I.S.F.A. École Doctorale Sciences Économiques et de Gestion

Dépendance et événements extrêmes en théorie de la ruine : étude univariée et multivariée, problèmes d'allocation optimale

THÈSE

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par

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Table des matières

Table des figures

Introduction générale

Introduction

En actuariat, la théorie de la ruine consiste en la modélisation mathématique et en l'étude de l'évolution des richesses d'une compagnie d'assurance. La problématique la plus souvent posée dans ce domaine est le calcul de la probabilité de ruine de la compagnie, c'est-à-dire de la probabilité que ses réserves financières passent sous la frontière fatidique du zéro. Ici, une compagnie dite en *ruine* ne mettra pas automatiquement la clé sous la porte ; le terme de *ruine* concerne davantage une situation critique où une compagnie ne peut satisfaire à ses engagements vis-à-vis de ses clients, de ses actionnaires ou d'une autorité de contrôle.
S'inscrivant dans ce cadre général, cette thèse a pour but d'étendre le modèle classique en relaxant certaines hypothèses (indépendance, stationnarité, unidimensionnalité) jugées parfois trop restrictives pour décrire correctement l'évolution complexe des réserves d'une compagnie d'assurance. L'étude est principalement axée sur une compagnie exposée à des risques extrêmes (tremblements de terre, inondations, tempête, attaques terroristes,...) et donc possédant une réserve initiale relativement élevée. Sous cette hypothèse, nous utilisons la partie de la théorie des valeurs extrêmes qui concerne la classe des distributions à variations régulières, dont l'exemple le plus connu est la distribution de Pareto. Ce travail prend en compte certaines spécificités de ces risques ; on peut en effet observer certaines formes de dépendance entre les montants des sinistres eux-même ainsi qu'entre les temps aléatoires séparant l'occurrence des sinistres et les montants de ces derniers. Le secteur assurantiel jouxte plusieurs autres domaines, comme le secteur financier ou le secteur juridique. Ces liens, ainsi que d'autres facteurs naturels (pluie, vent, neige...), influencent considérablement les corrélations observées entre les risques d'une compagnie d'assurance. C'est pourquoi nous nous sommes intéressés, dans ce travail, à des formes de dépendances dynamiques qui peuvent prendre en compte des pics de corrélation entre les risques, ce que nous appelons *crises de corrélation.*

De nombreuses études en théorie du risque considèrent un horizon de temps infini. Ceci permet d'avoir une première idée concernant le capital à posséder afin de pérenniser la compagnie. Dans un souci de contrôle du risque à court et moyen terme, d'une part imposé par les nouvelles règles standards de solvabilité (Solvabilité 2) qui fixent l'horizon à un an, d'autre part observé dans les modèles internes développés par les compagnies, qui travaillent avec un horizon variant de un à dix ans, cette thèse présente principalement des résultats en temps fini.

Une autre voie explorée dans ce travail est l'étude de la théorie de la ruine en dimension plus grande que une. Nous nous intéressons donc à une compagnie d'assurance composée de plusieurs lignes d'affaire. Ces différentes lignes peuvent modéliser soit des secteurs assurantiels différents, soit des branches d'un même secteur mais installées dans des régions géographiques différentes. L'analyse multivariée des richesses d'une compagnie d'assurance pose de nouvelles problématiques. Premièrement, s'il semble naturel de relier la notion de *ruine* à une position négative de la réserve de la compagnie en dimension une, définir la ruine en dimension supérieure est

plus difficile à appréhender. On peut en effet envisager différentes formes de ruine, comme par exemple l'insolvabilité d'une des branches de la compagnie ou encore l'insolvabilité de toutes les branches. De plus, certains transferts de capitaux peuvent être envisagés afin de compenser l'insolvabilité d'une des branches par une autre branche solvable. D'autres quantificateurs de risque sont étudiés ici comme l'espérance de l'intégrale temporelle de la partie négative du processus qui modélise les réserves de la compagnie.
Une autre problématique que nous traitons dans cette thèse, et qui concerne toujours la théorie du risque multivariée, est l'allocation de capital initial. Supposons que la compagnie d'assurance possède un capital initial fixe qu'elle doit allouer à ses diverses branches d'activité. Nous étudions dans ce travail l'allocation optimale qui minimise certains quantificateurs de risque, dont fait partie la probabilité de ruine multivariée. Nous analysons aussi l'effet de la dangerosité des risques de chaque branche ainsi que de la dépendance entre les branches sur cette allocation.

La suite de cette introduction va développer les points évoqués ci-dessus. Le modèle classique de la théorie de la ruine est d'abord introduit, puis les formes de dépendance modélisées dans cette thèse sont présentées. Nous formalisons alors le contexte multivarié et donnons des résultats utiles de la théorie des extrêmes sur lesquels s'est appuyée cette thèse.

Modèle classique de la théorie de la ruine

La théorie de la ruine a pris naissance en Suède au début du 20*ème* siècle dans les travaux de Lundberg (1903), relayés par Lundberg (1926) et Cramér (1930).
Le modèle classique de la théorie de ruine, le modèle de Cramér-Lundberg, représente l'évolution des réserves d'une compagnie d'assurance par un processus Poisson-composé avec dérive. Explicitement, si on note $(R(t))_{t\geq 0}$ le processus qui modélise les réserves, on a :

$$R(t) = u + ct - \sum_{i=0}^{N(t)} X_i \ , \tag{1}$$

où

- u, c sont deux réels supposés positifs, représentant respectivement les réserves initiales de la compagnie et le taux de cotisation demandé aux assurés, qui est supposé constant,
- $(N(t))_{t\geq 0}$ est un processus de Poisson homogène d'intensité λ représentant le nombre de sinistres,
- X_i est une variable aléatoire positive représentant le montant du $i^{\text{ème}}$ sinistre. Les $(X_i)_{i\geq 1}$ sont supposés indépendants, identiquement distribués (noté i.i.d par la suite) et indépendants de $(N(t))_{t\geq 0}$. On notera F_X leur fonction de répartition, f_X leur densité et μ leur moyenne communes.

On note par la suite $(S(t))_{t\geq 0}$ le processus qui représente les pertes agrégées de la compagnie avec

$$S(t) = \sum_{i=0}^{N(t)} X_i \ .$$

Par convention, $S(t) = 0$ si $N(t) = 0$. On note également

$$Y_t = S(t) - ct \ .$$

Pour $i \geq 1$, on définit par U_i le temps d'arrivée du $i$ème sinistre et on note $V_1 = U_1$ puis pour $i \geq 2$, $V_i = U_i - U_{i-1}$ les temps inter-sinistres.
Un exemple de trajectoire du processus $(R(t))_{t\geq 0}$ est tracé sur la Figure 1. Un modèle plus général est souvent utilisé en théorie de la ruine ; il s'agit du modèle de Sparre Andersen, introduit dans Andersen (1957), qui considère un processus de renouvellement pour $N(t)$.
Les paramètres u et c du modèle sont à considérer avec intérêt car ils sont à déterminer par la compagnie. Le taux de cotisation c, demandé aux assurés, ne doit pas être trop faible pour espérer couvrir les risques souscrits, et en même temps, il ne doit pas être non plus trop élevé pour éviter un exode des assurés chez les concurrents. Ce taux de prime est généralement déterminé grâce aux principes de prime (espérance, variance, écart-type). Étant donné que nous travaillons sous l'hypothèse d'une réserve initiale u relativement grande, c n'influe pas de manière significative dans nos résultats, étant négligeable devant u. Nous ne nous intéressons donc pas aux problématiques liées au taux de prime dans cette thèse. Pour plus de détails concernant les principes de prime, le lecteur peut par exemple consulter Rolski *et al.* (1999) (Chapitre 3).

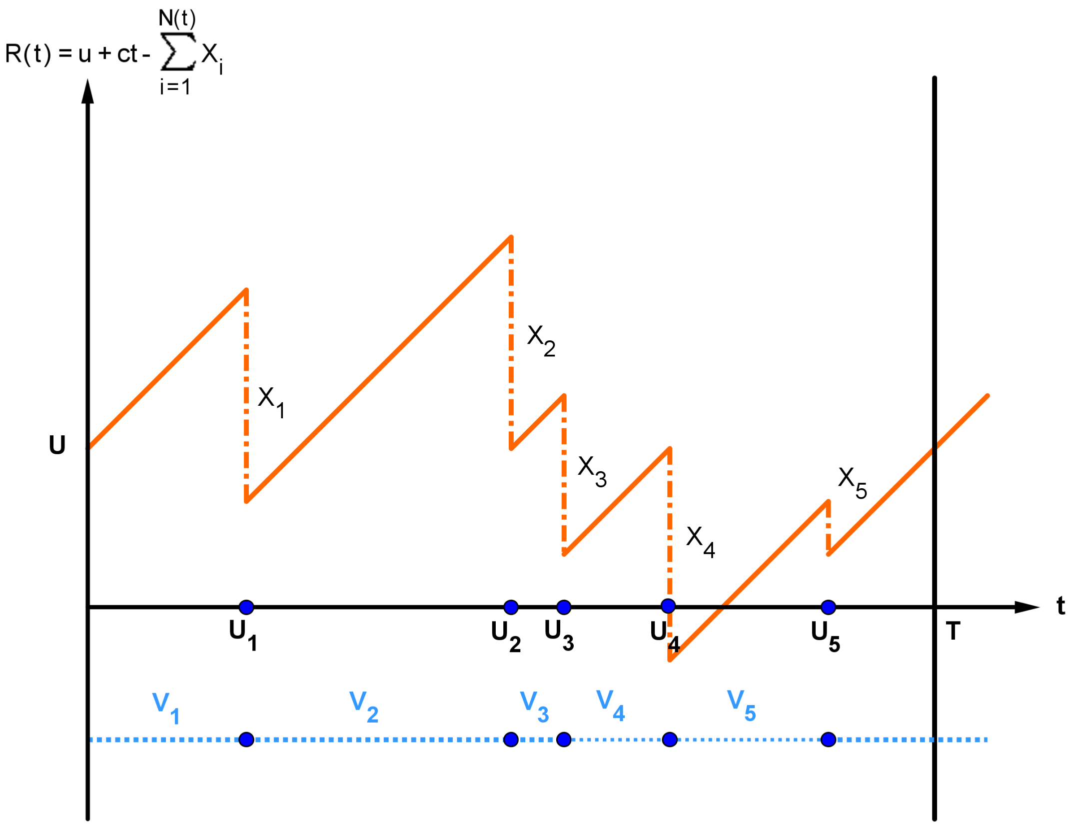


FIG. 1 – Un exemple de trajectoire du processus $(R(t))_{t\geq 0}$

Les travaux en théorie de la ruine s'intéressent majoritairement au calcul de la probabilité de ruine, en temps fini et infini dont voici les définitions.

Définition 0.0.1 *La probabilité de ruine en temps fini T avec réserves initiales u, notée $\psi(u, T)$ correspond à la probabilité que les réserves deviennent strictement négatives à un instant précédant*

T. Explicitement, pour $u, T > 0$,

$$\psi(u,T) = \mathbb{P}\left(\exists s \in [0,T], R(s) < 0\right) \ .$$

La probabilité de ruine en temps infini est définie naturellement, pour $u > 0$ par

$$\psi(u) = \lim_{T\to\infty} \psi(u,T) = \mathbb{P}\left(\exists s \geq 0, R(s) < 0\right) \ .$$

La probabilité de non-ruine en temps fini T avec réserves initiales u, notée $\phi(u,T)$, est définie par, pour $u, T > 0$

$$\phi(u,T) = 1 - \psi(u,T) \ .$$

On en déduit de manière équivalente la probabilité de non ruine en temps infini

$$\phi(u) = \lim_{T\to\infty} \phi(u,T) = 1 - \psi(u) \ .$$

Soit

$$\rho = c - \lambda\mu \ .$$

La constante ρ définit le chargement de sécurité de la compagnie. Il mesure la rentabilité de la compagnie dans le sens où :

Proposition 0.0.2

- *si $\rho > 0$, alors, presque sûrement,*

  $$\lim_{t\to+\infty} R_t = +\infty,$$

  et

  $$\phi(u) \neq 0.$$

  L'activité est donc rentable.

- *Si $\rho < 0$, alors, presque sûrement,*

  $$\lim_{t\to+\infty} R_t = -\infty,$$

  et

  $$\psi(u) = 1.$$

  L'activité n'est donc pas rentable.

Lorsqu'on travaille avec un horizon de temps infini, vu les résultats énoncés ci-dessus, on suppose logiquement $\rho > 0$. Cette hypothèse n'est pas nécessaire en temps fini, ce qui est généralement le cas ici.

Depuis maintenant quelques années, une autre fonction de risque est apparue et fait maintenant l'objet de nombreux papiers ; il s'agit de la fonction de pénalité de Gerber-Shiu. Cette fonction est apparue dans Gerber et Shiu (1998) et est définie comme suit :

$$m(u) = E\left[e^{-\delta\tau} w(R(\tau-), R(\tau)) \mathbb{1}_{\tau<\infty} | R(0) = u\right] \ ,$$

avec $\tau = \inf\{t \geq 0, R(t) < 0\}$,$\delta \geq 0$ un facteur d'actualisation, et $x, y \mapsto w(x,y)$ une fonction positive définie sur $[0,\infty]^2$ qui mesure la perte économique (par exemple une sanction financière du régulateur) liée à la ruine à travers deux quantités : la valeur du processus juste avant la ruine $R(\tau-)$ et le déficit au moment de la ruine $R(\tau)$. Pour $\delta = 0$ et $w(x, y) = 1$, on retrouve la probabilité de ruine en temps infini.

Je cite ici une liste non exhaustive d'ouvrages, Gerber (1979), De Vylder (1996), Rolski *et al.* (1999), Asmussen (2000) et Goovaerts *et al.* (2001), qui donnent un aperçu général des résultats en théorie du risque.

Formes de dépendance envisagées en théorie du risque

Le modèle classique de la théorie de la ruine est donc un modèle stationnaire et à accroissements indépendants. Il suppose l'indépendance entre :

- les montants de sinistres X_i, $i \geq 1$ entre eux ;
- les temps inter-sinistres V_i, $i \geq 1$ entre eux ;
- les montants de sinistres X_i, $i \geq 1$ et les temps inter-sinistre V_i, $i \geq 1$.

Selon les risques couverts par la compagnie d'assurance, ces hypothèses sont trop restrictives pour modéliser correctement l'évolution complexe des réserves de la compagnie.

Dépendance inter-sinistres, crises de corrélation

En pratique, l'indépendance entre les montants de sinistres n'est que rarement observée. De nombreux facteurs économiques, environnementaux, juridiques ou politiques créent de la corrélation entre les risques couverts par la compagnie. Il en résulte des formes plus ou moins complexes de dépendance, pouvant aller jusqu'à la dépendance totale. Prenons l'exemple d'une décision de justice qui à un type de sinistre associerait un montant forfaitaire ; les montants des sinistres de ce type seraient alors comonotones.
Le modèle étudié en théorie de la ruine est dynamique ; il faut donc prendre en compte l'évolution des corrélations au fil du temps. Des risques ne peuvent être que faiblement corrélés, voire indépendants lorsque les conditions extérieures (économiques, politiques,...) sont normales et devenir fortement corrélés lorsqu'un événement inhabituel survient. On peut faire ici un parallèle avec l'univers financier qui a connu ce phénomène lors de la crise des subprimes ; alors que les pertes financières étaient nombreuses, la corrélation entre les actifs augmentaient considérablement. En effet, le contexte économique défavorable a affecté la majorité des indices boursiers. Pour plus de détails sur les crises de corrélation en finance et en assurance, le lecteur peut se référer à Loisel *et al.* (2010). Pour des études sur les crises de corrélation en finance mais aussi dans d'autres domaines, il existe Gorban *et al.* (2010).
Les facteurs économiques, juridiques, politiques ou environnementaux qui entourent la compagnie d'assurance influent, comme nous venons de le voir sur les corrélations entre les risques ainsi que sur la fréquence et la dangerosité de ces risques ; c'est donc tous les paramètres du modèle qui évoluent. En effet, après les attentats du 11 septembre, il fut très difficile de trouver de la réassurance ; ne pouvant transférer les risques extrêmes, les compagnies d'assurance ont vu les marginales des montants de sinistres devenir beaucoup plus élevées. Pour modéliser les changements de l'environnement qui entoure la compagnie, nous avons choisi un processus d'environnement de Markov. Un processus de Markov est un processus dont le futur est indépendant du passé dès lors que l'on connaît le présent. Explicitement il satisfait la propriété suivante, appelée propriété de Markov :

Définition 0.0.3 (Propriété de Markov) *Soit* $(X_t)_{t\geq 0}$ *un processus sur l'espace probabilisé*

$(\Omega, \mathcal{F}, P)$ prenant ses valeurs dans un espace d'états S pour tout $t \geq 0$. Pour $t \geq 0$, notons $\mathcal{F}_t$ la tribu engendrée par $\{X_s; s \leq t\}$.
Pour tout $s, t \geq 0$ avec $s < t$, et pour tout $H \in \mathcal{F}_s$, et $x \in S$, la propriété de Markov dit que la distribution conditionnelle de X_t sachant H et $X_s = x$ est la même que la distribution conditionnelle de X_t sachant $X_s = x$ soit

$$P(X_t \in A | H, X_s = x) = P(X_t \in A | X_s = x).$$

Ce processus d'environnement, selon l'état dans lequel il se trouve, module la dépendance entre les montants de sinistres ainsi que leur distribution marginale, le paramètre d'intensité du processus de Poisson et le taux de prime demandé à l'assuré.
La dépendance entre les montants de sinistres pour un état de l'environnement donné est modélisée par une copule, qui est une fonction de répartition multivariée définie sur le cube unité en dimension d ($[0,1]^d$) et dont chaque marginale est distribuée uniformément sur $[0,1]$. Cet outil a le grand avantage de modéliser toutes les fonctions de distribution multivariée et de séparer les lois marginales de la structure de dépendance, comme l'a démontré Sklar (1959).

Théorème 0.0.4 (Sklar (1959)) *Soit $F : \mathbb{R}^d \to [0,1]$ une fonction de répartition multivariée avec des marginales $F_1, ..., F_d$, alors il existe une copule C telle que pour tout $(x_1, ..., x_d) \in \mathbb{R}^d$, $F(x_1, ..., x_n) = C(F_1(x_1), ..., F_d(x_d))$.*

De plus, si les marginales $F_1, ..., F_k$ sont continues alors C est unique. Pour une introduction plus complète sur la théorie des copules, le lecteur pourra se référer, par exemple, à Joe (1997) et Nelsen (2006).
Supposons, par exemple, qu'il existe deux états d'environnement. Le premier état pourrait correspondre à un état dit *normal*, dans lequel les montants de sinistres ne seraient que faiblement corrélés (ex : copule indépendante, copule gaussienne) avec une fréquence d'occurrence assez faible et des montants de sinistres dont la distribution aurait une queue relativement légère. Le deuxième état, état de *crise*, verrait la corrélation augmenter (copule de la borne supérieure de Fréchet, copule BLL[1]), la fréquence croître et la queue des distributions s'épaissir. C'est ce type de situations qui est étudié dans le Chapitre 1.

