obtains, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $B \in \mathcal{R}\left(F^0\right)$

$$\begin{aligned} F^0(u,B) &= \inf_{z_{\varepsilon_k} \overset{\tau}{\rightharpoonup} 0} \limsup_{k\to+\infty} \left(\Phi^{\varepsilon_k}(z_{\varepsilon_k}) + F^{\varepsilon_k}(u+z_{\varepsilon_k},B)\right) \\ &= \inf_{z_{\varepsilon_k} \overset{\tau}{\rightharpoonup} 0} \liminf_{k\to+\infty} \left(\Phi^{\varepsilon_k}(z_{\varepsilon_k}) + F^{\varepsilon_k}(u+z_{\varepsilon_k},B)\right). \end{aligned}$$

Let now $\varepsilon'$ denote any subsequence of $\varepsilon$. Thanks to the above method, there exist a subsequence $(\varepsilon'_k)_k$, a functional $\mathcal{F}^0$ and a rich family $\mathcal{R}\left(\mathcal{F}^0\right)$ such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $B \in \mathcal{R}\left(\mathcal{F}^0\right)$

$$\begin{aligned} \mathcal{F}^0(u,B) &= \inf_{z_{\varepsilon'_k} \overset{\tau}{\rightharpoonup} 0} \limsup_{k\to+\infty} \left(\Phi^{\varepsilon'_k}\left(z_{\varepsilon'_k}\right) + F^{\varepsilon'_k}\left(u+z_{\varepsilon'_k},B\right)\right) \\ &= \inf_{z_{\varepsilon'_k} \overset{\tau}{\rightharpoonup} 0} \liminf_{k\to+\infty} \left(\Phi^{\varepsilon'_k}\left(z_{\varepsilon'_k}\right) + F^{\varepsilon'_k}\left(u+z_{\varepsilon'_k},B\right)\right). \end{aligned}$$

Because $\mathcal{R}\left(F^0\right) \cap \mathcal{R}\left(\mathcal{F}^0\right)$ is still a rich family, one has

$$\forall u \in \mathbf{V}_{\Gamma_1}(\Omega), \forall B \in \mathcal{R} : F^0(u,.) = \mathcal{F}^0(u,.), \text{ on } \mathcal{R}\left(F^0\right) \cap \mathcal{R}\left(\mathcal{F}^0\right).$$

Because the countable intersection of rich families is a rich family too, one can repeat the above reasoning and deduce the existence of a rich family $\mathcal{R}$ in $\mathcal{B}\left(\mathbf{R}^N\right)$ on which the above limits coincide. One thus obtains, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $B \in \mathcal{R}$

$$\left(\Gamma\text{-}\lim_{\varepsilon\to 0} G^{\varepsilon}\right)(u,\omega) = \Phi^0(u) + F^0(u,B), \tag{14}$$

where the $\Gamma$-limit is taken with respect to the topology $\tau$.

Thanks to the above properties 1., 2., 3. and 4. and to the relations (12) and (13), $F^0$ belongs to $\mathbb{F}$. Because $\Phi^{\varepsilon}$ and $F^{\varepsilon}$ are quadratic, thanks to Corollary 7 and to Remark 8, there exist $\lambda \in \mathcal{M}_0$ finite, a symmetric matrix $(a_{ij})_{i,j=1,..,N}$ of Borel functions from $\mathbf{R}^N$ to $\mathbf{R}$ with $a_{ij}(x)\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbf{R}^N$ and for q.e. $x \in \mathbf{R}^N$, such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbf{R}^N)$

$$F^0(u,\omega) = \int_\omega u_i u_j d\mu_{ij},$$

with $\mu^\bullet = (\mu_{ij})_{i,j=1,..,N} = (a_{ij}\lambda)_{i,j=1,..,N} + \infty_{\mathbf{R}^N\setminus\Lambda} Id$, where $\Lambda$ is defined as in Remark 8.

Let us now precise the support of $\mu^\bullet$. For every $u, v \in \mathbf{H}^1_{\Gamma_1}(\mathbf{R}^N, \mathrm{div})$, such that $v_{|\Omega} = u_{|\Omega}$, one has

$$F^0(u,\mathbf{R}^N) = \int_{\mathbf{R}^N} v_i v_j d\mu_{ij},$$

because $F^0$ is local ($\mathbf{R}^N$ belongs to $\mathcal{R}$ because every rich family is dense, and every dense family contains $\mathbf{R}^N$). One deduces that *supp* $(\mu^\bullet) \subset \Omega \cup \Gamma_2$. Thanks to (14), one has

$$0 \leq \int_{\mathbf{R}^N} u_i u_j d\mu_{ij} + \Phi^0(u) \leq \liminf_{\varepsilon\to 0}\left(\Phi^\varepsilon(u) + F^\varepsilon(u,\mathbf{R}^N)\right). \tag{15}$$

Taking $u \in \mathbf{H}^1_0(\Omega,\mathrm{div}) = \{u \in \mathbf{H}^1_0(\Omega,\mathbf{R}^N) \mid \mathrm{div}(u) = 0\}$, then, for every $\varepsilon > 0$, $F^\varepsilon(u,\mathbf{R}^N) = 0$, and $\liminf_{\varepsilon\to 0}\Phi^\varepsilon(u) = \Phi^0(u)$. One deduces, using (15), that $\int_\Omega u_i u_j d\mu_{ij} = 0$, and thus that *supp* $(\mu^\bullet) \subset \Gamma_2$, which ends the proof. ■

**Remark 14** 1. *We thus get Navier's wall law at the zeroth-order limit of the problem (1).*

2. *Theorem 13 can be extended to every kind of obstacle functional in* $\mathbb{F}$*, using Theorem 6 for the integral representation. One can define, for example, sequences of obstacle functionals on* $\mathbf{H}^1(\mathbf{R}^N,\mathbf{R}^N)\times\mathcal{O}(\mathbf{R}^N)$ *of the kind*

$$(F^\varepsilon)^+(u,\omega) = \begin{cases} 0 & \text{if } \widetilde{u} \geq 0 \text{ q.e. on } \Gamma_{2,\varepsilon}\cap\omega, \\ +\infty & \text{otherwise,} \end{cases}$$

*the limit* $(F^0)^+$ *of which is defined on* $\mathbf{V}_{\Gamma_1}(\Omega)\times(\mathcal{R}^+\cap\mathcal{O}(\mathbf{R}^N))$ *(for some rich family* $\mathcal{R}^+$*) as*

$$(F^0)^+(u,\omega) = \int_{\omega\cap\Gamma_2} u_i^+ u_j^+ d\mu_{ij},$$

*where* $u_i^+ = \max(0,u_i)$*,* $i = 1,\ldots,N$*.*

3. *One proves that* $\mu_{ij} \in \mathbf{H}^{-1/2}(\Gamma_2)$*,* $\forall i,j = 1,\ldots,N$*, where* $\mu_{ij}$ *is the measure defined in Theorem 13. One first observes that the measure* $\lambda$ *defined in Theorem 6 belongs to* $\mathbf{H}^{-1/2}(\Gamma_2)^+$*.* $\lambda$ *is indeed finite. Because for every compact subset* $K \subset \Gamma_2$*, one has* $\lambda(K) < +\infty$*, hence* $\lambda$ *is a Radon nonnegative measure. Moreover, because* $\lambda$ *is absolutely continuous with respect to the capacity Cap, we deduce from [6, Theorem 2.2], the existence of a Radon measure* $\varkappa \in \mathbf{H}^{-1/2}(\Gamma_2)$ *and of a Borel function* $f : \Gamma_2 \to [0,+\infty[$ *such that* $f = \frac{d\lambda}{d\varkappa}$*.*

Let us come back to the study of problem (1). The solution $u^\varepsilon$ of (1), with the homogeneous Dirichlet boundary conditions on $\partial\Omega_\varepsilon$ is also the solution of the minimization problem

$$\inf_{v\in\mathbf{L}^2(\mathbf{R}^N,\mathbf{R}^N)}\left(G^\varepsilon(v,\mathbf{R}^N) + 2\int_{\Omega_\varepsilon}(u^\varepsilon\cdot\nabla)u^\varepsilon\cdot v dx - 2\int_{\Omega_\varepsilon} f\cdot v dx\right). \tag{16}$$

From Theorem 13, one deduces the following asymptotic behaviour of the solution of (1).

