Asymptotic analysis of pollution filtration through thin random fissures between two porous media*

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Abstract

We describe the asymptotic behavior of a filtration problem from a contaminated porous medium to a non-contaminated porous medium through thin vertical fissures of fixed height $h > 0$, of random thinness of order ε and which are ε-periodically distributed. We compute the limit velocity of the flow and the limit flux of pollutant at the interfaces between the two porous media and the intermediate one.

1 Introduction

We consider a porous medium which is contaminated by some pollutant and which communicates with another porous and non-contaminated medium through vertical fissures of height $h > 0$, of random thinness of order $\varepsilon > 0$, and which are periodically disposed. Each of these porous media is given a ε-periodic structure. Instead of considering a Stokes problem in each of these porous media, we assume that the velocity of the fluid is governed by a Darcy law with periodic permeability matrix.

The purpose of this work is to determine the influence of the fissures on the transport of the contaminant into the non-contaminated medium, computing the global flux of pollutant which penetrates in this non-contaminated medium and the asymptotic velocity of the fluid which flows through the fissures.

Let Ω be a bounded, smooth and open subset of $\mathbb{R}^3$, with boundary Γ, such that

$$\begin{cases} \Omega^+ & = & \Omega \cap \{x_3 > 0\} \neq \varnothing, \\ \Omega_h^- & = & \Omega \cap \{x_3 < -h\} \neq \varnothing. \end{cases}$$

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Let $\Sigma \times \{0\} = \partial\Omega \cap \{x_3 = 0\}$. Σ is a bounded and smooth subset of $\mathbb{R}^2$. We define

$$\begin{cases} \Gamma_0^+ &= \Sigma \times \{0\}, \\ \Gamma_h^- &= \Sigma \times \{-h\}, \\ Y_h &= \{x \in \Omega \mid (x_1, x_2) \in \Sigma,\ -h < x_3 < 0\}, \\ \Gamma^+ &= \partial\Omega^+ \backslash \Gamma_0^+, \\ \Gamma^- &= \partial\Omega_h^- \backslash \Gamma_h^-. \end{cases}$$

The domain Ω is thus equal to $\Omega^+ \cup \Gamma_0^+ \cup Y_h \cup \Gamma_h^- \cup \Omega_h^-$.

Let (Π, Υ, P) be some probability space and $(T(t))_{t\in\mathbb{R}}$ be a group of transformations on (Π, Υ), that is satisfying

$$\begin{cases} T(0) &= Id_\Pi, \\ T(t_1 + t_2) &= T(t_1) \circ T(t_2) \quad \forall t_1, t_2 \in \mathbb{R}, \\ P(T^{-1}(t) A) &= P(A) \quad \forall A \in \Upsilon,\ \forall t \in \mathbb{R}, \end{cases}$$

where Id_Π is the identity map on Π and the set $\{(t, \omega) \in \mathbb{R} \times \Pi \mid T(t)\omega \in A\}$ is $dt \times dP$ measurable, for every $A \in \Upsilon$. We suppose that T is ergodic (or metrically transitive), which means that every $A \in \Upsilon$ such that $T(t) A = A$, for every $t \in \mathbb{R}$, has a probability $P(A)$ equal to 0 or 1.

We introduce some random processes q and r defined on $\mathbb{R} \times \Pi$ and satisfying the following conditions:

1. $q(t, \omega)$ is a stationary random process, that is, for every positive integer n, every points $t_1, \cdots, t_n$, every t in $\mathbb{R}$, and every $B \in \mathcal{B}(\mathbb{R})$, one has

$$\begin{aligned} &P(\{\omega \mid q(t + t_1, \omega), \cdots, q(t + t_n, \omega) \in B\}) \\ &\quad = P(\{\omega \mid q(t_1, T(t)\omega), \cdots, q(t_n, T(t)\omega) \in B\}), \end{aligned}$$

   where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. Since T preserves the measure P, the above equality implies that the joint distribution of $\{q(t_1), \cdots, q(t_n)\}$ is the same as the joint distribution of $\{q(t_1 + t), \cdots, q(t_n + t)\}$, for every t in $\mathbb{R}$.

2. The derivatives $\frac{d^m q}{dt^m}$ and $\frac{d^m r}{dt^m}$ exist for $m = 1, 2, 3$ and there exist non-random constants c_1, c_2 and c_3 such that the following bounds hold true with probability 1

$$0 < c_1 \leq q(t, \omega) \leq c_2 < 1\ ;\ |r(t, \omega)| \leq 1\ ;\ \left|\frac{d^m q}{dt^m}\right|, \left|\frac{d^m r}{dt^m}\right| \leq c_3, \tag{1}$$

From the properties of T and q, we derive the ergodic theorem (see, for example, [10] and [18])

$$\forall n \in \mathbb{Z}^* : \langle q^n(0)\rangle = \lim_{\mathcal{T}\to\infty} \frac{1}{2\mathcal{T}} \int_{-\mathcal{T}}^{\mathcal{T}} q^n(t, \omega)\, dt, \tag{2}$$

almost surely, where the symbol $\langle . \rangle$ stands for the mathematical expectation with respect to the measure P.

Let $(\alpha_i(\omega))_{i\in\mathbb{Z}}$ and $(\beta_i(\omega))_{i\in\mathbb{Z}}$ be sequences of random variables satisfying

$$|\alpha_i(\omega)| \leq c_4 \ ; \ |\beta_i(\omega)| \leq c_4, \forall i \in \mathbb{Z}, \tag{3}$$

with probability 1, where c_4 is a non-random constant. We define, for every $i, j \in \mathbb{Z}$, the fissure $Y_{\varepsilon,ij}(\omega)$ as

$$Y_{\varepsilon,ij}(\omega) = \left\{ \begin{array}{l} x \in \mathbb{R}^3 \mid \varepsilon a_i^-\left(-\varepsilon^{-\theta}x_3\right) < x_1 - i\varepsilon < \varepsilon a_i^+\left(-\varepsilon^{-\theta}x_3\right), \\ \quad \varepsilon a_j^-\left(-\varepsilon^{-\theta}x_3\right) < x_2 - j\varepsilon < \varepsilon a_j^+\left(-\varepsilon^{-\theta}x_3\right), \ x_3 \in \left]-h, 0\right[\end{array} \right\},$$

with $0 < \varepsilon < 1$, $0 < \theta < 2/3$ and $a_i^\pm(z) = r(z + \beta_i(\omega), \omega) \pm q(z + \alpha_i(\omega), \omega)/2$. Let $I_\varepsilon(\omega) = \{(i,j) \in \mathbb{Z}^2 \mid Y_{\varepsilon,ij}(\omega) \subset Y_h\}$. We also define the sets

$$\left\{ \begin{array}{rclrcl} \Gamma^+_{0,\varepsilon,ij}(\omega) & = & \partial Y_{\varepsilon,ij}(\omega) \cap \Gamma^+_0, & \Gamma^+_{0,\varepsilon}(\omega) & = & \bigcup\limits_{(i,j)\in I_\varepsilon(\omega)} \Gamma^+_{0,\varepsilon,ij}(\omega), \\ \Gamma^-_{h,\varepsilon,ij}(\omega) & = & \partial Y_{\varepsilon,ij}(\omega) \cap \Gamma^-_h, & \Gamma^-_{h,\varepsilon}(\omega) & = & \bigcup\limits_{(i,j)\in I_\varepsilon(\omega)} \Gamma^-_{h,\varepsilon,ij}(\omega), \\ \Lambda_\varepsilon(\omega) & = & \partial Y_\varepsilon(\omega) \setminus \left(\Gamma^+_{0,\varepsilon} \cup \Gamma^-_{h,\varepsilon}\right)(\omega) & Y_\varepsilon(\omega) & = & \bigcup\limits_{(i,j)\in I_\varepsilon(\omega)} Y_{\varepsilon,ij}(\omega). \end{array} \right.$$

Let $Z = \left]-1/2, 1/2\right[^3$ be the unit cube of $\mathbb{R}^3$ and assume that it can be decomposed as $Z = Z^1 \cup S \cup Z^2$, where Z^1 and Z^2 are two disjoint, open and connected sets separated by the smooth surface S (Fig. 1).

Figure 1: A 2D view of the periodic structure of the porous media.

We assume that Ω^+ and Ω_h^- are two ε-periodic porous media which communicate through the fissures $Y_{\varepsilon,ij}(\omega)$.

Figure 2: The porous media and the fissures.

We set

$$\left\{\begin{array}{rclrcl} \Omega_f^{+,\varepsilon} & = & \Omega^+ \cap \left(\underset{k\in\mathbb{Z}^3}{\cup}\left(\varepsilon Z^1 + k\varepsilon\right)\right), & \Omega_{h,f}^{-,\varepsilon} & = & \Omega_h^- \cap \left(\underset{k\in\mathbb{Z}^3}{\cup}\left(\varepsilon Z^1 + k\varepsilon\right)\right), \\ \Omega_s^{+,\varepsilon} & = & \Omega^+ \cap \left(\underset{k\in\mathbb{Z}^3}{\cup}\left(\varepsilon Z^2 + k\varepsilon\right)\right), & \Omega_{h,s}^{-,\varepsilon} & = & \Omega_h^- \cap \left(\underset{k\in\mathbb{Z}^3}{\cup}\left(\varepsilon Z^2 + k\varepsilon\right)\right), \\ S_\varepsilon^+ & = & \Omega^+ \cap \left(\underset{k\in\mathbb{Z}^3}{\cup}\left(\varepsilon S + k\varepsilon\right)\right), & S_\varepsilon^- & = & \Omega_h^- \cap \left(\underset{k\in\mathbb{Z}^3}{\cup}\left(\varepsilon S + k\varepsilon\right)\right). \end{array}\right. \tag{4}$$

We suppose that $\Omega_f^{+,\varepsilon}$ (resp. $\Omega_{h,f}^{-,\varepsilon}$) is the portion of Ω^+ (resp. Ω_h^-) consisting of the pores which are filled in with some fluid and $\Omega_s^{+,\varepsilon}$ (resp. $\Omega_{h,s}^{-,\varepsilon}$) is the portion of Ω^+ (resp. Ω_h^-) consisting of the non-porous rocks. We suppose that

$$\left\{\begin{array}{rcl} \partial\Omega_f^{+,\varepsilon} \cap \partial Y_\varepsilon(\omega) & = & \Gamma_{0,\varepsilon}^+(\omega), \\ \partial\Omega_{h,f}^{-,\varepsilon} \cap \partial Y_\varepsilon(\omega) & = & \Gamma_{h,\varepsilon}^-(\omega). \end{array}\right.$$

We define the fluid part of the domain as

$$\Omega_f^\varepsilon(\omega) = \Omega_f^{+,\varepsilon} \cup \left(\Gamma_{0,\varepsilon}^+ \cup Y_\varepsilon \cup \Gamma_{h,\varepsilon}^-\right)(\omega) \cup \Omega_{h,f}^{-,\varepsilon}.$$

Let $f \in C(\Omega^+)$, $g^+ \in \mathbf{L}^2(\Omega^+;\mathbb{R}^3)$ and $g^- \in \mathbf{L}^2\left(\Omega_h^-;\mathbb{R}^3\right)$ be functions satisfying

$$supp\,(f) \subset \Omega^+ \text{ and } f \geq 0 \ \text{ in } \Omega^+ \ ; \ supp\left(g^+\right) \subset \Omega^+ \ ; \ supp\left(g^-\right) \subset \Omega_h^-.$$

We consider in $\Omega_f^\varepsilon(\omega)$ the reaction-diffusion problem with first-order reaction

$$\left\{\begin{array}{rcl} -D\Delta u_\varepsilon + v_\varepsilon \cdot \nabla u_\varepsilon + \mathcal{R} u_\varepsilon & = & f \quad \text{in } \Omega_f^\varepsilon(\omega), \\ u_\varepsilon & = & 0 \quad \text{on } \Gamma^+ \cup \Gamma^-, \\ \dfrac{\partial u_\varepsilon}{\partial n} & = & 0 \quad \text{on } S_\varepsilon^+ \cup S_\varepsilon^- \cup \Lambda_\varepsilon(\omega), \end{array}\right. \tag{5}$$

where u_ε is the concentration of the pollutant, $D = D_{mol}$ is the molecular diffusion coefficient, $\mathcal{R}$ is a nonnegative reaction coefficient, n is the unit outer normal and v_ε is the velocity of the fluid, which is the solution of the Darcy-Stokes problems

$$\left\{\begin{array}{rcl} \mu^+\left(K_\varepsilon^+\right)^{-1} v_{\varepsilon,d} - \nabla p_{\varepsilon,d} & = & g^+ \quad \text{in } \Omega_f^{+,\varepsilon}, \\ \mu^-\left(K_\varepsilon^-\right)^{-1} v_{\varepsilon,d} - \nabla p_{\varepsilon,d} & = & g^- \quad \text{in } \Omega_{h,f}^{-,\varepsilon}, \\ \operatorname{div}\left(v_{\varepsilon,d}\right) & = & 0 \quad \text{in } \Omega_f^{+,\varepsilon} \cup \Omega_{h,f}^{-,\varepsilon}, \\ v_{\varepsilon,d} \cdot n & = & 0 \quad \text{on } \partial\Omega_f^{+,\varepsilon} \cup \partial\Omega_{h,f}^{-,\varepsilon} \cup \Gamma, \\ -\mu\varepsilon^2 \Delta v_{\varepsilon,s} + \nabla p_{\varepsilon,s} & = & 0 \quad \text{in } Y_\varepsilon(\omega), \\ \operatorname{div}\left(v_{\varepsilon,s}\right) & = & 0 \quad \text{in } Y_\varepsilon(\omega), \\ v_{\varepsilon,s} & = & 0 \quad \text{on } \Lambda_\varepsilon(\omega), \end{array}\right. \tag{6}$$

with the following interface conditions

$$
\begin{cases}
(v_{\varepsilon,s})_3 = (v_{\varepsilon,d})_3 & \text{on } \Gamma^+_{0,\varepsilon}(\omega) \cup \Gamma^-_{h,\varepsilon}(\omega)\,, \\
\mu\varepsilon^2 \dfrac{\partial (v_{\varepsilon,s})_3}{\partial x_3}\,|_{x_3=0,-h} = p_{\varepsilon,d} - p_{\varepsilon,s} & \text{on } \Gamma^+_{0,\varepsilon}(\omega) \cup \Gamma^-_{h,\varepsilon}(\omega)\,, \\
\mu\varepsilon^2 \dfrac{\partial (v_{\varepsilon,s})_\tau}{\partial x_3} = -\gamma \left(K^+_\varepsilon\right)^{-1/2} (v_{\varepsilon,s})_\tau & \text{on } \Gamma^+_{0,\varepsilon}(\omega)\,, \\
\mu\varepsilon^2 \dfrac{\partial (v_{\varepsilon,s})_\tau}{\partial x_3} = \gamma \left(K^-_\varepsilon\right)^{-1/2} (v_{\varepsilon,s})_\tau & \text{on } \Gamma^-_{h,\varepsilon}(\omega)\,,
\end{cases}
\tag{7}
$$

where:

- $v_{\varepsilon,d}$ and $p_{\varepsilon,d}$ are respectively Darcy's velocity and pressure in $\Omega^{+,\varepsilon}_f$ and $\Omega^{-,\varepsilon}_{h,f}$,
- K^+_ε and K^-_ε are the absolute permeability matrices in $\Omega^{+,\varepsilon}_f$ and $\Omega^{-,\varepsilon}_{h,f}$ respectively,
- $v_{\varepsilon,s}$ and $p_{\varepsilon,s}$ are respectively the velocity and the pressure of the Stokes flow in the fissures,
- μ^+ (resp. μ^-, μ) is the viscosity coefficient in $\Omega^{+,\varepsilon}_f$ (resp. in $\Omega^{-,\varepsilon}_{h,f}$, $Y_\varepsilon(\omega)$),
- $(v_{\varepsilon,s})_\tau = \left((v_{\varepsilon,s})_1, (v_{\varepsilon,s})_2\right)$ is the tangential velocity.

We suppose that the 3×3 matrices K^+_ε and K^-_ε are defined through the Z-periodic construction: $K^+_\varepsilon = K^+(x/\varepsilon)$ and $K^-_\varepsilon = K^-(x/\varepsilon)$, where K^+ and K^- are bounded, symmetric and positive definite, and that $\mu^\pm$ and μ are positive constants. In the above interface conditions, $(7)_1$ means the continuity of the mass flux through the interfaces $\Gamma^+_{0,\varepsilon}(\omega)$ and $\Gamma^-_{h,\varepsilon}(\omega)$, $(7)_2$ represents the continuity of the normal stress through the corresponding interface, and the two last equalities of (7) represent the Beavers-Joseph-Saffman conditions on the tangential stress, with some nonnegative slippage coefficient γ (see [3], [6] and [15]).

We first describe the asymptotic behaviour of the solution v_ε of (6)-(7) using Γ-convergence methods (see [4] and [8], for the definition and the properties of this variational convergence). We prove that the asymptotic velocities $v^+_{0,d}$ (in Ω^+) and $v^-_{0,d}$ (in Ω^-_h) and the asymptotic pressures p^+_0 (in Ω^+) and p^-_0 (in Ω^-_h) are linked through the Darcy laws (25) in Ω^+ and Ω^-_h respectively. We also describe the asymptotic behaviour of the velocity $v_{\varepsilon,s}$ (see Corollary 10). We then describe the asymptotic behaviour of the solution u_ε of (5) using the energy method. We prove that the flux of pollutant through Γ^+_0 is given through $(39)_3$, while the flux through Γ^-_h is given through $(39)_5$.

Homogenization theory introduced in the few past decades (see for instance [7] and [12] and the references therein) gives powerful tools for the description of equivalent media which are heterogeneous at a microscopic level. The homogenization of transport problems of chemical products through porous media has been studied by many authors (see for example [11] and [13]). A model of random fissures has already been studied in [9] for a problem of radiophysics posed in $\mathbb{R}^+_2 \cup \left(\cup_{j\in\mathbb{Z}} Q^\varepsilon_j(\omega)\right) \cup \mathbb{R}^-_{2,h}$, where $Q^\varepsilon_j(\omega)$ is the j-th fissure and $\mathbb{R}^-_{2,h} = \{x \in \mathbb{R}^2 \mid x_2 < -h\}$, $\mathbb{R}^+_2 = \{x \in \mathbb{R}^2 \mid x_2 > 0\}$. We here adopt the shape of the fissures used in [9], that we extend to a 3D case, in order to model in a quite realistic way the constitution of the soil.

The paper is organized as follows. In the following section, we introduce and study the appropriate local coordinates inside the fissures. In the third section, we study the convergence of the velocities. The fourth part is concerned with the asymptotic analysis of the contaminant transport problem. We first deal with the case where $\mathcal{R} = 0$. We build the solutions of local problems in the neighborhood of the fissures in order to pass to the limit in the original problem and study their asymptotic properties. The fluxes across Γ_0^+ and Γ_h^- given in this section are those obtained in $(39)_{3,4}$, respectively, with $\mathcal{R} = 0$. In the last part of this section, we also consider the case of a dispersive contaminant ($\mathcal{R} > 0$) with random dispersion in the fissures (see Remark 17). We finally give the asymptotic behavior of the fully reaction-diffusion problem. We here again introduce the solutions of local problems.