Théorie du risque et environnement markovien

L'idée d'utiliser des processus markoviens pour modéliser des changements dans l'environnement dans lequel évolue la compagnie a été introduite par Asmussen (1989). Depuis, de nombreux auteurs se sont intéressés à cette problématique. Dans le cadre des distributions à queues lourdes qui nous intéresse, nous pouvons citer notamment Asmussen *et al.* (1994). Jusqu'à maintenant, l'environnement markovien modulait la distribution des montants de sinistres, le paramètre d'intensité du processus de Poisson ainsi que le taux de cotisation. Ce qui est nouveau dans le Chapitre 1 de cette thèse est la possibilité de moduler la structure de dépendance qui lie les montants de sinistres.
Le modèle utilisé est le suivant. Soit $(J(t))_{t \geq 0}$ un processus d'environnement de Markov d'états $1, \ldots, J \geq 2$, de mesure initiale π_0 et de matrice de transition Q. Définissons pour $1 \leq i \leq J$ le processus

$$Y^i(t) = c^i t - \sum_{m^i=1}^{N^i(t)} X^i_{m^i} \ ,$$

où

[1]La copule BLL correspond à la copule introduite dans le Chapitre 1, Sous-section 1.4.1

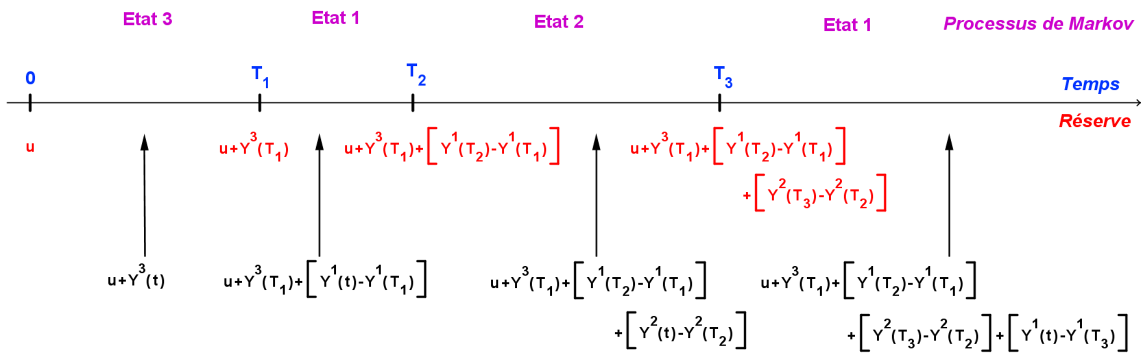
- $c^i \in \mathbb{R}_+$,
- $N^i(t)$ est un processus de Poisson de paramètre λ_i ,
- et $(X_{m^i})_{m^i \geq 0}$ est une suite de variables aléatoires identiquement distribuées, indépendante de $N^i(t)$.

Pour $1 \leq i \leq J$, $(Y^i(t))_{t\geq 0}$ est le processus qui modélise les gains moins les pertes de la compagnie dès lors que le processus d'environnement $(J(t))_{t\geq 0}$ se trouve dans l'état i. Nous pouvons ainsi définir le processus global $(R(t))_{t\geq 0}$ qui décrit l'évolution des réserves la compagnie,

$$\begin{aligned} R(t) &= u + \sum_{p\geq 1} \sum_{1\leq i \leq J} \left[Y^i(T_p) - Y^i(T_{p-1})\right] 1_{\{J_{T_{p-1}}=i, T_p \leq t\}} \\ &\quad + \sum_{p\geq 0} \sum_{1\leq i \leq J} \left[Y^i(t) - Y^i(T_p)\right] 1_{\{J_{T_p}=i, T_p \leq t < T_{p+1}\}}, \end{aligned}$$

où $u > 0$ et T_p est l'instant du $p^{\text{ème}}$ saut du processus $(J(t))_{t\geq 0}$. Un exemple est tracé sur la Figure 2.

Exemple 0.0.5 *Nous détaillons ci-dessous un deuxième exemple sous forme de frise pour bien comprendre l'évolution du processus.*



Le processus de risque part de la réserve initiale u. Le processus d'environnement est initialement dans l'état 3 ; le processus de risque évolue donc de la même manière que $Y^3(t)$ jusqu'au premier saut de $J(t)$, à savoir T_1. Le processus d'environnement, en T_1, passe dans l'état 1 ; le processus de risque va donc évoluer comme $Y^1(t)$ de T_1 à T_2 et ainsi de suite...

Remarque 0.0.6 (Changement d'état) *Comme nous le voyons sur l'exemple précédent, aux instants de saut du processus de Markov, le processus de risque qui décrit les réserves de la compagnie d'assurance est encore dans l'état précédent alors que le processus de Markov a déjà sauté. Ceci correspond à la situation où dans un premier temps, l'environnement change et une fois ce changement opéré on fait le bilan de la période précédente.*

Étant donné l'état dans lequel le processus d'environnement se trouve, on a à étudier un processus de risque avec montants de sinistres dépendants. Ce thème a été abordé par exemple dans Ignatov *et al.* (2001) et Lefèvre et Loisel (2009). Cet aspect de la théorie du risque dans le contexte des distributions à queues lourdes sera traitée Page 21 de cette introduction.

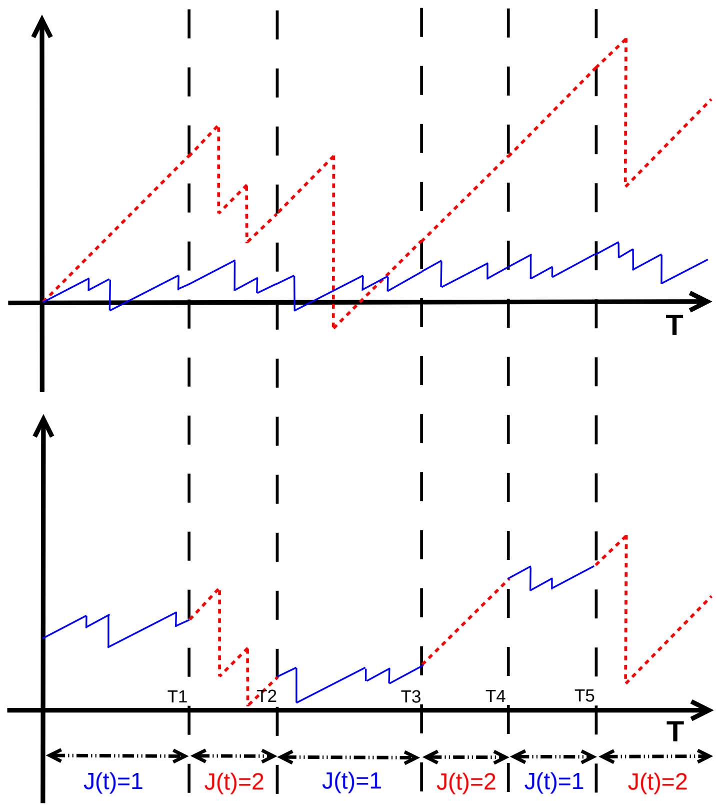


FIG. 2 – Exemple d'un processus de risque modulé, avec deux états (un rouge, l'autre bleu)

Dépendance entre temps inter-sinistres et montants de sinistres

Certains types de risques imposent de prendre en compte une dépendance entre les temps qui séparent deux sinistres et les montants des sinistres. Les catastrophes naturelles font partie de cette catégorie. Prenons l'exemple des tremblements de terre ; avec ce type de risque, on s'attend à un événement particulièrement dévastateur dès lors qu'un séisme de forte intensité n'a pas été observé depuis un certain temps. En effet, un séisme est le fruit d'un relâchement d'énergie qui s'est accumulée dans les failles entre les plaques tectoniques. Ce temps d'accumulation, conjugué au temps de reconstruction du bien assuré nous ont amené à penser qu'à la suite de plusieurs temps inter-sinistres suffisamment longs, un tremblement de terre sera plus destructeur. Nous nous intéressons aussi au phénomène inverse, illustré par les inondations. En effet, lorsque plusieurs inondations successives interviennent dans un laps de temps réduit, l'eau s'accumulant, le nombre d'habitations susceptibles d'être touchées croit et les dégâts matériels seront plus conséquents. Ainsi, une inondation succédant plusieurs temps inter-sinistres courts sera jugée plus dangereuse que les autres. Ces situations sont étudiées dans le Chapitre 2 de la thèse.

La dépendance entre les temps inter-sinistres et les montants des sinistres a été étudiée dans plusieurs travaux en théorie de la ruine. Dans Albrecher et Boxma (2004), le temps inter-sinistre est dépendant du montant du sinistre qui le précède. Explicitement, avec X_j qui modélise le montant du $j$ème sinistre, si X_j est plus grand qu'une certaine barrière T_j alors le temps d'attente du prochain sinistre est distribué selon une loi exponentielle de paramètre λ_1, dans le cas contraire, le temps d'attente est distribué selon une loi exponentielle de paramètre λ_2. Dans ce modèle, les auteurs obtiennent des résultats explicites de la transformée de Laplace de la probabilité de ruine. Dans Albrecher et Teugels (2006), le temps précédant un sinistre est dépendant avec le montant de celui-ci via une copule. Dans ce cadre, des résultats asymptotiques de la probabilité de ruine en temps fini et infini sont obtenus dans le cas de montants de sinistres ayant une distribution à queue légère. Considérant également dépendants le montant de sinistre et le temps inter-sinistre précédent, Boudreault *et al.* (2006) adopte une forme particulière de dépendance explicitant la densité $f_{X_j|V_j}$ du montant du $j$ème sinistre X_j connaissant le temps inter-sinistre précédent V_j,

$$f_{X_j|V_j}(x) = e^{-\beta V_j} x f_1(x) + (1 - e^{-\beta V_j}) f_2(x) \quad x \geq 0 \ ,$$

pour un certain $\beta > 0$, et certaines densités f_1 et f_2. La fonction de Gerber-Shiu est étudiée dans ce contexte. Dans Meng *et al.* (2008), le temps précédant un sinistre influence la distribution de son montant ; si le temps inter-sinistre V_i est plus grand qu'une certaine barrière M_i alors le montant du sinistre aura une certaine distribution, dans le cas contraire, le montant du sinistre aura une distribution différente. La probabilité de non-ruine en temps infini est calculée avec des montants de sinistres exponentiels. Enfin, Ambagaspitiya (2009) modélise la distribution jointe du temps inter-sinistre et du montant du sinistre suivant par deux distributions gamma bivariées. La probabilité de ruine en temps infini est calculée.
Le modèle que nous proposons au Chapitre 2 est original dans le sens où plusieurs temps inter-sinistres influent sur la distribution du montant de sinistre ; ceci permet de modéliser des phénomènes d'accumulation et de mieux prendre en considération le passé du processus.

Approche multivariée de la théorie de la ruine

Une compagnie d'assurance est généralement formée de multiples branches d'activité. La théorie de la ruine univariée consiste à étudier l'une des branches indépendamment des autres ou encore la somme agrégée des réserves des branches. Ces quantités sont intéressantes pour évaluer la solvabilité globale de la compagnie mais elles échouent dans l'analyse du comportement joint des branches d'activité afin de détecter celles qui portent majoritairement le risque. Cette analyse multivariée prend tout son sens compte tenu des nouvelles règles de solvabilité auxquelles seront bientôt soumises les compagnies d'assurance. Ces nouvelles normes proposent un modèle dans lequel les risques sont analysés de façon modulaire et ensuite agrégés au moyen de coefficients de corrélation. Les coefficients de corrélation n'en sont pas vraiment, ce sont des paramètres fixés par la directive et ils ne dépendent que de l'appartenance du risque à une certaine catégorie (catastrophe naturelle, marché, mortalité, longévité...). Ce modèle présente de grosses lacunes mais il met en avant la volonté des autorités de contrôle d'analyser tous les risques souscrits par la compagnie ; cela va dans le sens d'une étude multivariée qui pourra être intégrée dans les modèles internes, autorisés par ces nouvelles normes. Cette analyse multivariée peut permettre à la compagnie d'équilibrer son exposition aux risques, d'une part en transférant une partie d'un risque détecté trop dangereux vers le marché financier par le moyen de titrisation ou en achetant de la réassurance, d'autre part en allouant suffisamment de capital à chaque branche d'activité. Ce dernier problème est traité aux Chapitres 3 et 4 de la thèse ; le tout étant de définir le *suffisamment*. Étant donnée une réserve initiale globale fixe u, il faut trouver la meilleure répartition possible de ce capital entre chaque branche pour avoir une compagnie solvable ainsi que des lignes d'affaires équilibrées. Pour définir cette meilleure répartition ou allocation, nous choisissons de minimiser des critères de risque multivariés, que nous présentons juste après avoir défini le processus de risque multivarié.

Processus de risque multivarié

L'idée est de généraliser ce qu'il se fait en dimension 1. Nous étudions donc un vecteur de processus, soit

$$\mathbf{R}(\mathbf{t}) = \left(R^{(1)}(t), ..., R^{(d)}(t)\right) ,$$

avec $R^{(i)}$ un processus de risque univarié défini en (1).
On a donc

$$\mathbf{R}(\mathbf{t}) = \begin{pmatrix} u^{(1)} + c^{(1)}t - \sum\limits_{i=0}^{N^{(1)}(t)} X_i^{(1)} \\ \vdots \\ u^{(d)} + c^{(d)}t - \sum\limits_{i=0}^{N^{(d)}(t)} X_i^{(d)} \end{pmatrix} ,$$

où

- $u^{(j)}$, $c^{(j)}$ $(j = 1, ..., d)$ sont deux réels supposés positifs, représentant respectivement les réserves initiales de la branche j de la compagnie et le taux de cotisation demandé aux assurés de cette branche, qui est donc supposé constant,
- $N^{(j)}(t)$ $(j = 1, ..., d)$ est un processus de Poisson homogène d'intensité λ_j représentant le nombre de sinistres,
- $X_i^{(j)}$ $(j = 1, ..., d)$ est une variable aléatoire positive représentant le montant du $i^{\text{ème}}$ sinistre de la branche j. Pour $1 \leq j \leq d$, $\left(X_i^{(j)}\right)_{i \geq 1}$ est une suite de variables aléatoires i.i.d et indépendants des $N^{(j)}(t)$. On note $F_{X^{(j)}}$ leur fonction de répartition, $f_{X^{(j)}}$ leur densité et μ_j leur moyenne.

On note

- $\mathbf{c} = (c^{(1)}, ..., c^{(d)})$ le vecteur des taux de cotisation pour chaque branche,
- $\mathbf{u} = (u^{(1)}, ..., u^{(d)})$ le vecteur des réserves initiales de chaque branche,
- $\mathbf{N}(\mathbf{t}) = (N^{(1)}(t), ..., N^{(d)}(t))$ le vecteur des processus de Poisson,
- $\mathbf{S}(\mathbf{t}) = (S^{(1)}(t), ..., S^{(d)}(t))$ où $S^{(j)}(t) = \sum\limits_{i=0}^{N^{(j)}(t)} X_i^{(j)}$ $\;1 \leq j \leq d$ représente les pertes cumulées de la branche j,
- $\mathbf{Y}(\mathbf{t}) = (Y^{(1)}(t), ..., Y^{(d)}(t))$ où $Y^{(j)}(t) = S^{(j)}(t) - c^{(j)}t$ $\;1 \leq j \leq d$.

Critères de risque

Dans l'étude des processus de risque multivariés, une idée naturelle est de trouver un équivalent à la probabilité de ruine définie dans le cadre univarié par la Définition 0.0.1. La manière de définir la ruine multivariée n'est pas unique. Autant il semble normal de définir la ruine univariée comme une position négative du processus de risque à un instant donné, autant il est difficile de trouver une définition unanime dans le cadre multivarié. Par exemple, dans Cai et Li (2007), on trouve les probabilités de ruine multivariées suivantes, qui semblent être assez intuitives. Pour $T > 0$ et $\mathbf{u} = (u^{(1)}, \ldots, u^{(d)}) \in (0, \infty)^d$, on définit

$$\psi_{sum}(\mathbf{u}, T) = P\left(\sup_{[0,T]}\left\{\sum_{j=1}^{d} Y_t^{(j)}\right\} > \sum_{j=1}^{d} u_j\right) , \tag{2}$$

$$\psi_{and}(\mathbf{u}, T) = P\left(\bigcap_{j=1}^{d}\left\{\sup_{[0,T]} Y_t^{(j)} > u_j\right\}\right) , \tag{3}$$

$$\psi_{or}(\mathbf{u}, T) = P\left(\bigcup_{j=1}^{d}\left\{\sup_{[0,T]} Y_t^{(j)} > u_j\right\}\right) , \tag{4}$$

et

$$\psi_{sim}(\mathbf{u}, T) = P\left(\sup_{0 \leq t < T}(min\{Y^{(1)}(t) - u_1, ..., Y^{(d)}(t) - u_d\}) > 0\right). \tag{5}$$

On peut naturellement définir les probabilités de ruine correspondantes en temps infini en prenant $T = \infty$. La probabilité de ruine (3) est définie comme la probabilité que toutes les branches tombent en ruine (au sens où le processus tombe en dessous de 0). (4) définit la probabilité qu'une au moins des branches soit en ruine. (5) correspond à la probabilité que toutes les branches fassent faillite au même instant. Enfin (2) est la probabilité que la richesse agrégée de toutes les branches tombe en dessous de 0.

Remarque 0.0.7 *Pour $m = 1$, on retrouve, pour $T > 0$*

$$\psi(u, T) = \psi_{and}(u, T) = \psi_{or}(u, T) = \psi_{sim}(u, T) = \psi_{sum}(u, T) = P\left(\sup_{0 \leq t < T} Y(t) > u\right).$$

Ces diverses définitions montrent qu'il n'existe pas une façon unique d'envisager la ruine ; cela dépend surtout de ce qui nous intéresse. Dans le Chapitre 4 de cette thèse, nous étudions la probabilité de ruine définie par Hult et Lindskog (2006a). Pour $\beta \in [0, 1]$, soit

$$F_\beta = \left\{\mathbf{x} : \beta \sum_{k=1}^{d}\left(x^{(k)} \vee 0\right) < -\sum_{k=1}^{d}\left(x^{(k)} \wedge 0\right)\right\} , \tag{6}$$

où $\vee = \min$ et $\wedge = \max$. Pour $T > 0$, la probabilité de ruine multivariée en temps fini $\psi_{d,\beta}(\mathbf{u}, T)$ est définie comme la probabilité que le processus de risque $\mathbf{R}_t$ appartienne à F_β à un certain moment t avant T. Explicitement, pour $T > 0$, on a

$$\psi_{d,\beta}(\mathbf{u}, T) = P\left(\exists t \in [0, T], \;\; \mathbf{R}_t \in F_\beta\right) . \tag{7}$$

On a aussi, en temps infini

$$\psi_{d,\beta}(\mathbf{u}) = P\left(\exists t > 0, \;\; \mathbf{R}_t \in F_\beta\right) . \tag{8}$$

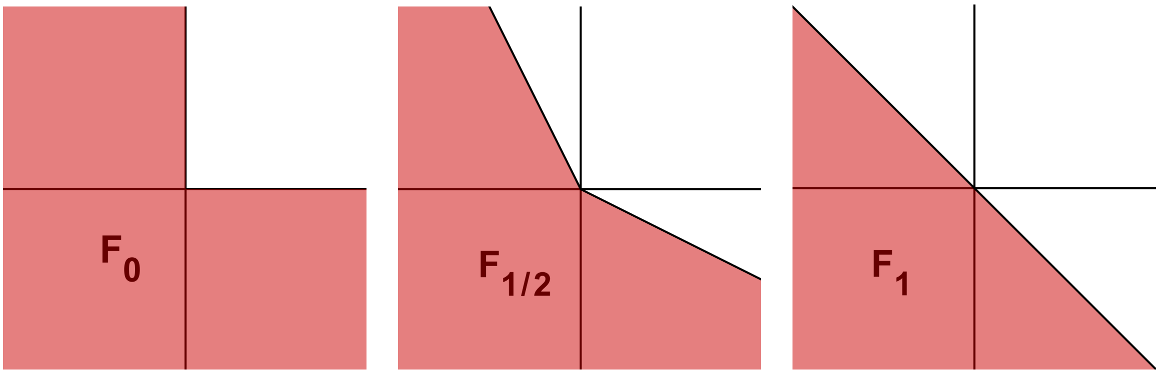


FIG. 3 – F_β pour β=0,1/2 et 1 en deux dimensions

Remarque 0.0.8 – *L'ensemble F_β correspond à la possibilité de transférer du capital entre les branches ; une branche d'activité qui est dans une position confortable au regard de sa solvabilité peut transférer une fraction β de sa richesse vers une autre branche qui est en position d'insolvabilité.*
- *Pour $\beta = 0$, les transferts de capitaux ne sont pas autorisés et donc $\psi_{d,0} = \psi_{or}$.*
- *Pour $\beta = 1$, les transferts sont autorisés sans aucune restriction et $\psi_{d,1} = \psi_{sum}$.*

La Figure 3 représente l'ensemble F_β pour β =0,1/2 et 1 en deux dimensions.