**Corollary 15** *The solution $(u^\varepsilon, p^\varepsilon)$ of (1), is such that $(u^\varepsilon)_\varepsilon$ converges to $u^0$ in the topology $\tau$ and $\left((p^\varepsilon)_{|\Omega}\right)_\varepsilon$ converges to $p^0$ in the strong topology of $\mathbf{L}^2(\Omega)/\mathbf{R}$, where $(u^0, p^0)$ belongs to $\mathbf{V}_{0,\Gamma_1}(\Omega) \times \mathbf{L}^2(\Omega)/\mathbf{R}$ and is the solution of the limit minimization problem*

$$\inf_{v \in \mathbf{L}^2(\mathbf{R}^N, \mathbf{R}^N)} \left( G^0\left(v, \mathbf{R}^N\right) + 2\int_\Omega \left(u^0 \cdot \nabla\right) u^0 \cdot v dx - 2\int_\Omega f \cdot v dx \right), \tag{17}$$

*or of the limit problem with Navier law*

$$\begin{cases} -\nu\Delta u^0 + \left(u^0 \cdot \nabla\right) u^0 + \nabla p^0 &= f \quad \text{in } \Omega, \\ \operatorname{div}\left(u^0\right) &= 0 \quad \text{in } \Omega, \\ u^0 &= 0 \quad \text{on } \Gamma_1, \\ u^0 \cdot n &= 0 \quad \text{on } \Gamma_2, \\ \left(I - n \otimes n\right) \nu \dfrac{\partial u^0}{\partial n} + \mu^\bullet u^0 &= 0 \quad \text{on } \Gamma_2. \end{cases} \tag{18}$$

**Proof.** We first observe that, for every sequence $(v_\varepsilon)_\varepsilon$ converging to $v$ in the topology $\tau$

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f \cdot v_\varepsilon dx = \int_\Omega f \cdot v dx,$$

Thanks to the properties of the Γ-convergence, $(u^\varepsilon)_\varepsilon$ converges to $u^0$ in the topology $\tau$, with $u^0 \in \mathbf{V}_{\Gamma_1}(\Omega)$, and

$$\lim_{\varepsilon \to 0} G^\varepsilon\left(u^\varepsilon, \mathbf{R}^N\right) = G^0\left(u^0, \mathbf{R}^N\right) = \nu \int_\Omega \left|\nabla u^0\right|^2 dx + \int_{\Gamma_2} \left(u^0\right)_i \left(u^0\right)_j d\mu_{ij}.$$

Then

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \left(u^\varepsilon \cdot \nabla\right) u^\varepsilon \cdot v_\varepsilon dx = \int_\Omega \left(u^0 \cdot \nabla\right) u^0 \cdot v dx,$$

for every sequence $(v_\varepsilon)_\varepsilon$ converging to $v$ in the topology $\tau$. For every $\varphi \in \mathbf{C}^1\left(\mathbf{R}^N\right)$, one has

$$\left|\int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla\varphi dx\right| \le \left(\int_{\Sigma_\varepsilon} |\nabla\varphi|^2 dx\right)^{1/2} \left(\int_{\mathbf{R}^N} |u^\varepsilon|^2 dx\right)^{1/2},$$

and thus $\lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla\varphi dx = 0$. Because $\operatorname{div}(u^\varepsilon) = \operatorname{div}\left(u^0\right) = 0$, and $u^\varepsilon = 0$, q.e. on $\Gamma_2$, one has

$$0 = \int_{\Omega_\varepsilon} u^\varepsilon \cdot \nabla\varphi dx = \int_\Omega u^\varepsilon \cdot \nabla\varphi dx + \int_{\Sigma_\varepsilon} u^\varepsilon \cdot \nabla\varphi dx.$$

Taking the limit of this equality, we obtain

$$0 = \int_\Omega u^0 \cdot \nabla\varphi dx = \int_{\Gamma_2} u^0 \cdot n\varphi d\Gamma_2,$$

which proves that $u^0 \cdot n = 0$ on $\Gamma_2$. Thus $u^0 \in \mathbf{V}_{0,\Gamma_1}(\Omega)$ is the solution of the problem (17). The variational formulation of (17) can be written as

$$\begin{aligned} \forall\varphi \in \mathbf{V}_{0,\Gamma_1}(\Omega) : &\int_\Omega \left(-\nu\Delta u^0 + \left(u^0 \cdot \nabla\right) u^0\right) \cdot \varphi dx \\ &+ \int_{\Gamma_2} \nu \frac{\partial u^0}{\partial n} \cdot \varphi d\Gamma_2 + \int_{\Gamma_2} \left(u^0\right)_i \varphi_j d\mu_{ij} = \int_\Omega f \cdot \varphi dx. \end{aligned}$$

There exists $p_0 \in L^2(\Omega)/\mathbf{R}$ such that $-\nu\Delta u^0 + \left(u^0 \cdot \nabla\right) u^0 - f = -\nabla p_0$. Thanks to Proposition 9, the sequence $\left((p^\varepsilon)_{|\Omega}\right)_\varepsilon$ converges to $p^0$ in the strong topology of $\mathbf{L}^2(\Omega)/\mathbf{R}$. Because $\varphi \cdot n = 0$ on $\Gamma_2$, with $n = (0, 0, 1)$, one has: $\nu \frac{\partial u^0}{\partial n} \cdot \varphi = (Id - n \otimes n)\, \nu \frac{\partial u^0}{\partial n} \cdot \varphi$, which ends the proof. ■

# 5 Special cases

We intend to specialize the general result obtained in Theorem 13, in two cases where the boundary $\Gamma_{2,\varepsilon}$ can be defined through some Lipschitz continuous function.

## 5.1 Periodic case

In this section, we suppose that $\Omega \subset \{x_3 > 0\}$ with $\partial\Omega \cap \{x_3 = 0\} = \Gamma_2$, $\Gamma_2$ containing 0. We define $Y = (-1/2, 1/2)^2$ and consider a $Y$-periodic function $h \in \mathbf{C}_c^2(Y, \mathbf{R}_+)$. For every $k \in \mathbf{Z}^2$, we define $Y_\varepsilon^k = (-\varepsilon/2, \varepsilon/2)^2 + (k_1\varepsilon, k_2\varepsilon)$, and let $I_\varepsilon = \{k \in \mathbf{Z}^2 \mid Y_\varepsilon^k \subset \Gamma_2\}$. We define $h_\varepsilon$ on $\Gamma_2$ through

$$h_\varepsilon(x') = \begin{cases} h\left(\dfrac{x'}{\varepsilon}\right) & \text{if there exists } k \in I_\varepsilon \text{ such that } x' = (x_1, x_2) \in Y_\varepsilon^k, \\ 0 & \text{otherwise} \end{cases}$$

and $\Sigma_\varepsilon$ through

$$\Sigma_\varepsilon = \left\{x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2,\ -\varepsilon h_\varepsilon(x') < x_3 < 0\right\}.$$

Thanks to Theorem 13, there exist a rich family $\mathcal{R} \subset \mathcal{B}(\mathbf{R}^3)$, a symmetric matrix $(\mu_{ij})_{i,j=1,\dots,N}$ of Borel measures having the same support contained in $\Gamma_2$, absolutely continuous with respect to the capacity $Cap$, and satisfying $\mu_{ij}(B)\zeta_i\zeta_j \geq 0$, $\forall\zeta \in \mathbf{R}^3$, $\forall B \in \mathcal{B}(\mathbf{R}^3)$, such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $\omega \in \mathcal{R} \cap \mathcal{O}(\mathbf{R}^3)$

$$\inf\left\{\begin{array}{l} \liminf\limits_{\varepsilon\to 0} \Phi^\varepsilon(z_\varepsilon) \mid u + z_\varepsilon = 0 \text{ on} \\ \qquad \{x_3 = -\varepsilon h_\varepsilon(x')\} \cap \omega \text{ and } z_\varepsilon \underset{\varepsilon\to 0}{\overset{\tau}{\rightharpoonup}} 0 \end{array}\right\} = \int_{\omega\cap\Gamma_2} u_i u_j d\mu_{ij}, \tag{19}$$

where $\Phi^\varepsilon$ is the energy functional defined in (10).