2 Local coordinates in the fissures

In the fissure $Y_{\varepsilon,ij}(\omega)$, $(i,j) \in I_\varepsilon(\omega)$, we define, for a fixed event ω for which the conditions (1) and (3) are satisfied, $\xi_1 = x_1 - i\varepsilon$, $\xi_2 = x_2 - j\varepsilon$, $z = -x_3$, and introduce the curvilinear coordinates $t \in (0,h)$ and $y_1, y_2 \in (-\varepsilon/2, \varepsilon/2)$. Thus doing, the lateral boundary of the fissure coincides with the planes $y_1, y_2 = \pm\varepsilon/2$. These coordinates are described through

$$\left\{ \begin{array}{rcl} \Phi_{1,\varepsilon}(\xi_1,\xi_2,z,y_1,y_2,t) & = & \xi_1 - a_i^+\left(\varepsilon^{-\theta}z\right)\left(\dfrac{\varepsilon}{2}+y_1\right) - a_i^-\left(\varepsilon^{-\theta}z\right)\left(\dfrac{\varepsilon}{2}-y_1\right) = 0, \\ \Phi_{2,\varepsilon}(\xi_1,\xi_2,z,y_1,y_2,t) & = & \xi_2 - a_j^+\left(\varepsilon^{-\theta}z\right)\left(\dfrac{\varepsilon}{2}+y_2\right) - a_j^-\left(\varepsilon^{-\theta}z\right)\left(\dfrac{\varepsilon}{2}-y_2\right) = 0, \\ \Phi_{3,\varepsilon}(\xi_1,\xi_2,z,y_1,y_2,t) & = & z - \varepsilon^{\theta}\psi_\varepsilon\left(\dfrac{\xi_1}{\varepsilon},\dfrac{\xi_2}{\varepsilon},\varepsilon^{-\theta}t\right) = 0. \end{array} \right. \quad (8)$$

Defining $\zeta_1 = \xi_1/\varepsilon, \zeta_2 = \xi_2/\varepsilon$ and $\tau = \varepsilon^{-\theta}t$, the orthogonality conditions of the coordinates given through

$$\left\{ \begin{array}{rcl} \dfrac{\partial\Phi_{1,\varepsilon}}{\partial\zeta_1}\dfrac{\partial\Phi_{3,\varepsilon}}{\partial\zeta_1} + \dfrac{\partial\Phi_{1,\varepsilon}}{\partial z}\dfrac{\partial\Phi_{3,\varepsilon}}{\partial z} & = & 0, \\ \dfrac{\partial\Phi_{2,\varepsilon}}{\partial\zeta_2}\dfrac{\partial\Phi_{3,\varepsilon}}{\partial\zeta_2} + \dfrac{\partial\Phi_{2,\varepsilon}}{\partial z}\dfrac{\partial\Phi_{3,\varepsilon}}{\partial z} & = & 0 \end{array} \right.$$

and the condition $\psi_\varepsilon\left(0,0,\varepsilon^{-\theta}t\right) = \tau$ imply the following Cauchy system

$$\left\{\begin{array}{rcl}\dfrac{\partial\psi_\varepsilon}{\partial\zeta_1} & = & \varepsilon^{2(1-\theta)}\left(\begin{array}{c}\left(a_i^-(\psi_\varepsilon)\right)'\dfrac{\zeta_1-a_i^+(\psi_\varepsilon)}{a_i^+(\psi_\varepsilon)-a_i^-(\psi_\varepsilon)}\\ -\left(a_i^+(\psi_\varepsilon)\right)'\dfrac{\zeta_1-a_i^-(\psi_\varepsilon)}{a_i^+(\psi_\varepsilon)-a_i^-(\psi_\varepsilon)}\end{array}\right),\\ \dfrac{\partial\psi_\varepsilon}{\partial\zeta_2} & = & \varepsilon^{2(1-\theta)}\left(\begin{array}{c}\left(a_j^-(\psi_\varepsilon)\right)'\dfrac{\zeta_2-a_j^+(\psi_\varepsilon)}{a_j^+(\psi_\varepsilon)-a_j^-(\psi_\varepsilon)}\\ -\left(a_j^+(\psi_\varepsilon)\right)'\dfrac{\zeta_2-a_j^-(\psi_\varepsilon)}{a_j^+(\psi_\varepsilon)-a_j^-(\psi_\varepsilon)}\end{array}\right),\\ \psi_\varepsilon\left(0,0,\varepsilon^{-\theta}t\right) & = & \tau,\end{array}\right. \tag{9}$$

which has to be solved.

Lemma 1 *1. The system (9) has a unique solution* $\psi_\varepsilon(\zeta_1,\zeta_2,t)=\tau+\psi_\varepsilon^1(\zeta_1,\tau)+\psi_\varepsilon^2(\zeta_2,\tau)$*, with* $\psi_\varepsilon^1(0,\tau)=\psi_\varepsilon^2(0,\tau)=0$.

2. For every $k\in\mathbb{N}^*$*, every* ζ_1*,* $\zeta_2\in[-k,k]$*, and every* $\tau\in\mathbb{R}$*, one has, when* ε *is close to* 0

$$\left\{\begin{array}{rcl}\psi_\varepsilon(\zeta_1,\zeta_2,t) & = & \tau+O\left(\varepsilon^{2(1-\theta)}\right),\\ \dfrac{\partial\psi_\varepsilon}{\partial\tau} & = & 1+O\left(\varepsilon^{2(1-\theta)}\right),\\ \dfrac{\partial\psi_\varepsilon}{\partial\zeta_\alpha},\dfrac{\partial^2\psi_\varepsilon}{\partial\tau^2},\dfrac{\partial^2\psi_\varepsilon}{\partial\zeta_\alpha\partial\tau} & = & O\left(\varepsilon^{2(1-\theta)}\right)\qquad\alpha=1,2.\end{array}\right.$$

Proof. 1. Observe that $a_i^+(\psi_\varepsilon)-a_i^-(\psi_\varepsilon)=q(\psi_\varepsilon+\alpha_i(\omega),\omega)\geq c_1>0$ and $\left|\left(a_i^\pm(\psi_\varepsilon)\right)'\right|\leq c_3$, thanks to (1). One deduces that, for every $k\in\mathbb{N}^*$ and every ζ_1, $\zeta_2\in[-k,k]$, the functions

$$\begin{array}{rcl}(\zeta_1,\psi_\varepsilon) & \longmapsto & \left(a_i^-(\psi_\varepsilon)\right)'\dfrac{\zeta_1-a_i^+(\psi_\varepsilon)}{a_i^+(\psi_\varepsilon)-a_i^-(\psi_\varepsilon)}-\left(a_i^+(\psi_\varepsilon)\right)'\dfrac{\zeta_1-a_i^-(\psi_\varepsilon)}{a_i^+(\psi_\varepsilon)-a_i^-(\psi_\varepsilon)},\\ (\zeta_2,\psi_\varepsilon) & \longmapsto & \left(a_j^-(\psi_\varepsilon)\right)'\dfrac{\zeta_2-a_j^+(\psi_\varepsilon)}{a_j^+(\psi_\varepsilon)-a_j^-(\psi_\varepsilon)}-\left(a_j^+(\psi_\varepsilon)\right)'\dfrac{\zeta_2-a_j^-(\psi_\varepsilon)}{a_j^+(\psi_\varepsilon)-a_j^-(\psi_\varepsilon)},\end{array}$$

are bounded and Lipschitz continuous with respect to $(\zeta_1,\psi_\varepsilon)$ and $(\zeta_2,\psi_\varepsilon)$ respectively. The Cauchy problem (9) thus has a unique solution ψ_ε. The condition $\psi_\varepsilon(0,0,\tau)=\tau$ implies $\psi_\varepsilon(\zeta_1,\zeta_2,t)=\tau+\psi_\varepsilon^1(\zeta_1,\tau)+\psi_\varepsilon^2(\zeta_2,\tau)$, with $\psi_\varepsilon^1(0,\tau)=\psi_\varepsilon^2(0,\tau)=0$.

2. Thanks to the preceding point and to the hypothesis (1), the quantities ψ_ε^1, ψ_ε^2, $\frac{\partial^2\psi_\varepsilon}{\partial^2\tau}$, $\frac{\partial\psi_\varepsilon}{\partial\zeta_\alpha}$ and $\frac{\partial^2\psi_\varepsilon}{\partial\zeta_\alpha\partial\tau}$ are equal to some $O\left(\varepsilon^{2(1-\theta)}\right)$, $\alpha=1,2$. ■

Lemma 2 *The system (8) has a unique solution* $\left(\bar{\xi}_1(y_1,y_2,t),\bar{\xi}_2(y_1,y_2,t),\bar{z}(y_1,y_2,t)\right)$.

Proof. The Jacobian $\overline{\Delta}$ of the matrix $\frac{\partial(\Phi_{1,\varepsilon},\Phi_{2,\varepsilon},\Phi_{3,\varepsilon})}{\partial(\xi_1,\xi_2,z)}$ can be computed as

$$\begin{array}{l}\overline{\Delta}=1+\dfrac{\partial\psi_\varepsilon}{\partial\zeta_1}\left(\left(a_i^-\right)'\dfrac{\zeta_1-a_i^+}{a_i^+-a_i^-}-\left(a_i^+\right)'\dfrac{\zeta_1-a_i^-}{a_i^+-a_i^-}\right)\\ \qquad+\dfrac{\partial\psi_\varepsilon}{\partial\zeta_2}\left(\left(a_j^-\right)'\dfrac{\zeta_2-a_j^+}{a_j^+-a_j^-}-\left(a_j^+\right)'\dfrac{\zeta_2-a_j^-}{a_j^+-a_j^-}\right).\end{array}$$

Thanks to Lemma 1, one has $\overline{\Delta} = 1 + O\left(\varepsilon^{2(1-\theta)}\right)$, when ε is close to 0. One deduces, using the implicit function theorem, that the system (8) has a unique solution $\left(\bar{\xi}_1, \bar{\xi}_2, \bar{z}\right)$.
■

Thanks to the implicit function theorem, one has

$$\left\{ \begin{array}{rcl} \dfrac{\partial \bar{\xi}_1}{\partial y_1} & = & \dfrac{1}{\overline{\Delta}} \left(a_i^+ - a_i^-\right) \left(1 - \dfrac{\partial \psi_\varepsilon}{\partial \zeta_2} \widetilde{\Phi}_{\varepsilon,j} + \dfrac{\partial \psi_\varepsilon}{\partial \zeta_1} \widetilde{\Phi}_{\varepsilon,i}\right), \\ \dfrac{\partial \bar{\xi}_1}{\partial y_2} & = & \dfrac{2}{\overline{\Delta}} \left(a_j^+ - a_j^-\right) \dfrac{\partial \psi_\varepsilon}{\partial \zeta_2} \widetilde{\Phi}_{\varepsilon,i}, \\ \dfrac{\partial \bar{\xi}_1}{\partial t} & = & \dfrac{2}{\overline{\Delta}} \varepsilon^{(1-\theta)} \dfrac{\partial \psi_\varepsilon}{\partial \tau} \widetilde{\Phi}_{\varepsilon,i}, \end{array} \right. \tag{10}$$

where

$$\begin{array}{rcl} \widetilde{\Phi}_{\varepsilon,i} & = & \left(a_i^-\right)' \dfrac{\zeta_1 - a_i^+}{a_i^+ - a_i^-} - \left(a_i^+\right)' \dfrac{\zeta_1 - a_i^-}{a_i^+ - a_i^-}, \\ \widetilde{\Phi}_{\varepsilon,j} & = & \left(a_j^-\right)' \dfrac{\zeta_2 - a_j^+}{a_j^+ - a_j^-} - \left(a_j^+\right)' \dfrac{\zeta_2 - a_j^-}{a_j^+ - a_j^-}. \end{array}$$

One also has

$$\left\{ \begin{array}{rclrcl} \dfrac{\partial \bar{\xi}_2}{\partial y_1} & = & \dfrac{2\left(a_i^+ - a_i^-\right)}{\overline{\Delta}} \dfrac{\partial \psi_\varepsilon}{\partial \zeta_2} \widetilde{\Phi}_{\varepsilon,j}, & \dfrac{\partial \bar{z}}{\partial y_1} & = & \dfrac{2}{\overline{\Delta}} \varepsilon^{\theta-1} \dfrac{\partial \psi_\varepsilon}{\partial \zeta_1} \left(a_i^+ - a_i^-\right), \\ \dfrac{\partial \bar{\xi}_2}{\partial y_2} & = & \dfrac{a_j^+ - a_j^-}{\overline{\Delta}} \left(\begin{array}{c} 1 - \dfrac{\partial \psi_\varepsilon}{\partial \zeta_1} \widetilde{\Phi}_{\varepsilon,i} \\ + \dfrac{\partial \psi_\varepsilon}{\partial \zeta_2} \widetilde{\Phi}_{\varepsilon,j} \end{array} \right), & \dfrac{\partial \bar{z}}{\partial y_2} & = & \dfrac{2}{\overline{\Delta}} \varepsilon^{\theta-1} \dfrac{\partial \psi_\varepsilon}{\partial \zeta_2} \left(a_j^+ - a_j^-\right), \\ \dfrac{\partial \bar{\xi}_2}{\partial t} & = & \dfrac{2}{\overline{\Delta}} \varepsilon^{1-\theta} \dfrac{\partial \psi_\varepsilon}{\partial \tau} \widetilde{\Phi}_{\varepsilon,j}, & \dfrac{\partial \bar{z}}{\partial t} & = & \dfrac{1}{\overline{\Delta}} \dfrac{\partial \psi_\varepsilon}{\partial \tau} \left(\begin{array}{c} 1 + \dfrac{\partial \psi_\varepsilon}{\partial \zeta_1} \widetilde{\Phi}_{\varepsilon,i} \\ + \dfrac{\partial \psi_\varepsilon}{\partial \zeta_2} \widetilde{\Phi}_{\varepsilon,j} \end{array} \right). \end{array} \right. \tag{11}$$

Let $\left(g_{\varepsilon,\alpha\beta}\right)_{\alpha,\beta=1,2,3}$ be the metric tensor associated to the local basis defined through the vectors (10) and (11). One has the following result.

Lemma 3 *1. The metric tensor $\left(g_{\varepsilon,\alpha\beta}\right)_{\alpha,\beta=1,2,3}$ satisfies a symmetry property and the following behaviour*

$$\begin{array}{rclrcl} g_{\varepsilon,11} & = & \left(q_i\right)^2 \left(\varepsilon^{-\theta} t\right) + O\left(\varepsilon^{2(1-\theta)}\right), & g_{\varepsilon,22} & = & \left(q_j\right)^2 \left(\varepsilon^{-\theta} t\right) + O\left(\varepsilon^{2(1-\theta)}\right), \\ g_{\varepsilon,12} & = & O\left(\varepsilon^{2(1-\theta)}\right), & g_{\varepsilon,23} & = & O\left(\varepsilon^{2(1-\theta)}\right), \\ g_{\varepsilon,13} & = & O\left(\varepsilon^{2(1-\theta)}\right), & g_{\varepsilon,33} & = & 1 + O\left(\varepsilon^{2(1-\theta)}\right), \end{array}$$

where $q_i\left(\varepsilon^{-\theta} t\right) = q\left(\varepsilon^{-\theta} t + \alpha_i(\omega), \omega\right)$, $\forall i \in \mathbb{Z}$.

2. The contravariant components $\left(g_\varepsilon^{\alpha\beta}\right)_{\alpha,\beta=1,2,3}$ of $\left(g_{\varepsilon,\alpha\beta}\right)_{\alpha,\beta=1,2,3}$ satisfy a symmetry *property and*

$$\begin{array}{rclrcl} g_\varepsilon^{11} & = & \dfrac{1}{(q_i)^2\left(\varepsilon^{-\theta}t\right)}+O\left(\varepsilon^{2(1-\theta)}\right), & g_\varepsilon^{23} & = & O\left(\varepsilon^{2(1-\theta)}\right), \\ g_\varepsilon^{12} & = & O\left(\varepsilon^{2(1-\theta)}\right), & g_\varepsilon^{13} & = & O\left(\varepsilon^{2(1-\theta)}\right), \\ g_\varepsilon^{22} & = & \dfrac{1}{(q_j)^2\left(\varepsilon^{-\theta}t\right)}+O\left(\varepsilon^{2(1-\theta)}\right), & g_\varepsilon^{33} & = & 1+O\left(\varepsilon^{2(1-\theta)}\right). \end{array}$$

Proof. Observing that

$$\begin{array}{rcl} \left|\dfrac{\partial}{\partial t}q_i\left(\varepsilon^{-\theta}z\right)-\dfrac{\partial}{\partial t}q_i\left(\varepsilon^{-\theta}t\right)\right| & = & O\left(\varepsilon^{2-3\theta}\right), \\ \left|q_i\left(\varepsilon^{-\theta}z\right)-q_i\left(\varepsilon^{-\theta}t\right)\right| & = & O\left(\varepsilon^{2(1-\theta)}\right), \\ \det\left(g_{\varepsilon,\alpha\beta}\right) & = & (q_i)^2\left(\varepsilon^{-\theta}t\right)(q_j)^2\left(\varepsilon^{-\theta}t\right)+O\left(\varepsilon^{2(1-\theta)}\right), \end{array}$$

these formulas are direct consequences of (10)-(11). ■

One deduces from the preceding results that the gradient of a function u expressed in the local coordinates (y_1, y_2, t) inside the fissure $Y_{\varepsilon,ij}(\omega)$ is of the form

$$\nabla u=\left(Id+O\left(\varepsilon^{2(1-\theta)}\right)\right)\left(\frac{1}{q_i}\frac{\partial u}{\partial y_1},\frac{1}{q_j}\frac{\partial u}{\partial y_2},\frac{\partial u}{\partial t}\right),$$

for some non-diagonal matrix $O\left(\varepsilon^{2(1-\theta)}\right)$.