Au Chapitre 3, nous nous intéressons à un autre critère de risque introduit par Loisel (2005). Ce critère de risque prend en considération le comportement du processus lorsque celui-ci est négatif. Le but est de différencier une compagnie en ruine avec un déficit petit et donc assez facile à combler d'une autre compagnie qui est en ruine de manière forte avec un déficit conséquent. Il est possible que les deux compagnies soient dans une situation positive en fin d'année, ce qui est demandé par les régulateurs ; cependant une compagnie qui se remet difficilement d'une situation inconfortable doit être surveillée de manière plus poussée qu'une compagnie qui n'a eu que de petits accidents. Pour mesurer cela, considérons un processus de risque univarié $(R(t))_{t\geq 0}$ et définissons, pour $u, T > 0$,

$$I_T(u) = \int_0^T \mathbb{1}_{\{R(t)<0\}} |R(t)| dt \,. \tag{9}$$

(9) correspond à l'aire délimitée d'une part par la droite d'équation $y = 0$ et d'autre part par la trajectoire du processus de risque $(R(t))_{t\geq 0}$ lorsque celui-ci est négatif. Sur la Figure 4, (9) correspond à l'aire en rouge. Nous pouvons ainsi définir comme indicateur de risque pour un processus univarié $(R(t))_{t\geq 0}$, l'espérance de $I_T(u)$. En considérant un horizon de temps infini, on peut obtenir $E[I_\infty(u)]$ à partir de la probabilité de ruine $\psi(u)$ en utilisant les résultats ci-dessous, démontrés dans Loisel (2005).

Théorème 0.0.9 *Soit $T \in \mathbb{R}^+$, $(Y_t)_{t\in[0,T)}$ un processus de renouvellement (qui peut-être modulé par un processus d'environnement avec un nombre d'états finis) intégrable par rapport au temps. Soit $u \in \mathbb{R}$, notons par $\tau(u, T)$ la variable aléatoire correspondant au temps passé par le processus*

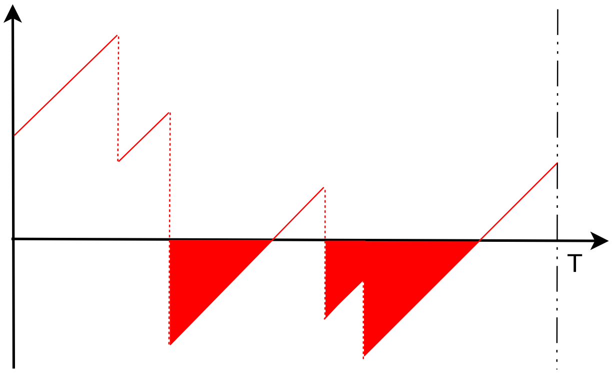


Fig. 4 – Partie négative du processus intégré par rapport au temps.

$u+Y_t$ sous la barrière zéro entre 0 et T :

$$\tau(u,T)=\int_0^T \mathbb{1}_{\{u+Y_t<0\}}dt,$$

Soit $\tau_0(u,T)$ le temps passé en zéro par le processus $u+Y_t$ entre 0 et T :

$$\tau_0(u,T)=\int_0^T \mathbb{1}_{\{u+Y_t=0\}}dt.$$

Enfin, soit $I_T(u)$ représentant l'intégrale par rapport au temps de la partie négative du processus $u+Y_t$ entre 0 et T :

$$I_T(u)=\int\limits_0^T \mathbb{1}_{\{u+X_t<0\}}|u+X_t|dt$$

et $f(u)=E(I_T(u))$.
Pour $u\in\mathbb{R}$, si $E\left(\tau_0(u,T)\right)=0$, alors f est dérivable en u et $f'(u)=-E\left(\tau(u,T)\right)$.

Théorème 0.0.10 *Soit $Y_t=ct-S(t)$, avec $S(t)$ un processus de Poisson composé. Soit $T<+\infty$ et h défini par $h(u)=E(\tau(u))$ pour $u\in\mathbb{R}$. h est dérivable sur $\mathbb{R}_*^+=(0,\infty)$, et pour $u>0$,*

$$h'(u)=-\frac{1}{c}E\left(N^0(u,T)\right),$$

où $N^0(u,T)=Card(\{t\in[0,T],\quad u+ct-S(t)=0\})$. De plus pour $T=\infty$, si $\tau(u)$ est intégrable pour tout $u>0$,

$$E\left(N^0(u,\infty)\right)=\frac{\psi(u)}{1-\psi(0)}. \tag{10}$$

On peut maintenant définir le critère de risque multivarié auquel nous nous sommes intéressés au Chapitre 3. Il s'agit de l'espérance de la somme des aires délimitées par la droite $y=0$ et par

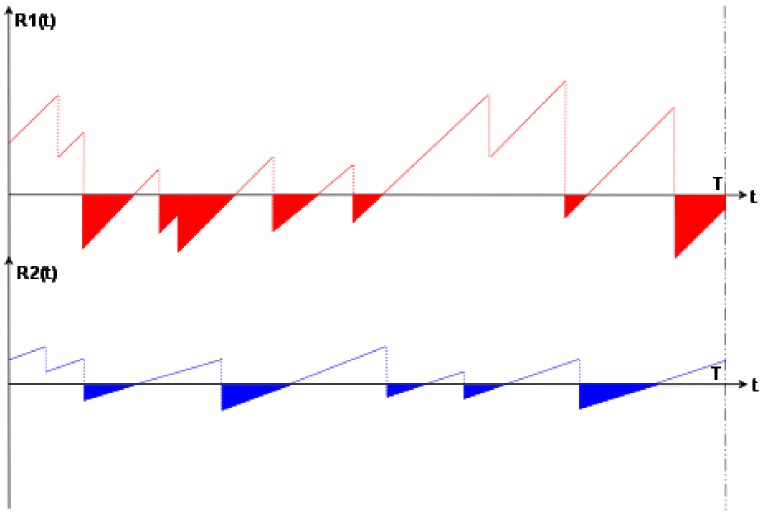


Fig. 5 – Critère de risque en dimension 2

la partie négative de chaque branche d'activité, (sur Figure 5, en dimension 2, cela correspond à l'espérance de l'aire en rouge plus l'espérance de l'aire en bleu), c'est-à-dire que nous nous intéressons à la fonction

$$I_T(\mathbf{u}) = \sum_{i=1}^{d} E[I_T^{(i)}(u_i)] \;, \tag{11}$$

où pour un processus de risque multivarié $\mathbf{R}(\mathbf{t}) = (R^{(1)}(t), \ldots, R^{(d)}(t))'$, on a, pour $1 \leq i \leq d$, et $u_i, T > 0$,

$$I_T^{(i)}(u_i) = \int_0^T \mathbb{1}_{\{R^{(i)}(t)<0\}} |R^{(i)}(t)| dt \;.$$

Une limite importante de ce critère de risque est le fait qu'il ne prenne pas en compte la dépendance entre les branches. Cependant, en le modifiant quelque peu, on peut obtenir un indicateur de risque qui prend en compte cette dépendance. Dans Loisel (2005) l'indicateur

$$\sum_{i=1}^{d} E\left(\int_0^T \mathbb{1}_{\{R^{(i)}(t)<0\}} \mathbb{1}_{\left\{\sum_{j=1}^{d} R^{(j)}(t)>0\right\}}\right)$$

pour $T > 0$ est proposé. Pour un processus discret multivarié $(\mathbf{R}_p)_{p\in\mathbb{N}}$, Cenac *et al.* (2010) étudie

$$\sum_{i=1}^{d} E\left(\sum_{p=1}^{n} g_i(R_p^{(i)}) \mathbb{1}_{\{R_p^{(i)}<0\}} \mathbb{1}_{\left\{\sum_{j=1}^{d} R_p^{(j)}>0\right\}}\right) \;.$$

pour $n > 0$ et des fonctions g_i dérivables, convexes et satisfaisant $g_i(x) \geq 0$ pour $x \leq 0$, $i = 1, \ldots, d$.

Ces indicateurs de risque vont servir de critère pour allouer de manière optimale une réserve initiale globale entre toutes les lignes d'affaires de la compagnie.

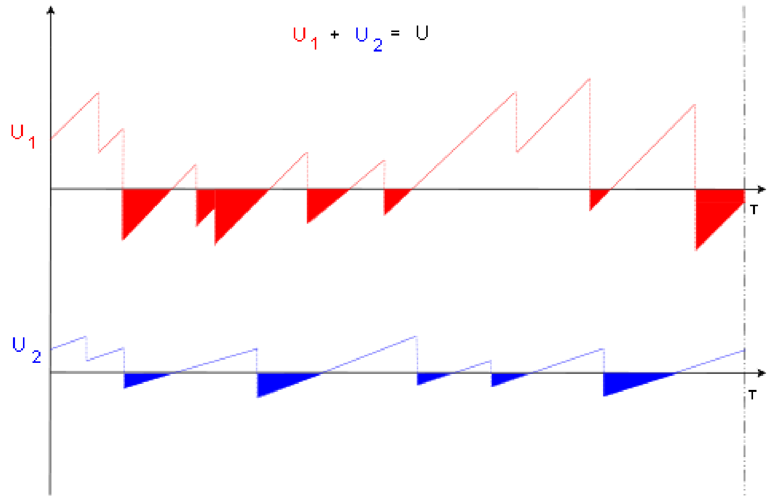


Fig. 6 – Un problème d'allocation optimale en dimension 2

Problèmes d'allocation optimale

Supposons ici que la compagnie d'assurance possède un capital initial global u. Ce capital doit être partagé entre toutes les branches d'activité qui composent la compagnie. Ce choix est primordial dans la bonne santé financière de la compagnie ; en effet, les transferts de capitaux entre les différentes branches sont très réglementés et il est difficile de combler une perte d'une branche par une autre. Étant donné un indicateur de risque, il est intéressant de chercher l'allocation qui permet de minimiser cet indicateur. Aux Chapitres 3 et 4, nous nous intéressons aux comportements asymptotiques de l'allocation optimale lorsque la réserve initiale est particulièrement grande, respectivement pour l'indicateur de risque $I_T(\mathbf{u})$ introduit en (11) et pour $\psi_{d,\beta=0}(\mathbf{u})$ défini en (7). Cela conduit aux deux problèmes d'optimisation suivant.

$$\begin{cases} \min_{\mathbf{u}\in\mathbb{R}_+^d} I_T(\mathbf{u}) \ , \\ \text{sous la contrainte } u^{(1)} + \ldots + u^{(d)} = u \ , \end{cases} \tag{12}$$

et

$$\begin{cases} \min_{\mathbf{u}\in\mathbb{R}_+^d} \psi_{d,0}(\mathbf{u}, T) \ , \\ \text{sous la contrainte } u^{(1)} + \ldots + u^{(d)} = u \ . \end{cases} \tag{13}$$

Le problème (12) a été traité dans Loisel (2005) dans le cas d'un horizon infini. L'allocation optimale dans Loisel (2005) consiste à ne pas allouer de capital aux branches les plus sûres et à répartir la réserve u de façon à rendre égal le temps que chacune des branches passe dans une position négative. Le problème (12) est illustré en dimension 2 par la Figure 6. Dans cette thèse, ces deux problèmes d'allocation optimale sont étudiés dans des contextes différents. Le Chapitre 3 traite le problème (12) et est davantage concerné par l'impact de la dangerosité des branches sur l'allocation optimale. Dans ce cas, les risques de chaque branche ont des distributions dont les queues sont différentes et la structure de dépendance importe peu vu que le critère de risque ne la prend pas en compte. Nous avons cependant proposé une structure de dépendance qui pourra être exploitée dans des travaux futurs. Le Chapitre 4 étudie l'allocation qui répond au problème (13) et étudie l'impact de la dépendance entre les branches sur l'allocation optimale et suppose dans un soucis de simplification que les risques souscrits par chaque branche ont des distributions dont les queues sont équivalentes. Les dépendances étudiées dans ces deux chapitres sont détaillées ci-après.

Dépendance inter-branches

Deux formes de dépendance sont utilisées dans cette thèse pour modéliser la corrélation entre les différence branches d'activité qui composent la compagnie.

La première est la dépendance par chocs communs. On appelle *choc commun* une occurrence de sinistre qui touche simultanément plusieurs branches de la compagnie. Prenons l'exemple d'un tremblement de terre ; il peut détruire des habitations, ce qui entraîne des pertes dans la branche multirisques-habitation, engendrer des dégâts sur des véhicules, donc amputer la branche auto et causer la perte de vies humaines, gérées par la branche vie de la compagnie. Dans cet exemple, un événement crée des pertes dans trois secteurs d'activité différents. Ces phénomènes de chocs communs peuvent survenir là où on s'y attend le moins. Un événement a touché simultanément la branche assurance-vie suédoise et la branche assurance dommage de l'Asie du sud-est, incroyable, non ? C'est pourtant ce qu'il s'est passé en décembre 2004 lorsque un tremblement de terre a causé un tsunami qui a ravagé le sud-est asiatique, endroit privilégié des Suédois pour y passer leurs vacances (cf Loisel *et al.* (2010)). Ces exemples montrent l'importance de ne pas négliger et donc de prendre en compte ce type de dépendance dans les modèles. Le modèle le plus utilisé pour ce type de dépendance en théorie du risque est le modèle Poisson à chocs communs. Outre ce modèle, une autre forme de chocs communs est étudié au Chapitre 4. À chaque survenance de sinistre, certaines branches d'activité auront parfois les mêmes pertes, les autres payant un montant indépendamment des premières. Cette situation peut correspondre, par exemple, à une compagnie qui couvre les mêmes risques mais dans des zones géographiques différentes, et parmi lesquelles certains contrats couvriraient une même somme sur un risque donné.

La deuxième forme de dépendance considérée dans la thèse est l'influence commune de l'environnement économique, politique, juridique ou environnemental sur les branches d'activité de la compagnie. L'exemple climatique est le plus parlant ; de mauvaises conditions climatiques augmentent la sévérité et la fréquence des sinistres dans de nombreux secteurs d'activité. L'idée pour modéliser ce phénomène est d'utiliser un processus d'environnement, de la même façon que dans le contexte univarié (cf Page 7 de cette introduction).

Modélisation d'événements extrêmes

L'étude des *événements extrêmes* est concernée par les phénomènes qui ont une probabilité assez grande de prendre une très grande valeur. Grossièrement, une variable aléatoire modélise un événement extrême s'il existe un réel $\alpha > 0$ tel que

$$P(X > x) \sim x^{-\alpha} \; , \quad x \to \infty \; , \tag{14}$$

où $\sim$ signifie que le ratio tend vers 1.
Nous disons dans ce cas que X est une variable aléatoire à queue épaisse (à droite). L'exemple type est la distribution Pareto ; nous pouvons citer également les distributions Cauchy ou Student. Pour appréhender un peu mieux la caractéristique de queue épaisse, nous pouvons mettre en parallèle une variable aléatoire satisfaisant (14), à comportement puissance avec une variable

aléatoire Y de distribution exponentielle ; en effet on a, pour un $\alpha > 0$ et un $\lambda > 0$

$$\lim_{x\to\infty} \frac{P(Y > x)}{P(X > x)} \sim \frac{e^{-\lambda x}}{x^{-\alpha}} = 0 \ ,$$

c'est-à-dire que pour un réel x suffisamment grand, $P(X > x) > P(Y > x)$ et donc X a une probabilité supérieure de prendre une très grande valeur comparé à Y.

Considérons une variable aléatoire $X \in [x_0, +\infty)$ satisfaisant (14) pour un $\alpha > 0$. On a

$$\int_{x_0}^{+\infty} x^{\beta-1} P(X > x)dx \begin{cases} < \infty & \text{si } \beta < \alpha \\ = \infty & \text{si } \beta \geq \alpha \end{cases}$$

Ceci signifie que tout moment de X d'ordre plus grand que α est infini ; les nombreuses méthodes statistiques qui se basent sur les moments échouent donc dans l'étude de cette classe, d'où la particularité de ces distributions.

Les phénomènes extrêmes en assurance

En théorie de la ruine, les variables aléatoires étudiées sont principalement les montants de sinistres auxquels est exposée la compagnie d'assurance. Nous nous intéressons donc ici à des phénomènes qui engendrent des pertes extrêmes. Nous utilisons ce type de variable aléatoire pour modéliser par exemple :

- les catastrophes naturelles ;
  - les inondations,
  - les tremblements de terre,
- les attaques terroristes.

Les tremblements de terre et les inondations, avec une dépendance entre les temps inter-sinistres et les montants de sinistres, sont particulièrement traités dans le Chapitre 2.

La classe des distributions à variations régulières

La classe des distributions à variations régulières est très largement utilisée pour décrire les phénomènes extrêmes. Elle est incluse dans la classe plus large des distributions sous-exponentielles. Pour plus de détails concernant ces classes et leur utilisation en théorie du risque, voir par exemple Embrechts *et al.* (1997).
Nous donnons ici les définitions de cette classe en contexte univarié puis multivarié avant de présenter divers résultats en théorie de la ruine dans des modèles où la distribution des montants de sinistres est à variations régulières.

Cas univarié

Définition 0.0.11 *Une fonction L définie sur $(0, \infty)$ est à variations lentes à l'infini si*

$$\lim_{u\to\infty} \frac{L(tu)}{L(u)} = 1, \quad \textit{pour tout } t > 0.$$

On note $L \in \mathcal{R}_0$

Définition 0.0.12 *Une variable aléatoire réelle X est à variations régulières s'il existe $\alpha > 0$, tel que*

$$\lim_{u\to\infty} \frac{P(X > tu)}{P(X > u)} = t^{-\alpha}, \quad \textit{pour tout } t > 0,$$

ou de manière équivalente,

$$P(X > u) = u^{-\alpha} L(u),$$

pour une fonction $L \in \mathcal{R}_0$.
On note $X \in \mathcal{R}_{-\alpha}$.

Comme mentionné ci-dessous, la classe de distributions à variations régulières appartient à la classe plus large des distributions sous-exponentielles dont voici la définition.

Définition 0.0.13 *Une variation aléatoire réelle X est dite sous-exponentielle si sa fonction de répartition F vérifie (avec $\overline{F} = 1 - F$)*

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n \; .$$

On note $X \in \mathcal{S}$.

Proposition 0.0.14 *Soit $(X_i)_{1\leq i\leq n}$ une suite de variables aléatoires indépendantes et toutes distribuées selon une variable aléatoire $X \in \mathcal{S}$. Soient $S_n = X_1 + \ldots + X_n$ et $M_n = \max(X_1, \ldots, X_n)$. On a*

$$P(S_n > x) \sim P(M_n > x) \sim nP(X > x) \; , \quad x \to \infty \; .$$

La Proposition 0.0.14 met en exergue le fait que sous l'hypothèse de distributions sous-exponentielles, la queue de la somme est déterminée par la queue du maximum. Cette caractéristique propre aux distributions à queue épaisse modélise bien les phénomènes type catastrophe naturelle ou attaque terroriste ; un des sinistres peut mettre en danger la solvabilité d'une compagnie d'assurance car il est rare et peut causer des pertes conséquentes.

Cas multivarié

Dans un cadre multivarié, plusieurs définitions ont été proposées dans la littérature pour étendre la notion de distributions à variations régulières. Ces définitions sont toutes équivalentes sous certaines hypothèses. Nous avons choisi de ne pas toutes les présenter par souci de clarté. Nous introduisons seulement celles qui sont manipulées dans cette thèse, au Chapitre 4. Cependant les autres définitions, ainsi que des discussions concernant les équivalences peuvent être trouvées par exemple dans Basrak (2000); Basrak *et al.* (2002); Resnick (2004b); Lindskog (2004); Resnick (2007). Avant d'énoncer ces définitions, la notion de *convergence vague* est introduite.

Définition 0.0.15 (Convergence vague) *Soient $\mu, \mu_1, \mu_2, \ldots$ des mesure de Radon sur $\mathcal{B}(\bar{\mathbb{R}}^d)$. μ_n converge vaguement vers μ, noté $\mu_n \xrightarrow{v} \mu$, si pour toutes fonctions f continues bornées à support compact,*

$$\int f d\mu_n \to \int f d\mu \; .$$

Proposition 0.0.16 (Hult et Lindskog (2006a)) *$\mathbf{X} \in \mathbb{R}^d$ est à variations régulières s'il vérifie une des assertions équivalentes suivantes.*

1. *Il existe un* $\alpha > 0$ *et une mesure de probabilité* S *sur la sphère unité* $\mathbb{S}^{d-1} = \{\mathbf{x} : |\mathbf{x}| = 1\}$ *telle que*
$$\frac{P(|\mathbf{X}| > xu, \mathbf{X}/|\mathbf{X}| \in A)}{P(|\mathbf{X}| > u)} \to x^{-\alpha} S(A) \quad u \to \infty ,$$
pour tout $x > 0$ *et borélien* $A \subset \mathbb{S}^{d-1}$ *avec* $S(\partial A) = 0$.
2. *il existe une mesure* μ *de Radon non nulle sur* $\mathcal{B}(\bar{\mathbb{R}}^d \backslash \{\mathbf{0}\})$ *avec* $\mu(\bar{\mathbb{R}}^d \backslash \mathbb{R}^d) = 0$ *telle que, sur* $\mathcal{B}(\bar{\mathbb{R}}^d \backslash \{\mathbf{0}\})$
$$\frac{P(u^{-1}\mathbf{X} \in \cdot)}{P(|\mathbf{X}| > u)} \xrightarrow{v} \mu(\cdot) \quad u \to \infty .$$

On note $\mathbf{X} \in \mathcal{MR}_{\mu,-\alpha}$. *De plus on a*
$$\mu(\mathbf{x} : |\mathbf{x}| > r, \mathbf{x}/|\mathbf{x}| \in A) = r^{-\alpha} S(A) ,$$
pour $r > 0$ *et* $A \in \mathbb{S}^{d-1}$.