Because the lower boundary $\Gamma_{2,\varepsilon}$ of $\Sigma_\varepsilon$, defined through the equality $\Gamma_{2,\varepsilon} = \{(x', x_3) \mid x_3 = -\varepsilon h_\varepsilon(x')\}$, has a periodic structure, the measures $\mu_{ij}$, $i, j = 1, \dots, N$, are invariant under translations on $\Gamma_2$. This implies $\mu_{ij} = K_{ij}dx'$, where $K_{ij}$, $i, j = 1, 2, 3$, are constants in $\overline{\mathbf{R}}$ satisfying $K_{ij}\zeta_i\zeta_j \geq 0$, $\forall\zeta \in \mathbf{R}^3$.

The purpose of this section is to identify these constants $K_{ij}$, $i, j = 1, 2, 3$. We observe that we do not have to determine $K_{i3}$, $i = 1, 2, 3$, because, in the limit problem, one has $u \cdot n = u \cdot e_3 = u_3 = 0$.

**Theorem 16** *The limit Navier wall law of the limit problem (18) is in this case*

$$\frac{\partial\left(u^0\right)_m}{\partial x_3} = c_m\left(u^0\right)_m, \text{ on } \Gamma_2,\ m = 1, 2,$$

*where the constants $c_m$ are defined in (21).*

**Proof.** We define the set $Z_h = \{x \mid x' \in Y,\ -h(x') < x_3 < 0\}$ and consider in $Z_h$ the local Stokes problems for $m = 1, 2$

$$\begin{cases} -\Delta w^m + \nabla q^m &= e^m \quad \text{in } Z_h, \\ \operatorname{div}(w^m) &= 0 \quad \text{in } Z_h, \\ w^m &= e^m \quad \text{on } \{x_3 = -h(x')\}, \\ w^m &= 0 \quad \text{on } \{x_3 = 0\}, \\ w^m, q^m & \qquad Y\text{-periodic}, \end{cases} \tag{20}$$

where $e^m$ is the $m$-th vector of the canonical basis of $\mathbf{R}^3$. Lax-Milgram' Theorem implies that (20) has a unique solution $(w^m, q^m)$ with

$$\begin{array}{lcl} w^m & \in & \mathbf{V}(Z_h) = \left\{\begin{array}{c} u \in \mathbf{H}^1(Z_h, \mathbf{R}^3) \mid \operatorname{div}(u) = 0 \text{ in } Z_h, \\ u = 0 \text{ on } \{x_3 = 0\},\ u\ Y\text{-periodic} \end{array}\right\} \\ q^m & \in & \mathbf{L}^2(Z_h)/\mathbf{R},\ q^m\ Y\text{-periodic.} \end{array}$$

Let $z_h = \max_{x' \in Y} h(x')$ and choose $H > z_h$. We define

$$\widetilde{Z}_h = \{x \mid x' \in Y, \; -H < x_3 < -h(x')\}$$

and consider in $\widetilde{Z}_h$ problems similar to (20) except that we impose $\widetilde{w}^m = e^m$ on $\{x_3 = -h(x')\}$ and $\widetilde{w}^m = 0$ on $\{x_3 = -H\}$. Let us define

$$\begin{array}{rcl} \widetilde{\Sigma}_\varepsilon & = & \left\{ x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, \; -\varepsilon H < x_3 < -\varepsilon h_\varepsilon(x') \right\}, \\ B_\varepsilon & = & \left\{ x \in \mathbf{R}^3 \mid x' = (x_1, x_2) \in \Gamma_2, \; -\varepsilon H < x_3 < 0 \right\} \end{array}$$

and the functions $(w^{\varepsilon m}, q^{\varepsilon m})$ and $(\widetilde{w}^{\varepsilon m}, \widetilde{q}^{\varepsilon m})$ through

$$\left\{ \begin{array}{rclrcl} w^{\varepsilon m}(x) & = & w^m\left(\dfrac{x}{\varepsilon}\right), & q^{\varepsilon m}(x) & = & q^m\left(\dfrac{x}{\varepsilon}\right), \\ \widetilde{w}^{\varepsilon m}(x) & = & \widetilde{w}^m\left(\dfrac{x}{\varepsilon}\right), & \widetilde{q}^{\varepsilon m}(x) & = & \widetilde{q}^m\left(\dfrac{x}{\varepsilon}\right). \end{array} \right.$$

We finally build the function $z_\varepsilon^{0m}$, on $B_\varepsilon$, through

$$z_\varepsilon^{0m}(x) = \left\{ \begin{array}{ll} w^{\varepsilon m}(x) & \text{if } x \in \Sigma_\varepsilon, \\ e^m & \text{on } \{x_3 = -\varepsilon h_\varepsilon(x')\}, \\ \widetilde{w}^{\varepsilon m}(x) & \text{on } \widetilde{\Sigma}_\varepsilon. \end{array} \right.$$

Because $h = 0$ on $\partial Y$, one can suppose that $z_\varepsilon^{0m} = 0$ on $\partial\Gamma_2 \times (-\varepsilon H, 0)$. This implies that $z_\varepsilon^{0m} \in \mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^3, \mathrm{div}\right)$ and $z_\varepsilon^{0m} = 0$ on $\partial B_\varepsilon$. Moreover

$$\begin{array}{rcl} \displaystyle\lim_{\varepsilon \to 0} \int_{\mathbf{R}^N} \left| z_\varepsilon^{0m} \right|^2 dx & = & \displaystyle\lim_{\varepsilon \to 0} \int_{B_\varepsilon} \left| z_\varepsilon^{0m} \right|^2 dx \\ & = & \displaystyle\lim_{\varepsilon \to 0} \sum_{k \in I_\varepsilon} \int_{Y_\varepsilon^k} \int_{-\varepsilon H}^0 \left| z_\varepsilon^{0m} \right|^2 dx \\ & = & \displaystyle\lim_{\varepsilon \to 0} \left( \begin{array}{l} \displaystyle\sum_{k \in I_\varepsilon} \varepsilon^3 \int_Y \int_{-H}^{-h(x')} \left| \widetilde{w}^m(x) \right|^2 dx \\ \qquad + \displaystyle\sum_{k \in I_\varepsilon} \varepsilon^3 \int_Y \int_{-h(x')}^0 \left| w^m(x) \right|^2 dx \end{array} \right) \\ & = & 0 \end{array}$$

and

$$\begin{array}{rcl} \displaystyle\lim_{\varepsilon \to 0} \Phi^\varepsilon\left(z_\varepsilon^{0m}\right) = & = & \displaystyle\lim_{\varepsilon \to 0} \nu\varepsilon \int_{\Sigma_\varepsilon} \left| \nabla z_\varepsilon^{0m} \right|^2 dx \\ & = & \displaystyle\nu \sum_{k \in I_\varepsilon} \varepsilon^2 \int_Y \int_{-h(x')}^0 \left| \nabla w^m(x) \right|^2 dx \\ & = & \nu \left| \Gamma_2 \right| c_m, \end{array}$$

with

$$c_m = \int_{Z_h} \left| \nabla w^m \right|^2 dx. \tag{21}$$

Taking $u = -e^m$ on $\Sigma_\varepsilon$, in (19), one obtains

$$\begin{array}{rcl} K_{mm} \left| \Gamma_2 \right| & = & \inf \left\{ \begin{array}{c} \displaystyle\liminf_{\varepsilon \to 0} \Phi^\varepsilon(z_\varepsilon) \mid z_\varepsilon = e^m \text{ on } \{x_3 = -\varepsilon h_\varepsilon(x')\}, \\ z_\varepsilon \underset{\varepsilon \to 0}{\overset{\tau}{\rightharpoonup}} 0 \end{array} \right\} \\ & \leq & \displaystyle\lim_{\varepsilon \to 0} \Phi^\varepsilon\left(z_\varepsilon^{0m}\right) = \nu c_m \left| \Gamma_2 \right|. \end{array}$$

This implies

$$K_{mm} \left| \Gamma_2 \right| \leq \nu c_m \left| \Gamma_2 \right|. \tag{22}$$