Hence, using the formula $\Delta u=\frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}g^{\alpha\beta}\partial_\beta u\right)$, for $\alpha,\beta=1,2,3$, with $|g|=|\det\left(g_{\alpha\beta}\right)|$, one has

$$\Delta u=\left(\frac{1}{(q_i)^2}\frac{\partial^2 u}{\partial (y_1)^2}+\frac{1}{(q_j)^2}\frac{\partial^2 u}{\partial (y_2)^2}+\frac{1}{q_iq_j}\frac{\partial}{\partial t}\left(q_iq_j\frac{\partial u}{\partial t}\right)\right)\left(1+O\left(\varepsilon^{2(1-\theta)}\right)\right).$$

3 Study of the fluid flow

3.1 Existence of a weak solution and a priori estimates

We define the functional space

$$\mathbf{V}_\varepsilon=\left\{\begin{array}{c} v\in\mathbf{L}^2\left(\Omega_f^\varepsilon(\omega);\mathbb{R}^3\right)\mid \operatorname{div}(v)=0 \text{ in } \Omega_f^\varepsilon(\omega),\ v\mid_{Y_\varepsilon(\omega)}\in\mathbf{H}^1\left(Y_\varepsilon(\omega);\mathbb{R}^3\right), \\ v=0 \text{ on } \Lambda_\varepsilon(\omega),\ v\cdot n=0 \text{ on } \partial\Omega_f^{+,\varepsilon}\cup\partial\Omega_{h,f}^{-,\varepsilon}\cup\Gamma \end{array}\right\}.$$

$\mathbf{V}_\varepsilon$ is a complete space when endowed with the norm

$$\|v\|_{\mathbf{V}_\varepsilon}=\left(\|v\|^2_{\mathbf{L}^2\left(\Omega_f^{+,\varepsilon};\mathbb{R}^3\right)}+\|v\|^2_{\mathbf{L}^2\left(\Omega_{h,f}^{-,\varepsilon};\mathbb{R}^3\right)}+\|\nabla v\|^2_{\mathbf{L}^2\left(Y_\varepsilon(\omega);\mathbb{R}^9\right)}\right)^{1/2}.$$

Multiplying (6) by $\Phi \in \mathbf{V}_\varepsilon$, using Green's formula and the conditions (7), we obtain the following variational formulation

$$\begin{aligned}
&\mu^+ \int_{\Omega_f^{+,\varepsilon}} \left(K_\varepsilon^+\right)^{-1} v_{\varepsilon,d} \cdot \Phi dx + \mu^- \int_{\Omega_{h,f}^{-,\varepsilon}} \left(K_\varepsilon^-\right)^{-1} v_{\varepsilon,d} \cdot \Phi dx + \mu\varepsilon^2 \int_{Y_\varepsilon(\omega)} \nabla v_{\varepsilon,s} \cdot \nabla \Phi dx \\
&\quad + \gamma \int_{\Gamma_{0,\varepsilon}^+} \left(K_\varepsilon^+\right)^{-1/2} \left(v_{\varepsilon,s}\right)_\tau \cdot (\Phi)_\tau \, dx' + \gamma \int_{\Gamma_{h,\varepsilon}^-} \left(K_\varepsilon^-\right)^{-1/2} \left(v_{\varepsilon,s}\right)_\tau \cdot (\Phi)_\tau \, dx' \\
&\quad = \int_{\Omega_f^{+,\varepsilon}} g^+ \cdot \Phi dx + \int_{\Omega_{h,f}^{-,\varepsilon}} g^- \cdot \Phi dx,
\end{aligned} \tag{12}$$

where $x' = (x_1, x_2)$. Using standard arguments, we immediately deduce from this variational formulation that the system (6)-(7) has a unique weak solution $(v_\varepsilon, p_\varepsilon) \in \mathbf{V}_\varepsilon \times L^2\left(\Omega_\varepsilon(\omega)\right)/\mathbb{R}$.

Lemma 4 *There exists a non-random constant C independent of ε such that*

$$\begin{aligned}
\int_{Y_\varepsilon(\omega)} |v_\varepsilon|^2 dx &\leq C\varepsilon^2 \int_{Y_\varepsilon(\omega)} |\nabla v_\varepsilon|^2 dx, & \int_{\Omega_f^\varepsilon(\omega)} |v_\varepsilon|^2 dx &\leq C, \\
\int_{\Gamma_{0,\varepsilon}^+} |v_\varepsilon|^2 dx' + \int_{\Gamma_{h,\varepsilon}^-} |v_\varepsilon|^2 dx' &\leq C\varepsilon^2, & \varepsilon^2 \int_{Y_\varepsilon(\omega)} |\nabla v_\varepsilon|^2 dx &\leq C.
\end{aligned}$$

Proof. Define the normalized fissure

$$Z_{\varepsilon,ij}(\omega) = \left\{ \begin{array}{c} (z_1, z_2, x_3) \in \mathbb{R}^2 \mid a_i^-\left(-\varepsilon^{-\theta} x_3\right) < z_1 < a_i^+\left(-\varepsilon^{-\theta} x_3\right), \\ a_j^-\left(-\varepsilon^{-\theta} x_3\right) < z_2 < a_j^+\left(-\varepsilon^{-\theta} x_3\right), \, x_3 \in (-h, 0) \end{array} \right\}. \tag{13}$$

For every $\psi \in C^1\left(Z_{\varepsilon,ij}(\omega)\right)$ such that $\psi = 0$ on the lateral boundary of $Z_{\varepsilon,ij}(\omega)$, one has inside $Z_{\varepsilon,ij}(\omega)$

$$\left(\psi(s, z_2, x_3)\right)^2 = \left(\int_{a_i^-}^s \frac{\partial \psi}{\partial z_1} dz_1\right)^2 \leq q_i\left(-\varepsilon^{-\theta} x_3\right) \int_{a_i^-}^{a_i^+} \left(\frac{\partial \psi}{\partial z_1}\right)^2 dz_1.$$

Thanks to the hypothesis (1), one has the following Poincaré estimate

$$\int_{Z_{\varepsilon,ij}(\omega)} \psi^2 dz_1 dz_2 \leq C \int_{Z_{\varepsilon,ij}(\omega)} |\nabla \psi|^2 dz_1 dz_2.$$

Defining $z_1 = (x_1 - \varepsilon i)/\varepsilon$ and $z_2 = (x_2 - \varepsilon j)/\varepsilon$, one gets the first estimate.

We replace Φ by v_ε in the variational formulation (12). Since the matrices K_ε^+ and K_ε^- are bounded, symmetric and positive definite, we deduce, using the first part of this Lemma, that

$$\begin{aligned}
&\int_{\Omega_f^{+,\varepsilon}} |v_{\varepsilon,d}|^2 dx + \int_{\Omega_{h,f}^{-,\varepsilon}} |v_{\varepsilon,d}|^2 dx + \mu\varepsilon^2 \int_{Y_\varepsilon(\omega)} |\nabla v_{\varepsilon,s}|^2 dx + \gamma \int_{\Gamma_{0,\varepsilon}^+} \left|(v_{\varepsilon,s})_\tau\right|^2 dx' \\
&\quad + \gamma \int_{\Gamma_{h,\varepsilon}^-} \left|(v_{\varepsilon,s})_\tau\right|^2 dx' \leq C \left(\int_{\Omega_f^{+,\varepsilon}} g^+ \cdot v_{\varepsilon,d} dx + \int_{\Omega_{h,f}^{-,\varepsilon}} g^- \cdot v_{\varepsilon,d} dx \right),
\end{aligned}$$

hence, using Cauchy-Schwarz inequality

$$\int_{\Omega_f^\varepsilon(\omega)} |v_\varepsilon|^2\,dx \le C\left(\int_{\Omega_f^{+,\varepsilon}} |v_\varepsilon|^2\,dx\right)^{1/2} + C\left(\int_{\Omega_{h,f}^{-,\varepsilon}} |v_\varepsilon|^2\,dx\right)^{1/2} \le C\left(\int_{\Omega_f^\varepsilon(\omega)} |v_\varepsilon|^2\,dx\right)^{1/2}$$

and

$$\varepsilon^2\int_{Y_\varepsilon(\omega)} |\nabla v_\varepsilon|^2\,dx \le C\left(\int_{\Omega_f^\varepsilon(\omega)} |v_\varepsilon|^2\,dx\right)^{1/2}.$$

Using a trace theorem in the normalized fissure $Z_{\varepsilon,ij}(\omega)$, we have

$$\int_{\partial Z_{\varepsilon,ij}(\omega)} \psi^2 dz_1 dz_2 \le C\int_{Z_{\varepsilon,ij}(\omega)} |\nabla\psi|^2\,dz_1dz_2,$$

which ends the proof. ■

We now deal with the pressure. We define the extension $\widetilde{p}_{\varepsilon,d}$ of the pressure $p_{\varepsilon,d}$ in the solid parts of the porous media

$$\widetilde{p}_{\varepsilon,d} = \begin{cases} p_{\varepsilon,d} & \text{in } \Omega_f^{+,\varepsilon}\cup\Omega_{h,f}^{-,\varepsilon}, \\ \dfrac{1}{|Z^1|}\displaystyle\int_{Z^1} p_{\varepsilon,d}\left(a_\varepsilon z + a_\varepsilon l\right)dz & \forall z\in\varepsilon Z^2 + l\varepsilon\subset\Omega^+\cup\Omega_h^-,\ l\in\mathbb{Z}^3. \end{cases}$$

We define the zero mean-value pressures

$$\begin{cases} p_\varepsilon^+ &=& \widetilde{p}_{\varepsilon,d} - \dfrac{1}{|\Omega^+|}\displaystyle\int_{\Omega^+}\widetilde{p}_{\varepsilon,d}dx & \text{in } \Omega^+, \\ p_\varepsilon^- &=& \widetilde{p}_{\varepsilon,d} - \dfrac{1}{|\Omega_h^-|}\displaystyle\int_{\Omega_h^-}\widetilde{p}_{\varepsilon,d}dx & \text{in } \Omega_h^-, \\ \overline{p_\varepsilon} &=& p_{\varepsilon,s} - \dfrac{1}{|Y_{\varepsilon,ij}(\omega)|}\displaystyle\int_{Y_{\varepsilon,ij}(\omega)} p_{\varepsilon,s}dx & \text{in } Y_{\varepsilon,ij}(\omega),\ (i,j)\in I_\varepsilon(\omega). \end{cases}$$

Lemma 5 *There exists a non-random constant C independent of ε such that*

$$\int_{\Omega^+}|\nabla p_\varepsilon^+|^2\,dx \le C,\quad \int_{\Omega_h^-}|\nabla p_\varepsilon^-|^2\,dx \le C,\quad \int_{Y_\varepsilon(\omega)}(\overline{p_\varepsilon})^2\,dx \le C,$$
$$\int_{\Omega^+}(p_\varepsilon^+)^2\,dx \le C,\quad \int_{\Omega_h^-}(p_\varepsilon^-)^2\,dx \le C.$$

Proof. We multiply $(6)_{1,2}$ by ∇p_ε and obtain

$$\int_{\Omega_f^{+,\varepsilon}}|\nabla p_{\varepsilon,d}|^2\,dx = -\int_{\Omega_f^{+,\varepsilon}} g^+\cdot\nabla p_{\varepsilon,d}dx + \mu^+\int_{\Omega_f^{+,\varepsilon}}\left(K_\varepsilon^+\right)^{-1} v_{\varepsilon,d}\cdot\nabla p_{\varepsilon,d}dx,$$
$$\int_{\Omega_{h,f}^{-,\varepsilon}}|\nabla p_{\varepsilon,d}|^2\,dx = -\int_{\Omega_{h,f}^{-,\varepsilon}} g^-\cdot\nabla p_{\varepsilon,d}dx + \mu^-\int_{\Omega_{h,f}^{-,\varepsilon}}\left(K_\varepsilon^-\right)^{-1} v_{\varepsilon,d}\cdot\nabla p_{\varepsilon,d}dx.$$

The boundary conditions $(6)_4$, the smoothness of $g^{\pm}$ and Lemma 4 imply

$$\int_{\Omega^+} \left|\nabla p_\varepsilon^+\right|^2 dx \le C \; ; \; \int_{\Omega_h^-} \left|\nabla p_\varepsilon^-\right|^2 dx \le C.$$

Using Poincaré-Wirtinger' inequality, we have

$$\begin{array}{rcccl} \int_{\Omega^+} (p_\varepsilon^+)^2 dx & \le & C(\Omega^+) \int_{\Omega^+} |\nabla p_\varepsilon|^2 dx & \le & C, \\ \int_{\Omega_h^-} (p_\varepsilon^-)^2 dx & \le & C\left(\Omega_h^-\right) \int_{\Omega_h^-} |\nabla p_\varepsilon|^2 dx & \le & C. \end{array}$$

In order to get estimates on $\overline{p_\varepsilon}$, we consider the problem

$$\left\{ \begin{array}{rcll} \operatorname{div}_z(\Phi_\varepsilon^0) & = & \overline{p_\varepsilon}(\varepsilon z_1, \varepsilon z_2, x_3) & \text{in } Z_{\varepsilon,ij}(\omega), \\ \Phi_\varepsilon^0 & = & 0 & \text{on } \partial Z_{\varepsilon,ij}(\omega), \end{array} \right.$$

where the fissure $Z_{\varepsilon,ij}(\omega)$ is defined in (13). This problem has a unique solution Φ_ε^0 in the space $\left\{(-\Delta)^{-1}\nabla w \mid w \in \mathbf{L}^2(Z_{\varepsilon,ij}(\omega))\right\}$ (see [17] for example), such that

$$\int_{Z_{\varepsilon,ij}(\omega)} \left|\nabla \Phi_\varepsilon^0\right|^2 dz dx_3 \le C \int_{Z_{\varepsilon,ij}(\omega)} (\overline{p_\varepsilon})^2 dz dx_3, \tag{14}$$

where, due to the hypothesis (1), the constant C is non-random and independent of ε. Let us define

$$\Phi_\varepsilon(x) = \begin{pmatrix} \varepsilon(\Phi_\varepsilon^0)_1((x_1 - i\varepsilon)/\varepsilon, (x_2 - j\varepsilon)/\varepsilon, x_3) \\ \varepsilon(\Phi_\varepsilon^0)_2((x_1 - i\varepsilon)/\varepsilon, (x_2 - j\varepsilon)/\varepsilon, x_3) \\ (\Phi_\varepsilon^0)_3((x_1 - i\varepsilon)/\varepsilon, (x_2 - j\varepsilon)/\varepsilon, x_3) \end{pmatrix}.$$

Then

$$\left\{ \begin{array}{rcll} \operatorname{div}(\Phi_\varepsilon)(x) & = & \operatorname{div}_z(\Phi_\varepsilon^0)(x) = \overline{p_\varepsilon}(x) & \text{in } Y_{\varepsilon,ij}(\omega), \\ \Phi_\varepsilon & = & 0 & \text{on } \partial Y_{\varepsilon,ij}(\omega) \end{array} \right.$$

and, thanks to (14)

$$\int_{Y_\varepsilon(\omega)} |\nabla \Phi_\varepsilon|^2 dx \le \int_{Z_{\varepsilon,ij}(\omega)} \left|\nabla \Phi_\varepsilon^0\right|^2 dz dx_3 \le C \int_{Z_{\varepsilon,ij}(\omega)} (\overline{p_\varepsilon})^2 dz dx_3 = \frac{C}{\varepsilon^2} \int_{Y_\varepsilon(\omega)} (\overline{p_\varepsilon}(x))^2 dx. \tag{15}$$

Multiplying $(6)_5$ by Φ_ε in $Y_\varepsilon(\omega)$, we get

$$\mu\varepsilon^2 \int_{Y_\varepsilon(\omega)} \nabla v_\varepsilon \cdot \nabla \Phi_\varepsilon dx - \int_{Y_\varepsilon(\omega)} (\overline{p_\varepsilon}(x))^2 dx = 0.$$

Using the inequality (15), we have

$$\int_{Y_\varepsilon(\omega)} (\overline{p_\varepsilon}(x))^2 dx \le C \left(\int_{Y_\varepsilon(\omega)} (\overline{p_\varepsilon}(x))^2 dx \right)^{1/2} \left(\varepsilon^2 \int_{Y_\varepsilon(\omega)} |\nabla v_\varepsilon|^2 dx \right)^{1/2}.$$

Thanks to Lemma 4, we derive the last estimate in $Y_\varepsilon(\omega)$. ■

3.2 Convergence

Observe the following result.

Lemma 6 *1. For every $\varphi \in C_0^1(Y_h)$, we have*

$$\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{Y_{\varepsilon,ij}(\omega)}\varphi(x_1,x_2,x_3)\,dx = h\left\langle q^2\right\rangle\int_\Sigma\varphi(x',0)\,dx',\ \textit{almost surely.}$$

2. Let $(w_\varepsilon)_\varepsilon$ be a sequence such that $\sup_\varepsilon\int_{Y_\varepsilon(\omega)}(w_\varepsilon)^2\,dx < +\infty$. There exists a subsequence, still denoted in the same way, such that

$$\lim_{\varepsilon\to 0}\int_{Y_\varepsilon(\omega)}w_\varepsilon\varphi dx = h\left\langle q^2(0)\right\rangle\int_\Sigma w(x',0)\,\varphi(x',0)\,dx',\ \forall\varphi\in C_0\left(\mathbb{R}^3\right),\ \textit{almost surely.}$$

Proof. 1. Define $\xi_{i,\varepsilon}(y) = \varepsilon i - \varepsilon r_i\left(\varepsilon^{-\theta}t\right) - y_1 q_i\left(\varepsilon^{-\theta}t\right)$ and $\xi_{j,\varepsilon}(y) = \varepsilon j - \varepsilon r_j\left(\varepsilon^{-\theta}t\right) - y_1 q_j\left(\varepsilon^{-\theta}t\right)$ where $r_i\left(\varepsilon^{-\theta}t\right) = r\left(\varepsilon^{-\theta}t + \beta_i(\omega),\omega\right)$ and $q_i\left(\varepsilon^{-\theta}t\right) = q\left(\varepsilon^{-\theta}t + \alpha_i(\omega),\omega\right)$. According to the properties of the above-defined curvilinear coordinates, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{Y_{\varepsilon,ij}(\omega)}\varphi(x_1,x_2,x_3)\,dx\\
&\quad= \lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{-\varepsilon/2}^{\varepsilon/2}\int_{-\varepsilon/2}^{\varepsilon/2}\int_0^h\varphi\left(\xi_{i,\varepsilon}(y),\xi_{j,\varepsilon}(y),-t\right)q_i\left(\varepsilon^{-\theta}t\right)q_j\left(\varepsilon^{-\theta}t\right)dydt\\
&\quad= \lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\varepsilon^2\int_0^h\varphi(\varepsilon i,\varepsilon j,-t)\,q^2\left(\varepsilon^{-\theta}t\right)dt\\
&\quad= \lim_{\varepsilon\to 0}\frac{h}{\varepsilon^{-\theta}h}\int_0^{\varepsilon^{-\theta}h}\left(\int_\Sigma\varphi\left(x_1,x_2,-\varepsilon^\theta s\right)dx_1dx_2\right)q^2(s)\,ds\\
&\quad= h\left\langle q^2\right\rangle\int_\Sigma\varphi(x',0)\,dx',
\end{aligned}$$

where we have used the ergodicity result (2).