Dans Hult et Lindskog (2006b), on trouve une définition équivalente aux assertions de la Proposition 0.0.16.

Définition 0.0.17 (Hult et Lindskog (2006b)) *Un vecteur aléatoire* $\mathbf{X} \in \mathbb{R}^d$ *de support non borné est à variations régulières s'il existe* $\alpha > 0$, *une fonction* L *à variations lentes et une mesure de Radon non nulle sur* $\mathcal{B}(\bar{\mathbb{R}}^d \backslash \{\mathbf{0}\})$ *avec* $\mu(\bar{\mathbb{R}}^d \backslash \mathbb{R}^d) = 0$ *tels que, quand* $u \to \infty$, *sur* $\mathcal{B}(\bar{\mathbb{R}}^d \backslash \{\mathbf{0}\})$
$$u^{\alpha} L(u) P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot) .$$
On note $\mathbf{X} \in \mathcal{MR}(\alpha, L, \mu)$.

La théorie des distributions multivariées à variations régulières nous permet d'avoir des résultats sur la somme de variables aléatoires dépendantes comme nous le verrons par la suite. De plus, que ce soit dans le contexte univarié ou multivarié, l'étude des trajectoires des processus de risque avec des distributions des montants de sinistres à variations régulières est équivalente (pour une réserve initiale suffisamment grande) à l'étude de ce processus à échéance. Ces résultats, qui seront utilisés tout au long de cette thèse, sont énoncés ci-dessous.

Somme de variables, vecteurs aléatoires à variations régulières et probabilités de ruine

En dimension 1

La théorie de la ruine étudie des processus Poisson composé avec dérive ; il est donc pertinent de s'intéresser dans un premier temps au comportement de la distribution de la somme des montants de sinistres que paient la compagnie. La somme de variables aléatoires à variations régulières a fait l'objet de plusieurs études ces dernières années. On doit donc étudier le comportement asymptotique, quand $u \to \infty$ de la variable aléatoire
$$S_d = \sum_{i=1}^{d} X_d ,$$
où pour $1 \leq i \leq d$, $X_i \in \mathcal{R}_{-\alpha}$ pour un certain $\alpha > 0$. Le cas où les X_i , $1 \leq i \leq d$ sont indépendants et identiquement distribués (voir par exemple Embrechts *et al.* (1997)) est bien connu, on a, quand $u \to \infty$,
$$P(S_d > u) \sim dP(X_1 > u) .$$

Lorsque les $X_i \in \mathcal{R}_{-\alpha}$, $1 \leq i \leq d$ sont comonotones et identiquement distribués, donc par exemple tous égaux à X_1, on a, vue la Définition 0.0.12,

$$P(S_d > u) = P(dX_1 > u) \sim d^\alpha P(X_1 > u) \ .$$

Ces deux cas sont très particuliers. Ils sont utiles lorsqu'on envisage par exemple le modèle où $X_i = W_0$ avec probabilité $p \in (0,1)$ et $X_i = W_i$ avec probabilité $1-p$, où $(W_i)_{i\geq 0}$ forme une suite i.i.d. ; dans ce cas, certains montants de sinistres sont comonotones, les autres indépendants. Ce modèle est développé dans le Chapitre 1 de la thèse puis étendu dans un contexte multivarié dans le Chapitre 4.
Concernant la somme de variables aléatoires dans des formes de dépendance plus générales, Barbe *et al.* (2006) énonce le résultat suivant.

Proposition 0.0.18 (Barbe *et al.* (2006)) *: Soit $\mathbf{X} = (X^{(1)}, \ldots, X^{(d)})$ un vecteur aléatoire de $\mathbb{R}^d$ à variations régulières pour un certain $\alpha > 0$. Alors,*

$$P(X^{(1)} + \ldots + X^{(d)} > u) \sim q_{d,\alpha} P(X^{(1)} > u) \ .$$

avec

$$q_{d,\alpha} = \int_{\mathbb{S}_{d-1}} \left({w^{(1)}}^{1/\alpha} + \ldots + {w^{(d)}}^{1/\alpha} \right) H(dw) \ ,$$

où H est une mesure spectrale (cf Barbe et al. *(2006)) définie sur l'ensemble*

$$\mathbb{S}_{d-1} = \{\mathbf{w} = (w^{(1)}, \ldots, w^{(d)}) \ : \ w_1 + \ldots + w_d = 1\} \ .$$

Certains auteurs ont travaillé sur des cas particuliers. On peut citer notamment Wüthrich (2003a); Alink *et al.* (2004, 2005) qui ont étudié des variables aléatoires dont la structure de dépendance est décrite par une copule archimédienne. En dimension 2, on peut citer Albrecher *et al.* (2006) qui étudient différentes copules (archimédienne, FGM). Enfin, Kortschak et Albrecher (2009) envisagent la classe plus large des distributions sous-exponentielles.

En théorie du risque, Embrechts *et al.* (1979) se sont intéressés au comportement d'une variable aléatoire Poisson composée $Y = \sum_{i=1}^N X_i$ avec $(X_i)_{i\geq 1}$ une suite i.i.d. de distribution sous-exponentielle et N une distribution Poisson de paramètre $\lambda > 0$. Ils obtiennent le résultat suivant.

Proposition 0.0.19 (Embrechts *et al.* (1979)) *Les assertions suivantes sont équivalentes :*
- $Y \in \mathcal{S}$.
- $X \in \mathcal{S}$.
- $\lim_{u\to\infty} \frac{P(Y>u)}{P(X>u)} = \lambda$.

Ce résultat nous permet d'obtenir la probabilité de ruine en temps infini lorsque les montants de sinistres X_i , $i \geq 1$ sont indépendants et identiquement distribués, de distribution commune $X \in \mathcal{R}_{-\alpha}$ pour $\alpha > 0$ (voir par exemple Embrechts *et al.* (1997))

$$\psi(u) \sim \frac{\lambda}{\rho} \int_u^\infty P(X > x) \ , \quad u \to \infty \ .$$

En temps fini, sous les même hypothèses, nous obtenons (voir Chapitre 1), avec $T > 0$

$$\psi(u,T) \sim P(X_1 + \ldots + X_{N(T)} > u) \sim \lambda T P(X > u) \ .$$

Dans le Chapitre 1 de cette thèse, nous obtenons en réalité un résultat plus fort puisque nous relâchons l'hypothèse de l'indépendance entre les montants de sinistres.

En dimension d>1

À l'heure actuelle, peu de résultats existent dans la littérature sur la théorie de la ruine multivariée en générale, et encore moins sur le cas particulier des distributions des montants de sinistres à queue épaisse. La littérature n'est toutefois pas vierge sur ce domaine, nous pouvons citer par exemple Picard *et al.* (2003) qui étudie un cas discret et Collamore (1996, 2002) qui obtiennent des résultats dans le cas de distributions à queue légère. Concernant les distributions à variations régulières, la probabilité de ruine multivariée a été étudiée par Hult et Lindskog (2006a), s'appuyant sur les résultats de Hult *et al.* (2005) et Hult et Lindskog (2006b). Ces résultats, qui sont le point de départ du Chapitre 4 sont énoncés ci-dessous.

Nous rappelons brièvement le modèle et le problème étudiés ici :

$$\mathbf{R}_t = u\mathbf{a} + \mathbf{c}t - \mathbf{S}_t \ , \tag{15}$$

avec $\mathbf{S}_t = \sum_{i=1}^{N(t)} \mathbf{X}_i$ un processus de Poisson composé multivarié avec $(\mathbf{X}_i)_{i\geq 1}$ une suite de vecteurs i.i.d. à variations régulières multivariées, ($\mathbf{X}_i \in \mathcal{MR}_{-\alpha,\mu}$ pour tout $i \geq 1$) et indépendante de $N(t)$ un processus de Poisson. Nous supposons de plus que $\alpha > 1$. Nous étudions la probabilité de ruine

$$\psi_{d,\beta}(u,T) = P(\mathbf{R}_t \in F_\beta \text{ pour un } t \in [0,T]) \ ,$$

où

$$F_\beta = \left\{ \mathbf{x} : \beta \sum_{k=1}^{d} \left(x^{(k)} \vee 0 \right) < - \sum_{k=1}^{d} \left(x^{(k)} \wedge 0 \right) \right\} \ .$$

Le résultat suivant étudie la somme de deux vecteur indépendants $\mathbf{X}$ et $\tilde{\mathbf{X}}$ qui sont à variations régulières. Comme dans le cas univarié, ce résultat va nous permettre ensuite d'étudier un processus de Poisson composé puis la probabilité de ruine.

Proposition 0.0.20 (Hult et Lindskog (2006b)) *Soit $\mathbf{X}$ un vecteur aléatoire de $\mathbb{R}^d$ satisfaisant $\mathbf{X} \in \mathcal{MR}(\alpha, L, \mu)$.*

- *Soit $\tilde{\mathbf{X}}$ un vecteur aléatoire de $\mathbb{R}^d$ indépendant de $\mathbf{X}$. Si $\tilde{\mathbf{X}}$ satisfait $u^\alpha L(u) P(u^{-1}\tilde{\mathbf{X}} \in \cdot) \xrightarrow{v} \tilde{\mu}(\cdot)$ sur $\mathcal{B}(\bar{\mathbb{R}}^d \backslash \{\mathbf{0}\})$ quand $u \to \infty$ pour une certaine mesure de Radon $\tilde{\mu}$ ($\tilde{\mu}$ peut être la mesure nulle) avec $\tilde{\mu}(\bar{R}^d \backslash R^d) = 0$, alors $\mathbf{X} + \tilde{\mathbf{X}} \in \mathcal{MR}(\alpha, L, \mu + \tilde{\mu})$.*
- *Si il existe un $k \geq 1$ pour lequel il existe des vecteurs aléatoires i.i.d. $\mathbf{X}_1, \ldots, \mathbf{X}_k$ tels que $\mathbf{X} \stackrel{d}{=} \mathbf{X}_1 + \ldots + \mathbf{X}_d$, alors $\mathbf{X}_1 \in \mathcal{MR}(\alpha, L, \mu/k)$.*

Le précédent résultat peut s'étendre à des sommes aléatoires comme le montre le résultat suivant.

Proposition 0.0.21 *Soit $(\mathbf{X}_k)_{k\geq 1}$ une suite i.i.d. de vecteurs aléatoires de $\mathbb{R}^d$ et soit $N \geq 0$ une variable aléatoire discrète satisfaisant $\sum_{n=1}^{\infty} P(N = n)(1+\epsilon)^n < \infty$ pour un $\epsilon > 0$. Supposons de plus que N et $\mathbf{X}_{k\geq 1}$ sont indépendants et que N n'est pas nulle presque surement.*

- *Si $\mathbf{X}_1 \in \mathcal{MR}(\alpha, L, \mu)$ alors $\sum_{k=1}^{N} \mathbf{X}_k \in \mathcal{MR}(\alpha, L, E(N)\mu)$.*
- *Si $\sum_{k=1}^{N} \mathbf{X}_k \in \mathcal{MR}(\alpha, L, \mu)$, alors $\mathbf{X}_1 \in \mathcal{MR}(\alpha, L, \mu/E(N))$.*

Le résultat précédent permet d'étudier un processus de risque multivarié à un instant t donné. Le résultat suivant permet de lier l'étude du processus à un temps fixé à la probabilité de ruine ; en effet, étudier un processus de risque multivarié avec des montants de sinistres de distribution à variations régulières est équivalent, pour une réserve initiale grande, à l'étudier à échéance.

Proposition 0.0.22 (Hult et Lindskog (2006b)) *Soit* $\mathbf{S_t} = \sum_{i=1}^{N(t)} \mathbf{X}_i$ *un processus de Poisson composé multivarié comme défini ci-dessus. Si* $A \in \mathcal{B}(\mathbb{R}^d)$ *est borné loin de* $\mathbf{0}$ *avec* $\mu(A) > 0$ *et* $\mu(\partial A) = 0$ *pour un* $T > 0$ *on a*

$$\lim_{u\to\infty} \frac{P(\mathbf{S}_t \in uA \textit{ pour un } t \in [0,T])}{P(\mathbf{S}_T \in uA)} = 1 \ .$$

Dans Hult et Lindskog (2006b), le résultat démontré est plus général puisqu'il concerne des processus additifs.

Nous pouvons maintenant énoncer le résultat sur la probabilité de ruine en temps fini.

Proposition 0.0.23 (Hult et Lindskog (2006a)) *Pour le processus de risque défini en (15) nous avons,*

$$\psi_{d,F_\beta}(u,T) \sim E(N_T)\mu(\mathbf{a} - F_\beta)P(|\mathbf{X}| > u) \ .$$

Ce dernier résultat se démontre grâce aux deux précédents. En effet, pour $T > 0$,

$$\begin{aligned}
\lim_{u\to\infty} \frac{\psi_{d,\beta}(u,T)}{P(|\mathbf{X}| > u)} &= \lim_{u\to\infty} \frac{P(\mathbf{R}_t \in F_\beta \text{ pour un } t \in [0,T])}{P(|\mathbf{X}| > u)} \ , \\
&= \lim_{u\to\infty} \frac{P(\mathbf{S}_t \in u(\mathbf{a} - F_\beta) \text{ pour un } t \in [0,T])}{P(|\mathbf{X}| > u)} \ , \\
&= \lim_{u\to\infty} \frac{P(\mathbf{S}_T \in u(\mathbf{a} - F_\beta))}{P(|\mathbf{X}| > u)} \ , \\
&= E(N_T)\mu(\mathbf{a} - F_\beta) \ .
\end{aligned}$$

Bien qu'il ne sert pas dans cette thèse, il me paraît important de mentionner le résultat suivant, qui donne l'équivalent de la même probabilité de ruine, mais pour un temps infini. En effet ce résultat peut d'une part intéresser le lecteur et d'autre part être à la base de travaux futurs.

Proposition 0.0.24 (Hult et Lindskog (2006a)) *Pour le processus de risque défini en (15), en supposant de plus* $\mathbf{p} = \lambda\mathbf{c} - E(\mathbf{X_1}) \in (0,\infty)^d$*, nous avons,*

$$\lim_{u\to\infty} \frac{\psi_{d,\beta}(u,\infty)}{uP(|\mathbf{X}| > u)} = \int_0^\infty \mu(v\mathbf{p} + \mathbf{a} - F_\beta)dv \ .$$

Principaux résultats

Cette thèse est divisée en deux parties, composées chacune de deux chapitres. La première partie étudie des modèles particuliers de la théorie de la ruine où les hypothèses fortes de dépendance et de stationnarité sont relâchées. La deuxième partie investit des mesures de risque multidimensionnelles servant de base à des problèmes d'allocation optimale. Chaque chapitre correspond à un article.

Crises de corrélation en théorie de la ruine

Le Chapitre 1 de cette thèse est constitué de l'article Biard *et al.* (2008). Dans ce premier chapitre, nous introduisons de la dépendance entre les montants de sinistres et étudions son impact sur l'équivalent au premier ordre de la probabilité de ruine. Nous montrons premièrement qu'une corrélation plus forte entre les montants des sinistres n'implique pas nécessairement une augmentation de la probabilité de ruine. Nous remarquons aussi que dans le cas où les montants de sinistres ont une distribution à variations régulières, la probabilité de ruine est équivalente à la fonction de survie du processus de Poisson à l'échéance lorsque la réserve initiale tend vers l'infini. Nous introduisons également une nouvelle copule et obtenons des équivalents de la probabilité de ruine pour cette structure de dépendance ainsi que pour d'autres formes déjà connues. Dans la dernière section de ce chapitre, nous présentons un modèle où la structure de dépendance ainsi que les paramètres du modèle évoluent en fonction de l'état dans lequel se trouve un processus de Markov. Ce processus de Markov décrit l'environnement (économique, politique, environnemental, juridique) dans lequel évolue la compagnie.

Dépendance entre montants de sinistres et temps inter-sinistres

L'article Biard *et al.* (2010a) est le Chapitre 2. Nous relâchons ici une autre hypothèse forte du modèle classique : la dépendance entre les montants de sinistres et les temps qui séparent les sinistres. Nous gardons de la dépendance entre les montants de sinistres utilisant les résultats du Chapitre 1. Nous construisons deux modèles qui prennent en compte la spécificité des risques tels que les tremblements de terre et les inondations. Dans ces modèles, la distribution d'un montant de sinistre dépend de plusieurs temps inter-sinistre qui le précèdent. Nous obtenons des équivalents de la probabilité de ruine ainsi que les distributions de variables aléatoires $M_+(n,k,t,\tau) = \{$*Nombre de séquences entre* 0 *et* t *de* k *temps inter-sinistres plus grands que* $\tau\}$ et $M_-(n,k,t,\tau) = \{$*Nombre de séquences entre* 0 *et* t *d'au moins* k *temps inter-sinistres plus petits que* τ $\}$. Nous étudions l'impact des paramètres de ce modèle sur ces équivalents et nous construisons une méthode récursive pour évaluer les distributions des variables aléatoires $M_+(n,k,t,\tau)$ et $M_-(n,k,t,\tau)$.

Mesurer le risque avec la partie négative du processus

L'article Biard *et al.* (2010b), Chapitre 3 de cette thèse, étudie la mesure de risque $E[I_T(u)]$ où

$$I_T(u) = \int_0^T 1_{\{R(t)<0\}} |R(t)| dt \ ,$$

pour un processus de risque $R(t)$ introduite dans Loisel (2005). Nous obtenons des équivalents de cette mesure de risque pour des montants de sinistres avec des distributions sous-exponentielles, à variations régulières, super-exponentielles et dans le cas où l'exposant de Cramér-Lundberg existe. Dans une seconde partie de cet article, la répartition d'une grande réserve initiale entre deux lignes d'affaires est étudiée. Nous recherchons l'allocation qui minimise l'équivalence de la somme du critère de risque $E[I_T(u)]$ calculée pour chaque branche. Nous montrons dans un premier temps que donner la totalité de la réserve à une branche n'est pas optimale puis nous donnons le comportement asymptotique de cette allocation optimale. Nous remarquons alors que la proportion de la réserve initiale qu'on attribue à la branche la moins risquée tend vers zéro lorsque u tend vers l'infini, ce qui est assez intuitif puisque nous sommes concernés par la solvabilité de la compagnie.

Probabilité de ruine multivariée

Le Chapitre 4, Biard (2010), s'intéresse à la probabilité de ruine multivariée. Contrairement à la mesure de risque du chapitre précédent, la structure de dépendance entre les différentes branches d'activité de la compagnie joue un rôle. Nous envisageons différents modèles de chocs communs et nous permettons le transfert d'une portion de la richesse d'une branche vers une autre branche. Dans le cas du modèle de Poisson à chocs communs, nous trouvons les allocations optimales qui minimisent l'équivalent de la probabilité de ruine de la compagnie lorsque la réserve initiale est grande. Lorsque la dépendance est trop forte, pour deux branches d'activité, on alloue la moitié de la réserve à chaque branche. Lorsque toutes les branches de la compagnie sont indépendantes, la part allouée dépend de la fréquence des sinistres de chaque branche.

Première partie

Théorie de la ruine univariée : modèles non-stationnaires avec dépendance

Chapitre 1

Crises de corrélation en théorie de la ruine
Impact of correlation crises in risk theory : asymptotics of finite-time ruin probabilities for heavy-tailed claim amounts when some independence and stationarity assumptions are relaxed

In the renewal risk model, several strong hypotheses may be found too restrictive to model accurately the complex evolution of the reserves of an insurance company. In the case where claim sizes are heavy-tailed, we relax independence and stationarity assumptions and extend some asymptotic results on finite-time ruin probabilities, to take into account possible correlation crises like the one recently bred by the sub-prime crisis: claim amounts, in general assumed to be independent, may suddenly become strongly positively dependent. The impact of dependence and non-stationarity is analyzed and several concrete examples are given.