Take any sequence $(z_\varepsilon)_\varepsilon \subset \mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^3, \mathrm{div}\right)$ such that $z_\varepsilon = e^m$ on the surface $\{x_3 = -\varepsilon h_\varepsilon(x')\}$ and $(z_\varepsilon)_\varepsilon$ converges to 0 in the topology $\tau$. We write the subdifferential inequality

$$\Phi^\varepsilon(z_\varepsilon) \geq \Phi^\varepsilon\left(z_\varepsilon^{0m}\right) + 2\nu\varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0m} \cdot \nabla\left(z_\varepsilon - z_\varepsilon^{0m}\right) dx. \tag{23}$$

We observe that

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0m} \cdot \nabla\left(z_\varepsilon - z_\varepsilon^{0m}\right) dx &= -\varepsilon \int_{\Sigma_\varepsilon} \Delta z_\varepsilon^{0m} \cdot \left(z_\varepsilon - z_\varepsilon^{0m}\right) dx \\ &\quad -\varepsilon \int_{\Gamma_2} \frac{\partial z_\varepsilon^{0m}}{\partial n} \cdot \left(z_\varepsilon - z_\varepsilon^{0m}\right) d\Gamma_2. \end{aligned}$$

Using the regularity (at least $\mathbf{H}^2$) of $w^m$, we obtain

$$\varepsilon \Delta z_\varepsilon^{0m} \underset{\varepsilon\to 0}{\rightharpoonup} \mathbf{1}_{\Gamma_2} \int_{Z_h} \Delta w^m(x)\, dx,$$

where the convergence takes place in the weak topology of $\mathbf{L}^2\left(\mathbf{R}^3, \mathbf{R}^3\right)$ and $\mathbf{1}_{\Gamma_2}$ is the characteristic function of $\Gamma_2$. Then

$$\begin{aligned} &\left| \varepsilon \int_{\Gamma_2} \frac{\partial z_\varepsilon^{0m}}{\partial n} \cdot \left(z_\varepsilon - z_\varepsilon^{0m}\right) d\Gamma_2 \right| \\ &\quad \leq \left( \int_{\Gamma_2} \left| \frac{\partial w^m}{\partial n} \right|^2 d\Gamma_2 \right)^{1/2} \left( \int_{\mathbf{R}^3} \left| z_\varepsilon - z_\varepsilon^{0m} \right|^2 dx \right)^{1/2}. \end{aligned}$$

Because $(z_\varepsilon - z_0^{\varepsilon m})_\varepsilon$ converges to 0 in the strong topology $\mathbf{L}^2\left(\mathbf{R}^3, \mathbf{R}^3\right)$, we have

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \nabla z_\varepsilon^{0m} \cdot \nabla\left(z_\varepsilon - z_\varepsilon^{0m}\right) dx = 0.$$

Taking the $\liminf$ in (23), one obtains

$$\liminf_{\varepsilon\to 0} \Phi^\varepsilon(z_\varepsilon) \geq \liminf_{\varepsilon\to 0} \Phi^\varepsilon\left(z_\varepsilon^{0m}\right) = \nu c_m |\Gamma_2|.$$

In this last inequality, taking the infimum with respect to all sequences $(z_\varepsilon)_\varepsilon$ satisfying the imposed conditions, one obtains: $K_{mm} |\Gamma_2| \geq \nu c_m |\Gamma_2|$. This inequality and (22) imply: $K_{mm} = \nu c_m$. Taking now $u = -\left(e^1 + e^2\right)$ on $\Sigma_\varepsilon$ in (19), one obtains

$$\begin{aligned} (K_{11} + 2K_{12} + K_{22}) |\Gamma_2| &= \inf \left\{ \begin{array}{c} \liminf_{\varepsilon\to 0} \Phi^\varepsilon(z_\varepsilon) \mid z_\varepsilon = e^1 + e^2 \text{ on} \\ \{x_3 = -\varepsilon h_\varepsilon(x')\},\ z_\varepsilon \underset{\varepsilon\to 0}{\overset{\tau}{\rightharpoonup}} 0 \end{array} \right\} \\ &\leq \lim_{\varepsilon\to 0} \Phi^\varepsilon\left(z_\varepsilon^{01} + z_\varepsilon^{02}\right). \end{aligned}$$

Because $\int_{Z_h} \nabla w^1 \cdot \nabla w^2 dz = 0$, we have

$$\lim_{\varepsilon\to 0} \Phi^\varepsilon\left(z_\varepsilon^{01} + z_\varepsilon^{02}\right) = \nu |\Gamma_2| (c_1 + c_2).$$

This implies: $K_{12} \leq 0$, through the above expression of $K_{mm}$. Writing a subdifferential inequality as in (23), one obtains: $K_{12} \geq 0$, which implies: $K_{12} = 0$. ■

## 5.2 Case where $h_\varepsilon$ is independent of $\varepsilon$

As in the previous section, we still suppose that $\Omega \subset \{x_3 > 0\}$ and $\partial\Omega \cap \{x_3 = 0\} = \Gamma_2$. But, we here suppose that the boundary $\Gamma_{2,\varepsilon}$ is given as

$$\Gamma_{2,\varepsilon} = \{(x', x_3) \mid x_3 = -\varepsilon h(x')\}$$

where $h$ is a Lipschitz continuous function satisfying $h(x') > 0$, $\forall x' \in \Gamma_2$. We have the following result.

**Theorem 17** *Under the preceding hypothesis, the Navier wall law is in this case*

$$(Id - n \otimes n) \frac{\partial u^0}{\partial n} + \frac{u^0}{h} = 0, \ on\ \Gamma_2.$$

**Proof.** Thanks to Theorem 13, there exist a rich family $\mathcal{R}_{\Gamma_2} \subset \mathcal{B}(\Sigma)$, a symmetric matrix $(\mu_{ij})_{i,j=1,\dots,N}$ of Borel measures having their support contained in $\Gamma_2$, which are absolutely continuous with respect to the capacity $Cap$, and satisfying $\mu_{ij}(B)\zeta_i\zeta_j \geq 0$, $\forall \zeta \in \mathbf{R}^3$, $\forall B \in \mathcal{B}(\Sigma)$, such that, for every $u \in \mathbf{V}_{\Gamma_1}(\Omega)$ and every $\omega \in \mathcal{R}_{\Gamma_2} \cap \mathcal{O}(\Gamma_2)$

$$\int_\omega u_i u_j d\mu_{ij} = \inf \left\{ \begin{array}{c} \liminf\limits_{\varepsilon\to 0} \Phi^\varepsilon(z_\varepsilon) \mid u + z_\varepsilon = 0 \text{ on} \\ \{x_3 = -\varepsilon h(x')\} \cap \omega,\ z_\varepsilon \underset{\varepsilon\to 0}{\overset{\tau}{\rightharpoonup}} 0 \end{array} \right\}. \tag{24}$$

Take $u = -e^1$ on $\{x_3 = -\varepsilon h(x')\}$. Then choose $\omega \in \mathcal{R}_{\Gamma_2} \cap \mathcal{O}(\Gamma_2)$, an open subset $\omega^\varepsilon$ of $\mathbf{R}^2$ such that $\omega^\varepsilon \setminus \overline{\omega} = \{x' \in \mathbf{R}^2 \mid 0 < d(x', \partial\omega) < \varepsilon\}$ and $\varphi^\varepsilon \in \mathbf{C}^1(\mathbf{R}^2)$ with $0 \leq \varphi^\varepsilon \leq 1$ such that

$$\left\{ \begin{array}{lcl} \varphi^\varepsilon & = & 1 \quad \text{in } \omega, \\ \varphi^\varepsilon & = & 0 \quad \text{on } \partial\omega_\varepsilon. \end{array} \right.$$

We define the function $w^{1\varepsilon}$ through

$$\left\{ \begin{array}{lcl} (w^{1\varepsilon})_1(x) & = & \dfrac{x_3}{\varepsilon h(x')} \varphi^\varepsilon(x'), \\ (w^{1\varepsilon})_2(x) & = & 0, \\ (w^{1\varepsilon})_3(x) & = & \dfrac{\varepsilon}{2}\left(\dfrac{\partial h}{\partial x_1}(x')\varphi^\varepsilon(x') - \dfrac{\partial \varphi^\varepsilon}{\partial x_1}(x') h(x')\right) \\ & & \quad + \dfrac{(x_3)^2}{2}\left(\dfrac{1}{\varepsilon h(x')}\dfrac{\partial \varphi^\varepsilon}{\partial x_1}(x') - \dfrac{\varphi^\varepsilon(x')}{\varepsilon h^2(x')}\dfrac{\partial h}{\partial x_1}(x')\right). \end{array} \right.$$