2. The sequence of measures $(\nu_\varepsilon)_\varepsilon$, with $\nu_\varepsilon = \mathbf{1}_{Y_\varepsilon(\omega)}dx$, $\mathbf{1}_A$ being the characteristic function of the set A, thus converges in the weak sense of measures, when ε goes to 0, to the measure $\nu = h\left\langle q^2\right\rangle\mathbf{1}_\Sigma(x')\,dx'$. Using the hypothesis on $(w_\varepsilon)_\varepsilon$, we deduce that the sequence of measures $(v_\varepsilon w_\varepsilon)_\varepsilon$ has bounded variation. Up to some subsequence, the sequence $(v_\varepsilon w_\varepsilon)_\varepsilon$ thus converges to some χ_0 in the weak sense of measures. For every $\varphi\in C_0(\mathbb{R}^3)$, one has, thanks to Fenchel's inequality

$$\int_{\mathbb{R}^3}(w_\varepsilon)^2\,d\nu_\varepsilon \geq 2\int_{\mathbb{R}^3}w_\varepsilon\varphi d\nu_\varepsilon - \int_{\mathbb{R}^3}\varphi^2 d\nu_\varepsilon.$$

Passing to the limit, we get

$$+\infty > \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^3}(w_\varepsilon)^2\,d\nu_\varepsilon \geq 2\left\langle\chi_0,\varphi\right\rangle - \int_{\mathbb{R}^2}\varphi^2(x',0)\,d\nu.$$

Thus

$$\sup\left\{\langle\chi_0,\varphi\rangle \mid \varphi\in C_0\left(\mathbb{R}^3\right),\ \int_{\mathbb{R}^2}\varphi^2\left(x',0\right)d\nu\leq 1\right\}<+\infty.$$

Using Riesz' representation theorem, we can identify χ_0 with $w\nu$, for some $w\in L^2\left(\mathbb{R}^2\right)$.
■

Remark 7 *1. From Lemma 4, using the above result, we deduce that, up to some subsequence*

$$\lim_{\varepsilon\to 0}\int_{Y_\varepsilon(\omega)}v_\varepsilon\cdot\Phi dx=h\left\langle q^2\left(0\right)\right\rangle\int_\Sigma v_f\cdot\Phi dx',\ \forall\Phi\in\mathbf{C}_0\left(\Sigma;\mathbb{R}^3\right),$$

almost surely, and from Lemma 5, we deduce the existence of $\pi_0\in L^2\left(\Sigma\right)$ *such that, up to some subsequence*

$$\lim_{\varepsilon\to 0}\int_{Y_\varepsilon(\omega)}\overline{p_\varepsilon}\varphi dx=\int_\Sigma\pi_0\varphi d\nu=h\left\langle q^2\right\rangle\int_\Sigma\pi_0\left(x'\right)\varphi\left(x'\right)dx',\ \forall\varphi\in C_0\left(\Sigma\right),$$

almost surely, where $x'=\left(x_1,x_2\right)$.

In the rest of the paper, we will no more indicate this almost surely convergence where there is no doubt.

2. From Lemma 4 and the above computations, we deduce the existence of $v_0\in\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)$ *such that, up to some subsequence*

$$\begin{array}{rcll} v_\varepsilon\mid_{\Omega_f^{+,\varepsilon}} & \underset{\varepsilon\to 0}{\to} & v_0\mid_{\Omega^+}=:v_{0,d}^+ & w\text{-}\mathbf{L}^2\left(\Omega^+;\mathbb{R}^3\right), \\ v_\varepsilon\mid_{\Omega_{h,f}^{-,\varepsilon}} & \underset{\varepsilon\to 0}{\to} & v_0\mid_{\Omega_h^-}=:v_{0,d}^- & w\text{-}\mathbf{L}^2\left(\Omega_h^-;\mathbb{R}^3\right), \\ \lim_{\varepsilon\to 0}\int_{Y_\varepsilon(\omega)}v_\varepsilon\cdot\Phi dx & = & h\left\langle q^2\right\rangle\int_\Sigma v_0\cdot\Phi dx' & \forall\Phi\in\mathbf{C}_0\left(\Sigma;\mathbb{R}^3\right). \end{array}$$

3. We set $v_0\mid_\Sigma=:v_{0,f}$. *For every* $\varphi\in C_0^1\left(\Sigma\right)$, *one has*

$$\begin{aligned} 0=\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{Y_{\varepsilon,ij}(\omega)}\operatorname{div}\left(v_\varepsilon\right)\varphi dx &= -\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{Y_{\varepsilon,ij}(\omega)}v_\varepsilon\cdot\nabla\varphi dx \\ &= -h\left\langle q^2\right\rangle\int_\Sigma v_{0,f}\cdot\nabla\varphi dx' \\ &= h\left\langle q^2\right\rangle\int_\Sigma\operatorname{div}\left(v_{0,f}\right)\varphi dx', \end{aligned}$$

thanks to the estimates of Lemma 4. Thus $\operatorname{div}\left(v_{0,f}\right)=0$ *in* Σ.

4. It is easily seen that $\operatorname{div}\left(v_{0,d}^+\right)=0$ *in* Ω^+ *and* $\operatorname{div}\left(v_{0,d}^-\right)=0$ *in* Ω_h^-. *For every* $\varphi\in C^1\left(\Omega^+\right)$, *one has*

$$0=\lim_{\varepsilon\to 0}\int_{\Omega_f^{+,\varepsilon}}\operatorname{div}\left(v_\varepsilon\right)\varphi dx=-\lim_{\varepsilon\to 0}\int_{\Omega_f^{+,\varepsilon}}v_\varepsilon\cdot\nabla\varphi dx+\lim_{\varepsilon\to 0}\int_{\Gamma_{0,\varepsilon}^+}\left(v_\varepsilon\right)_3\varphi dx'.$$

Observe that

$$\lim_{\varepsilon\to 0}\int_{\Gamma^+_{0,\varepsilon}} (v_\varepsilon)_3\,\varphi dx' = \left\langle q^2(0)\right\rangle \int_\Sigma (v_{0,f})_3\,\varphi dx',$$

whence $\left(v^+_{0,d}\right)_3|_{\Sigma\times\{0\}} = \langle q^2(0)\rangle\,(v_{0,f})_3$. *In a similar way, but working in* $\Omega^{-,\varepsilon}_{h,f}$, *instead of* $\Omega^{+,\varepsilon}_f$, *we have* $\left(v^-_{0,d}\right)_3|_{\Sigma\times\{-h\}} = \langle q^2(0)\rangle\,(v_{0,f})_3$.

5. *From Lemmas 5 and 6, we get, up to some subsequence*

$$\begin{array}{rcll} p^+_\varepsilon & \underset{\varepsilon\to 0}{\to} & p^+_0 & s\text{-}L^2(\Omega^+), \\ p^-_\varepsilon & \underset{\varepsilon\to 0}{\to} & p^-_0 & s\text{-}L^2\left(\Omega^-_h\right), \\ \lim_{\varepsilon\to 0}\int_{Y_\varepsilon(\omega)} \overline{p_\varepsilon}\varphi dx & = & h\langle q^2(0)\rangle\int_\Sigma \pi_0\varphi dx' & \forall\varphi\in C_0(\Sigma). \end{array} \tag{16}$$

We now set

$$\mathbf{V}_0 = \left\{ \begin{array}{l} v\in \mathbf{L}^2\left(\Omega^+\cup\Omega^-_h;\mathbb{R}^3\right) \mid \operatorname{div}(v)=0 \text{ in } \Omega^+\cup\Omega^-_h, \\ \quad v\cdot n = 0 \text{ on } \Gamma^+\cup\Gamma^-,\ v_3|_{\Sigma\times\{0\}} = \langle q^2(0)\rangle (v_{,f})_3 = v_3|_{\Sigma\times\{-h\}} \end{array}\right\}.$$

Every function $v\in\mathbf{V}_0$ can be extended in a function of $\mathbf{L}^2(Y_\varepsilon(\omega);\mathbb{R}^3)$ independent of x_3 in $Y_\varepsilon(\omega)$.

We define the appropriate notion of convergence for the problem (6).

Definition 8 *A sequence* $(V_\varepsilon)_\varepsilon$, *with* $V_\varepsilon\in\mathbf{V}_\varepsilon$ *for every* ε, τ_0*-converges to some* $V\in\mathbf{V}_0$ *if*

$$\left\{\begin{array}{rcll} V_\varepsilon|_{\Omega^{+,\varepsilon}_f} & \underset{\varepsilon\to 0}{\rightharpoonup} & V|_{\Omega^+} =: V^+_d & w\text{-}\mathbf{L}^2(\Omega^+;\mathbb{R}^3), \\ V_\varepsilon|_{\Omega^{-,\varepsilon}_{h,f}} & \underset{\varepsilon\to 0}{\rightharpoonup} & V|_{\Omega^-_h} =: V^-_d & w\text{-}\mathbf{L}^2\left(\Omega^-_h;\mathbb{R}^3\right), \\ \lim_{\varepsilon\to 0}\int_{Y_\varepsilon(\omega)} V_\varepsilon\cdot\Phi dx & = & h\langle q^2(0)\rangle\int_\Sigma V_f\cdot\Phi dx' & \forall\Phi\in\mathbf{C}_0(\Sigma;\mathbb{R}^3), \end{array}\right.$$

with $V_f := V|_\Sigma$.

We define the functional F_ε on $\mathbf{L}^2\left(\Omega^\varepsilon_f(\omega);\mathbb{R}^3\right)$ through

$$F_\varepsilon(v) = \left\{\begin{array}{ll} \mu^+\int_{\Omega^{+,\varepsilon}_f} (K^+_\varepsilon)^{-1} v\cdot v dx + \mu^-\int_{\Omega^{-,\varepsilon}_{h,f}} (K^-_\varepsilon)^{-1} v\cdot v dx + \mu\varepsilon^2\int_{Y_\varepsilon(\omega)} |\nabla v|^2 dx & \\ \quad +\gamma\int_{\Gamma^+_{0,\varepsilon}} (K^+_\varepsilon)^{-1/2} v_\tau\cdot v_\tau dx' + \gamma\int_{\Gamma^-_{h,\varepsilon}} (K^-_\varepsilon)^{-1/2} v_\tau\cdot v_\tau dx' & \\ & \text{if } v\in\mathbf{V}_\varepsilon, \\ +\infty & \text{otherwise.} \end{array}\right.$$

In order to describe the asymptotic behaviour of this functional, we consider the Z-periodic solution $\Phi^\pm_k$ of the local Darcy systems

$$\left\{\begin{array}{rcl} (K^\pm)^{-1}\Phi^\pm_k - \nabla\pi^\pm_k & = & e_k \quad \text{in } Z^1,\ k=1,2,3, \\ \operatorname{div}\left(\Phi^\pm_k\right) & = & 0 \quad \text{in } Z^1, \\ \Phi^\pm_k\cdot n & = & 0 \quad \text{on } S, \end{array}\right.$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. We then consider the Stokes system

$$\left\{ \begin{array}{rcl} -\Delta\eta_k + \nabla\xi_k &=& e_k \quad \text{in } Z' = (-1/2, 1/2)^2, \ k = 1, 2, \\ \operatorname{div}(\eta_k) &=& 0 \quad \text{in } Z', \\ \eta_k &=& 0 \quad \text{on } \partial Z', \end{array} \right. \tag{17}$$

where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$, and the local scalar problem

$$\left\{ \begin{array}{rcl} -\Delta\eta_0 &=& 1 \quad \text{in } Z' = (-1/2, 1/2)^2, \\ \eta_0 &=& 0 \quad \text{on } \partial Z'. \end{array} \right. \tag{18}$$

We define the 3×3 matrices $\widehat{K}^+$, $\widehat{K}^-$, the 2×2 matrix K_f and the constant k_0 through

$$\left\{ \begin{array}{rcll} \widehat{K}^+_{ml} &=& \displaystyle\int_{Z^1} (\Phi^+_m)_l \, dz & m, l = 1, 2, 3, \\ \widehat{K}^-_{ml} &=& \displaystyle\int_{Z^1} (\Phi^-_m)_l \, dz & m, l = 1, 2, 3, \\ (K_f)_{ml} &=& \displaystyle\int_{Z'} (\eta_m)_l \, dz & m, l = 1, 2, \\ k_0 &=& \displaystyle\int_{Z'} |\nabla\eta_0|^2 \, dz = \int_{Z'} \eta_0 dz. & \end{array} \right. \tag{19}$$

One can prove that the matrices $\widehat{K}^\pm$ and K_f are symmetric and positive definite (see [16]). Our main result of this part reads as follows.

Theorem 9 *Suppose that r is also a stationary random process. The sequence $(F_\varepsilon)_\varepsilon$ Γ-converges in the topology τ_0 to the functional F_0 defined through*

$$F_0(v) = \left\{ \begin{array}{ll} \mu^+ \displaystyle\int_{\Omega^+} \left(\widehat{K}^+\right)^{-1} v_d^+ \cdot v_d^+ dx + \mu^- \int_{\Omega_h^-} \left(\widehat{K}^-\right)^{-1} v_d^- \cdot v_d^- dx & \\ \quad + \mu \dfrac{h\langle q^2\rangle}{\langle q\rangle^2} \displaystyle\int_\Sigma (K_f)^{-1} (v_f)_\tau \cdot (v_f)_\tau \, dx' + \mu \frac{h\langle q^2\rangle \langle 1/q^2\rangle}{k_0} \int_\Sigma \left((v_f)_3\right)^2 dx' & \\ \quad + \langle q^2\rangle \gamma \displaystyle\int_\Sigma \left(\left(K^{*+}\right)^{-1/2} + \left(K^{*-}\right)^{-1/2}\right) (v_f)_\tau \cdot (v_f)_\tau \, dx' & \\ & \textit{if } v \in \mathbf{V}_0, \\ +\infty & \textit{otherwise,} \end{array} \right.$$

where $K^{*\pm} = \int_{\langle a^-(0)\rangle}^{\langle a^+(0)\rangle} \int_{\langle a^-(0)\rangle}^{\langle a^+(0)\rangle} K^\pm(z_1, z_2, 0) \, dz_1 dz_2.$

Proof. For the definition and the properties of the Γ-convergence, we refer to [4] and [8].

Upper Γ-limit. Choose any smooth $v \in \mathbf{C}^1\left(\overline{\Omega^+ \cup \Omega_h^-}; \mathbb{R}^3\right) \cap \mathbf{V}_0$. We define $v|_{\Omega^+} =: v_d^+$, $v|_{\Omega_h^-} =: v_d^-$, $v|_\Sigma =: v_f$ and build in $Y_{\varepsilon,ij}(\omega)$

$$\left\{ \begin{array}{rcl} \left(v^0_{\varepsilon,f}\right)_\tau &=& \dfrac{\left((K_f)^{-1} v_f\right)_k (i\varepsilon, j\varepsilon, 0)}{h\langle q\rangle} \displaystyle\int_{-h}^0 \eta_{\varepsilon,k,ij}(x) \, dx_3, \\ \left(v^0_{\varepsilon,f}\right)_3 &=& \dfrac{(v_f)_3 (i\varepsilon, j\varepsilon, 0)}{hk_0} \displaystyle\int_{-h}^0 \eta_{\varepsilon,0,ij}(x) \, dx_3, \end{array} \right. \tag{20}$$

where

$$\begin{aligned}\eta_{\varepsilon,k,ij}(x) &= \begin{pmatrix} q_i\left(-\varepsilon^{-\theta}x_3\right)(\eta_k)_1\left(z\left(x_1,x_2,x_3\right)\right) \\ q_j\left(-\varepsilon^{-\theta}x_3\right)(\eta_k)_2\left(z\left(x_1,x_2,x_3\right)\right)\end{pmatrix}, \\ \eta_{\varepsilon,0,ij}(x) &= \eta_0\left(z\left(x_1,x_2,x_3\right)\right),\end{aligned}$$

with

$$\begin{aligned} z\left(x_1,x_2,x_3\right) &= \begin{pmatrix} \dfrac{x_1 - i\varepsilon - \varepsilon\left(a_i^-\left(-\varepsilon^{-\theta}x_3\right) + a_i^+\left(-\varepsilon^{-\theta}x_3\right)\right)/2}{\varepsilon q_i\left(-\varepsilon^{-\theta}x_3\right)} \\ \dfrac{x_2 - j\varepsilon - \varepsilon\left(a_j^-\left(-\varepsilon^{-\theta}x_3\right) + a_j^+\left(-\varepsilon^{-\theta}x_3\right)\right)/2}{\varepsilon q_j\left(-\varepsilon^{-\theta}x_3\right)}\end{pmatrix}, \\ q_i(s) &= q\left(s+\alpha_i(\omega),\omega\right),\end{aligned} \tag{21}$$

η_k being the solution of (17), η_0 the solution of (18) and K_f and k_0 being defined in (19). We then define the test-function v_ε^0 through

$$\begin{cases} v_\varepsilon^0 = \Phi_j^+\left(\dfrac{x}{\varepsilon}\right)\left(\left(\widehat{K}^+\right)^{-1}v_d^+\right)_j & \text{in } \Omega_f^{+,\varepsilon}, \\ v_\varepsilon^0 = \Phi_j^-\left(\dfrac{x}{\varepsilon}\right)\left(\left(\widehat{K}^-\right)^{-1}v_d^-\right)_j & \text{in } \Omega_{h,f}^{-,\varepsilon}, \\ v_\varepsilon^0 = v_{\varepsilon,f}^0 & \text{in } Y_\varepsilon(\omega).\end{cases} \tag{22}$$

One deduces from this construction that $v_\varepsilon^0\,|_{\Omega^+}\in \mathbf{L}^2\left(\Omega_f^{+,\varepsilon};\mathbb{R}^3\right)$, $v_\varepsilon^0\,|_{\Omega_h^-}\in \mathbf{L}^2\left(\Omega_{h,f}^{-,\varepsilon};\mathbb{R}^3\right)$, $v_\varepsilon^0\,|_{Y_\varepsilon(\omega)}\in \mathbf{H}^1\left(Y_\varepsilon(\omega);\mathbb{R}^3\right)$, $v_\varepsilon^0=0$ on $\Lambda_\varepsilon(\omega)$, $\operatorname{div}\left(v_\varepsilon^0\right)=0$ in $\Omega_f^{+,\varepsilon}\cup\Omega_{h,f}^{-,\varepsilon}$, $v_\varepsilon^0\cdot n=0$ on $\partial\Omega_f^\varepsilon(\omega)$ and

$$\operatorname{div}\left(v_\varepsilon^0\right) = \frac{\left(\left(K_f\right)^{-1}v_f\right)_k(i\varepsilon,j\varepsilon,0)}{h\langle q\rangle}\int_{-h}^0 \operatorname{div}_z\left(\eta_{\varepsilon,k,ij}\right)(x)\,dx_3 = 0, \text{ in } Y_{\varepsilon,ij}(\omega).$$

Therefore $v_\varepsilon^0\in\mathbf{V}_\varepsilon$. Moreover v_ε^0 is independent of x_3 in each fissure $Y_{\varepsilon,ij}(\omega)$. Using the ergodic result (2) and making some computations, one easily proves that $\left(v_\varepsilon^0\right)_\varepsilon$ τ_0-converges to v.