1.1 Introduction

In the Solvency II framework, computing the Solvency Capital Requirement (SCR) and the risk margin often involves approximation of finite-time ruin probabilities in internal models. For several kinds of insurance risks, heavy-tailed distributions may be used to model individual claim amounts. Pareto distributions, or more generally regular variation distributions are often preferred to log-normal distributions to fit empirical data. A natural way to study these risks could be to use the classical compound renewal risk model with heavy-tailed claim size distribution. In the classical Sparre Andersen risk model, the classical risk process $(R_t)_{t\geq 0}$ is defined as follows: for $t \geq 0$,

$$R(t) = u + ct - S(t),$$

where u is the non-negative amount of initial reserves, $c > 0$ is the premium income rate. The cumulated claim amount up to time t is described by the compound renewal process

$$S(t) = \sum_{i=1}^{N(t)} X_i,$$

where amounts of claims X_i, $i = 1, 2, ...$ are non-negative independent, identically distributed random variables, distributed as X. As usual $S_t = 0$ if $N(t) = 0$. The number of claims $N(t)$ until $t \geq 0$ is modeled by a renewal process $(N(t))_{t\geq 0}$ defined from the inter-occurrence times $(T_k)_{k\geq 1}$ by $N_t = \sum_{k\geq 1} 1_{\{T_k \leq t\}}$. Claim amounts and inter-occurrence times times are assumed to be mutually independent.
What we want to compute is the probability of ruin before time t with initial reserve u denoted by $\psi(u,t)$:

$$\psi(u,t) = P(\exists s \in [0,t], \quad R(s) < 0 \mid R(0) = u), \quad u \geq 0, \, t > 0.$$

Note that as we consider finite-time ruin probabilities, no profit condition has to be satisfied from a theoretical point of view. In this framework it is possible to adapt directly properties of sums of independent random variables with regular variation of index $-\alpha$ with $\alpha > 0$ to derive the asymptotics of $\psi(u,t)$ as u tends to infinity (see Section 2 for definitions and details):

$$\psi(u,t) \sim \frac{1}{E(T_1)} u^{-\alpha} \quad \text{as } u \to +\infty. \tag{1.1}$$

The problem is that in the real world, the mutual independence of $X_1, \ldots, X_n, \ldots, T_1, \ldots, T_n, \ldots$ is not realistic for a certain number of reasons. First, the claim amounts X_k, $k \geq 1$ are not independent in practice, and may present complex forms of positive dependence: some factors may have an impact on those amounts; some claims of a certain type may have identical (in the sense of comonotonicity) severities depending on the outcomes of trials at the court. Second, weather or economic conditions can create as well strong positive dependence on claim amounts, which can be weakly dependent and independent in the usual regime, and suddenly become strongly positively dependent if a so-called correlation crisis breaks out. The marginal distribution of the claim amount may be modified as well, or remain identical. The most remarkable recent example of such a crisis is certainly the sub-prime crisis. Not only did the number of losses increase, but correlation was also raised.

Dependence between claim amounts has been investigated by Ignatov et al. (2001) and by Lefèvre and Loisel (2009) who provide recursive formulas that involve Appell-type properties for the finite-time ruin probability with dependent claim amounts. Sums of dependent random variables have been studied by many authors, in particular by Barbe et al. (2006), Alink et al. (2005), Wüthrich (2003) and by Kortschak and Albrecher (2009). Dependence and non-stationarity have been partially taken into account for infinite-time ruin probabilities by Boudreault et al. (2006), and Albrecher and Boxma (2004) who assume that the sequence (T_k, X_k), $k \geq 1$ is i.i.d. or that the (X_k, T_{k_1}), $k \geq 1$ are i.i.d., and by Asmussen (1989) and many others who assume that the risk process is modulated by a Markovian environment process.

The natural question that arises is the following: what is the impact of this dependence on the level of SCR in internal models of Solvency II, that is can we derive expressions equivalent to (1.1) when those independence and stationarity assumptions are relaxed? What is the impact of potential correlation crises on the asymptotic behavior of finite-time ruin probabilities for heavy-tailed claim amounts? This is the question we address in this paper, in the case where the X_k, $k \geq 1$ have distributions with regular variation.

Our paper is organized as follows: in Section 2, we study a first example to show that the effect of positive dependence between claim amounts may vary a lot. In Section 3, we study a basic model with dependent claim amounts. We also consider stochastic correlation between claim amounts and use stochastic orderings to study the impact of stochastic correlation. In Section 4, we consider more complex dependence structures between the claim amounts. In Section 5, we use the results of the first Sections to analyze our main model, with possible outbreaks of correlation crises. We derive the corresponding of the finite-time ruin probability as the initial reserve tends to infinity. We illustrate our method on two examples in which the stationarity and independence properties mentioned above are relaxed.

Throughout the paper, we assume that claim amount distributions belong to the regular variation class:

Definition 1.1.1 *(Regular variation). A distribution function F is regularly varying of index $-\alpha$ with $\alpha > 0$ (written $F \in \mathcal{R}_{-\alpha}$) if*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \quad \textit{for } y > 0.$$

Before studying the correlation crisis model, let us first discuss preliminary models with static or stochastic dependence between claim amounts.

1.2 Varying effects of positive dependence

It is often believed that positive dependence between risks increases the probability of ruin over any given time horizon. This seems to be natural, for example, if the different claims are subjected to some exterior environment. Conclusions in that direction are indeed pointed out, e.g., in Cossette and Marceau (2000), Frostig (2003) and Picard et al. (2003).

In this section, we show through a simple illustration that ruin probabilities can not only increase, but also decrease owing to the presence of positive dependence between claim amounts. Such a decreasing effect is possible in a different model where each claim size depends on the previous claim interval (as, e.g., in Albrecher and Boxma (2004) and Boudreault et al. (2006). In that case, positive dependence corresponds to a kind of mutualisation that plays a protective role. For the present example, the decreasing effect obtained comes rather from the claim size distribution itself as it is a consequence of the max-sum-equivalence property for heavy-tailed distributions.

Specifically, let us consider two particular risk models in which the successive claim amounts $(X_n)_{n\geq 1}$ have the same distributions but are dependent in a comonotonic way. For both models, we will compare the ruin probability $\psi(u,t)$ in the independent case, i.e. when the X_n are independent identically distributed, and in a comonotonic case when all the $X_n = X_1$ almost surely, i.e. under an extremal positive dependence. To simplify, throughout this Section, we assume that the counting process $(N(t))_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ and that it is independent from the sequence of claim amounts.

(i) Let us assume that the successive claim amounts have a common biatomic distribution given by

$$P(X_1 = 1) = 0.99 \quad \text{and} \quad P(X_1 = 1000) = 0.01. \tag{1.2}$$

Note that this law may be considered as heavy-tailed. Let us take $\lambda = c = 1$, an horizon of length $t = 10$ and $u = 990$ as initial surplus.

Intuitively, as the average number of claims up to t is equal to $\lambda t = 10$, ruin will occur when $u = 990$ if there arises (at least) one large claim (of size 1000) before time t, or if there arise sufficiently many small claims (of size 1), this event being however of small probability. In addition, the probability of getting at least one large claim is clearly smaller in the comonotonic case than in the independent case. Thus, one expects that the ruin probability before time $t = 10$ will be also smaller in the comonotonic case.

Let us show this rigourously. By definition,

$$\psi(990, 10) = P[S(\tau) \leq 990 + \tau \ \text{ for some } \tau \leq 10].$$

From (1.2) and since $P[N(10) \geq 991] < 10^{-500}$ is negligible, we can approximate $\psi(990, 10)$ by a quantity $\psi_a(990, 10)$ given by

$$\psi_a(990, 10) = \sum_{j=1}^{990} P[N(10) = j \ \text{ and at least one these j claims is of size 1000}]. \tag{1.3}$$

In the comonotonic case, (1.3) yields the approximation $\psi_a^{com}(990, 10)$ given by

$$\psi_a^{com}(990, 10) = P(X_1 = 1000)\ P[1 \leq N(10) \leq 990],$$

while in the independent case, the corresponding approximation $\psi_a^{\perp}(990, 10)$ is

$$\psi_a^{\perp}(990, 10) = \sum_{j=1}^{990} \left[1 - P(X_1 = 1)^j\right]\ P[N(10) = j].$$

So we see that

$$\begin{aligned}
[\psi_a^{\perp} - \psi_a^{com}](990, 10) \quad &> \quad P[2 \leq N(10) \leq 990]\ \left\{\left[1 - P(X_1 = 1)^2\right] - P(X_1 = 1000)\right\} \\
&\simeq \quad 0.00227 \ >> \ 10^{-500} > P[N(10) \geq 991].
\end{aligned}$$

Thus, as for the exact ruin probabilities $\psi^{\perp}$ and ψ^{com}, we get the inequality $\psi^{\perp}(990, 10) > \psi^{com}(990, 10)$.

(ii) Let us consider another situation where the common claim amount distribution is still a biatomic law but now given by

$$P(X_1 = 1) = 0.99 \quad \text{and} \quad P(X_1 = 10) = 0.01. \tag{1.4}$$

In comparison with (1.2), this law may be viewed as light-tailed. Let us take $\lambda = c = 1$, a n horizon of length $t = 10$ and $u = 100$ as initial surplus.

This time, large claims (of size 10) will cause ruin before time $t = 10$ only if they are also relatively numerous, which is more probable in the comonotonic case. So, one expects intuitively that the comonotonic case could provide a higher ruin probability than the independent case.

Let us establish this result. First, we observe that ruin is sure when there arise 11 large claims before $t = 10$. Thus, the ruin probability for the comonotonic case, $\psi^{com}(100, 10)$, satisfies

$$\psi^{com}(100, 10) > P[N(10) \geq 11]\ P(X_1 = 10) \simeq 0.00417. \tag{1.5}$$

On the other hand, occurrence of ruin before $t = 10$ implies necessarily that the total claim amount at t is larger than $u = 100$. Using (1.4), we then have

$$\psi(100, 10) \leq P[S(10) > 100] = 1 - \sum_{j=0}^{100} P[N(10) = j,\ S(10) \leq 100].$$

For $0 \leq j \leq 10$, the event $[N(10) = j,\ S(10) \leq 100]$ is equivalent to $[N(10) = j]$. For $11 \leq j \leq 100$, the event $[N(10) = j,\ S(10) \leq 100]$ means that the number of large claims, k say, satisfies the relation $10k + (j - k) \leq 100$; so, in the independent case,

$$P[N(10) = j,\ S(10) \leq 100] = P[N(10) = j] \sum_{k=0}^{\lfloor (100-j)/9 \rfloor} \binom{j}{k} [P(X_1 = 10)]^k\ [P(X_1 = 1)]^{j-k}.$$

Thus, the ruin probability for the independent model, $\psi^{\perp}(100, 10)$, satisfies

$$\begin{aligned} \psi^{\perp}(100, 10) \leq 1 - &\sum_{j=0}^{100} P[N(10) = j] \\ &\left(\sum_{k=0}^{\lfloor (100-j)/9 \rfloor} \binom{j}{k} [P(X_1 = 10)]^k\ P(X_1 = 1)]^{j-k} \right) \simeq\ 10^{-14}. \end{aligned} \tag{1.6}$$

Comparing (1.5) with (1.6) then gives the inequality $\psi^{\perp}(100, 10) < \psi^{com}(100, 10)$.

Clearly, in general positive dependence will not affect ruin probabilities in a monotone way. Nevertheless, the two examples above show that asymptotically as $u \to \infty$, such a property could be true for certain classes of dependent claim amounts with heavy-tailed distributions, as in example (i), or with light-tailed distributions, as in example (ii).

1.3 A basic situation with heavy-tailed claims

Following the previous illustration, we are going to establish that for certain heavy-tailed claim amount laws, positive dependence affects ruin probabilities in a monotone way, increasing or decreasing, when the initial surplus is large enough. For related questions on the asymptotic tail behaviour of sums of dependent risks, the reader is referred to, e.g., Alink et al. (2004), Albrecher et al. (2006), Kortschak and Albrecher (2009) and Barbe et al. (2006).

In Sections 3 and 4, the counting process $(N(t))_{t\geq 0}$ is a renewal process that is independent from the sequence of claim amounts. In the model under study, the different claim amounts have the same law but they are either independent or with a common level. Clearly, this assumption makes them exchangeable and positively correlated. A notation $\sim$ will mean that the ratio tends to 1 as $u \to \infty$.

Proposition 1.3.1 *Suppose that premiums arrive at a constant rate c and claims occur according to some point process $N(t)_{t\geq 0}$. Moreover, independently of this arrival claim process, the successive claim amounts $(X_n)_{n\geq 1}$ are described by*

$$X_n = I_n W_0 + (1 - I_n) W_n, \quad n \geq 1, \tag{1.7}$$

where $(W_n)_{n\geq 0}$ is a sequence of i.i.d positive random variables of distribution function F_W with

$$F_W \in \mathcal{R}_{-\alpha}, \ \alpha \geq 0, \tag{1.8}$$

and $(I_n)_{n\geq 1}$ is a sequence of i.i.d Bernoulli random variables with

$$P(I_1 = 1) = p \ \in [0, 1], \tag{1.9}$$

these two sequences being mutually independent. Let u be the initial reserve and denote by $\psi_p(u,t)$ the corresponding ruin probability over any fixed finite-time horizon $(0,t)$. Then, asymptotically for u large enough,

$$\psi_p(u,t) \ \sim \ \{(1-p)\, E[N(t)] + E\left[Z_p(t)\right]^\alpha\} \ \overline{F_W}(u + ct), \tag{1.10}$$

where $Z_p(t)$ denotes a mixed binomial random variable $Bin[N(t), p]$.

Proof. Let

$$S_p(t) = \sum_{n=1}^{N(t)} X_n$$

be the aggregate claim amount. To establish (1.10), we first calculate $P[S_p(t) > x]$ for large x, and we then approximate $\psi_p(u,t)$ by $P[S_p(t) > u]$.

Step 1. A key point is the convolution closure property and the max-sum-equivalence property of the regular variation class (see, e.g., Cai and Tang (2004)). Specifically, when F_1 and F_2 belong to $\mathcal{R}_{-\alpha}$, $\alpha \geq 0$, the convolution closure states that

$$F_1 * F_2 \ \text{ belongs to } \mathcal{R}_{-\alpha},$$

and the max-sum-equivalence means that

$$\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) + \overline{F_2}(x) \ \text{ for large } x.$$

Since $F_W \in \mathcal{R}_{-\alpha}$ by assumption, these properties allow us to write that for any $k \geq 1$ and any pairwise distinct $n_1, ..., n_{k-j} \geq 1$ with $0 \leq j \leq k-1$,

$$\begin{aligned} P\left(W_{n_1} + ... + W_{n_{k-j}} + jW_0 > x\right) \ &\sim \ (k-j)\overline{F_W}(x) + \overline{F_W}\,(x/j) \\ &\sim \ \left(k - j + \frac{\overline{F_W}\,(x/j)}{\overline{F_W}(x)}\right) \overline{F_W}(x) \\ &\sim \ (k - j + j^\alpha)\ \overline{F_W}(x). \end{aligned} \tag{1.11}$$

Thus, (1.11) yields, for any $k \geq 1$ and $0 \leq j \leq k-1$,

$$P\left[S_p(t) > x | N(t) = k, \sum_{i=1}^{k} I_i = j\right] \sim (k - j + j^{\alpha})\, \overline{F_W}(x). \tag{1.12}$$

We also have, for $k \geq 1$ and $j = k$,

$$P\left[S_p(t) > x | N(t) = k, \sum_{i=1}^{k} I_i = k\right] = P(kW_0 > x) = \frac{\overline{F_W}\,(x/k)}{\overline{F_W}(x)}\, \overline{F_W(x)} \sim k^{\alpha}\, \overline{F_W}(x), \tag{1.13}$$

and for $k = j = 0$,

$$P[S_p(t) > x | N(t) = 0] = 0. \tag{1.14}$$

From (1.12), (1.13) and (1.14), and since $N(t)_{t\geq 0}$, $(W_n)_{n\geq 0}$, $(I_n)_{n\geq 0}$ are mutually independent, we then get

$$P[S_p(t) > x] \sim \left\{ \sum_{k=1}^{\infty} P[N(t) = k] \sum_{j=0}^{k} \binom{k}{j} p^j (1-p)^{k-j} (k - j + j^{\alpha}) \right\} \overline{F_W}(x). \tag{1.15}$$

Obviously, (1.15) can be rewritten as

$$P[S_p(t) > x] \sim \{(1-p)E[N(t)] + E\,[Z_p(t)]^{\alpha}\}\, \overline{F_W}(x), \tag{1.16}$$

Step 2. Let us show that for any $c, t > 0$,

$$\psi_p(u,t) \sim P[S_p(t) > u + ct], \tag{1.17}$$

as $u \to \infty$. Indeed, we observe that

$$\begin{aligned} 0 &\leq \frac{\psi_p(u,t) - P[S_p(t) > u + ct]}{\psi_p(u,t)} \\ &\leq \frac{\psi_p(u,t) - P[S_p(t) > u + ct]}{P[S_p(t) > u + ct]} \\ &\leq \frac{P[S_p(t) > u] - P[S_p(t) > u + ct]}{P[S_p(t) > u + ct]} \\ &\sim \frac{\overline{F_W}(u)}{\overline{F_W}(u + ct)} - 1, \end{aligned} \tag{1.18}$$

using (1.16). For any $x \in \mathbb{R}$, one knows that $\overline{F_W}(u)/\overline{F_W}(u+x) \to 1$ as $u \to \infty$ (see Lemma 1.3.5 in Embrechts et al. (1997)). Therefore, the approximation (1.17) follows from (1.18). Finally, combining (1.16) and (1.17) yields the formula (1.10). $\diamond$

By the approximation (1.10), $\psi_p(u,t)$ is simply given as a product of two distinct factors, the former in terms of $N(t)$, p and α, and the latter in terms of F_W, u and c. Note that the claim amount distribution plays a role in both factors. For example, choose $\overline{F_W}(x) \sim l(x)\, x^{-\alpha}$, $x > 0$, where $\alpha > 1$ and $l(x)$ is slowly varying. This covers the Pareto law and the loggamma law, inter alia. From (1.10), if α increases, $\overline{F_W}(u + ct)$ decreases while $E[Z_p(t)]^{\alpha}$ increases.

Let us observe that in particular, (1.10) gives, if $p = 0$ (i.e. when $X_n = W_n$ are i.i.d.),

$$\psi_p(u,t) \sim E[N(t)]\, \overline{F_W}(u + ct),$$

while if $p = 1$ (i.e. when $X_n = W_0$ for all n),

$$\psi_p(u,t) \;\sim\; E[N(t)]^\alpha \, \overline{F_W}(u+ct).$$

Let us also indicate that by (1.7), any pair (X_n, X_m) of claim amounts, with $n \neq m$, has a correlation $corr(X_n, X_m) = p^2 \, var(W)$, which is positive and increasing in p as expected.

Proposition 1.3.1 can be extended to the case where the type of claim amount, either W_n or W_0, is influenced by a random environment. More precisely, suppose that the indicators $(I_n)_{n\geq 1}$ have now a common random parameter P, with some distribution on $[0,1]$. Let $\psi_P(u,t)$ be the ruin probability over $(0,t)$. Then, as $u \to \infty$,

$$\psi_P(u,t) \;\sim\; \{[1-E(P)]\, E[N(t)] + E\,[Z_P(t)]^\alpha\}\; \overline{F_W}(u+ct), \tag{1.19}$$

$Z_P(t)$ being of mixed law $MBin[N(t), P]$. One easily checks that now, $corr(X_n, X_m) = E(P^2)\, var(W)$, $n \neq m$, which is again positive but increasing in $E(P^2)$ (i.e. in $var(P)$ if the mean $E(P)$ is fixed).