One has $\operatorname{div}(w^{1\varepsilon}) = 0$, $\forall \varepsilon > 0$, and $w^{1\varepsilon} = e^1$ on $\{x_3 = -\varepsilon h(x')\} \cap (\omega \times (-\infty, 0))$. We now consider the problem

$$\left\{ \begin{array}{rcll} -\Delta\zeta^{1\varepsilon} + \nabla\varpi^{1\varepsilon} & = & e^1 & \text{in } \Omega, \\ \operatorname{div}(\zeta^{1\varepsilon}) & = & 0 & \text{in } \Omega, \\ \zeta^{1\varepsilon} & = & 0 & \text{on } \Gamma_1, \\ \zeta^{1\varepsilon} & = & \left(0, 0, \dfrac{1}{2}\left(\dfrac{\partial h}{\partial x_1}(x')\varphi^\varepsilon(x') - \dfrac{\partial \varphi_\varepsilon}{\partial x_1}(x') h(x')\right)\right) & \text{on } \Gamma_2. \end{array} \right. \tag{25}$$

The problem (25) has a unique solution $(\zeta^{1\varepsilon}, \varpi^{1\varepsilon}) \in \mathbf{H}^1_{\Gamma_1}(\Omega, \operatorname{div}) \times \mathbf{L}^2(\Omega)/\mathbf{R}$, satisfying

$$\int_\Omega |\nabla\zeta^{1\varepsilon}|^2 dx \leq C\ ;\ \int_\Omega |\zeta^{1\varepsilon}|^2 dx \leq C,$$

where $C$ is a constant independent of $\varepsilon$. Let $H > z_h$, with $z_h = \max_{\Gamma_2} h$. We define the function $\widetilde{w}^{1\varepsilon}$ in $D_\varepsilon = \{x \mid -H < x_3 < -\varepsilon h(x')\}$ through

$$\begin{cases} \left(\widetilde{w}^{1\varepsilon}\right)_1(x) &= \dfrac{x_3 + H}{\varepsilon (H - h(x'))}\varphi^\varepsilon(x'), \\ \left(\widetilde{w}^{1\varepsilon}\right)_2(x) &= 0, \\ \left(\widetilde{w}^{1\varepsilon}\right)_3(x) &= \dfrac{\varepsilon}{2}\left(\dfrac{\partial h}{\partial x_1}(x')\varphi^\varepsilon(x') - \dfrac{\partial \varphi^\varepsilon}{\partial x_1}(x')(H - h(x'))\right) \\ & \qquad -\dfrac{(x_3+H)^2}{2}\left(\begin{array}{c} \dfrac{1}{\varepsilon(H-h(x'))}\dfrac{\partial\varphi^\varepsilon}{\partial x_1}(x') \\ +\dfrac{\varphi^\varepsilon(x')}{\varepsilon(H-h(x'))^2}\dfrac{\partial h}{\partial x_1}(x') \end{array}\right). \end{cases}$$

We consider the bounded, smooth and open subset $\Omega_H = \{x \mid x_3 > -H\}$ and $\partial\Omega_H \cap \{x \mid x_3 = -H\} = \Gamma_2$, and the solution $\left(\zeta_H^{1\varepsilon}, \omega_H^{1\varepsilon}\right)$ of the problem

$$\begin{cases} -\Delta\zeta_H^{1\varepsilon} + \nabla\varpi_H^{1\varepsilon} &= e^1 & \text{in } \Omega_H, \\ \operatorname{div}\left(\zeta_H^{1\varepsilon}\right) &= 0 & \text{in } \Omega_H, \\ \zeta_H^{1\varepsilon} &= 0 & \text{on } \Omega_H \setminus \Gamma_2, \\ \zeta_H^{1\varepsilon} &= \left(0, 0, \dfrac{1}{2}\left(\begin{array}{c} \dfrac{\partial h}{\partial x_1}(x')\varphi^\varepsilon(x') \\ -\dfrac{\partial\varphi^\varepsilon}{\partial x_1}(x')(H - h(x')) \end{array}\right)\right) & \text{on } \Gamma_2. \end{cases}$$

Let us define the function $z_0^{1,\varepsilon}$ through

$$z_\varepsilon^{0,1} = \begin{cases} \varepsilon\zeta^{1\varepsilon} & \text{in } \Omega, \\ w^{1\varepsilon} & \text{in } \Sigma_\varepsilon, \\ \widetilde{w}^{1\varepsilon} & \text{in } D_\varepsilon, \\ \varepsilon\zeta_H^{1\varepsilon} & \text{in } \Omega_H. \end{cases}$$

One immediately verifies that $z_\varepsilon^{0,1} \in \mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^3, \operatorname{div}\right)$, $z_\varepsilon^{0,1} = e^1$ on the surface $\{x_3 = -\varepsilon h(x')\} \cap (\omega \times (-\infty, 0))$, $\left(z_\varepsilon^{0,1}\right)_\varepsilon$ converges to 0 in the strong topology of $\mathbf{L}^2\left(\mathbf{R}^3, \mathbf{R}^3\right)$ and

$$\lim_{\varepsilon\to 0}\Phi^\varepsilon\left(z_\varepsilon^{0,1}\right) = \lim_{\varepsilon\to 0}\nu\varepsilon\int_{\omega^\varepsilon\times(-\varepsilon h(x'),0)}\left|\nabla z_\varepsilon^{0,1}\right|^2 dx = \nu\int_\omega\frac{dx'}{h(x')}.$$

One thus deduces from (24) within this context

$$\mu_{11}(\omega) \le \nu\int_\omega\frac{dx'}{h(x')}.$$

Furthermore, taking $(z_\varepsilon)_\varepsilon \subset \mathbf{H}^1_{\Gamma_1}\left(\mathbf{R}^3, \operatorname{div}\right)$, $z_\varepsilon = e^1$ on $\{x_3 = -\varepsilon h(x')\} \cap (\omega \times (-\infty, 0))$, $(z_\varepsilon)_\varepsilon$ converges to 0 in the topology $\tau$, and using the subdifferential inequality

$$\begin{aligned} \Phi^\varepsilon(z_\varepsilon) &\ge \Phi^\varepsilon\left(z_\varepsilon^{0,1}\right) \\ &+\nu\varepsilon\int_{\Sigma_\varepsilon}\nabla z_\varepsilon^{0,1}\cdot\nabla\left(z_\varepsilon - z_\varepsilon^{0,1}\right)dx + \nu\int_\Omega\nabla z_\varepsilon^{0,1}\cdot\nabla\left(z_\varepsilon - z_\varepsilon^{0,1}\right)dx, \end{aligned}$$

we prove that $\mu_{11}(\omega) \ge \nu\int_\omega dx'/h(x')$. This implies the equality: $\mu_{11}(\omega) = \nu\int_\omega dx'/h(x')$ and, since this equality is true for every $\omega \in \mathcal{R}_{\Gamma_2} \cap \mathcal{O}(\Gamma_2)$, we obtain $\mu_{11} = \nu dx'/h(x')$.

Choosing now $u = -e^2$ on $\Sigma_\varepsilon$, we can build a test-function $z_\varepsilon^{0,2}$ in a similar way and prove: $\mu_{22} = \nu dx'/h(x')$.