We compute the limit $\lim_{\varepsilon\to0}F_\varepsilon\left(v_\varepsilon^0\right)$. We have

$$\begin{aligned}\lim_{\varepsilon\to0}\mu\varepsilon^2\int_{Y_\varepsilon(\omega)}\left|\nabla v_\varepsilon^0\right|^2dx &= \lim_{\varepsilon\to0}\mu\varepsilon^2\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{Y_{\varepsilon,ij}(\omega)}\left|\nabla\left(v_\varepsilon^0\right)_\tau\right|^2dx \\ &\quad+\lim_{\varepsilon\to0}\mu\varepsilon^2\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{Y_{\varepsilon,ij}(\omega)}\left|\nabla\left(v_\varepsilon^0\right)_3\right|^2dx.\end{aligned}$$

Using the expression $(20)_1$, we have

$$\begin{aligned}\lim_{\varepsilon\to0}\mu\varepsilon^2\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{Y_{\varepsilon,ij}(\omega)}\left|\nabla\left(v_\varepsilon^0\right)_\tau\right|^2dx &= \lim_{\varepsilon\to0}\mu\varepsilon^2\sum_{(i,j)\in I_\varepsilon(\omega)}\frac{1}{h^2\langle q\rangle^2} \\ \times\int_{Y_{\varepsilon,ij}(\omega)}&\begin{pmatrix}\left(\left(K_f\right)^{-1}v_f\right)_k(i\varepsilon,j\varepsilon,0)\displaystyle\int_{-h}^0 q_i\left(-\varepsilon^{-\theta}x_3\right)\nabla_z\eta_k\left(z\left(x_1,x_2,x_3\right)\right)dx_3 \\ \times\left(\left(K_f\right)^{-1}v_f\right)_l(i\varepsilon,j\varepsilon,0)\displaystyle\int_{-h}^0 q_i\left(-\varepsilon^{-\theta}x_3\right)\nabla_z\eta_l\left(z\left(x_1,x_2,x_3\right)\right)dx_3\end{pmatrix}.\end{aligned}$$

We introduce the change of variables $(z_1, z_2) = z(x_1, x_2, x_3)$, where $z(x_1, x_2, x_3)$ has been defined in $(21)_1$, and get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mu\varepsilon^2 \sum_{(i,j)\in I_\varepsilon(\omega)} \int_{Y_{\varepsilon,ij}(\omega)} \left|\nabla\left(v_\varepsilon^0\right)_\tau\right|^2 dx \\
&\quad = \mu\lim_{\varepsilon\to 0} \sum_{(i,j)\in I_\varepsilon(\omega)} \frac{\varepsilon^2}{\langle q\rangle^2} \int_{-h}^{0}\int_{Z'} \left(\begin{array}{c} \left((K_f)^{-1} v_f\right)_k (i\varepsilon, j\varepsilon, 0) \left((K_f)^{-1} v_f\right)_l (i\varepsilon, j\varepsilon, 0) \\ \times\nabla_z \eta_k(z)\cdot\nabla_z\eta_l(z)\, q_i\left(-\varepsilon^{-\theta}x_3\right) q_j\left(-\varepsilon^{-\theta}x_3\right) dz dx_3 \end{array}\right).
\end{aligned}$$

Using the ergodicity result (2) and the definition $(19)_1$ of K_f, the above limit is equal to

$$\begin{aligned}
&\mu\frac{h\langle q^2\rangle}{\langle q\rangle^2}\int_\Sigma \left(\left((K_f)^{-1} v_f\right)_k \left((K_f)^{-1} v_f\right)_l\right)(x')\, dx' \int_{Z'} \nabla_z\eta_k(z)\cdot\nabla_z\eta_l(z)\, dz \\
&\quad = \mu\frac{h\langle q^2\rangle}{\langle q\rangle^2}\int_\Sigma (K_f)^{-1}(v_f)_\tau\cdot(v_f)_\tau\, dx',
\end{aligned}$$

because v_f is independent of x_3 in Y_h. Using a similar argument, we have

$$\lim_{\varepsilon\to 0}\mu\varepsilon^2 \sum_{(i,j)\in I_\varepsilon(\omega)} \int_{Y_{\varepsilon,ij}(\omega)} \left|\nabla\left(v_\varepsilon^0\right)_3\right|^2 dx = \mu h \frac{\langle q^2\rangle\langle 1/q^2\rangle}{k_0}\int_\Sigma \left((v_f)_3\right)^2 dx'.$$

On the other hand, observe that, for every $\psi\in\mathbf{C}_c^1(\Omega;\mathbb{R}^3)$ we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega_f^{+,\varepsilon}} v_\varepsilon^0\cdot\psi dx &= \int_{\Omega^+}\left(\left(\widehat{K}^+\right)^{-1} v_d^+\right)_j \int_{Z^1}\Phi_j^+(z)\, dz\cdot\psi dx \\
&= \int_{\Omega^+} v_d^+\cdot\psi dx, \\
\lim_{\varepsilon\to 0}\int_{\Gamma_{0,\varepsilon}^+} K_\varepsilon^+(x)\cdot\psi(x)\, dx &= \lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\varepsilon^2\int_{a_i^-(0)}^{a_i^+(0)}\int_{a_j^-(0)}^{a_j^+(0)} K^+(y',0)\, dy'\cdot\psi(i\varepsilon, j\varepsilon, 0) \\
&= \int_\Sigma\left(\int_{\langle a^-(0)\rangle}^{\langle a^+(0)\rangle}\int_{\langle a^-(0)\rangle}^{\langle a^+(0)\rangle} K^+(z_1, z_2, 0)\, dz_1 dz_2\right)\cdot\psi(x', 0)\, dx'.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(\begin{array}{c} \mu^+\int_{\Omega^+}\left(K_\varepsilon^+\right)^{-1} v_\varepsilon^0\cdot v_\varepsilon^0 dx + \mu^-\int_{\Omega_h^-}\left(K_\varepsilon^-\right)^{-1} v_\varepsilon^0\cdot v_\varepsilon^0 dx \\ +\gamma\int_{\Gamma_{0,\varepsilon}^+}\left(K_\varepsilon^+\right)^{-1/2}\left(v_\varepsilon^0\right)_\tau\cdot\left(v_\varepsilon^0\right)_\tau dx' + \gamma\int_{\Gamma_{h,\varepsilon}^-}\left(K_\varepsilon^-\right)^{-1/2}\left(v_\varepsilon^0\right)_\tau\cdot\left(v_\varepsilon^0\right)_\tau dx' \end{array}\right) \\
&= \mu^+\int_{\Omega^+}\left(\widehat{K}^+\right)^{-1} v_d^+\cdot v_d^+ dx + \mu^-\int_{\Omega_h^-}\left(\widehat{K}^-\right)^{-1} v_d^-\cdot v_d^- dx \\
&\quad + \langle q^2\rangle\gamma\int_\Sigma\left(\left(K^{*+}\right)^{-1/2} + \left(K^{*-}\right)^{-1/2}\right)(v_f)_\tau\cdot(v_f)_\tau\, dx',
\end{aligned}$$

whence $\lim_{\varepsilon\to 0} F_\varepsilon\left(v_\varepsilon^0\right) = F_0(v)$.

For every $v \in \mathbf{V}_0$, there exists a sequence $(v_m)_m \subset \mathbf{C}^1\left(\overline{\Omega^+ \cup \Omega_h^-};\mathbb{R}^3\right) \cap \mathbf{V}_0$ such that

$$v_m \underset{m\to+\infty}{\to} v, \text{ s-}\mathbf{L}^2\left(\Omega^+ \cup \Omega_h^-;\mathbb{R}^3\right). \tag{23}$$

Building the sequence $\left((v_m)_\varepsilon^0\right)_\varepsilon$ associated to v_m through (22), the sequence $\left((v_m)_\varepsilon^0\right)_\varepsilon$ τ_0-converges to v_m, and using the above computations for smooth functions, we have: $\lim_{\varepsilon\to 0} F_\varepsilon\left((v_m)_\varepsilon^0\right) = F_0(v_m)$. Hence $\lim_{m\to\infty}\lim_{\varepsilon\to 0} F_\varepsilon\left((v_m)_\varepsilon^0\right) = F_0(v)$. Using the diagonalization argument of [4, Corollary 1.18], there exists a sequence $(v_\varepsilon^0)_\varepsilon$, $v_\varepsilon^0 = \left(v_{m(\varepsilon)}\right)_\varepsilon^0$ ($m(\varepsilon) \to_{\varepsilon\to 0} +\infty$), such that $(v_\varepsilon^0)_\varepsilon$ τ_0-converges to v, and $\limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon^0) \le F_0(v)$.

Lower Γ-limit. Let $(v_\varepsilon^1)_\varepsilon$ be a sequence such that $v_\varepsilon^1 \in \mathbf{V}_\varepsilon$ for every ε, and $(v_\varepsilon^1)_\varepsilon$ τ_0-converges to v. We write the subdifferential inequality

$$\mu\varepsilon^2\int_{Y_\varepsilon(\omega)}\left|\nabla v_\varepsilon^1\right|^2 dx \ge \mu\varepsilon^2\int_{Y_\varepsilon(\omega)}\left|\nabla (v_m)_\varepsilon^0\right|^2 dx + 2\mu\varepsilon^2\int_{Y_\varepsilon(\omega)}\nabla (v_m)_\varepsilon^0\cdot\left(\nabla v_\varepsilon^1 - \nabla (v_m)_\varepsilon^0\right) dx, \tag{24}$$

where $\left((v_m)_\varepsilon^0\right)_\varepsilon$ is the sequence associated to v_m through (22) and where the sequence $(v_m)_m$ satisfies the conditions (23). Observe that

$$\int_{Y_{\varepsilon,ij}(\omega)}\nabla (v_m)_\varepsilon^0\cdot\left(\nabla v_\varepsilon^1 - \nabla (v_m)_\varepsilon^0\right) dx = -\int_{Y_{\varepsilon,ij}(\omega)}\Delta (v_m)_\varepsilon^0\cdot\left(v_\varepsilon^1 - (v_m)_\varepsilon^0\right) dx,$$

because $v_\varepsilon^1 - (v_m)_\varepsilon^0 = 0$ on $\partial Y_\varepsilon(\omega)\setminus\left(\Gamma_{0,\varepsilon}^+ \cup \Gamma_{h,\varepsilon}^-\right)(\omega)$, and $\frac{\partial (v_m)_\varepsilon^0}{\partial x_3}\,|_{x_3=0} = \frac{\partial (v_m)_\varepsilon^0}{\partial x_3}\,|_{x_3=-h} = 0$. Then, using the ergodicity result (2), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} 2\mu\varepsilon^2\int_{Y_\varepsilon(\omega)}\nabla (v_m)_\varepsilon^0\cdot\left(\nabla v_\varepsilon^1 - \nabla (v_m)_\varepsilon^0\right) dx\\
&= -\mu\frac{h\langle q^2\rangle\langle 1/q^2\rangle}{\langle q\rangle}\int_\Sigma\int_{Z'}\Delta_z\eta_k(z)\left((K_f)^{-1}(v_m)_\tau\right)_k\cdot(v - v_m)_\tau(x')\,dz dx'\\
&\quad -\mu\frac{h\langle q^2\rangle\langle 1/q^2\rangle}{k_0}\int_\Sigma\int_{Z'}\Delta_z\eta_0(z)(v_m)_3(v - v_m)_3(x')\,dz dx',
\end{aligned}$$

whence

$$\lim_{m\to\infty}\lim_{\varepsilon\to 0} 2\mu\varepsilon^2\int_{Y_\varepsilon(\omega)}\nabla (v_m)_\varepsilon^0\cdot\left(\nabla v_\varepsilon^1 - \nabla (v_m)_\varepsilon^0\right) dx = 0.$$

Recalling the inequality (24) and the computations built in the above case of smooth functions, we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\mu\varepsilon^2\int_{Y_\varepsilon(\omega)}\left|\nabla v_\varepsilon^1\right|^2 dx\\
&\ge \mu\frac{h\langle q^2\rangle}{\langle q\rangle^2}\int_\Sigma (K_f)^{-1}(v_f)_\tau\cdot(v_f)_\tau\,dx' + \mu h\frac{\langle q^2\rangle}{k_0}\langle 1/q^2\rangle\int_\Sigma\left((v_f)_3\right)^2 dx'.
\end{aligned}$$

Thus, computing in an easy way the lim inf of the other terms in $F_\varepsilon(v_\varepsilon^1)$, we obtain: $\liminf_\varepsilon F_\varepsilon(v_\varepsilon^1) \ge F_0(v)$. ■

Let us write the problem associated to the limit functional F_0.

Corollary 10 *The solution $(v_\varepsilon, p_\varepsilon)$ of the problem (6)-(7) verifies the following properties:*

- $(v_\varepsilon)_\varepsilon$ *τ_0-converges to $v_0 \in \mathbf{V}_0$, and set $v_0\,|_{\Omega^+}= v_{0,d}^+$, $v_0\,|_{\Omega_h^-}= v_{0,d}^-$, $v_0\,|_\Sigma = v_{0,f}$.*
- $(p_\varepsilon^+)_\varepsilon$ *converges to p_0^+, s-$L^2(\Omega^+)$, $(p_\varepsilon^-)_\varepsilon$ converges to p_0^-, s-$L^2\left(\Omega_h^-\right)$ and $(16)_3$ holds true.*
- $v_{0,d}^+$, $v_{0,d}^-$, $v_{0,f}$, p_0^+, p_0^- *and π_0 are solutions of the following problems:*

i) in the regions Ω^+ and Ω_h^-, one has the Darcy laws

$$\left\{\begin{array}{rcll} \mu^+\left(\widehat{K}^+\right)^{-1} v_{0,d}^+ - \nabla p_0^+ &=& g^+ & \text{in } \Omega^+,\\ \operatorname{div}\left(v_{0,d}^+\right) &=& 0 & \text{in } \Omega^+,\\ \mu^-\left(\widehat{K}^-\right)^{-1} v_{0,d}^- - \nabla p_0^- &=& g^- & \text{in } \Omega_h^-,\\ \operatorname{div}\left(v_{0,d}^-\right) &=& 0 & \text{in } \Omega_h^-,\\ \left(v_{0,d}^+\right)_3 |_{\Sigma\times\{0\}} &=& \langle q^2\rangle \left(v_{0,f}\right)_3 & \text{on } \Sigma\times\{0\},\\ \left(v_{0,d}^-\right)_3 |_{\Sigma\times\{-h\}} &=& \langle q^2\rangle \left(v_{0,f}\right)_3 & \Sigma\times\{-h\}, \end{array}\right. \tag{25}$$

ii) on Σ, the velocity $(v_{0,f})_3$ is given through

$$(v_{0,f})_3(x') = \left(p_0^+(x',0) - p_0^-(x',-h)\right)\frac{k_0}{\mu h \langle q^2(0)\rangle \langle 1/q^2(0)\rangle} \tag{26}$$

and the tangential velocity $(v_{0,f})_\tau$ satisfies the modified Darcy law

$$\left\{\begin{array}{rcll} \dfrac{\mu}{\langle q\rangle^2}(K_f)^{-1}(v_{0,f})_\tau + \dfrac{\gamma}{h}\left((K^{*+})^{-1/2} + (K^{*-})^{-1/2}\right)(v_{0,f})_\tau + \nabla\pi_0 &=& 0 & \text{in } \Sigma,\\ \operatorname{div}(v_{0,f})_\tau &=& 0 & \text{in } \Sigma,\\ (v_{0,f})_\tau\cdot n &=& 0 & \text{on } \partial\Sigma. \end{array}\right.$$

Proof. Thanks to the properties of the Γ-convergence, the sequence $(v_\varepsilon)_\varepsilon$ τ_0-converges to $v_0 \in \mathbf{V}_0$ and $\lim_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) = F_0(v_0)$, where v_0 is the minimizer of the problem

$$\inf_{v\in\mathbf{V}^0}\left(F_0(v) - 2\int_{\Omega^+} g^+\cdot v dx - 2\int_{\Omega_h^-} g^-\cdot v dx\right).$$

For every $V\in\mathbf{V}_0$, we have the following identity

$$\begin{aligned} &\mu^+\int_{\Omega^+}\left(\widehat{K}^+\right)^{-1} v_{0,d}\cdot V dx + \mu^-\int_{\Omega_h^-}\left(\widehat{K}^-\right)^{-1} v_{0,d}\cdot V dx\\ &+\mu h\frac{\langle q^2\rangle}{\langle q\rangle^2}\int_\Sigma (K_f)^{-1}(v_{0,f})_\tau\cdot V_\tau dx' + \mu h\frac{\langle q^2\rangle}{k_0}\langle 1/q^2\rangle\int_\Sigma (v_{0,f})_3 V_3 dx'\\ &+\langle q^2\rangle\gamma\int_\Sigma\left((K^{*+})^{-1/2} + (K^{*-})^{-1/2}\right)(v_{0,f})_\tau\cdot V_\tau dx'\\ &=\int_{\Omega^+} g^+\cdot V dx + \int_{\Omega_h^-} g^-\cdot V dx. \end{aligned}$$

We infer the existence of a pressure p_0^+ (resp. p_0^-, π_0) in Ω^+ (resp. Ω_h^-, Σ) such that

$$\int_{\Omega^+} \nabla p_0^+ \cdot V dx + \int_{\Omega_h^-} \nabla p_0^- \cdot V dx + h \int_\Sigma \nabla \pi_0 \cdot V_\tau dx' + \mu h \frac{\langle q^2 \rangle \langle 1/q^2 \rangle}{k_0} \int_\Sigma (v_{0,f})_3 V_3 dx' = 0,$$

which implies, because V (resp. V_τ) is divergence-free in $\Omega^+ \cup \Omega_h^-$ (resp. Σ)

$$\int_\Sigma \left(- \left(p_0^+ (x', 0) - p_0^- (x', -h) \right) + \mu h \frac{\langle q^2 \rangle \langle 1/q^2 \rangle}{k_0} (v_{0,f})_3 \right) V_3 dx' + h \int_{\partial \Sigma} \pi_0 n \cdot V_\tau d\sigma.$$

This gives the result. ■

4 Study of the transport problem

Let us now consider the transport problem (5). In this section, we will describe the asymptotic behaviour of the solution u_ε of (5), when ε goes to 0, distinguishing between the cases $\mathcal{R} = 0$ and $\mathcal{R} \neq 0$.

4.1 Existence of a weak solution and a priori estimates

We define the space

$$H^1_{\Gamma^+ \cup \Gamma^-} \left(\Omega_f^\varepsilon (\omega) \right) = \left\{ u \in H^1 \left(\Omega_f^\varepsilon (\omega) \right) \mid u = 0 \text{ on } \Gamma^+ \cup \Gamma^- \right\}.$$

Lemma 11 *1. The problem (5) has a unique weak solution $u_\varepsilon \in H^1_{\Gamma^+ \cup \Gamma^-} \left(\Omega_f^\varepsilon (\omega) \right)$ which is nonnegative in $\Omega_f^\varepsilon (\omega)$.*

2. There exists a non-random constant C which is independent of ε such that

$$\int_{\Omega_f^\varepsilon (\omega)} (u_\varepsilon)^2 dx \leq C \ ; \ \int_{\Omega_f^\varepsilon (\omega)} |\nabla u_\varepsilon|^2 dx \leq C.$$

3. There exists a linear and bounded extension operator $P^\varepsilon : H^1 \left(\Omega_f^\varepsilon (\omega) \right) \to H^1 (\Omega)$ and two non-random positive constants C_1 and C_2 such that

$$\begin{aligned} P^\varepsilon u_\varepsilon &= u_\varepsilon & \text{in } \Omega_f^\varepsilon (\omega), \\ \int_\Omega |P^\varepsilon u_\varepsilon|^2 dx &\leq C_1 \int_{\Omega_f^\varepsilon (\omega)} |u_\varepsilon|^2 dx, \\ \int_\Omega |\nabla P^\varepsilon u_\varepsilon|^2 dx &\leq C_2 \int_{\Omega_f^\varepsilon (\omega)} |\nabla u_\varepsilon|^2 dx. \end{aligned}$$

Proof. 1. Using the standard variational methods, one proves that the problem (5) has a unique solution $u_\varepsilon \in H^1_{\Gamma^+\cup\Gamma^-}\left(\Omega^\varepsilon_f(\omega)\right)$. Multiplying (5) par $(u_\varepsilon)^- = \min(0, u_\varepsilon)$ and using Green's formula, one has

$$D\int_{\Omega^\varepsilon_f(\omega)} \left|\nabla (u_\varepsilon)^-\right|^2 dx + \int_{\Omega^\varepsilon_f(\omega)} \left(v_\varepsilon \cdot \nabla (u_\varepsilon)^-\right)(u_\varepsilon)^- dx + \mathcal{R}\int_{\Omega^\varepsilon_f(\omega)} \left((u_\varepsilon)^-\right)^2 dx$$
$$= \int_{\Omega^{+,\varepsilon}_f} f\,(u_\varepsilon)^- dx \le 0,$$

because f is nonnegative. Because $\operatorname{div}(v_\varepsilon) = 0$ in $\Omega^\varepsilon_f(\omega)$ and $v_\varepsilon \cdot n = 0$ on $\partial\Omega^\varepsilon_f(\omega)$, one has

$$\int_{\Omega^\varepsilon_f(\omega)} (v_\varepsilon \cdot \nabla u_\varepsilon)\, u_\varepsilon dx = -\int_{\Omega^\varepsilon_f(\omega)} (v_\varepsilon \cdot \nabla u_\varepsilon)\, u_\varepsilon dx = 0.$$

We deduce that $\int_{\Omega^\varepsilon_f(\omega)} \left|\nabla (u_\varepsilon)^-\right|^2 dx + \mathcal{R}\int_{\Omega^\varepsilon_f(\omega)} \left((u_\varepsilon)^-\right)^2 dx \le 0$, thus $(u_\varepsilon)^- = 0$, in $\Omega^\varepsilon_f(\omega)$, hence u_ε is nonnegative in $\Omega^\varepsilon_f(\omega)$.