Let us examine how the law of P can affect $\psi_p(u,t)$. For that, we use the concept of $s-$convex stochastic ordering (see Lefèvre and Utev (1996) and Denuit et al. (1998)). By definition, given two random variables Y and Z, then for any $s = 1, 2, \ldots$,

$$X \leq^{\mathcal{D}}_{s-cx} Y \;\text{ if }\; E[\phi(Y)] \leq E[\phi(Z)] \;\text{ for all } s\text{-convex function } \phi : \mathcal{D} \to \mathbb{R}, \tag{1.20}$$

i.e. in short, for any function ϕ on $\mathcal{D}$ whose s-th derivative exists and satisfies $\phi^{(s)} \geq 0$. Note that the first $s-1$ moments of Y and Z are then necessarily equal. The order $\leq^{\mathcal{D}}_{1-cx}$ is just the stochastic order, $\leq^{\mathcal{D}}_{2-cx}$ is the usual convex order (which implies $var(Y) \leq var(Z)$) and $\leq^{\mathcal{D}}_{3-cx}$ is also very popular (it means that Y has smaller right-side risk than Z). Put $\alpha_{[s]} = \alpha(\alpha-1)\ldots(\alpha-s+1)$, and $\delta_{s,1} = 1$ (0) if $s = 1$ $(\neq 1)$.

Property 1.3.2 *For u large enough, given any $s = 1, 2, \ldots$,*

$$\textit{if }\; \alpha_{[s]} \leq \;(\textit{resp.} \geq)\; \delta_{s,1}, \;\; \textit{then } P \leq^{[0,1]}_s Q \;\; \textit{implies }\; \psi_P(u,t) \geq \;(\textit{resp.} \leq)\; \psi_Q(u,t). \tag{1.21}$$

Proof. From (1.10), we have

$$\psi_P(u,t) - \psi_Q(u,t) \;\sim\; \left(E\,\{[Z_P(t)]^\alpha - Z_P(t)\} - E\,\{[Z_Q(t)]^\alpha - Z_Q(t)\}\right) \overline{F_W}(u+ct). \tag{1.22}$$

A binomial law $Bin(n,p)$ is stochastically $s-$convex in the parameter p (see Denuit and Lefèvre (2001)). Thus, if $P \leq^{[0,1]}_s Q$, then $MBin(n,P) \leq^{\{0,\ldots,n\}}_s MBin(n,Q)$, so that $Z_P(t) \leq^{\mathbb{N}}_s Z_Q(t)$. Now, consider the function $f(x) \equiv x^\alpha - x$, $x \in \{0,1,\ldots\}$. We see that $f(x)$ (resp. $-f(x)$) is $s-$convex when $\alpha_{[s]} \geq$ (resp. $\leq$) $\delta_{s,1}$. Therefore, from (1.22), we deduce the announced implication (1.21). $\diamond$

So, Property 3.2 states that if $P \leq^{[0,1]}_1 Q$, then

$$\psi_P(u,t) \geq \;(\leq)\; \psi_Q(u,t) \;\text{ for }\; \alpha \leq \;(\geq)\; 1.$$

In particular, this is true if P and Q reduce to two constants p and q such that $p \leq q$.

An identical conclusion holds under the condition $P \leq^{[0,1]}_2 Q$, but remember that $E(P) = E(Q)$ here. For instance, one has $P_2^{min} \leq^{[0,1]}_2 P \leq^{[0,1]}_2 P_2^{max}$, where $P_2^{min} = E(P)$ and P_2^{max} is a variable with two atoms, 0 and 1, such that $P(P_2^{max} = 1) = E(P)$. This yields

$$\psi_{P_2^{min}}(u,t) \geq \;(\leq)\; \psi_P(u,t) \geq \;(\leq)\; \psi_{P_2^{max}}(u,t) \;\text{ for }\; \alpha \leq \;(\geq)\; 1,$$

where

$$\psi_{P_2^{max}}(u,t) \;\sim\; [1-E(P)]\;\{E[N(t)]+E[N(t)]^{\alpha}\}\;\overline{F_W}(u+ct).$$

If $P \leq_3^{[0,1]} Q$, then

$$\psi_P(u,t) \geq \;(\leq)\;\psi_Q(u,t)\;\text{ for }\;\alpha \leq 1 \text{ or } \alpha \geq 2\;(1 \leq \alpha \leq 2).$$

Let us recall that this time, $E(P) = E(Q)$ and $E(P^2) = E(Q^2)$. For instance, one might now use the inequality $P_3^{min} \leq_3^{[0,1]} P \leq_3^{[0,1]} P_3^{max}$, where P_3^{min} is a variable with two atoms, 0 and $E(P^2)/E(P)$, such that $P(P_3^{min}=0) = var(P)/E(P^2)$, and P_3^{max} is a variable with two atoms, $[E(P)-E(P^2)]/[1-E(P)]$ and 1, such that $P(P_3^{max}=1) = var(P)/\{[(1-E(P)]^2+var(P)\}$.

1.4 More complex dependent cases

In this part, the framework of Section 3 is retained except that the claim amounts $(X_i)_{i\geq 1}$ are no longer of the form (1.3.1), but with a dependence structure described by a specific copula. Denote by $C^{(n)}$ the copula of the vector $(X_1, X_2, ..., X_n)$. To satisfy the condition of independence, the following relation must be hold

$$C^{(n)}(u_1,...,u_{n-1},1) = C^{(n-1)}(u_1,...,u_{n-1})\;,\quad n \geq 2. \tag{1.23}$$

1.4.1 A non-standard class of copulas

In section 1.3 we studied the asymptotic behavior of the finite-time ruin probability $\psi(u,t)$ from initial reserve u within time t for regular variation claim amounts X_i, $i \geq 0$ such that each X_i is equal to W_0 with probability $p \in [0,1]$ or equal to W_i with probability $1-p$, where the W_k, $k \geq 0$ are i.i.d. regular variation random variables. We extend here the result to the more general model case where for $n \geq 2$,

$$(X_1,\ldots,X_n)$$

has again identical marginals but with copula $C^{(n)}$ defined below. Let $\mathfrak{P}([1,n])$ denote the set of partitions of $[1,n]$. For $A^{(n)} \in \mathfrak{P}([1,n])$, consider all the subsets a in $A^{(n)}$ and define the associated copula

$$C_{A^{(n)}}(u_1,\ldots,u_n) = \prod_{a\in A^{(n)}} \min_{k\in a}(u_k). \tag{1.24}$$

Then, the copula $C^{(n)}$ is represented as a weighted average of these copulas over all possible partitions $A^{(n)}$, i.e.

$$C^{(n)} = \sum_{A^{(n)}\in\mathfrak{P}([1,n])} \lambda_{A^{(n)}} C_{A^{(n)}}, \tag{1.25}$$

where the nonnegative weights must satisfies, for all $n \geq 1$ and $A^{(n)} \in \mathfrak{P}[1,n]$:

$$\begin{cases} \displaystyle\sum_{A^{(n)}\in\mathfrak{P}([1,n])} \lambda_{A^{(n)}} = 1, \\ \displaystyle\sum_{\substack{A^{(n+1)}\in\mathfrak{P}[1,n+1] \\ A^{(n+1)}\setminus\{n+1\}=A^{(n)}}} \lambda_{A^{(n+1)}} = \lambda_{A^{(n)}}, \end{cases} \tag{1.26}$$

$A^{(n+1)}\setminus\{n+1\}$ meaning that $\{n+1\}$ is removed from all subsets of $A^{(n+1)}$ that contain $\{n+1\}$.

Proposition 1.4.1 *For all $n \geq 2$, $C^{(n)}$ defined by (1.25) satisfies the condition (1.23).*

Proof. By (1.24) and (1.25), we get

$$\begin{aligned}
C^{(n)}(u_1, ..., u_{n-1}, 1) &= \sum_{A^{(n)} \in \mathfrak{P}([1,n])} \lambda_{A^{(n)}} C_{A^{(n)}}(u_1, ..., u_{n-1}, 1) \\
&= \sum_{A^{(n)} \in \mathfrak{P}([1,n])} \lambda_{A^{(n)}} \prod_{a \in A^{(n)}} \min_{k \in a \setminus \{n\}} (u_k) \\
&= \sum_{A^{(n-1)} \in \mathfrak{P}([1,n-1])} \sum_{\substack{A^{(n)} \in \mathfrak{P}[1,n] \\ A^{(n)} \setminus \{n\} = A^{(n-1)}}} \lambda_{A^{(n)}} \prod_{a \in A^{(n)}} \min_{k \in a \setminus \{n\}} (u_k) \\
&= \sum_{A^{(n-1)} \in \mathfrak{P}([1,n-1])} \sum_{\substack{B \in \mathfrak{P}[1,n] \\ B \setminus \{n\} = A^{(n-1)}}} \lambda_{A^{(n)}} \prod_{a \in A^{(n-1)}} \min_{k \in a} (u_k) \\
&= \sum_{A^{(n-1)} \in \mathfrak{P}([1,n-1])} \lambda_{A^{(n-1)}} \prod_{a \in A^{(n-1)}} \min_{k \in a} (u_k) \\
&= C_{n-1}(u_1, ..., u_{n-1}),
\end{aligned}$$

hence (1.23). $\diamond$

Proposition 1.4.2 *For claim amounts $(X_i)_{i \geq 1}$ that have the same marginal distribution F with $F \in \mathcal{R}_{-\alpha}$, $\alpha > 0$, and such that $(X_1, ..., X_n)$, $n \geq 2$ has a copula $C^{(n)}$ given by (1.25), then for large u and any $t > 0$, we have*

$$\psi(u,t) \sim \left\{ \sum_{k=1}^{\infty} P\left[N(t) = k\right] \sum_{A^{(k)} \in \mathfrak{P}([1,k])} \lambda_{A^{(k)}} \left(\sum_{a \in A^{(k)}} Card(a)^{\alpha} \right) \right\} \overline{F}(u + ct). \tag{1.27}$$

Proof. As in the proof of Proposition 1.3.1, we start by computing $P\left(S_t > x\right)$. For all $n \geq 1$ and $A^{(n)} \in \mathfrak{P}[1,n]$ denote $(X_1^{(A^{(n)})}, ..., X_n^{(A^{(n)})})$ a copy of the vector $(X_1, \ldots, X_n)$ with a dependence structure described by the copula $C_{A^{(n)}}$ defined in (1.24).
For all $k \geq 1$ and $x > 0$, we have

$$P\left(X_1 + ... + X_k > x\right) = \sum_{A^{(k)} \in \mathfrak{P}[1,k]} \lambda_{A^{(k)}} P\left(X_1^{(A^{(k)})} + ... + X_k^{(A^{(k)})} > x\right)$$

For all subsets $a \in A^{(n)}$, denote $X^{(A^{(n)},a)}$ the common variable the variables in $(X_1^{(A^{(n)})}, ..., X_n^{(A^{(n)})})$ whose index belongs to a; then,

$$P\left(X_1 + ... + X_k > x\right) = \sum_{A^{(k)} \in \mathfrak{P}[1,k]} \lambda_{A^{(k)}} P\left(\sum_{a \in A} Card(a) X^{(A^{(k)},a)} > x\right).$$

Arguing as in Step 1 of Proposition 1.3.1, we obtain that

$$\begin{aligned}
P\left(X_1 + ... + X_k > x\right) &\sim \sum_{A^{(k)} \in \mathfrak{P}[1,k]} \lambda_{A^{(k)}} \sum_{a \in A^{(k)}} P\left(Card(a) X^{(A^{(k)},a)} > x\right) \\
&\sim \left[\sum_{A^{(k)} \in \mathfrak{P}[1,k]} \lambda_{A^{(k)}} \left(\sum_{a \in A^{(k)}} Card(a)^{\alpha} \right) \right] \overline{F}(x).
\end{aligned}$$

Thus, we may conclude in the same way as in Step 2 of Proposition 1.3.1. $\diamond$
We note that the sequence $(\lambda_{A^{(k)}})_{k\geq 1}$ can be built using a tree and appropriate transition probabilities. For all $k \geq 1$, $A^{(k)} \in \mathfrak{P}([1,k])$ and $A^{(k+1)} \in \mathfrak{P}([1,k+1])$, let $p_{A^{(k)} \to A^{(k+1)}}$ be the probability that $(X_1, ..., X_{k+1})$ has its dependence structure described by $C_{A^{(k+1)}}$ knowing that $(X_1, ..., X_k)$ has its dependence structure described by $C_{A^{(k)}}$. This is illustrated in Figure 1.1. Then, for all $k \geq 1$, $A^{(k)} \in \mathfrak{P}([1,k])$, we have

$$\lambda_{A^{(k)}} = \prod_{i=1}^{k-1} p_{A^{(i)} \to A^{(i+1)}}.$$

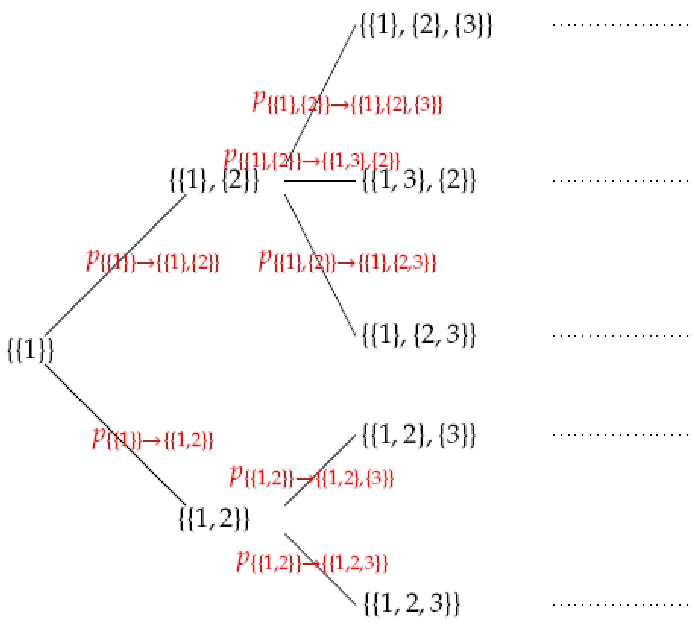


Figure 1.1: Step-by-step construction of the sequence $(\lambda_{A^{(k)}})_{k\geq 1}$ with transition probabilities $p_{A^{(k)} \to A^{(k+1)}}$.

1.4.2 Classical copulas

A general result

Definition 1.4.3 *(Multivariate regular variation). A random vector $\mathbf{X} = (X_1, ..., X_n)$ belongs to $\mathcal{MR}_{-\alpha}$, $\alpha \geq 0$ if there exists a $\theta \in \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere with respect to a norm $|\cdot|$, such that*

$$\frac{P\left(|\mathbf{X}| > tu, \mathbf{X}/\left|\mathbf{X}\right| \in \cdot\right)}{P\left(|\mathbf{X}| > u\right)} \xrightarrow{v} t^{-\alpha} P_{\mathbb{S}^{n-1}}\left(\theta \in \cdot\right),$$

where $\xrightarrow{v}$ denotes vague convergence on $\mathbb{S}^{n-1}$.

Definition 1.4.4 *(Extreme value copula). A copula such that*

$$C(u_1^t, ..., u_k^t) = C^t(u_1, ..., u_k)\ , \quad \forall t > 0,$$

is called an extreme value copula.

Definition 1.4.5 *(Domain of attraction of copula). Let C be a copula and let C^* be an extreme value copula. The copula C belongs to the domain of attraction of C^*, written $C \in CDA(C^*)$, if for all* $\mathbf{u}$

$$\lim_{m\to\infty} C^m(\mathbf{u}^{1/m}) = C^*(\mathbf{u}).$$

Proposition 1.4.6 *For claim amounts $(X_i)_{i\geq 1}$, such that for all $n \geq 1$, $(X_1, ..., X_n) \in \mathcal{MR}_{-\alpha}$ with the same marginal distribution F and their dependence structure is described by a copula $C^{(n)}$ which belongs to the domain of attraction of ${C^{(n)}}^*$, then for large u and any $t > 0$,*

$$\psi(u,t) \sim \left\{ \sum_{k=1}^{\infty} P\left[N(t) = k\right]\; q_{k,\alpha} \right\} \overline{F}(u + ct), \tag{1.28}$$

where

$$q_{k,\alpha} = \int_{\mathbb{T}_k} \left(p_1^{1/\alpha} + ... + p_k^{1/\alpha}\right)^{\alpha} dU_k(\mathbf{p}),$$

where $\mathbb{T}_k$ denotes the k-dimensional unit simplex and U_k is the measure such that :

$${C^{(k)}}^*(x_1, .., x_k) = exp\left(-\int_{\mathbb{T}_k} \max_{1\leq j\leq k} \left\{-p_j log(x_j)\right\} dU_k(\mathbf{p})\right).$$

Proof. This result follows directly from a theorem of Barbe et al. (2006). Indeed under the assumptions made above, these authors show that

$$P\left(\sum_{p=1}^{k} X_p > x\right) \sim q_{k,\alpha}\overline{F}(x).$$

With the same method as in Proposition 1.3.1 (step 1 and step 2), we then obtain the approximation 1.28. $\diamond$

Remark. Another relation links C^* with $q_{k,\alpha}$. From Resnick (2004) and Barbe et al. (2006), there exists a Radon measure ν on the punctured space $\mathbb{E} = [0,\infty]^k \backslash \{0\}$ such that

$$q_{k,\alpha} = \nu(\Omega), \tag{1.29}$$

where $\Omega = \left\{(p_1, ..., p_k) \in [0,\infty)^k : p_1^{1/\alpha} + ... + p_k^{1/\alpha} > 1\right\}$ and

$$\nu([\mathbf{0}, \mathbf{x}]^c) = -log\left\{C^*(e^{-\mathbf{x}})\right\}, \quad \forall \mathbf{x} \in [0,\infty)^k. \tag{1.30}$$

Property 1.4.7 *Under the assumptions of Proposition 1.4.6, comparing the case where $(X_1, \ldots, X_n) \in \mathcal{MR}_{-\alpha}$, with the case where $(X_1, \ldots, X_n) \in \mathcal{MR}_{-\beta}$ yield for large u*

$$\alpha \leq \ (resp.\ \geq\ )\ \beta \Rightarrow \psi_\alpha(u,t) \leq \ (resp.\ \geq\ )\ \psi_\beta(u,t).$$

Proof. Directly by checking that for fixed $\mathbf{p} \in \mathbb{T}_k$, $\left(p_1^{1/\alpha} + ... + p_k^{1/\alpha}\right)^{\alpha}$ is an increasing function of α. $\diamond$

Independent copula

Definition 1.4.8 *The independent copula is*

$$C_{ind}(u_1, ..., u_k) = \prod_{i=1}^{k} u_i. \tag{1.31}$$

It is easily seen that (1.28) simplifies here as follows.

Proposition 1.4.9 *Under the assumptions of Proposition 1.4.6, with in addition $C_{ind}^{(n)}$ as copula, then for large u*

$$\psi(u,t) \sim \lambda t \overline{F}(u+ct).$$

Fréchet upper bound

Definition 1.4.10 *The comonotonic copula is*

$$C_{com}(u_1, ..., u_k) = min(u_1, ..., u_k). \tag{1.32}$$

Proposition 1.4.11 *Under the assumptions of Proposition 1.4.6, with in addition $C_{com}^{(n)}$ as copula, then for large u*

$$\psi(u,t) \sim E[N(t)]^{\alpha} \overline{F}(u+ct).$$

Gaussian copula

Definition 1.4.12 *The Gaussian or normal copula is*

$$C_{Ga,\Sigma}(u_1, ..., u_k) = \Phi_{\Sigma}^{k}\left(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_k)\right), \tag{1.33}$$

where Φ_{Σ}^{k} denotes the joint distribution function of the $k-$variate standard normal distribution with correlation matrix Σ and Φ^{-1} denotes the inverse of the distribution function of the univariate standard normal distribution.

Remark. The Gaussian copula satisfies the condition (1.23) if the correlation matrix is adapted, that is if

$$C_{Ga,\Sigma_k}^{(k)}(u_1, ..., u_{k-1}, u_k = 1) = C_{Ga,\Sigma_{k-1}}^{(k-1)}(u_1, ..., u_{k-1}), \tag{1.34}$$

with Σ_{k-1} is formed with the first $k-1$ rows and columns of Σ_k.

Lemma 1.4.13 *(Marshall and Olkin (1983)) The Gaussian copula (when the correlation matrix does not contain any 1 except in its diagonal) belongs to the domain of attraction of the independent copula.*

Proposition 1.4.14 *Under the assumptions of Proposition 1.4.6, with in addition $C_{Ga,\Sigma_n}^{(n)}$ as copula, such 1.34 is satisfied, then for large u*

$$\psi(u,t) \sim \lambda t \overline{F}(u+ct).$$

Archimedean copulas

Definition 1.4.15 *Let $\phi : [0,1] \to [0,\infty]$ be continuous and strictly decreasing with $\phi(0) \leq \infty$ and $\phi(1) = 0$. A pseudo inverse of ϕ is defined as*

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & 0 \leq t \leq \phi(0) \\ 0 & \phi(0) \leq t \leq \infty \end{cases},$$

where ϕ^{-1} is the classical inverse of ϕ. Functions ϕ are called generators of Archimedean copulas. If $\phi(0) = \infty$, then ϕ is called a strict generator.