Finally, taking $u=-\left(e^1+e^2\right)$ on $\Sigma_\varepsilon$, we consider the sequence $\left(z_\varepsilon^0\right)_\varepsilon$ defined through: $z_\varepsilon^0=z_\varepsilon^{0,1}+z_\varepsilon^{0,2}$. One deduces from the above computations that

$$\lim_{\varepsilon\to 0}\Phi^\varepsilon\left(z_\varepsilon^0\right)=\lim_{\varepsilon\to 0}\Phi^\varepsilon\left(z_\varepsilon^{0,1}+z_\varepsilon^{0,2}\right)=2\nu\int_\omega\frac{dx'}{h\left(x'\right)}$$

and, as in the periodic case, that $\mu_{12}=0$. The boundary conditions on $\Gamma_2$ can thus be written as

$$\left\{\begin{array}{rcl}\left(u^0\right)_3&=&0,\\ \dfrac{\partial\left(u^0\right)_m}{\partial x_3}&=&\dfrac{1}{h}\left(u^0\right)_m,\ m=1,2,\end{array}\right.$$

which ends the proof. ■

**Remark 18** *In a general way, if* $\Sigma_\varepsilon=\{\sigma+tn\mid\sigma\in\Gamma_2,\ -\varepsilon h(\sigma)<t<0\}$, *with* $h$ *positive and Lipschitz continuous on* $\Gamma_2$, *we can prove that the limit law is*

$$\left\{\begin{array}{rcl}\left(Id-n\otimes n\right)\dfrac{\partial u^0}{\partial n}+\dfrac{u^0}{h}&=&0,\\ u^0\cdot n&=&0.\end{array}\right.$$

# 6 Optimal control problem

For a given real $m>0$, we consider the set $\Xi_m$ of all matrices $\mathbf{h}=Diag\left(h_i\right)_{i=1,..,N}$ of functions $h_i:\Gamma_2\to[0,+\infty]$, $d\Gamma_2$-measurable and such that

$$\int_{\Gamma_2}h_i d\Gamma_2=m,\ \forall i=1,\ldots,N.$$

We suppose that $\partial\Omega$ is $\mathbf{C}^2$ and consider the Navier-Stokes problem, with Navier wall law, according to Theorem 17

$$\left\{\begin{array}{rcll}-\nu\Delta u^h+\left(u^h\cdot\nabla\right)u^h+\nabla p^h&=&f&\text{in }\Omega,\\ \operatorname{div}\left(u^h\right)&=&0&\text{in }\Omega,\\ \mathbf{h}\left(Id-n\otimes n\right)\dfrac{\partial u^h}{\partial n}+u^h&=&0&\text{on }\Gamma_2,\\ u^h\cdot n&=&0&\text{on }\Gamma_2,\\ u^h&=&0&\text{on }\Gamma_1,\end{array}\right.\tag{26}$$

which has a unique solution $\left(u^h,p^h\right)\in\mathbf{V}_{0,\Gamma_1}(\Omega)\times\mathbf{L}^2(\Omega)/\mathbf{R}$. We define the functional $\mathbf{F}$ defined on $\Xi_m\times\mathbf{H}^1_{\Gamma_1}(\Omega,\operatorname{div})$ and associated to (26) through

$$\mathbf{F}(\mathbf{h},u)=\left\{\begin{array}{ll}\dfrac{\nu}{2}\displaystyle\int_\Omega|\nabla u|^2dx+\frac{1}{2}\sum_{i=1}^N\int_{\Gamma_2}\frac{(u_i)^2}{h_i}d\Gamma_2 & \\ \qquad+\displaystyle\int_\Omega\left(u^h\cdot\nabla\right)u^h\cdot u dx-\int_\Omega f\cdot u dx & \text{if }u\in\mathbf{V}_{0,\Gamma_1}(\Omega),\\ +\infty & \text{otherwise.}\end{array}\right.$$

We consider the optimal control problem (3), which means that the cost functional is here taken as the global energy. We observe that

$$\mathbf{F}\left(\mathbf{h},u^h\right)=-\int_\Omega f\cdot u^h dx.$$

This implies that the minimization of $\mathbf{F}$, with respect to $u$ on the set $\mathbf{V}_{0,\Gamma_1}(\Omega)$, is equivalent to the maximization of the work of the external forces on this set. The problem (3) has a unique minimizer when Poincaré's inequality

$$\left(\int_{\Gamma_2}|u_i|\,d\Gamma_2\right)^2\leq\int_{\Gamma_2}h_i d\Gamma_2\int_{\Gamma_2}\frac{(u_i)^2}{h_i}d\Gamma_2,$$

becomes an equality, for every $i = 1, \ldots, N$, that is when

$$h_i^m = m \frac{|u_i^m|_{\Gamma_2}}{\int_{\Gamma_2} |u_i^m| \, d\Gamma_2},$$

where $(u^m, p^m)$ is the solution of

$$\left\{ \begin{array}{rcll} -\nu\Delta u^m + (u^m \cdot \nabla) u^m + \nabla p^m & = & f & \text{in } \Omega, \\ \operatorname{div}(u^m) & = & 0 & \text{in } \Omega, \\ u^m \cdot n & = & 0 & \text{on } \Gamma_2, \\ u^m & = & 0 & \text{on } \Gamma_1, \\ (Id - n \otimes n) \dfrac{\partial u^m}{\partial n} & & & \\ +\dfrac{1}{m} \begin{pmatrix} sign\left((u^m)_1(x)\right) \displaystyle\int_{\Gamma_2} |(u^m)_1| \, d\Gamma_2 \\ \vdots \\ sign\left((u^m)_N(x)\right) \displaystyle\int_{\Gamma_2} |(u^m)_N| \, d\Gamma_2 \end{pmatrix} & = & 0 & \text{on } \Gamma_2. \end{array} \right.$$

Trivially, the study of the $\Gamma$-convergence of the sequence of the energies associated to (3), when $m$ goes to 0 and relatively to the weak topology of $\mathbf{H}^1\left(\Omega, \mathbf{R}^N\right)$, will lead to the following conclusions: $(u^m)_m$ converges to $u^0$ in the weak topology of $\mathbf{H}^1\left(\Omega, \mathbf{R}^N\right)$, $(p^m)_m$ converges to $p^0$ in the strong topology of $\mathbf{L}^2(\Omega)/\mathbf{R}$, where $\left(u^0, p^0\right)$ is the solution of the problem

$$\left\{ \begin{array}{rcll} -\nu\Delta u^0 + \left(u^0 \cdot \nabla\right) u^0 + \nabla p^0 & = & f & \text{in } \Omega, \\ \operatorname{div}\left(u^0\right) & = & 0 & \text{in } \Omega, \\ u^0 & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{27}$$

In order to study the asymptotic behavior of $\left((u^m/m)_{|\Gamma_2}\right)_m$, we introduce the following linearized perturbation of the Navier-Stokes problem (27)

$$\left\{ \begin{array}{rcll} -\nu\Delta u^{0,m} + \nabla p^{0,m} & = & f - (u^m \cdot \nabla) u^m & \text{in } \Omega, \\ \operatorname{div}\left(u^{0,m}\right) & = & 0 & \text{in } \Omega, \\ u^{0,m} & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{28}$$

The problem (28) is a Stokes system, the source term of which is $f - (u^m \cdot \nabla) u^m$. Consider now the functional $I_m$ defined on $\mathbf{V}_{0,\Gamma_1}(\Omega)$ through

$$\begin{array}{rcl} I_m(v) & = & \dfrac{m\nu}{2} \displaystyle\int_\Omega |\nabla v|^2 \, dx + \frac{1}{2} \sum_{i=1}^N \left( \int_{\Gamma_2} |v_i| \, d\Gamma_2 \right)^2 \\ & & + \displaystyle\int_{\Gamma_2} (Id - n \otimes n) \frac{\partial u^{0,m}}{\partial n} \cdot v d\Gamma_2. \end{array}$$

$I_m$ has a unique minimizer $(v^m, q^m) \in \mathbf{V}_{0,\Gamma_1}(\Omega) \times \mathbf{L}^2(\Omega)/\mathbf{R}$ which is the solution of the problem

$$\left\{\begin{array}{rcll} -\nu m \Delta v^m + \nabla q^m & = & 0 & \text{in } \Omega, \\ \operatorname{div}(v^m) & = & 0 & \text{in } \Omega, \\ v^m \cdot n & = & 0 & \text{on } \Gamma_2, \\ v^m & = & 0 & \text{on } \Gamma_1, \\ (Id - n\otimes n)\dfrac{\partial u^{0,m}}{\partial n} + m\,(Id - n\otimes n)\dfrac{\partial v^m}{\partial n} & & & \\ + \begin{pmatrix} sign\left((v^m)_1\right)\displaystyle\int_{\Gamma_2} |(v^m)_1|\, d\Gamma_2 \\ \vdots \\ sign\left((v^m)_N\right)\displaystyle\int_{\Gamma_2} |(v^m)_N|\, d\Gamma_2 \end{pmatrix} & = & 0 & \text{on } \Gamma_2. \end{array}\right.$$