2. As already observed, we have

$$D\int_{\Omega^\varepsilon_f(\omega)} |\nabla u_\varepsilon|^2 dx + \mathcal{R}\int_{\Omega^\varepsilon_f(\omega)} (u_\varepsilon)^2 dx = \int_{\Omega^{+,\varepsilon}_f} f u_\varepsilon dx.$$

For $\mathcal{R} \neq 0$, one deduces from this equality, using Cauchy-Schwarz' inequality, that $\int_{\Omega^\varepsilon_f(\omega)} (u_\varepsilon)^2 dx \le C$ and $\int_{\Omega^\varepsilon_f(\omega)} |\nabla u_\varepsilon|^2 dx \le C$.

In the case $\mathcal{R} = 0$, one can prove, using [2, Lemma 3.4], that there exists a non-random constant C independent of ε such that

$$\int_{\Omega^\varepsilon_f(\omega)} (u_\varepsilon)^2 dx \le C \int_{\Omega^\varepsilon_f(\omega)} |\nabla u_\varepsilon|^2 dx.$$

Thus, using the above equality, we get the desired estimates.

3. This is a particular case of the result given in [1]. ■

We will still denote by u_ε its extension $P^\varepsilon u_\varepsilon$ to the whole Ω. From Lemmas 11 and 6, we deduce the existence of $u_0^+ \in H^1_{\Gamma^+}(\Omega^+)$ and $u_0^- \in H^1_{\Gamma^-}\left(\Omega^-_h\right)$, such that, up to some subsequence

$$\begin{cases} u_\varepsilon\,|_{\Omega^+} \underset{\varepsilon\to 0}{\rightharpoonup} u_0^+ & \text{w-}H^1_{\Gamma^+}(\Omega^+)\,, \\ u_\varepsilon\,|_{\Omega^-_h} \underset{\varepsilon\to 0}{\rightharpoonup} u_0^- & \text{w-}H^1_{\Gamma^-}\left(\Omega^-_h\right), \\ \lim\limits_{\varepsilon\to 0}\int_{Y_\varepsilon(\omega)} u_\varepsilon \varphi dx = h\left\langle q^2(0)\right\rangle \int_\Sigma u_0^+(x',0)\,\varphi(x')\,dx' & \forall\varphi \in C_0(\Sigma)\,. \end{cases} \tag{27}$$

We intend to describe the problems satisfied by u_0^+ and u_0^-, in their respective domains. We now define the notion of convergence associated to sequences satisfying the above convergences.

Definition 12 *A sequence $(U_\varepsilon)_\varepsilon$, with $U_\varepsilon \in H^1_{\Gamma^+\cup\Gamma^-}\left(\Omega^\varepsilon_f(\omega)\right)$ for every ε, τ_1-converges to U, with $U\,|_{\Omega^+} =: U^+ \in H^1_{\Gamma^+}(\Omega^+)$ and $U\,|_{\Omega^-_h} =: U^- \in H^1_{\Gamma^-}\left(\Omega^-_h\right)$, if the convergences (27) are satisfied, replacing u_ε by U_ε.*

4.2 The asymptotic behaviour in the case $\mathcal{R} = 0$

In this subsection, we deal with the case $\mathcal{R} = 0$. Using the boundary conditions $(5)_{2,3}$, we consider the variational formulation of the problem (5)

$$\forall u \in H^1_{\Gamma^+ \cup \Gamma^-}\left(\Omega^\varepsilon_f(\omega)\right) : D\int_{\Omega^\varepsilon_f(\omega)} \nabla u_\varepsilon \cdot \nabla u dx + \int_{\Omega^\varepsilon_f(\omega)} (v_\varepsilon \cdot \nabla u_\varepsilon)\, u dx = \int_{\Omega^{+,\varepsilon}_f} f u dx,$$

where v_ε is the velocity of the fluid flow, that is the solution of (6)-(7). We consider the Z-periodic solution b_j of the cell problem

$$\left\{\begin{array}{rcll} \Delta b_j & = & 0 & \text{in } Z^1 \quad j = 1,2,3, \\ (\nabla b_j + e_j)\cdot n & = & 0 & \text{on } S \end{array}\right. \tag{28}$$

and the problem

$$\left\{\begin{array}{rcll} \Delta c_m & = & 0 & \text{in } Z' \qquad m = 1,2, \\ (\nabla c_m + e_m)\cdot n & = & 0 & \text{on } \partial Z'. \end{array}\right.$$

We define the tensors $\widehat{D}$ and D^* through

$$\left\{\begin{array}{rcl} \widehat{D}_{ij} & = & D\left(\left|Z^1\right|\delta_{ij} + \displaystyle\int_{Z^1}\frac{\partial b_j}{\partial z_i}dz\right), \\ D^*_{ml} & = & D\left(\delta_{ml} + \displaystyle\int_{Z'}\frac{\partial c_l}{\partial z_m}dz'\right). \end{array}\right.$$

Let χ^ε_+ (resp. χ^ε_-) be the characteristic function of $\Omega^{+,\varepsilon}_f(\omega)$ (resp. $\Omega^{-,\varepsilon}_{h,f}(\omega)$). We have the following result.

Lemma 13 *One has, up to some subsequence:*

1. $\chi^\varepsilon_+ D\nabla u_\varepsilon \rightharpoonup_{\varepsilon\to 0} \widehat{D}\nabla u^+_0$, *in* $\mathbf{L}^2(\Omega^+;\mathbb{R}^3)$*-weak,*
2. $\chi^\varepsilon_- D\nabla u_\varepsilon \rightharpoonup_{\varepsilon\to 0} \widehat{D}\nabla u^-_0$, *in* $\mathbf{L}^2\left(\Omega^-_h;\mathbb{R}^3\right)$*-weak,*
3. $\lim_{\varepsilon\to 0}\int_{Y_\varepsilon(\omega)} D\nabla_\tau u_\varepsilon\cdot\varphi dx = h\left\langle q^2(0)\right\rangle\int_\Sigma D^*\nabla_\tau u^+_0(x',0)\cdot\varphi dx'$, $\forall\varphi\in\mathbf{C}^\infty_c(\Sigma;\mathbb{R}^2)$, *where* $\nabla_\tau u^+_0 = \left(\frac{\partial u^+_0}{\partial x_1},\frac{\partial u^+_0}{\partial x_2}\right)$.

Proof. 1. Let $\varphi\in C^\infty_c(\Omega^+)$, $b^\varepsilon_j = \varepsilon b_j(x/\varepsilon)$. Multiplying (5) (for $\mathcal{R}=0$) by $\chi^\varepsilon_+ b^\varepsilon_j\varphi$, we get

$$\int_{\Omega^+}\chi^\varepsilon_+ D\nabla u_\varepsilon\cdot\left(\varphi\nabla b^\varepsilon_j + b^\varepsilon_j\nabla\varphi\right)dx + \int_{\Omega^+}\chi^\varepsilon_+ v_\varepsilon\cdot\nabla u_\varepsilon b^\varepsilon_j\varphi dx = \int_{\Omega^+}\chi^\varepsilon_+ f b^\varepsilon_j\varphi dx,$$

from which we deduce that

$$\lim_{\varepsilon\to 0}\int_{\Omega^+}\chi^\varepsilon_+ D\nabla u_\varepsilon\varphi\cdot\nabla b^\varepsilon_j dx = 0. \tag{29}$$

Observe now that, through (28)

$$\begin{aligned}\int_{\Omega^+}\chi_+^\varepsilon D\nabla\left(\varphi u_\varepsilon\right)\cdot\left(\nabla b_j^\varepsilon+e_j\right)dx &= D\int_{\Omega^+}\chi_+^\varepsilon\left(\nabla b_j^\varepsilon+e_j\right)\varphi\cdot\nabla u_\varepsilon dx\\ &\quad +D\int_{\Omega^+}\chi_+^\varepsilon\left(\nabla b_j^\varepsilon+e_j\right)u_\varepsilon\cdot\nabla\varphi dx\\ &= 0.\end{aligned}$$

Thus, taking into account (29), one has, up to some subsequence

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega^+}\chi_+^\varepsilon D\varphi\left(e_j\cdot\nabla u_\varepsilon\right)dx &= -\lim_{\varepsilon\to 0}D\int_{\Omega^+}\chi_+^\varepsilon\left(\nabla b_j^\varepsilon+e_j\right)u_\varepsilon\cdot\nabla\varphi dx\\ &= -D\sum_{i=1}^{3}\left(\left|Z^1\right|\delta_{ij}+\int_{Z^1}\frac{\partial b_j}{\partial z_i}dz\right)\int_{\Omega^+}\frac{\partial\varphi}{\partial x_i}u_0^+dx\\ &= D\sum_{i=1}^{3}\left(\left|Z^1\right|\delta_{ij}+\int_{Z^1}\frac{\partial b_j}{\partial z_i}dz\right)\int_{\Omega^+}\frac{\partial u_0^+}{\partial x_i}\varphi dx.\end{aligned}$$

2. In a similar way than above, we get, for every $\varphi\in C_c^\infty\left(\Omega_h^-\right)$

$$\lim_{\varepsilon\to 0}\int_{\Omega_h^-}\chi_-^\varepsilon D\varphi\left(e_j\cdot\nabla u_\varepsilon\right)dx = D\sum_{i=1}^{3}\left(\left|Z^1\right|\delta_{ij}+\int_{Z^1}\frac{\partial b_j}{\partial z_i}dz\right)\int_{\Omega_h^-}\frac{\partial u_0^-}{\partial x_i}\varphi dx.$$

3. We define the quantity c_m^ε, $m=1,2$, through

$$c_m^\varepsilon\left(x\right)=\varepsilon\left\langle q\left(0\right)\right\rangle c_m\begin{pmatrix}\dfrac{x_1-i\varepsilon-\varepsilon\left(a_i^-\left(-\varepsilon^{-\theta}x_3\right)+a_i^+\left(-\varepsilon^{-\theta}x_3\right)\right)/2}{\varepsilon q_i\left(-\varepsilon^{-\theta}x_3\right)}\\ \dfrac{x_2-j\varepsilon-\varepsilon\left(a_j^-\left(-\varepsilon^{-\theta}x_3\right)+a_j^+\left(-\varepsilon^{-\theta}x_3\right)\right)/2}{\varepsilon q_j\left(-\varepsilon^{-\theta}x_3\right)}\end{pmatrix},$$

Then, using curvilinear coordinates, the ergodic result (2) and making some computations as before, we get the result. ■

Our main result in this subsection reads as follows.

Theorem 14 *The sequence $\left(u_\varepsilon\right)_\varepsilon$, where u_ε is the solution of (5), converges in the topology τ_1 to the solution $\left(u_0^+,u_0^-\right)$ of the variational formulation*

$$\begin{aligned}&\forall\left(u^+,u^-\right)\in H^1_{\Gamma^+}\left(\Omega^+\right)\times H^1_{\Gamma^-}\left(\Omega_h^-\right): \int_{\Omega^+}\widehat{D}\nabla u_0^+\cdot\nabla u^+dx+\int_{\Omega_h^-}\widehat{D}\nabla u_0^-\cdot\nabla u^-dx\\ &\quad+\int_{\Omega^+}\left(v_{0,d}^+\cdot\nabla u_0^+\right)u^+dx+\int_{\Omega_h^-}\left(v_{0,d}^-\cdot\nabla u_0^-\right)u^-dx\\ &\quad+h\left\langle q^2\left(0\right)\right\rangle\int_\Sigma D^*\nabla_\tau u_0^+\cdot\nabla_\tau u^+dx'-h\left\langle q^2\left(0\right)\right\rangle\int_\Sigma\left(\left(v_0\right)_\tau\cdot\nabla_\tau u^+\right)u_0^+dx'\\ &\quad+\frac{D}{h\left\langle 1/q^2\right\rangle}\int_\Sigma\left(u_0^+\left(u^--u^+\right)-u_0^-\left(u^--u^+\right)\exp\left(\frac{p_0^+-p_0^-}{D\langle q^2\rangle\left\langle 1/q^2\right\rangle}\frac{k_0}{\mu}\right)\right)dx'\\ &\quad=\left|Z^1\right|\int_{\Omega^+}fu^+dx,\end{aligned}\tag{30}$$

where $v_{0,d}^{+}$ and $v_{0,d}^{-}$ are the limit velocities appearing in Remark 7 and p_0^+ and p_0^- are the pressures appearing in Corollary 10.

Before starting the proof of Theorem 14, let us introduce the constant "vertical" velocity $(v_{\varepsilon,ij})_3$ in the fissure $Y_{\varepsilon,ij}(\omega)$, $(i,j) \in I_\varepsilon(\omega)$, defined as

$$(v_{\varepsilon,ij})_3 = \left(p_{0,\varepsilon,ij}^{+} - p_{0,\varepsilon,ij}^{-}\right) \frac{k_0}{\mu h \langle q^2 \rangle \langle 1/q^2 \rangle}, \tag{31}$$

where $p_{0,\varepsilon,ij}^{+} = p_0^+(i\varepsilon, j\varepsilon, 0)$, $p_{0,\varepsilon,ij}^{-} = p_0^-(i\varepsilon, j\varepsilon, -h)$, p_0^+ and p_0^- being the pressures defined in Corollary 10 (compare to (26)). Inside the fissure $Y_{\varepsilon,ij}(\omega)$, for every $(i,j) \in I_\varepsilon(\omega)$, we define, for every $u \in C^2\left(\overline{\Omega}\right)$ satisfying $u = 0$ on $\Gamma = \Gamma^+ \cup \Gamma^-$

$$\begin{aligned} \bar{u}_{\varepsilon,ij}(x_1,x_2,x_3) &= u^+(x_1,x_2,0) \\ &+ \frac{\left(u^-(x_1,x_2,-h) - u^+(x_1,x_2,0)\right) \displaystyle\int_{x_3}^{0} \frac{1}{q_i(-\varepsilon^{-\theta}t)\, q_j(-\varepsilon^{-\theta}t)} \exp\left(-\frac{t\,(v_{\varepsilon,ij})_3}{D}\right) dt}{\displaystyle\int_{-h}^{0} \frac{1}{q_i(-\varepsilon^{-\theta}t)\, q_j(-\varepsilon^{-\theta}t)} \exp\left(-\frac{t\,(v_{\varepsilon,ij})_3}{D}\right) dt}, \end{aligned} \tag{32}$$

where $q_i(s)$ is defined in $(21)_2$. We finally define the test-function $u_{0,\varepsilon}$ through

$$u_{0,\varepsilon}(x) = \begin{cases} u^+(x) & \text{in } \Omega^+, \\ \bar{u}_{\varepsilon,ij}(x) & \text{in } Y_{\varepsilon,ij}(\omega),\ \forall (i,j) \in I_\varepsilon(\omega), \\ u^-(x) & \text{in } \Omega_h^-. \end{cases} \tag{33}$$

The properties of this test-function are gathered in the following result.

Lemma 15 *1. One has:*

$$\begin{cases} -D\dfrac{\partial}{\partial x_3}\left(q_i q_j\left(-\varepsilon^{-\theta}x_3\right)\dfrac{\partial \bar{u}_{\varepsilon,ij}}{\partial x_3}\right)(x) & & \\ \quad -(q_i q_j)\left(-\varepsilon^{-\theta}x_3\right)(v_{\varepsilon,ij})_3 \dfrac{\partial \bar{u}_{\varepsilon,ij}}{\partial x_3}(x) = 0 & & \text{in } Y_{\varepsilon,ij}(\omega), \\ \bar{u}_{\varepsilon,ij}(x_1,x_2,0) = u^+(x_1,x_2,0), & & \text{on } \Gamma_{0,\varepsilon,ij}^{+}(\omega), \\ \bar{u}_{\varepsilon,ij}(x_1,x_2,-h) = u^-(x_1,x_2,-h) & & \text{on } \Gamma_{h,\varepsilon,ij}^{-}(\omega). \end{cases} \tag{34}$$

2. *For every $\varepsilon > 0$, $u_{0,\varepsilon} \in H^1_{\Gamma^+ \cup \Gamma^-}(\Omega_\varepsilon(\omega))$.*

3. *The sequence $(u_{0,\varepsilon})_\varepsilon$ τ_1-converges to u.*

Proof. 1. This is an immediate consequence of the definition (32) of $\bar{u}_{\varepsilon,ij}$.

2. This is an immediate consequence of the construction (33) of $u_{0,\varepsilon}$, in Ω^+ and in Ω_h^-, and through the "boundary conditions" $(34)_{2,3}$ satisfied by $u_{0,\varepsilon}$ at the ends of the fissure $Y_{\varepsilon,ij}(\omega)$.

3. From this construction, we deduce that $(u_{0,\varepsilon}\mid_{\Omega^+})_\varepsilon$ (resp. $\left(u_{0,\varepsilon}\mid_{\Omega_h^-}\right)_\varepsilon$) converges to u^+ (resp. u^-) in $H^1(\Omega^+)$-strong (resp. $H^1\left(\Omega_h^-\right)$-strong).