Definition 1.4.16 *A decreasing (resp. increasing) function $f : \mathbb{R} \to \mathbb{R}$ is completely monotonic on an interval I if it is continuous on I and satisfies*

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0, \quad \left(\text{resp. } (-1)^{k-1} \frac{d^k}{dt^k} f(t) \geq 0 \right),$$

for all t in the interior of I and any $k \geq 1$.

As a consequence, if f is completely monotonic on $[0,\infty]$ and $f(c) = 0$ for some $c > 0$, then f must be identically zero on $[0,\infty]$. So if the pseudo-inverse $\phi^{[-1]}$ of an Archimedean generator ϕ is completely monotonic, it must be positive on $[0,\infty]$, i.e. ϕ is a strict generator and $\phi^{[-1]} = \phi^{-1}$.

Lemma 1.4.17 *(see Nelsen (2006)) Let ϕ be a continuous strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\phi(0) = \infty$ and $\phi(1) = 0$. If C is the function from $[0,1]^d$ to $[0,1]$ given by*

$$C_\phi(u_1, ..., u_k) = \phi^{[-1]} \left(\phi(u_1) + ... + \phi(u_k) \right), \tag{1.35}$$

then C is a $k-$copula for all $k \geq 2$ if and only if $\phi^{[-1]}$ is completely monotonic on $[0,\infty]$.

C_ϕ given by (1.35) is named an Archimedean copula with generator ϕ. Since $\phi(1) = 0$, the condition (1.23) is clearly satisfied.

Proposition 1.4.18 *Under the assumptions of Proposition 1.4.6, with in addition $C_\phi^{(n)}$ as copula, such that $\phi(1-1/t) \in \mathcal{R}_{-\beta}$ for some $\beta > 1$, then for large u*

$$\psi(u,t) \sim \left\{ \sum_{k=1}^{\infty} P\left[N(t) = k\right] \int_{\mathbb{T}_k} \left(\sum_{i=1}^{k} p_i^{1/\alpha} \right)^{\alpha} u_{k,\beta}(\mathbf{p}) d\mathbf{p} \right\} \overline{F}(u + ct),$$

where $u_{k,\beta}(\mathbf{p}) = \left\{ \prod_{i=1}^{k-1} (i\beta - 1) \right\} \left(\prod_{i=1}^{k} p_i \right)^{-\beta-1} \left(\sum_{i=1}^{k} p_i^{-\beta} \right)^{1/\beta - k}$.

Proof. This follows from Barbe et al. (2006) who proved that

$$P\left(X_1 + ... + X_k > x\right) \sim \left\{ \sum_{k=1}^{\infty} P\left[N(t) = k\right] \int_{\mathbb{T}_k} \left(\sum_{i=1}^{k} p_i^{1/\alpha} \right)^{\alpha} u_{k,\beta}(\mathbf{p}) d\mathbf{p} \right\} \overline{F}(x),$$

in the above notation. $\diamond$

1.4.3 Mixture of copulas

Proposition 1.4.19 *Under the assumptions of Proposition 1.4.6, with in addition* $\tilde{C} = \sum_{i=1}^{n} \gamma_i C_i$ *as copula with* $\gamma_i \in \mathbb{R}_+$ *and* $\sum_{i=1}^{n} \gamma_i = 1$ *and if we assume that* $C_i \in CDA(C_i^*)$ *for* $i = 1, ..., n$*, and* C_i^* *is linked with a* $q_{k,\alpha}^{(i)}$ *like in (1.29) and (1.30). Then, for large* u*,*

$$\psi(u,t) \sim \left\{ \sum_{k=1}^{\infty} P\left[N(t) = k\right] \sum_{i=1}^{n} \gamma_i q_{k,\alpha}^{(i)} \right\} \overline{F}(u+ct). \tag{1.36}$$

Proof. This is immediate since if $(X_1^{(i)}, ..., X_k^{(i)})$ is a copy of $(X_1, ..., X_k)$ with dependence structure described by C_i, then, for all $k \geq 1$ and large x,

$$\begin{aligned} P\left(X_1 + ... + X_k > x\right) &= \sum_{i=1}^{n} \gamma_i P(X_1^{(i)} + ... + X_k^{(i)} > x) \\ &\sim \left(\sum_{i=1}^{n} \gamma_i q_{k,\alpha}^{(i)} \right) \overline{F}(x). \end{aligned}$$

$\diamond$

Example. Let us consider a sequence of r.v. $(X_i)_{i\geq 1}$ such that for $i \geq 1$,

- $P(X_i = Y_i) = p$,
- $P(X_i = Z_i) = 1 - p$,

where $p \in [0,1]$ and for all $k \geq 1$, $(Z_1, ..., Z_k)$ and $(Y_1, ..., Y_k)$ belongs to $\mathcal{MR}_{-\alpha}$ with common cdf F. The dependence structure of $(Y_1, ..., Y_k)$ is described by a Gaussian copula which satisfies (1.23) and the dependence structure of $(Z_1, ..., Z_k)$ is described by the copula (1.25).
Now, consider the risk process $R(t) = u + ct - \sum_{i=1}^{N(t)} X_i$. By (1.27) and (1.36) we get large u that

$$\psi(u,t) \sim \left\{ p\lambda t + (1-p) \sum_{k=1}^{\infty} P\left[N(t) = k\right] \sum_{A^{(k)} \in \mathfrak{P}([1,k])} \lambda_{A^{(k)}} \left(\sum_{a \in A^{(k)}} Card(a)^{\alpha} \right) \right\} \overline{F}(u+ct).$$

In particular we deduce that

$$\alpha > 1 \text{ resp. } \alpha < 1 \ \Rightarrow \psi(u,t) \text{increases (resp. decreases) with } p,$$

and if $\alpha = 1$, $\psi(u,t) = \lambda t \overline{F}(u+ct)$. Indeed,

$$\frac{\partial \psi}{\partial p} = \left\{ \lambda t - \sum_{k=1}^{\infty} P\left[N(t) = k\right] \sum_{A^{(k)} \in \mathfrak{P}([1,k])} \lambda_{A^{(k)}} \left(\sum_{a \in A^{(k)}} Card(a)^{\alpha} \right) \right\} \overline{F}(u+ct),$$

and we see that for all $k \geq 1$ and $A \in \mathfrak{P}([1,k])$, $\sum_{a \in A^{(k)}} Card(a)^{\alpha} \left\{ \begin{array}{c} > \\ < \\ = \end{array} \right. k$ if $\left\{ \begin{array}{l} \alpha > 1 \\ \alpha < 1 \\ \alpha = 1 \end{array} \right.$, and $\sum_{k=1}^{\infty} P\left[N(t) = k\right] \sum_{A^{(k)} \in \mathfrak{P}([1,k])} \lambda_{A^{(k)}} k = \lambda t$.

1.5 Dependence through an environment process

In this Section, we aim at taking into account the fact that one or several correlation crises may occur: claim amounts, in general independent or weakly dependent, may suddenly become comonotone or strongly positively dependent. The claim size distribution may become more dangerous as well. To this end, the dependence between claim amounts and the claim size distribution and intensity are modulated by a Markovian environment process. More precisely,

- there exists a Markovian environment process $(J(t))_{t\geq 0}$ with states $i = 1, \ldots, J \geq 2$
  - with initial distribution π_0,
  - and transition rate matrix Q.
- for $i = 1, ..., J$ the claim amounts $(X^i_n)_{n\geq 1}$ are J independent sequences defined as in Proposition 1.3.1, i.e.
$$X^i_n = I^i_n W^i_0 + (1 - I^i_n) W^i_n \ , \quad n \geq 1,$$
  - where the $(W^i_n)_{n\geq 0}$ are i.i.d. r.v.'s with cdf $F^i_W \in \mathcal{R}_{-\alpha^i}$,
  - the $(I^i_n)_{n\geq 1}$ are i.i.d. Bernoulli r.v.'s with parameter $p^i \in [0, 1]$,
  - the $(W^i_n)_{n\geq 0}$, are independent from the $(I^i_k)_{k\geq 1}$,
  - and the $(W^i_n)_{n\geq 0}$ and $(I^i_k)_{k\geq 1}$ are independent from a Poisson process $N^i(t)$ with parameter λ^i.

Let us define the J independent processes

$$Y^i(t) = c^i t - \sum_{m^i=1}^{N^i(t)} X^i_{m^i} \ , \quad i = 1, ..., J.$$

Let T_p be the instant of the p^{th} jump of the process $(J(t))_{t\geq 0}$, and define $(R(t))_{t\geq 0}$ by

$$\begin{aligned} R(t) &= u + \sum_{p\geq 1} \sum_{1\leq i\leq J} \left[Y^i(T_p) - Y^i(T_{p-1})\right] 1_{\{J_{T_{p-1}}=i, T_p\leq t\}} \\ &\quad + \sum_{p\geq 1} \sum_{1\leq i\leq J} \left[Y^i(t) - Y^i(T_{p-1})\right] 1_{\{J_{T_{p-1}}=i, T_{p-1}\leq t<T_p\}}. \end{aligned}$$

Thus, we have built a modulated risk process. For an illustration see figure 1.2.

We now discuss our model with two situations of special interest. In the first case, the crisis causes the claim amounts to be more dangerous. In the second case, one pure correlation crisis is considered: the dependence between claim amounts increases, but the claim size distribution remains unchanged.

1.5.1 Correlation and severity crisis: case where one state dominates

In this Subsection, we suppose that state 1 is more hazardous than the other states, that is to say, for all $i \geq 2$, we have $\alpha^1 < \alpha^i$.

Proposition 1.5.1 *As $u \to +\infty$, we have for any $t > 0$*

$$\psi(u, t) \sim \left(\sum_{i=1}^{J} \pi_0(i) E\left(M_i^{\perp} + \left[\left([M_i^{com}]^{\alpha^1} \right) \right] \right) \right) \bar{F}^1(u), \tag{1.37}$$

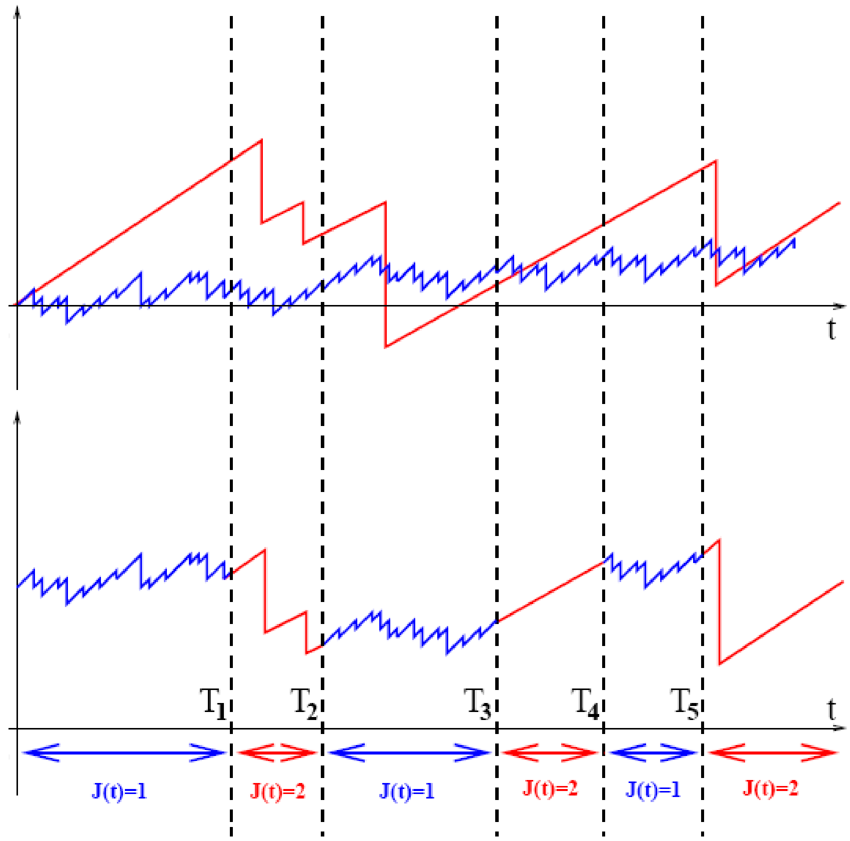


Figure 1.2: A typical modulated risk process with two states (red and blue).

- *where* $W_i^1(t)$ *is the time spent by the environment process in state* 1 *during* $[0,t]$ *given that* $J(0)=i$,
- $M_i^{\perp}$ *follows a mixed Poisson distribution with random parameter* $\lambda^1(1-p^1)W_i^1(t)$,
- *and* M_i^{com} *follows a mixed Poisson distribution with random parameter* $\lambda^1 p^1 W_i^1(t)$.

Proof. First, rewrite $R(t)$ as follows,

$$R(t) = u + C(t) - S(t),$$

where

$$C(t) = \sum_{p\geq 1}\sum_{1\leq i\leq J}\left(c^i(T_p - T_{p-1})\right)\mathbb{1}_{\{J_{T_{p-1}}=i, T_p\leq t\}} + \sum_{p\geq 1}\sum_{1\leq i\leq J}\left(c^i(t - T_{p-1})\right)\mathbb{1}_{\{J_{T_{p-1}}=i, T_{p-1}\leq t\leq T_p\}},$$

and where

$$S(t) = \sum_{p\geq 1}\sum_{1\leq i\leq J}\left(\sum_{m^i=1}^{N^i(T_p)} X^i_{m^i} - \sum_{m^i=1}^{N^i(T_{p-1})} X^i_{m^i}\right)\mathbb{1}_{\{J_{T_{p-1}}=i, T_p\leq t\}}$$
$$+ \sum_{p\geq 1}\sum_{1\leq i\leq J}\left(\sum_{m^i=1}^{N^i(t)} X^i_{m^i} - \sum_{m^i=1}^{N^i(T_{p-1})} X^i_{m^i}\right)\mathbb{1}_{\{J_{T_{p-1}}=i, T_{p-1}\leq t\leq T_p\}}.$$

Then, notice that, for all $t > 0$, $S(t)$ has the same distribution as

$$\tilde{S}(t) = \sum_{j=1}^{J} \sum_{m^j=1}^{N^j(W^j(t))} X^j_{m^j},$$

where for $j = 1, ..., J$ and $t > 0$, $W^j(t)$ is the time spent by the environment process in state j during $[0, t]$. Thus,

$$\begin{aligned} P(R(t) < 0) &= \sum_{j=1}^{J} \pi_0(i) P(S(t) > u + C(t) \,|J(0) = i) \\ &= \sum_{j=1}^{J} \pi_0(i) P(\tilde{S}(t) > u + C(t) \,|J(0) = i). \end{aligned}$$

Let us define for $i, j = 1, ..., J$ and $t > 0$, $W^j_i(t)$ as the time spent by the environment process in state j during $[0, t]$ given $J(0) = i$. We have, for all $j = 1, ..., J$, $t > 0$ and large u,

$$\begin{aligned} P\left(\sum_{m^j=1}^{N^j(W^j(t))} X^j_{m^j} > u \,|J(0) = i\right) &= P\left(\sum_{m^j=1}^{N^j(W^j_i(t))} X^j_{m^j} > u\right) \\ &= P\left(\sum_{m_j=1}^{N_j(W^j_i(t))} (I^j_{m^j} W^j_0 + (1 - I^j_{m^j}) W^j_{m_j}) > u\right) \\ &= P\left(\sum_{m_j=1}^{N^{com}_j(W^j_i(t))} W^j_0 + \sum_{m_j=1}^{N^{\perp}_j(W^j_i(t))} W^j_{m_j} > u\right), \end{aligned}$$

using the thinning property of the Poisson process and where $N^{com}_j(W^j_i(t))$ is Poisson distributed with parameter $\lambda^j p^j W^j_i(t)$ and $N^{\perp}_j(W^j_i(t))$ is Poisson distributed with parameter $\lambda^j p^j W^j_i(t)$. For $i, j = 1, ...J$, $t > 0$ and large u, we have

$$\begin{aligned} P\left(\sum_{m_j=1}^{N^{com}_j(W^j_i(t))} W^j_0 > u\right) &= P\left(N^{com}_j(W^j_i(t)) W^j_0 > u\right) \\ &= \sum_{n=0}^{\infty} P\left(N^{com}_j(W^j_i(t)) = n\right) P\left(W^j_0 > \frac{u}{n}\right) \\ &\sim \sum_{n=0}^{\infty} P\left(N^{com}_j(W^j_i(t)) = n\right) n^{\alpha^j} \overline{F}(u) \\ &\sim E(N^{com}_j(W^j_i(t)))^{\alpha^j} \overline{F^j}(u) \in \mathcal{R}_{-\alpha^j}, \end{aligned}$$

and

$$P\left(\sum_{m_j=1}^{N_j^{\perp}(W_i^j(t))} W_{m_j}^j > u\right) = \sum_{n=0}^{\infty} P\left(N_j^{\perp}(W_i^j(t)) = n\right) P\left(\sum_{m_j=1}^{n} W_{m_j}^j > u\right) \tag{1.38}$$

$$\sim \sum_{n=0}^{\infty} P\left(N_j^{\perp}(W_i^j(t)) = n\right) n\overline{F}(u)$$

$$\sim E(N_j^{\perp}(W_i^j(t)))\overline{F^j}(u) \in \mathcal{R}_{\alpha^j}. \tag{1.39}$$

Since for all $i, j = 1, ...J$ $\sum_{m_j=1}^{N_j^{com}(W_i^j(t))} W_0^j$ and $\sum_{m_j=1}^{N_j^{\perp}(W_i^j(t))} W_{m_j}^j$ are independent and belong to $\mathcal{R}_{-\alpha^j}$, we have for $t > 0$ and large u,

$$P\left(\sum_{m^j=1}^{N^j(W^j(t))} X_{m^j}^j > u \,|J(0) = i\right) \sim \left(E(N_j^{\perp}(W_i^j(t))) + E(N_j^{com}(W_i^j(t)))^{\alpha^j}\right) \overline{F^j(u)} \in \mathcal{R}_{-\alpha^j}$$

Since for all $j = 2, ..., J$, $\alpha^1 < \alpha^j$, and for all $i, j = 1, ..., J$, $i \neq j$ $(X_{m_i}^i)_{m_i \geq 1}$ and $(X_{m_j}^j)_{m_j \geq 1}$ we have for $t > 0$ and large u,

$$\begin{aligned}
P\left(S(t) > u\right) &= \sum_{j=1}^{J} \pi_0(i) P(\tilde{S}(t) > u \,|J(0) = i) \\
&= \sum_{j=1}^{J} \pi_0(i) P(\sum_{j=1}^{J} \sum_{m^j=1}^{N^j(W^j(t))} X_{m^j}^j > u \,|J(0) = i) \\
&\sim \sum_{j=1}^{J} \pi_0(i) P(\sum_{m^1=1}^{N^1(W^1(t))} X_{m^1}^1 > u \,|J(0) = i) \\
&\sim \sum_{j=1}^{J} \pi_0(i) \left(E(N_1^{\perp}(W_i^1(t))) + E(N_1^{com}(W_i^1(t)))^{\alpha^1}\right) \overline{F^1(u)} \\
&\sim \left(\sum_{i=1}^{J} \pi_0(i) E\left(M_i^{\perp} + \left[\left([M_i^{com}]^{\alpha^1}\right)\right]\right)\right) \bar{F}^1(u).
\end{aligned}$$

and since for all $s \in \mathbb{R}$, we have $\overline{F^1}(u + s) \sim \overline{F^1(u)}$ for large u,

$$P\left(S(t) > u + \sup_{j=1,...,J} c^j t\right) \sim \left(\sum_{i=1}^{J} \pi_0(i) E\left(M_i^{\perp} + \left[\left([M_i^{com}]^{\alpha^1}\right)\right]\right)\right) \bar{F}^1(u).$$

We conclude with the inequality:

$$P\left(S(t) > u + \sup_{j=1,...,J} c^j t\right) \leq P(R(t) < 0) \leq P\left(S(t) > u\right).$$

$\diamond$

The next Theorem gives a way to compute the moments of $N^{com}(t)$ when $\alpha^1 \in \mathbb{N}$ using a result in Castella et al. (2007).