We observe that the couple $(v^m, q^m)$ defined through

$$v^m = \frac{u^m - u^{0,m}}{m} \;;\; q^m = p^m - p^{0,m},$$

is the minimizer of $I_m$. For every $\varphi \in \mathbf{H}^{1/2}\left(\Gamma_2, \mathbf{R}^N\right)$, there exists a unique extension $v_\varphi \in \mathbf{V}_{0,\Gamma_1}(\Omega)$ of $\varphi$ defined through

$$\int_\Omega |\nabla v_\varphi|^2\, dx = \inf_{\left\{w \in \mathbf{V}_{0,\Gamma_1}(\Omega) | w|_{\Gamma_2} = \varphi\right\}} \int_\Omega |\nabla w|^2\, dx.$$

Let us denote $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$ the space of finite Radon measures on $\Gamma_2$ with values in $\mathbf{R}^N$. We consider the functional $J_m$ defined on $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$ through

$$J_m(\varphi) = \left\{\begin{array}{ll} \dfrac{m\nu}{2}\displaystyle\int_\Omega |\nabla v_\varphi|^2\, dx + \frac{1}{2}\sum_{i=1}^N \left(\int_{\Gamma_2} |\varphi_i|\, d\Gamma_2\right)^2 & \\ \quad + \displaystyle\int_{\Gamma_2} (Id - n\otimes n)\frac{\partial u^{0,m}}{\partial n}\cdot \varphi d\Gamma_2 & \text{if } \varphi \in \mathbf{H}^{1/2}\left(\Gamma_2, \mathbf{R}^N\right) \\ & \text{and } \varphi\cdot n = 0 \text{ on } \Gamma_2, \\ +\infty & \text{otherwise.} \end{array}\right.$$

Then $(v^m)_{|\Gamma_2}$ is the unique minimizer of $J_m$.

**Proposition 19** *One has the following properties.*

*1.* $\sup_m \sum_{i=1}^n \left(\int_{\Gamma_2} |v_i^m|\, d\Gamma_2\right) < +\infty.$

*2. The sequence* $(J_m)_m$ $\Gamma$*-converges, when* $m$ *tends to* $0$ *and with respect to the weak* topology of* $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$*, to the functional* $J$ *defined from* $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$ *to* $\mathbf{R}$ *through*

$$J(\lambda) = \sum_{i=1}^N \left(|\lambda_i|(\Gamma_2)\right)^2 + \int_{\Gamma_2} (Id - n\otimes n)\frac{\partial u^0}{\partial n} d\lambda,$$

*where* $|\lambda_i|(\Gamma_2)$ *is the total variation of* $\lambda_i$ *on* $\Gamma_2$*.*

**Proof.** 1. Remark that a regularity property of the boundary $\partial\Omega$ implies that

$$\sup_m \left\| (Id - n\otimes n)\frac{\partial u^{0,m}}{\partial n} \right\|_{\mathbf{L}^\infty(\Gamma_2, \mathbf{R}^N)} < +\infty.$$

One thus obtains

$$J_m\left((v^m)_{|\Gamma_2}\right) \geq \frac{1}{2}\sum_{i=1}^{N}\left(\int_{\Gamma_2}|v_i^m|\,d\Gamma_2\right)^2 - \frac{C}{2}\sum_{i=1}^{N}\left(\int_{\Gamma_2}|v_i^m|\,d\Gamma_2\right).$$

Moreover

$$\sup_m J_m\left((v^m)_{|\Gamma_2}\right) \leq \sup_m J_m(0) = 0 \Rightarrow \sup_m \sum_{i=1}^{N}\left(\int_{\Gamma_2}|v_i^m|\,d\Gamma_2\right) \leq C.$$

This implies the existence of a subsequence of $\left((v^m)_{|\Gamma_2}\right)_m$, still denoted $\left((v^m)_{|\Gamma_2}\right)_m$, which converges to some $\lambda$ in the weak* topology of $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$.

2. Choose any sequence $(\varphi^m)_m \subset \mathbf{H}^{1/2}\left(\Gamma_2, \mathbf{R}^N\right)$, satisfying $\varphi^m \cdot n = 0$, on $\Gamma_2$ and converging to $\lambda$ in the weak* topology of $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$. The functional $\mu \mapsto |\mu|$, where $|\mu|$ is the total variation of $\mu$, being lower semi-continuous on $\mathcal{M}(\Gamma_2)$, one has

$$\liminf_{m\to 0}\int_{\Gamma_2}|\varphi_i^m|\,d\Gamma_2 \geq |\lambda_i|(\Gamma_2).$$

Thanks to the regularity of the boundary, $\left((Id - n\otimes n)\frac{\partial u^{0,m}}{\partial n}\right)_m$ uniformly converges to $(Id - n\otimes n)\frac{\partial u^0}{\partial n}$, hence

$$\liminf_{m\to 0}\int_{\Gamma_2}(Id - n\otimes n)\frac{\partial u^{0,m}}{\partial n}\cdot\varphi^m d\Gamma_2 \geq \int_{\Gamma_2}\left((Id - n\otimes n)\frac{\partial u^0}{\partial n}\right)_i d\lambda_i.$$

This implies

$$\liminf_{m\to 0} J_m(\varphi^m) \geq J(\lambda). \tag{29}$$

In order to prove the $\Gamma$-lim sup property, let us suppose that $\Omega \subset \{x_N < 0\}$ and $\partial\Omega \cap \{x_N = 0\} = \Gamma_2$ (in fact using a system of local coordinates, one can then study the case of every smooth surface $\Gamma_2$). We define $x' = (x_1, \ldots, x_{N-1})$ and the nonnegative and smooth function $\rho_\varepsilon$ through

$$\rho_\varepsilon(x') = \begin{cases} \dfrac{C}{\varepsilon^{N-1}}\exp\left(-\dfrac{\varepsilon^2}{\varepsilon^2 - |x'|^2}\right) & \text{if } |x'| < \varepsilon, \\ 0 & \text{if } |x'| \geq \varepsilon, \end{cases}$$

where

$$C = \left(\int_{B_{N-1}(0,1)}\exp\left(\frac{-1}{1-|\zeta|^2}\right)d\zeta\right)^{-1}.$$

Let $\left(\omega_{[1/\varepsilon]}\right)_\varepsilon$, where $[1/\varepsilon]$ denotes the entire part of $1/\varepsilon$, be a sequence of open subsets of $\Gamma_2$ such that

$$\begin{cases} \omega_1 \subset \omega_2 \subset \ldots \subset \omega_{[1/\varepsilon]} \subset \ldots \subset \Gamma_2, \\ \bigcup_\varepsilon \omega_{[1/\varepsilon]} = \Gamma_2, \\ d\left(\omega_{[1/\varepsilon]}, \partial\Gamma_2\right) = \varepsilon. \end{cases}$$

We associate the partition of unity $(\eta_\varepsilon)_\varepsilon$ through

$$\begin{cases} \eta_\varepsilon \in \mathbf{C}_c^\infty\left(\omega_{[1/\varepsilon]}\right), \\ \eta_\varepsilon(x') = 1 \text{ in } \omega_{[1/\varepsilon]-1} \ \left([1/\varepsilon] - 1 = [1/\varepsilon'], \text{ with } \varepsilon' = \dfrac{\varepsilon}{1-\varepsilon}\right), \\ 0 \leq \eta_\varepsilon(x') \leq 1, \forall x' \in \Gamma_2, \forall \varepsilon > 0. \end{cases}$$

For $\lambda = (\lambda_1, \ldots, \lambda_{N-1}, 0) \in \mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$, we define the vectorial measure $\lambda^\varepsilon$ through $\lambda^\varepsilon = (\lambda * \rho_\varepsilon)\, \eta_\varepsilon$. We observe that $\lambda^\varepsilon \in \mathbf{C}_c^\infty\left(\Gamma_2, \mathbf{R}^N\right)$ and

$$\begin{cases} \lambda^\varepsilon & \underset{\varepsilon \to 0}{\rightharpoonup} & \lambda & w^*\text{-}\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right), \\ \left|\nabla \lambda^\varepsilon\right|(x') & \leq & \dfrac{C}{\varepsilon^N} & \forall x' \in \Gamma_2. \end{cases}$$