Moreover, for every $\varphi \in C_0(\mathbb{R}^3)$, we define

$$A_\varepsilon = \int_{Y_\varepsilon(\omega)} \varphi \frac{\displaystyle\int_{x_3}^0 \frac{1}{q_i(-\varepsilon^{-\theta}t)\, q_j(-\varepsilon^{-\theta}t)} \exp\left(-\frac{t\,(v_{\varepsilon,ij})_3}{D}\right) dt}{\displaystyle\int_{-h}^0 \frac{1}{q_i(-\varepsilon^{-\theta}t)\, q_j(-\varepsilon^{-\theta}t)} \exp\left(-\frac{t\,(v_{\varepsilon,ij})_3}{D}\right) dt} dx.$$

One has

$$\begin{aligned}\lim_{\varepsilon\to 0} A_\varepsilon &= \lim_{\varepsilon\to 0} \sum_{(i,j)\in I_\varepsilon(\omega)} \frac{1}{\displaystyle\int_{-h}^0 \frac{1}{q_i(-\varepsilon^{-\theta}t)\, q_j(-\varepsilon^{-\theta}t)} \exp\left(-\frac{t\,(v_{\varepsilon,ij})_3}{D}\right) dt} \\ &\quad \times \int_{-h}^0 \varphi(\varepsilon i, \varepsilon j, x_3) \left(\begin{array}{c}\displaystyle\int_{x_3}^0 \frac{1}{q_i(-\varepsilon^{-\theta}t)\, q_j(-\varepsilon^{-\theta}t)} \\ \times \exp\left(-\dfrac{t\,(v_{\varepsilon,ij})_3}{D}\right) dt\end{array}\right) q^2\left(-\varepsilon^{-\theta}x_3\right) dx_3\end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0} A_\varepsilon &= \lim_{\varepsilon\to 0} \sum_{(i,j)\in I_\varepsilon(\omega)} \frac{1}{\displaystyle\int_{-h}^0 \frac{1}{q_i(-\varepsilon^{-\theta}t)\, q_j(-\varepsilon^{-\theta}t)} \exp\left(-\frac{t\,(v_{\varepsilon,ij})_3}{D}\right) dt} \\ &\quad \times \varepsilon^\theta \int_{-h\varepsilon^{-\theta}}^0 \varphi\left(\varepsilon i, \varepsilon j, \varepsilon^\theta \xi\right) \left(\begin{array}{c}\displaystyle\int_{\varepsilon^\theta \xi}^0 \frac{1}{q_i(-\varepsilon^{-\theta}t)\, q_j(-\varepsilon^{-\theta}t)} \\ \times \exp\left(-\dfrac{t\,(v_{\varepsilon,ij})_3}{D}\right) dt\end{array}\right) q^2(\xi)\, d\xi \\ &= 0.\end{aligned}$$

Thus

$$\begin{aligned}&\lim_{\varepsilon\to 0} \int_{Y_\varepsilon(\omega)} \varphi \frac{u^-(x_1,x_2,-h) - u^+(x_1,x_2,0)}{\displaystyle\int_{-h}^0 \frac{1}{q_i(-\varepsilon^{-\theta}t)\, q_j(-\varepsilon^{-\theta}t)} \exp\left(-\frac{t\,(v_{\varepsilon,ij})_3}{D}\right) dt} \\ &\quad \times \int_{x_3}^0 \frac{1}{q_i(-\varepsilon^{-\theta}t)\, q_j(-\varepsilon^{-\theta}t)} \exp\left(-\frac{t\,(v_{\varepsilon,ij})_3}{D}\right) dt dx = 0,\end{aligned}$$

from which we deduce that

$$\lim_{\varepsilon\to 0} \int_{Y_\varepsilon(\omega)} \varphi u_{0,\varepsilon} dx = h\left\langle q^2(0)\right\rangle \int_\Sigma u^+(x',0)\, \varphi(x',0)\, dx'.$$

Thus the sequence $(u_{0,\varepsilon})_\varepsilon$ τ_1-converges to u. ■

Proof of Theorem 14. Thanks to the boundary conditions $(6)_{4,7}$, we have

$$\int_{\Omega_f^\varepsilon(\omega)} (v_\varepsilon \cdot \nabla u_\varepsilon)\, u_{0,\varepsilon} dx = -\int_{\Omega_f^\varepsilon(\omega)} (v_\varepsilon \cdot \nabla u_{0,\varepsilon})\, u_\varepsilon dx.$$

Using Lemma 15 and the "compensated compactness" result (see [14]), we immediately deduce the following limits

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega^+}(v_\varepsilon\cdot\nabla u_\varepsilon)\,u_{0,\varepsilon}dx &= \int_{\Omega^+}\left(v^+_{0,d}\cdot\nabla u^+_0\right)u^+dx,\\ \lim_{\varepsilon\to 0}\int_{\Omega^-_h}(v_\varepsilon\cdot\nabla u_\varepsilon)\,u_{0,\varepsilon}dx &= \int_{\Omega^-_h}\left(v^-_{0,d}\cdot\nabla u^-_0\right)u^-dx,\end{aligned}$$

$u_{0,\varepsilon}$ being independent of ε in $\Omega^+\cup\Omega^-_h$. We then write

$$\begin{aligned}&D\int_{Y_\varepsilon(\omega)}\nabla u_\varepsilon\cdot\nabla u_{0,\varepsilon}dx-\int_{Y_\varepsilon(\omega)}(v_\varepsilon\cdot\nabla u_{0,\varepsilon})\,u_\varepsilon dx\\ &\quad= D\int_{Y_\varepsilon(\omega)}\nabla_\tau u_\varepsilon\cdot\nabla_\tau u_{0,\varepsilon}dx-\int_{Y_\varepsilon(\omega)}\left((v_\varepsilon)_\tau\cdot\nabla_\tau u_{0,\varepsilon}\right)u_\varepsilon dx\\ &\qquad+D\int_{Y_\varepsilon(\omega)}\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial u_{0,\varepsilon}}{\partial x_3}dx-\int_{Y_\varepsilon(\omega)}(v_{\varepsilon,s})_3\frac{\partial u_{0,\varepsilon}}{\partial x_3}u_\varepsilon dx.\end{aligned}$$

A direct computation gives

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(D\int_{Y_\varepsilon(\omega)}\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial u_{0,\varepsilon}}{\partial x_3}dx-\int_{Y_\varepsilon(\omega)}\left((v_{\varepsilon,s})_3\frac{\partial u_{0,\varepsilon}}{\partial x_3}\right)u_\varepsilon dx\right)\\ &\quad=\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{Y_{\varepsilon,ij}(\omega)}\left(D\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial \bar u_{\varepsilon,ij}}{\partial x_3}q_iq_j-(v_{\varepsilon,s})_3\frac{\partial \bar u_{\varepsilon,ij}}{\partial x_3}u_\varepsilon q_iq_j\right)dx\\ &\quad=\lim_{\varepsilon\to 0}\left(\begin{array}{l}\displaystyle\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{Y_{\varepsilon,ij}(\omega)}\left(-D\frac{\partial}{\partial x_3}\left(q_iq_j\frac{\partial \bar u_{\varepsilon,ij}}{\partial x_3}\right)-(v_{\varepsilon,s})_3\,q_iq_j\frac{\partial \bar u_{\varepsilon,ij}}{\partial x_3}\right)u_\varepsilon dx\\ \displaystyle\quad -D\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^+_{0,\varepsilon,ij}(\omega)}\frac{\partial \bar u_{\varepsilon,ij}}{\partial x_3}\,|_{x_3=0}\;q_iq_j\,(0)\,u_\varepsilon dx'\\ \displaystyle\quad +D\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^-_{h,\varepsilon,ij}(\omega)}\left(\frac{\partial \bar u_{\varepsilon,ij}}{\partial x_3}\,|_{x_3=-h}\right)q_iq_j\left(-h\varepsilon^{-\theta}\right)u_\varepsilon dx'\end{array}\right).\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(D\int_{Y_\varepsilon(\omega)}\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial u_{0,\varepsilon}}{\partial x_3}dx-\int_{Y_\varepsilon(\omega)}\left((v_{\varepsilon,s})_3\frac{\partial u_{0,\varepsilon}}{\partial x_3}\right)u_\varepsilon dx\right)\\ &\quad=\lim_{\varepsilon\to 0}\left(\begin{array}{l}\displaystyle\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{Y_{\varepsilon,ij}(\omega)}\left((v_{\varepsilon,ij})_3-(v_{\varepsilon,s})_3\right)\frac{\partial \bar u_{\varepsilon,ij}}{\partial x_3}q_iq_ju_\varepsilon dx\\ \displaystyle\quad -D\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^+_{0,\varepsilon,ij}(\omega)}\left(q_iq_j\,(0)\,\frac{\partial \bar u_{\varepsilon,ij}}{\partial x_3}\,|_{x_3=0}\;u^+_\varepsilon\right)dx'\\ \displaystyle\quad -D\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^-_{h,\varepsilon,ij}(\omega)}\left(\frac{\partial \bar u_{\varepsilon,ij}}{\partial x_3}\,|_{x_3=-h}\right)q_iq_j\left(-h\varepsilon^{-\theta}\right)u^-_\varepsilon dx'\end{array}\right).\end{aligned}$$

Using $(34)_1$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(D\int_{Y_\varepsilon(\omega)}\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial u_{0,\varepsilon}}{\partial x_3}dx-\int_{Y_\varepsilon(\omega)}\left((v_{\varepsilon,s})_3\frac{\partial u_{0,\varepsilon}}{\partial x_3}\right)u_\varepsilon dx\right)\\
&\quad= D\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^+_{0,\varepsilon,ij}(\omega)}\frac{\left(u^-\left(x',-h\right)-u^+\left(x',0\right)\right)}{\displaystyle\int_{-h}^0\frac{1}{q_i\left(-\varepsilon^{-\theta}t\right)q_j\left(-\varepsilon^{-\theta}t\right)}\exp\left(-\frac{t\left(v_{\varepsilon,ij}\right)_3}{D}\right)dt}u_\varepsilon^+\left(x',-h\right)dx'\\
&\qquad-D\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^-_{h,\varepsilon,ij}(\omega)}\frac{\left(u^-\left(x',-h\right)-u^+\left(x',0\right)\right)\exp\left(\dfrac{h\left(v_{\varepsilon,ij}\right)_3}{D}\right)}{\displaystyle\int_{-h}^0\frac{1}{q_i\left(-\varepsilon^{-\theta}t\right)q_j\left(-\varepsilon^{-\theta}t\right)}\exp\left(-\frac{t\left(v_{\varepsilon,ij}\right)_3}{D}\right)dt}u_\varepsilon^-\left(x',0\right)dx'.
\end{aligned}$$

Introducing the change of variables $s=-\varepsilon^{-\theta}t$, we get

$$\begin{aligned}
&\int_{-h}^0\frac{1}{q_i\left(-\varepsilon^{-\theta}t\right)q_j\left(-\varepsilon^{-\theta}t\right)}\exp\left(-\frac{t\left(v_{\varepsilon,ij}\right)_3}{D}\right)dt\\
&\quad=\frac{h}{\varepsilon^{-\theta}h}\int_0^{\varepsilon^{-\theta}h}\frac{1}{q_i(s)\,q_j(s)}\exp\left(\varepsilon^\theta s\frac{\left(v_{\varepsilon,ij}\right)_3}{D}\right)ds\underset{\varepsilon\to 0}{\to}h\left\langle 1/q^2\right\rangle,
\end{aligned}$$

using the ergodicity property (2). Thus, using the proof of Lemma 6, we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(D\int_{Y_\varepsilon(\omega)}\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial u_{0,\varepsilon}}{\partial x_3}dx-\int_{Y_\varepsilon(\omega)}\left((v_{\varepsilon,s})_3\frac{\partial u_{0,\varepsilon}}{\partial x_3}\right)u_\varepsilon dx\right)\\
&\quad=\frac{D}{h\left\langle 1/q^2\right\rangle}\int_\Sigma\left(u_0^+\left(u^--u^+\right)-u_0^-\left(u^--u^+\right)\exp\left(\frac{p_0^+-p_0^-}{D\langle q^2\rangle\left\langle 1/q^2\right\rangle}\frac{k_0}{\mu}\right)\right)dx'.
\end{aligned}$$

We now compute, using Lemma 13

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(D\int_{Y_\varepsilon(\omega)}\nabla_\tau u_\varepsilon\cdot\nabla_\tau u_{0,\varepsilon}dx-\int_{Y_\varepsilon(\omega)}\left(\left(v_\varepsilon\right)_\tau\cdot\nabla_\tau u_{0,\varepsilon}\right)u_\varepsilon dx\right)\\
&\quad=\lim_{\varepsilon\to 0}\left(\int_{Y_\varepsilon(\omega)}D^*\nabla_\tau u_\varepsilon\cdot\left(\nabla_\tau u\right)_{0,\varepsilon}dx-\int_{Y_\varepsilon(\omega)}\left(\left(v_\varepsilon\right)_\tau\cdot\left(\nabla_\tau u\right)_{0,\varepsilon}\right)u_\varepsilon dx\right)\\
&\quad=h\left\langle q^2(0)\right\rangle\int_\Sigma D^*\nabla_\tau u_0^+\cdot\left(\nabla_\tau u^+\right)dx'-h\left\langle q^2(0)\right\rangle\int_\Sigma\left(\left(v_0\right)_\tau\cdot\left(\nabla_\tau u^+\right)\right)u_0^+dx',
\end{aligned}$$

which leads to the limit variational formulation (30). ■

The problem associated to this limit variational formulation (30) is given in the following Corollary.

Corollary 16 *The sequence $(u_\varepsilon)_\varepsilon$, where u_ε is the solution of (5), τ_1-converges to the* *solution u_0 of the problem*

$$\left\{\begin{array}{rcll} -\operatorname{div}\left(\widehat{D}\nabla u_0^+\right)+v_{0,d}^+\cdot\nabla u_0^+ & = & \left|Z^1\right| f & in\ \Omega^+, \\ -\operatorname{div}\left(\widehat{D}\nabla u_0^-\right)+v_{0,d}^-\cdot\nabla u_0^- & = & 0 & in\ \Omega_h^-, \\ -\widehat{D}\nabla u^+\cdot e_3 & & & \\ -h\left\langle q^2(0)\right\rangle \operatorname{div}_\tau\left(D^*\nabla u_0^+\right) & & & \\ +h\left\langle q^2(0)\right\rangle \left(v_{0,f}\right)_\tau\cdot\nabla_\tau u_0^+ & = & \dfrac{D}{h\left\langle 1/q^2(0)\right\rangle}\left(\begin{array}{c} u_0^+(.,0) \\ -u_0^-(.,-h)A \end{array}\right) & on\ \Gamma_0^+, \\ D^*\nabla_\tau u_0^+\cdot n_\tau & = & 0 & on\ \partial\Sigma, \\ \widehat{D}\nabla u_0^-\cdot e_3 & = & \dfrac{-D}{h\left\langle 1/q^2(0)\right\rangle}\left(\begin{array}{c} u_0^+(.,0) \\ -u_0^-(.,-h)A \end{array}\right) & on\ \Gamma_h^-, \\ u_0^+ & = & 0 & on\ \Gamma^+, \\ u_0^- & = & 0 & on\ \Gamma^-, \end{array}\right. \quad (35)$$

where $A=\exp\left(\frac{p_0^+-p_0^-}{D\langle q^2\rangle\langle 1/q^2\rangle}\frac{k_0}{\mu}\right)$, $u_0\mid_{\Omega^+}=: u_0^+$ *and* $u_0\mid_{\Omega_h^-}=: u_0^-$.

Proof. This is an immediate consequence of the limit variational formulation (30). ■

Remark 17 *Consider the case of a dispersive contaminant with a diffusion coefficient $D(x,\omega)$ defined through $D(x,\omega)=D_{mol}+D_{disp}(x,\omega)$ with*

$$D_{disp}(x,\omega)=\left\{\begin{array}{ll} D_{disp}(x) & in\ \Omega^+\cup\Omega_h^-, \\ D_{disp}\left(x_1,x_2,-\varepsilon^{-\theta}x_3+\alpha_{ij}(\omega),\omega\right) & in\ Y_{\varepsilon,ij}(\omega), \end{array}\right.$$

where $(\alpha_{ij}(\omega))_{i,j\in\mathbb{Z}}$ is a sequence of random variables such that $|\alpha_{ij}(\omega)|\leq C$, $\forall i,j\in\mathbb{Z}$, with probability 1, C being some non-random constant. We suppose that D is continuous with respect to the variable x and, with probability 1, $d_0\leq D(x,\omega)\leq d_1$, where d_0 and d_1 are positive and non-random constants. We suppose that D_{disp} is a stationary random process.

Let $u\in C^2\left(\overline{\Omega}\right)$ be such that $u=0$ on Γ. We build the modified test-function $\bar{u}_{\varepsilon,ij}$ inside the fissure $Y_{\varepsilon,ij}(\omega)$

$$\bar{u}_{\varepsilon,ij}(x_1,x_2,x_3)=u^+(x_1,x_2,0) + \frac{\left(u^-(x_1,x_2,-h)-u^+(x_1,x_2,0)\right)\displaystyle\int_{x_3}^0\frac{\exp\left(-t\left(v_{\varepsilon,ij}\right)_3\right)}{q_i\left(-\varepsilon^{-\theta}t\right)q_j\left(-\varepsilon^{-\theta}t\right)D_*\left(-\varepsilon^{-\theta}t\right)}dt}{\displaystyle\int_{-h}^0\frac{\exp\left(-t\left(v_{\varepsilon,ij}\right)_3\right)}{q_i\left(-\varepsilon^{-\theta}t\right)q_j\left(-\varepsilon^{-\theta}t\right)D_*\left(-\varepsilon^{-\theta}t\right)}dt},$$

where $(D_*)(s)=\left(D_{mol}+D_{disp}\left(i\varepsilon,j\varepsilon,s+\alpha_{ij}(\omega),\omega\right)\right)$. Implementing this test-function in the above process, one gets at the limit a problem similar to (35), except that $\left\langle 1/q^2(0)\right\rangle$ is now replaced by $\left\langle 1/D_*(.)\,q^2(0)\right\rangle$, where $\left\langle 1/\left(D_*q^2(0)\right)\right\rangle$ is the mathematical expectation *of* $1/\left(q^2(t)\,D_*(.,t)\right)$*, with respect to the measure probability* P*, and* b_j *is replaced by the* Z*-periodic solution* b_j *of the problem*

$$\begin{cases} \operatorname{div}\left(D(x)\left(e_j+\nabla b_j\right)\right) &= 0 \quad \text{in } Z^1 \quad j=1,2,3,\\ \left(\nabla b_j+e_j\right)\cdot n &= 0 \quad \text{on } S,\end{cases}$$

and c_m *by*

$$\begin{cases} \operatorname{div}\left(D(x,\omega)\left(e_m+\nabla c_m\right)\right) &= 0 \quad \text{in } Z' \qquad m=1,2,\\ \left(\nabla c_m+e_m\right)\cdot n &= 0 \quad \text{on } \partial Z'.\end{cases}$$

4.3 The asymptotic behaviour in the case of a reactive contaminant ($\mathcal{R}>0$)

In this subsection, we consider the reaction-diffusion equation (5) with first-order reaction, that is with $\mathcal{R}>0$ (see for example [5], [11]). We denote $w_{\varepsilon,ij}$ the solution of the differential equation

$$\begin{cases} -D\dfrac{\partial}{\partial x_3}\left(\left(q_iq_j\right)\left(-\varepsilon^{-\theta}x_3\right)\dfrac{\partial w_{\varepsilon,ij}}{\partial x_3}\right)(x_3)-\left(q_iq_j\right)\left(-\varepsilon^{-\theta}x_3\right)v_{\varepsilon,ij}\dfrac{\partial w_{\varepsilon,ij}}{\partial x_3}(x_3) & \\ \qquad +\mathcal{R}\left(q_iq_j\right)\left(-\varepsilon^{-\theta}x_3\right)w_{\varepsilon,ij}(x_3) &= 0,\\ w_{\varepsilon,ij}(0) &= 1,\\ w'_{\varepsilon,ij}(0) &= 0\end{cases} \tag{36}$$

and $z_{\varepsilon,ij}$ the solution of

$$\begin{cases} -D\dfrac{\partial}{\partial x_3}\left(\left(q_iq_j\right)\left(-\varepsilon^{-\theta}x_3\right)\dfrac{\partial z_{\varepsilon,ij}}{\partial x_3}\right)(x_3)-\left(q_iq_j\right)\left(-\varepsilon^{-\theta}x_3\right)v_{\varepsilon,ij}\dfrac{\partial z_{\varepsilon,ij}}{\partial x_3}(x_3) & \\ \qquad +\mathcal{R}\left(q_iq_j\right)\left(-\varepsilon^{-\theta}x_3\right)z_{\varepsilon,ij}(x_3) &= 0,\\ z_{\varepsilon,ij}(0) &= 0,\\ z'_{\varepsilon,ij}(0) &= \dfrac{1}{q_iq_j(0)},\end{cases} \tag{37}$$

where $v_{\varepsilon,ij}$ is the velocity defined in (31). We have the following estimates.

Proposition 18 *There exist non-random positive constants* C_0 *and* C_1 *independent of* ε *and of* i *and* j*, such that:*

1. $\forall\varepsilon>0$, $\forall(i,j)\in I_\varepsilon(\omega)$, $\forall x_3\in[-h,0]$: $1\le w_{\varepsilon,ij}(x_3)\le C_1$ *and* $-C_0^{-1}x_3\le z_{\varepsilon,ij}(x_3)\le C_1$.

2. $\forall (i,j) \in I_\varepsilon(\omega)$, $\forall x_3 \in [-h,0]$

$$\left\{ \begin{array}{rcl} \lim_{\varepsilon\to 0} \left| w_{\varepsilon,ij}(x_3) - \cosh\left(\widehat{\mathcal{R}} x_3\right) \right| & = & 0, \\ \lim_{\varepsilon\to 0} \left| z_{\varepsilon,ij}(x_3) - \frac{\langle 1/q^2(0)\rangle}{\widehat{\mathcal{R}}} \sinh\left(\widehat{\mathcal{R}} x_3\right) \right| & = & 0, \\ \lim_{\varepsilon\to 0} \left| w'_{\varepsilon,ij}(x_3)(q_i q_j)\left(-\varepsilon^{-\theta} x_3\right) \exp\left(\frac{x_3 v_{\varepsilon,ij}}{D}\right) - \frac{\widehat{\mathcal{R}}}{\langle 1/q^2(0)\rangle} \sinh\left(\widehat{\mathcal{R}} x_3\right) \right| & = & 0, \\ \lim_{\varepsilon\to 0} \left| z'_{\varepsilon,ij}(x_3)(q_i q_j)\left(-\varepsilon^{-\theta} x_3\right) \exp\left(\frac{x_3 v_{\varepsilon,ij}}{D}\right) - \cosh\left(\widehat{\mathcal{R}} x_3\right) \right| & = & 0, \end{array} \right.$$

where $\widehat{\mathcal{R}} = \sqrt{\mathcal{R}\langle q^2(0)\rangle \langle 1/q^2(0)\rangle / D}$.

Proof. 1. Multiplying the equations (36) and (37) by $\exp(x_3 v_{\varepsilon,ij}/D)$ and integrating by parts the first term of these equations, we obtain the Voltera type integral equations

$$\left\{ \begin{array}{rcl} w_{\varepsilon,ij}(x_3) & = & \frac{\mathcal{R}}{D} \int_0^{x_3} (q_i q_j)\left(-\varepsilon^{-\theta} s\right) \exp\left(s \frac{v_{\varepsilon,ij}}{D}\right) w_{\varepsilon,ij}(s) \\ & & \times \left(\int_s^{x_3} \frac{\exp\left(-\zeta \frac{v_{\varepsilon,ij}}{D}\right)}{(q_i q_j)\left(-\varepsilon^{-\theta}\zeta\right)} d\zeta \right) ds + 1, \\ z_{\varepsilon,ij}(x_3) & = & \frac{\mathcal{R}}{D} \left(\int_0^{x_3} (q_i q_j)\left(-\varepsilon^{-\theta} s\right) \exp\left(s \frac{v_{\varepsilon,ij}}{D}\right) z_{\varepsilon,ij}(s) \right) \\ & & \times \left(\int_s^{x_3} \frac{\exp\left(-\zeta \frac{v_{\varepsilon,ij}}{D}\right)}{(q_i q_j)\left(-\varepsilon^{-\theta}\zeta\right)} d\zeta \right) ds + \int_0^{x_3} \frac{\exp\left(-\zeta \frac{v_{\varepsilon,ij}}{D}\right)}{(q_i q_j)\left(-\varepsilon^{-\theta}\zeta\right)} d\zeta, \end{array} \right. \tag{38}$$

(which can be solved by the method of successive approximations). Taking into account the hypothesis (1), we obtain the first point of the Proposition.

2. Consider the integral equations

$$\left\{ \begin{array}{rcl} w_{ij}(x_3) & = & \frac{\mathcal{R}}{D} \langle q^2(0)\rangle \langle 1/q^2(0)\rangle \int_0^{x_3} (x_3 - s) w_{ij}(s)\, ds + 1, \\ z_{ij}(x_3) & = & \frac{\mathcal{R}}{D} \langle q^2(0)\rangle \langle 1/q^2(0)\rangle \int_0^{x_3} (x_3 - s) z_{ij}(s)\, ds + x_3 \langle 1/q^2(0)\rangle, \end{array} \right.$$

whose solutions are $w_{ij}(x_3) = \cosh\left(\widehat{\mathcal{R}} x_3\right)$ and $z_{ij}(x_3) = \langle 1/q^2(0)\rangle \sinh\left(\widehat{\mathcal{R}} x_3\right) / \widehat{\mathcal{R}}$ respectively. The differences $w_{\varepsilon,ij} - w_{ij}$ and $z_{\varepsilon,ij} - z_{ij}$ satisfy integral equations of Voltera type. Using the ergodic result (2), we prove the convergences

$$\lim_{\varepsilon\to 0} \left| w_{\varepsilon,ij}(x_3) - w_{ij}(x_3) \right| = \lim_{\varepsilon\to 0} \left| z_{\varepsilon,ij}(x_3) - z_{ij}(x_3) \right| = 0,$$

uniformly with respect to $x_3 \in [-h,0]$.

The last estimates for the derivatives follow from the derivation of the equations (38). ■

Let $u \in C^2\left(\overline{\Omega}\right)$ be such that $u = 0$ on $\Gamma^+ \cup \Gamma^-$. We here define the test-function $\bar{u}_{\varepsilon,ij}$ inside the fissure $Y_{\varepsilon,ij}(\omega)$ through

$$\bar{u}_{\varepsilon,ij}(x) = u^+(x_1,x_2,0)\,w_{\varepsilon,ij}(x_3) + \frac{u^-(x_1,x_2,-h) - u^+(x_1,x_2,0)\,w_{\varepsilon,ij}(-h)}{z_{\varepsilon,ij}(-h)} z_{\varepsilon,ij}(x_3)\,.$$

We finally define the test-function $u_{0,\varepsilon}$ in the same way as (33). It is easily proved that the sequence $\left(u_{0,\varepsilon}\right)_\varepsilon$ τ_1-converges to u. On the other hand, we compute

$$\begin{cases} u_{0,\varepsilon}(x',0) &= u^+(x_1,x_2,0)\,, \\ \bar{u}_{\varepsilon,ij}(x',-h) &= u^-(x_1,x_2,-h)\,, \\ \dfrac{\partial u_{0,\varepsilon}}{\partial x_3}\,|_{x_3=0} &= \dfrac{u^-(x_1,x_2,-h) - u^+(x_1,x_2,0)\,w_{\varepsilon,ij}(-h)}{z_{\varepsilon,ij}(-h)} \dfrac{1}{q_i q_j(0)}, \\ \dfrac{\partial u_{0,\varepsilon}}{\partial x_3}\,|_{x_3=-h} &= u^+(x_1,x_2,0)\,w'_{\varepsilon,ij}(-h) \\ & \quad + \dfrac{u^-(x_1,x_2,-h) - u^+(x_1,x_2,0)\,w_{\varepsilon,ij}(-h)}{z_{\varepsilon,ij}(-h)} z'_{\varepsilon,ij}(-h)\,. \end{cases}$$

We have the following result.

Lemma 19 *One has:*

$$\lim_{\varepsilon\to 0} \sum_{(i,j)\in I_\varepsilon(\omega)} \int_{\Gamma^+_{0,\varepsilon,ij}(\omega)} \frac{\partial u_{0,\varepsilon}}{\partial x_3}\,|_{x_3=0}\,(q_i q_j)(0)\,u^+_\varepsilon d\sigma$$
$$= \frac{-1}{\langle 1/q^2(0)\rangle} \int_\Sigma \frac{\widehat{\mathcal{R}}}{\sinh\left(\widehat{\mathcal{R}}h\right)} \left(u^- - u^+\cosh\left(\widehat{\mathcal{R}}h\right)\right) u^+_0 dx',$$
$$\lim_{\varepsilon\to 0} \sum_{(i,j)\in I_\varepsilon(\omega)} \int_{\Gamma^-_{h,\varepsilon,ij}(\omega)} \frac{\partial u_{0,\varepsilon}}{\partial x_3}\,|_{x_3=-h}\,(q_i q_j)\left(h\varepsilon^{-\theta}\right) u^-_\varepsilon d\sigma$$
$$= \frac{-1}{\langle 1/q^2(0)\rangle} \int_\Sigma \frac{\widehat{\mathcal{R}}}{\sinh\left(\widehat{\mathcal{R}}h\right)} \left(u^-\cosh\left(\widehat{\mathcal{R}}h\right) - u^+\right) \exp\left(\frac{p_0^+ - p_0^-}{D\langle 1/q^2\rangle}\frac{k_0}{\mu}\right) u^-_0 dx',$$

Proof. Using the estimates of Proposition 18, we can replace, when ε is small enough, $w_{\varepsilon,ij}(-h)$ by $\cosh\left(\widehat{\mathcal{R}}h\right)$, $z_{\varepsilon ij}(-h)$ by $-\langle 1/q^2(0)\rangle \sinh\left(\widehat{\mathcal{R}}h\right)/\widehat{\mathcal{R}}$, $w'_{\varepsilon,ij}(-h)$ by the quantity $-\frac{\exp(hv_{\varepsilon,ij}/D)}{(q_iq_j)\left(\varepsilon^{-\theta}h\right)}\frac{\widehat{\mathcal{R}}}{\langle 1/q^2(0)\rangle}\sinh\left(\widehat{\mathcal{R}}h\right)$ and $z'_{\varepsilon,ij}(-h)$ by the quantity $\frac{\exp(hv_{\varepsilon,ij}/D)}{(q_iq_j)\left(\varepsilon^{-\theta}h\right)}\cosh\left(\widehat{\mathcal{R}}h\right)$.

Thus

$$
\begin{aligned}
&\frac{\partial u_{0,\varepsilon}}{\partial x_3}\,|_{x_3=-h}\,(q_iq_j)\left(h\varepsilon^{-\theta}\right)\\
&= \left(\begin{array}{l} u^+(x_1,x_2,0)\,w'_{\varepsilon,ij}(-h) \\ \quad +\dfrac{u^-(x_1,x_2,-h)-u^+(x_1,x_2,0)\,w_{\varepsilon,ij}(-h)}{z_{\varepsilon,ij}(-h)}z'_{\varepsilon,ij}(-h) \end{array}\right)(q_iq_j)\left(h\varepsilon^{-\theta}\right)\\
&\underset{\varepsilon\to 0}{\sim} \left(\begin{array}{l} -u^+(x_1,x_2,0)\,\dfrac{\exp\left(hv_{\varepsilon,ij}/D\right)}{(q_iq_j)\left(\varepsilon^{-\theta}h\right)}\dfrac{\widehat{\mathcal{R}}}{\langle 1/q^2(0)\rangle}\sinh\left(\widehat{\mathcal{R}}h\right) \\ -\widehat{\mathcal{R}}\dfrac{u^-(x_1,x_2,-h)-u^+(x_1,x_2,0)\cosh\left(\widehat{\mathcal{R}}h\right)}{\langle 1/q^2(0)\rangle\sinh\left(\widehat{\mathcal{R}}h\right)} \\ \quad\times\dfrac{\exp\left(hv_{\varepsilon,ij}/D\right)}{(q_iq_j)\left(\varepsilon^{-\theta}h\right)}\cosh\left(\widehat{\mathcal{R}}h\right) \end{array}\right)(q_iq_j)\left(h\varepsilon^{-\theta}\right)\\
&= \frac{\widehat{\mathcal{R}}}{\langle 1/q^2(0)\rangle}\exp\left(hv_{\varepsilon,ij}/D\right)\left(\begin{array}{l} -u^+(x_1,x_2,0)\sinh\left(\widehat{\mathcal{R}}h\right) \\ -\dfrac{u^-(x_1,x_2,-h)-u^+(x_1,x_2,0)\cosh\left(\widehat{\mathcal{R}}h\right)}{\sinh\left(\widehat{\mathcal{R}}h\right)} \\ \quad\times\cosh\left(\widehat{\mathcal{R}}h\right) \end{array}\right)\\
&= \frac{\widehat{\mathcal{R}}}{\langle 1/q^2(0)\rangle\sinh\left(\widehat{\mathcal{R}}h\right)}\exp\left(hv_{\varepsilon,ij}/D\right)\\
&\quad\times\left(\begin{array}{l} u^+(x_1,x_2,0)\left(-(\sinh)^2\left(\widehat{\mathcal{R}}h\right)+(\cosh)^2\left(\widehat{\mathcal{R}}h\right)\right) \\ -u^-(x_1,x_2,-h)\cosh\left(\widehat{\mathcal{R}}h\right) \end{array}\right)\\
&= \frac{\widehat{\mathcal{R}}}{\langle 1/q^2(0)\rangle\sinh\left(\widehat{\mathcal{R}}h\right)}\exp\left(hv_{\varepsilon,ij}/D\right)\left(u^+(x_1,x_2,0)-u^-(x_1,x_2,-h)\cosh\left(\widehat{\mathcal{R}}h\right)\right),
\end{aligned}
$$

whence

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^-_{h,\varepsilon,ij}(\omega)}\frac{\partial u_{0,\varepsilon}}{\partial x_3}\,|_{x_3=-h}\,(q_iq_j)\left(h\varepsilon^{-\theta}\right)u^-_\varepsilon\,d\sigma\\
&\quad=\frac{-1}{\langle 1/q^2(0)\rangle}\int_\Sigma\frac{\widehat{\mathcal{R}}}{\sinh\left(\widehat{\mathcal{R}}h\right)}\left(u^-\cosh\left(\widehat{\mathcal{R}}h\right)-u^+\right)\exp\left(\frac{p_0^+-p_0^-}{D\langle 1/q^2\rangle}\frac{k_0}{\mu}\right)u_0^-\,dx'.
\end{aligned}
$$

Similarly, we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^+_{0,\varepsilon,ij}(\omega)}\frac{\partial u_{0,\varepsilon}}{\partial x_3}\mid_{x_3=0}(q_iq_j)(0)\,u^+_\varepsilon d\sigma\\
=&\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^+_{0,\varepsilon,ij}(\omega)}\frac{u^-(x_1,x_2,-h)-u^+(x_1,x_2,0)\,w_{\varepsilon,ij}(-h)}{z_{\varepsilon,ij}(-h)}\frac{1}{q_iq_j(0)}(q_iq_j)(0)\,u^+_\varepsilon d\sigma\\
=&\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^+_{0,\varepsilon,ij}(\omega)}\frac{u^-(x_1,x_2,-h)-u^+(x_1,x_2,0)\,w_{\varepsilon,ij}(-h)}{z_{\varepsilon,ij}(-h)}u^+_\varepsilon d\sigma\\
=&\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon(\omega)}\int_{\Gamma^+_{0,\varepsilon,ij}(\omega)}-\widehat{\mathcal{R}}\frac{u^-(x_1,x_2,-h)-u^+(x_1,x_2,0)\cosh\left(\widehat{\mathcal{R}}h\right)}{\langle 1/q^2(0)\rangle\sinh\left(\widehat{\mathcal{R}}h\right)}u^+_\varepsilon d\sigma\\
=&\frac{-1}{\langle 1/q^2(0)\rangle}\int_\Sigma\frac{\widehat{\mathcal{R}}}{\sinh\left(\widehat{\mathcal{R}}h\right)}\left(u^- - u^+\cosh\left(\widehat{\mathcal{R}}h\right)\right)u^+_0\,dx',
\end{aligned}$$

which leads to the desired limit. ■

Now, using the same methods as in the above subsection, we obtain the following Theorem.

Theorem 20 *The sequence $(u_\varepsilon)_\varepsilon$, where u_ε is the solution of (5), τ_1-converges to the solution u_0 of the problem*

$$\begin{cases}
-\operatorname{div}\left(\widehat{D}\nabla u_0^+\right)+v^+_{0,d}\cdot\nabla u_0^+ +\mathcal{R}u_0^+ = |Z^1|\,f & \text{in } \Omega^+,\\
-\operatorname{div}\left(\widehat{D}\nabla u_0^-\right)+v^-_{0,d}\cdot\nabla u_0^- +\mathcal{R}u_0^- = 0 & \text{in } \Omega^-_h,\\
-\widehat{D}\nabla u_0^+.e_3 - h\langle q^2(0)\rangle\operatorname{div}_\tau\left(D^*\nabla u_0^+\right) & \\
\quad +h\langle q^2(0)\rangle\left(v_{0,f}\right)_\tau\cdot\nabla_\tau u_0^+ = \left(\begin{array}{c}\dfrac{D}{\langle 1/q^2(0)\rangle}\dfrac{\widehat{\mathcal{R}}}{\sinh\left(\widehat{\mathcal{R}}h\right)}\\ \times\left(u_0^+\cosh\left(\widehat{\mathcal{R}}h\right)-u_0^-A\right)\end{array}\right) & \text{on } \Gamma_0^+,\\
D^*\nabla_\tau u_0^+\cdot n_\tau = 0 & \text{on } \partial\Sigma,\\
\widehat{D}\nabla u_0^-\cdot e_3 = \left(\begin{array}{c}\dfrac{D}{\langle 1/q^2(0)\rangle}\dfrac{\widehat{\mathcal{R}}}{\sinh\left(\widehat{\mathcal{R}}h\right)}\\ \times\left(u_0^-\cosh\left(\widehat{\mathcal{R}}h\right)A-u_0^+\right)\end{array}\right) & \text{on } \Gamma_h^-,\\
u_0^+ = 0 & \text{on } \Gamma^+,\\
u_0^- = 0 & \text{on } \Gamma^-,
\end{cases} \tag{39}$$

with $A=\exp\left(\frac{p_0^+-p_0^-}{D\langle q^2\rangle\langle 1/q^2\rangle}\frac{k_0}{\mu}\right)$, $u_0^+ := u_0\mid_{\Omega^+}$ *and* $u_0^- := u_0\mid_{\Omega_h^-}$.

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