We build the function $w^\varepsilon$

$$\begin{cases} \left(w^\varepsilon\right)_i(x) & = & \dfrac{\varepsilon - x_N}{\varepsilon}\left(\lambda^\varepsilon\right)_i(x') & i = 1, \ldots, N-1,\ \forall x \in \Omega, \\ \left(w^\varepsilon\right)_N(x) & = & \dfrac{\operatorname{div}\left(\lambda^\varepsilon(x')\right)}{2}\left(\dfrac{\left(\varepsilon - x_N\right)^2}{\varepsilon} - \varepsilon\right). \end{cases}$$

We immediately observe that $w^\varepsilon \in \mathbf{H}^1\left(\Omega, \mathbf{R}^N\right)$ and

$$\begin{cases} \operatorname{div}\left(w^\varepsilon\right) & = & 0 & \text{in } \Omega, \\ \left(w^\varepsilon\right)_N & = & 0 & \text{on } \Gamma_2, \\ w^\varepsilon & = & 0 & \text{on } \Gamma_1, \end{cases}$$

that is $w^\varepsilon \in \mathbf{V}_{0,\Gamma_1}(\Omega)$, for every $\varepsilon > 0$. We now define

$$\begin{cases} \varepsilon & = & m^{\frac{1}{4N}}, \\ w^m & = & w^{m^{\frac{1}{4N}}}, \\ \lambda^m & = & \lambda^{m^{\frac{1}{4N}}}. \end{cases}$$

One has

$$\begin{cases} m\displaystyle\int_\Omega \left|\nabla w^m\right|^2 dx & \leq & C\sqrt{m}, \\ J_m\left(\lambda^m\right) & = & I_m\left(v_{\lambda^m}\right) \leq I_m\left(w^m\right), \end{cases}$$

hence

$$\limsup_{m \to 0} J_m\left(\lambda^m\right) \leq \limsup_{m \to 0} I_m\left(w^m\right) = J(\lambda).$$

This inequality and (29) end the proof. ∎

One has the following result.

**Theorem 20** *Let*

$$\begin{aligned} M_i &= \max_{\sigma \in \Gamma_2}\left|\left((Id - n \otimes n)\frac{\partial u^0}{\partial n}\right)_i(\sigma)\right|, \\ K_i^\pm &= \left\{\sigma \in \Gamma_2 \mid \left((Id - n \otimes n)\frac{\partial u^0}{\partial n}\right)_i(\sigma) = \pm M_i\right\}. \end{aligned}$$

*We have the following properties.*

1. *When $m$ goes to $0$, the sequence $\left(\left(u^m/m\right)_{|\Gamma_2}\right)_m$ converges in the weak* topology of the space $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$ to a vectorial measure $\lambda = (\lambda_i)_{i=1,\ldots,N}$ such that $supp\,(\lambda_i) \subseteq K_i^+ \cup K_i^-$, with $\lambda_i$ positive on $K_i^-$ and negative on $K_i^+$, $i = 1, \ldots, N$.*

2. *$\int_{\Gamma_2}\left((Id - n \otimes n)\frac{\partial u^0}{\partial n}\right)_i d\lambda_i = -M_i$, $i = 1, \ldots, N$.*

3. *$\lim_{m \to 0}\int_{\Gamma_2}\left|u_i^m/m\right| d\Gamma_2 = \left|\lambda_i\right|(\Gamma_2) = M_i$, $i = 1, \ldots, N$.*

4. *When $m$ goes to $0$, the sequence $\left(h_i^m/m\right)_m$ converges in the weak* topology of $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$ to a measure $\overline{\lambda}_i$ such that supp $\left(\overline{\lambda}_i\right) \subseteq K_i^+ \cup K_i^-$, $\overline{\lambda}_i$ is positive on $K_i^-$ and negative on $K_i^+$, and $\left|\overline{\lambda}_i\right|(\Gamma_2) = 1$, $i = 1, \ldots, N$.*

**Proof.** One deduces from Proposition 19 and from the properties of the Γ-convergence that $\left((v^m)_{|\Gamma_2}\right)_m = \left((u^m/m)_{|\Gamma_2}\right)_m$ converges in the weak* topology of $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$, when $m$ goes to 0, to a measure $\lambda = (\lambda_i)_{i=1,\ldots,N}$ such that $J(\lambda) = \min_{v \in \mathcal{M}(\Gamma_2, \mathbf{R}^N)} J(v)$. Define

$$\mathcal{M}_1\left(\Gamma_2, \mathbf{R}^N\right) = \left\{\mu \in \mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right) \mid |\mu_i|(\Gamma_2) = 1, \, i = 1, \ldots, N\right\}$$

and consider the functional $\widetilde{J}$ defined from $[0, +\infty[^N \times \mathcal{M}_1\left(\Gamma_2, \mathbf{R}^N\right)$ to $\mathbf{R}$ through

$$\begin{aligned} &\widetilde{J}\left((t_1, \ldots, t_N), (\mu_1, \ldots, \mu_N)\right) \\ &\quad = J\left((t_1\mu_1, \ldots, t_N\mu_N)\right) \\ &\quad = \frac{1}{2}\sum_{i=1}^{N}(t_i)^2 + \sum_{i=1}^{N} t_i \int_{\Gamma_2} \left((Id - n \otimes n)\frac{\partial u^0}{\partial n}\right)_i d\mu_i. \end{aligned}$$

One has

$$\min_{v \in \mathcal{M}(\Gamma_2, \mathbf{R}^N)} J(v) = \min_{\mu \in \mathcal{M}_1(\Gamma_2, \mathbf{R}^N)} \min_{\substack{t_i \geq 0 \\ i=1,\ldots,N}} \widetilde{J}\left((t_1, \ldots, t_N), (\mu_1, \ldots, \mu_N)\right). \tag{30}$$

The minimum of (30) with respect to $t = (t_1, \ldots, t_N)$ exists if

$$\int_{\Gamma_2} \left((Id - n \otimes n)\frac{\partial u^0}{\partial n}\right)_i d\mu_i \leq 0, \, \forall i = 1, \ldots, N.$$

Let us now find the minimum with respect to $\mu \in \mathcal{M}_1\left(\Gamma_2, \mathbf{R}^N\right)$. One has

$$-\int_{\Gamma_2} \left((Id - n \otimes n)\frac{\partial u^0}{\partial n}\right)_i d\mu_i \geq -M_i,$$

for every $\mu \in \mathcal{M}_1\left(\Gamma_2, \mathbf{R}^N\right)$ such that

$$\int_{\Gamma_2} \left((Id - n \otimes n)\frac{\partial u^0}{\partial n}\right)_i d\mu_i \leq 0, \, \forall i = 1, \ldots, N,$$

the minimum being reached in the case of equality, that is if and only if *supp* $(\mu_i) \subset K_i^+ \cup K_i^-$. One has $\lambda_i = M_i\mu_i$, $i = 1, \ldots, N$. Remarking that $\overline{\lambda}_i = \mu_i$, one observes that $\left(h_i^m/m\right)_m$ converges in the weak* topology of $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$, when $m$ tends to 0, to $\overline{\lambda}_i$, and the same result occurs for the sequence $\left(\left(|u_i^m|_{\Gamma_2}\right) / \int_{\Gamma_2} |u_i^m| \, d\Gamma_2\right)_m$. The sequence $\left(h_i^m/m\right)_m$ converges in $\mathcal{M}\left(\Gamma_2, \mathbf{R}^N\right)$-weak* to a probability measure $\overline{\lambda}_i$ ($\overline{\lambda}_i(\Gamma_2) = 1$) with support in the set of points of $\Gamma_2$ where the shear motions, given through $(Id - n \otimes n)\frac{\partial u^0}{\partial n}$, are large for the limit flow described through (27). ■

**Remark 21** *We thus think that, inside this flow, a thin boundary layer of thinness $mh_i$ occurs in the $i$-th direction with a probability $\overline{\lambda}_i$ (for every $i$).*

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