Proposition 1.5.2 *If besides $\alpha^1 \in \mathbb{N}$, we have*

$$E\left(M_i^{\perp}\right) = \lambda^1(1-p^1)E\left[W_i^1(t)\right] = \lambda^1(1-p^1)D_i^1(1,t),$$

and

$$E\left([M_i^{com}]^{\alpha^1}\right) = E\left[\sum_{k=0}^{\alpha^1} S(\alpha^1,k)(\lambda^1 p^1)^k \left(W_i^1(t)\right)^k\right]$$
$$= \sum_{k=0}^{\alpha^1} S(\alpha^1,k)(\lambda^1 p^1)^k D_i^1(k,t),$$

where $S(\alpha^1,k)$ is the (α^1,k) Stirling number of the second kind, and where

$$\text{for } m \geq 1, \quad D_i^1(m,t) = E\left[\left(W_i^1(t)\right)^m \mid J(0)=i\right]$$

is the i^{th} component of vector $D^1(m,t)$ defined by $D^1(0,t)=1$ and for $m \geq 1$,

$$D^1(m,t) = r\int_0^t e^{Q(t-u)}A_{11}D^1(m-1,u)du \text{ where } A_{11} \text{ is } J\times J \text{ with coeff. } \delta_{i1}\delta_{j1}.$$

Proof. Castella et al. (2007) show how to compute $D_i^1(m,t)$ for $m \geq 1$ and $i = 1,...,J$. Moreover we know that the moments of a Poisson distribution X with parameter λ are given by, for $m \geq 1$,

$$EX^m = \sum_{k=1}^{m} S(m,k)\lambda^k.$$

$\diamond$

1.5.2 Pure correlation crisis: a typical case

In this Subsection, let us assume that there exist three states such that

- in state 1, the X_n^1, $n \geq 1$ are i.i.d.,
- in state 2, there is a light correlation: the X_n^2, $n \geq 1$ have Gaussian copulas,
- and in state 3, the X_n^3, $n \geq 1$ are given by the basic dependence model with parameter p_3.

Moreover claim amounts are identically distributed with common c.d.f. F such that $\overline{F} \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$ and for all $n \geq 1$ $(X_1^2,...,X_n^2)$ belongs to $\mathcal{MR}_{-\alpha}$.

Proposition 1.5.3 *We have for any $t > 0$, as $u \to +\infty$,*

$$\psi(u,t) \sim \left(\sum_{i=1}^{3} \pi_0(i)[\lambda^1 D_i^1(1,t) + \lambda^2 D_i^2(1,t) + \lambda^3\left(1-p^3\right)D_i^3(1,t) + \sum_{k=0}^{\alpha} S(\alpha,k)(\lambda^3 p^3)^k D_i^3(k,t)]\right)\bar{F}(u).$$

Proof. Since $(X_n^1)_{n\geq 1}$, $(X_n^2)_{n\geq 1}$ and $(X_n^3)_{n\geq 1}$ are independent, we have for $t > 0$ and large u,

$$P(S(t) > u) = P\left(\sum_{j=1}^{3}\sum_{m^j=1}^{N^j(W^j(t))} X_{m^j}^j > u\right) \qquad \text{see the proof of Proposition 1.5.1,}$$
$$\sim \sum_{m^j=1}^{3} P\left(\sum_{m^j=1}^{N^j(W^j(t))} X_{m^j}^j > u\right).$$

We have for $t > 0$ and large u, we have

$$P\left(\sum_{m^1=1}^{N^1(W^1(t))} X^1_{m^1} > u\right) \sim \lambda^1 E(W^1(t))\overline{F}(u),$$

from Proposition 1.4.14, we have

$$P\left(\sum_{m^2=1}^{N^2(W^2(t))} X^2_{m^2} > u\right) \sim \lambda^2 E(W^2(t))\overline{F}(u),$$

and from (1.39), we have

$$P\left(\sum_{m^3=1}^{N^3(W^3(t))} X^3_{m^3} > u\right) \sim \lambda^3 \left(E(W^3(t)) + E\left[(N^3(W^3(t)))^\alpha\right]\right)\overline{F}(u),$$

We conclude with Proposition 1.5.2. $\diamond$
Similar results may be obtained with classical copulas, and dependence between different state processes.

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Chapitre 2

Dépendance entre montants et temps inter-sinistres
Asymptotic Finite-Time Ruin Probabilities for a Class of Path-Dependent Heavy-Tailed Claim Amounts Using Poisson Spacings

2.1 Introduction

The compound Poisson risk model is the central model proposed in insurance theory. An abundant literature has been devoted to its analysis and applications. The reader is referred e.g. to the comprehensive books by Rolski et al. (1999), Asmussen (2000) and Goovaerts et al. (2001). It is well recognized, however, that the simplifying hypotheses at the basis of the model may be too restrictive for certain insurance coverages, especially with rare but extreme risks like earthquake or flooding. The present paper is concerned with such situations.

2.1.1 Framework and motivations

Let us begin by recalling the model in its standard version. The reserves of the company, $\{R(t),\ t \geq 0\}$, are given by

$$R(t) = u + ct - S(t), \quad t \geq 0, \tag{2.1}$$

where $u \geq 0$ is the amount of initial reserves, $c > 0$ is the premium income rate and $S(t)$ is the total claim amount up to time t. $\{S(t),\ t \geq 0\}$ is a compound Poisson process, i.e.

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{2.2}$$

where $\{N(t),\ t \geq 0\}$ is a Poisson process, with parameter λ, which counts the claim occurrences until time t, and $\{X_i,\ i = 1, 2, \ldots\}$ are the successive claim amounts which are represented by non-negative independent identically distributed random variables. Let $\{V_i,\ i = 1, 2, \ldots\}$ be the interarrival times of successive claims. By construction, these random variables are independent with a common exponential distribution. Moreover, the inter-occurrence times are assumed to be independent of the claim amounts.

A statistics of great interest is the probability of (non-)ruin over any fixed horizon of finite length. Denote by $\psi(u,t)$ the probability of ruin before time t, $t \geq 0$, for initial reserves u:

$$\psi(u,t) = P(\exists\ s \in [0,t],\ R(s) < 0 \mid R(0) = u), \quad u, t \geq 0. \tag{2.3}$$

Different methods have been proposed to evaluate $\psi(u,t)$; see, e.g., Lefèvre and Loisel (2008).

Recently, much research has been devoted to the evaluation of ruin probabilities, over finite or infinite horizon, when some independence and stationarity assumptions of the model are relaxed. A number of references will be mentioned later in the section (the list being non-exhaustive, of course).

In practice, the independence assumptions on, and between, the sequences $\{X_i\}$ and $\{V_i\}$ may be too unrealistic. Typically, this arises in the case of natural disasters like earthquake (or flooding). The occurrence of an earthquake often increases the probability of by-claims in a near future. If the last earthquake occurred a long time ago, the next earthquake is likely to be more severe. If two earthquakes occur in a short time span, the second one may cause unusual damages like flooding.

This question has already been raised and discussed in the literature. Albrecher and Boxma (2004) consider that the time between two claim occurrences depends on the previous claim amount. Exact expressions for the Laplace transform of the (ultimate) survival probability are derived. In Albrecher and Teugels (2006), the claim interarrival time and the subsequent claim size are dependent through an arbitrary copula structure. Asymptotic results for both the finite and infinite-time ruin probabilities are then derived. Boudreault et al. (2006) consider a particular form of dependence among the claim interarrival time and the subsequent claim size: if the current claim interarrival time exceeds a certain threshold, the distribution of the next claim is modified. The defective renewal equation satisfied by the expected Gerber-Shiu discounted penalty function is then obtained. In Meng et al. (2008), the time between two claim occurrences determines the distribution of the next claim. Some exact and approximation results are derived for the survival probability. Ambagaspitiya (2009) determines the ruin probability for two forms of dependence between claim size and occurrence in the Sparre Andersen model. See also the references given in these papers.

The present paper discusses several scenarios of dependence between claim amounts and claim interarrival time. Such scenarios are motivated by earthquake or flooding-type risks. Our purpose is to provide approximations to the finite-time ruin probabilities, for large initial reserves, when the claim amounts have heavy-tailed distributions. This work can also be seen as a sequel to a recent paper by Biard et al. (2008). Here the claim amount distributions are allowed to depend, to some extent, on the history of the claim arrival process.

2.1.2 Basic assumptions and implications

As in the classical model, claims occur according to a Poisson process $\{N(t),\ t \geq 0\}$. Let $U_i = \sum_{j=1}^{i} V_j,\ i \geq 1$, be the claim arrival times.

Let us turn to the claim amounts $\{X_i,\ i \geq 1\}$. Firstly, the X_i is allowed to be one of two different types of random variables Y_i or Z_i depending on the behavior of past interarrival times V_j, $j \leq i$. Three different models that describe this dependence will be examined.

Secondly, all the severities Y_i have the same distribution function (d.f.) F, and each vector $\mathbf{Y}^{(j)} = (Y_1, ..., Y_j)$, $j \geq 1$, is of multivariate regular variation of index $-\alpha$ with $\alpha > 0$, i.e. there exists a $\theta \in \mathbf{S}^{j-1}$, where $\mathbf{S}^{j-1}$ is the unit sphere with respect to a norm $|\bullet|$, such that

$$\frac{P\left(|\mathbf{Y}^{(j)}| > tx, \mathbf{Y}^{(j)}/|\mathbf{Y}^{(j)}| \in \bullet\right)}{P\left(|\mathbf{Y}^{(j)}| > x\right)} \xrightarrow{v} t^{-\alpha} P_{\mathbf{S}^{j-1}}(\theta \in \bullet),$$

where $\xrightarrow{v}$ denotes vague convergence on $\mathbf{S}^{j-1}$ (see e.g. Resnick (2004) and Basrak et al. (2002)). Analogously, the amounts Z_i too have a common distribution function G, and each vector $\mathbf{Z}^{(j)} = (Z_1, ..., Z_j)$, $j \geq 1$, is again multivariate regularly varying but of index $-\beta$ with $\beta > 0$. The sequences $\{Y_i,\ i \geq 1\}$, $\{Z_i,\ i \geq 1\}$ and $\{V_i,\ i \geq 1\}$ are independent of each other.

A key tool in the analysis will be the following well-known result (see e.g. Barbe et al. (2006)). If $(Y_1, ..., Y_j)$ is regularly varying of index $-\alpha$ with common marginal d.f. F, then the right tail of the partial sum $S_j = Y_1 + \ldots + Y_j$ can be approximated as

$$P(Y_1 + ... + Y_j > x) \sim q_{j,\alpha}\, \overline{F}(x) \quad \text{for large } x, \tag{2.4}$$

where $\sim$ means that the ratio tends to 1 as $x \to \infty$. The parameter $q_{j,\alpha}$ in (2.4) depends on j, α and on the dependence structure inside the vector $(Y_1, ..., Y_j)$.

Table 2.1 provides examples of $q_{j,\alpha}$ for some classical multivariate copulas (see Nelsen (2006) for a nice introduction to copulas). The independent copula corresponds to the special case where the Y_i, $1 \leq i \leq j$, are independent. The Fréchet upper bound represents the case where the Y_i's are comonotonic. The non-degenerate Gaussian copula is the case where the dependence between the Y_i's is drawn from a multivariate Gaussian distribution with a correlation matrix with coefficients strictly other than ± 1 outside of the diagonal. A copula introduced in Biard et al. (2008), denoted BLL in the sequel, is built as follows:

$$Y_i = I_i W_0 + (1 - I_i) W_i, \quad 1 \leq i \leq j, \tag{2.5}$$

where W_i, $0 \leq i \leq j$, are i.i.d. random variables, and I_i, $1 \leq i \leq j$, are i.i.d. Bernoulli random variables with parameter p, these two sequences being mutually independent.

Special copulas	$q_{j,\alpha}$
Independent	j
Fréchet upper bound	j^α
Non-degenerate Gaussian	j
BLL	$\sum_{i=0}^{j} \binom{j}{i} p^i (1-p)^{j-i} (j - i + i^\alpha)$

Table 2.1: Values of $q_{j,\alpha}$ in (2.4) for different copulas.

Let us mention that the study of the sum of dependent random variables has received much attention in actuarial sciences. See e.g. Wüthrich (2003), Alink et al. (2004), Alink et al. (2005), Barbe et al. (2006), Albrecher et al. (2006), Biard et al. (2008) and Kortschak and Albrecher (2009), among many others.

The paper is organized as follows. In Section 2, we obtain the asymptotic finite-time ruin probabilities for regularly varying claim sizes in the risk model introduced by Boudreault et al. (2006). In Section 3, we derive such ruin probabilities in two other risk models, for earthquake or flooding-type risks, that take consecutive gauge-loading effects into account. The methods of proof will rely on the approximation (2.4) and the analysis of spacings in a conditioned Poisson process. Finally, some numerical illustrations are presented in Section 4.

2.2 Direct effects of each claim interarrival time

Our starting point is the model of Boudreault et al. (2006) where claim amounts are of two different types depending on the length of the previous claim interarrival time. Hereafter, we are going to assume that the claim amounts may be dependent and they have heavy tailed distributions.

Specifically, if an interarrival period V_i is larger than a fixed threshold τ, then the next claim amount X_i is given by a random variable Y_i, and if not, X_i corresponds to another random variable Z_i. As stipulated before, each vector $\mathbf{Y}^{(j)} = (Y_1, ..., Y_j)$ is of multivariate regular variation of index $-\alpha$ and common d.f. F, while each vector $\mathbf{Z}^{(j)} = (Z_1, ..., Z_j)$ is multivariate regularly varying of index $-\beta$ and common d.f. G.

For instance, for earthquake-type risks, one would expect that $\alpha < \beta$: the longer a period without any earthquake, the more serious will be the next earthquake. For flooding risks, the inverse situation where $\alpha > \beta$ seems to be quite plausible.

Consider the random variable $M(t, \tau)$ that gives the number of spacings of the Poisson process $\{N(s),\ 0 \leq s \leq t\}$ which are larger than τ. Conditioning by the number of events $N(t)$, define the following.

Let $M(n, t, \tau)$ be the random variable that counts the number, during $(0, t)$, of Poisson spacings which are larger than τ, given that $N(t) = n$ (≥ 1) and $0 < \tau < t$.

Proposition 2.2.1 *If $\alpha < \beta$, for $t > 0$ and large u, the ruin probability*

$$\psi(u,t) \sim \left\{ \sum_{n=1}^{\infty} P[N(t) = n] \sum_{j=1}^{\min(n, \lfloor t/\tau \rfloor)} P[M(n,t,\tau) = j]\ q_{j,\alpha} \right\} \overline{F}(u + ct), \tag{2.6}$$

while if $\alpha > \beta$, for $t > 0$ and large u,

$$\psi(u,t) \sim \left\{ \sum_{n=1}^{\infty} P[N(t)=n] \sum_{j=\max(1,n-\lfloor t/\tau \rfloor)}^{n} P[M(n,t,\tau)=n-j]\, q_{j,\beta} \right\} \overline{G}(u+ct), \qquad (2.7)$$

with, for $0 \le j \le n$,

$$P[M(n,t,\tau)=j] = \sum_{i=j}^{n} (-1)^{i-j} \binom{n}{j} \binom{n-j}{i-j} P(V_1^* > \frac{\tau}{t}, \dots, V_i^* > \frac{\tau}{t}), \qquad (2.8)$$

where

$$P(V_1^* > v, \dots, V_i^* > v) = \begin{cases} 1, & v \le 0, \\ (1-iv)^n, & 0 < v < 1/i, \\ 0, & 1/i \le v < 1. \end{cases} \qquad (2.9)$$

Proof. Consider $S(t)$, the aggregate claim amount (2.2). Following Biard et al. (2008), we know that if $x \mapsto P[S(t) > x]$ is regularly varying, then for large u,

$$\psi(u,t) \sim P[S(t) > u + ct]. \qquad (2.10)$$

To begin with, we find that

$$\begin{aligned} P[S(t) > x] &= \sum_{n=1}^{\infty} P[N(t)=n] P\left(X_1 + ... + X_n > x\right) \\ &= \sum_{n=1}^{\infty} P[N(t)=n] \sum_{j=1}^{n} P[M(t,\tau)=j | N(t)=n] \\ &\qquad P[X_1 + ... + X_n > x | N(t)=n, M(t,\tau)=j], \end{aligned}$$

and by the model assumptions,

$$\begin{aligned} P[S(t) > x] \quad &= \quad \sum_{n=1}^{\infty} P[N(t)=n] \sum_{j=1}^{n} P[M(n,t,\tau)=j] \\ &\qquad P(Y_1 + ... + Y_j + Z_1 + ... + Z_{n-j} > x). \end{aligned} \qquad (2.11)$$

Note that

$$P[M(n,t,\tau)=j] = 0 \ \text{ for } \ j > t/\tau,$$

so that the summation over j in (2.11) goes from 1 to $\min(n, \lfloor t/\tau \rfloor)$. Now, suppose that $\alpha < \beta$, so that $G(x) = o(F(x))$ for large x. Putting $S_k = Y_1 + ... + Y_k$ and $T_k = Z_1 + ... + Z_k$, $k \ge 1$, and using Proposition 1.1 in Cai and Tang (2004) and (2.4), we obtain

$$\begin{aligned} P(S_j + T_{n-j} > x) \quad &\sim \quad P(S_j > x) + P(T_{n-j} > x) \\ &\sim \quad P(S_j > x) \ \sim \ q_{j,\alpha}\, \overline{F}(x), \quad 1 \le j \le n. \end{aligned} \qquad (2.12)$$

Inserting (2.12) in (2.11) then yields

$$P[S(t) > x] \sim \left\{ \sum_{n=1}^{\infty} P[N(t)=n] \sum_{j=1}^{\min(n,\lfloor t/\tau \rfloor)} P[M(n,t,\tau)=j]\, q_{j,\alpha} \right\} \overline{F}(x),$$

that is, the function $P[S(t) > x]$ is regularly varying with parameter α. Therefore, (2.10) is applicable and gives the announced result (2.6). The approximation (2.7) is derived in a similar way.

Now, it remains to evaluate the p.m.f. of the random variable $M(n,t,\tau)$. As $\{N(t),\ t \geq 0\}$ is a Poisson process, it is well-known that given $N(t) = n$, the claim instants $U_1, \ldots, U_n$ are distributed as the order statistics of n uniform random variables on $[0,t]$, and the claim interarrival times $V_1, \ldots, V_n$ (i.e. the associated spacings) are exchangeable random variables such that the p.d.f. of the vector $(V_1^*, \ldots, V_i^*)$, where $V_i^* = V_i/t, 1 \leq i \leq n$, is given by

$$\begin{aligned} f(v_1, \ldots, v_i) &= \frac{n!}{(n-i)!}(1 - v_1 - \ldots - v_i)^{n-i}, \\ &\qquad \text{for } v_1, \ldots, v_i \geq 0,\ v_1 + \ldots + v_i \leq 1; \end{aligned}$$

see e.g. David and Nagaraja (2003), Sec. 6.4. Consequently, whenever $v_1, \ldots, v_i \geq 0$ with $v_1 + \ldots + v_i \leq 1$,

$$P(V_1^* > v_1, \ldots, V_i^* > v_i) = (1 - v_1 - \ldots - v_i)^n. \tag{2.13}$$

In particular, choosing $v_i = \tau/t = v$ for all i in (2.13), we find that $P(V_1 > \tau, \ldots, V_i > \tau)$ is provided by the formula (2.9) above. Let us consider the event that exactly j claim interarrival times are larger than τ, given that $N(t) = n$, with $1 \leq j \leq n$. Following e.g. David and Nagaraja (2003) (p. 129-130), we then get

$$\begin{aligned} P[M(n,t,\tau) = j] &= \frac{1}{j!}\sum_{l=0}^{n-j} \frac{(-1)^l}{l!}(j+l)! \sum_{i_1 < \cdots < i_{j+l}} P(V_{i_1} > \tau, \ldots, V_{i_{j+l}} > \tau | N(t) = n) \\ &= \frac{1}{j!}\sum_{l=0}^{n-j} \frac{(-1)^l}{l!}(j+l)! \binom{n}{j+l} P(V_1 > \tau, \ldots, V_{j+l} > \tau | N(t) = n) \\ &= \sum_{i=j}^{n} (-1)^{i-j} \frac{n!}{j!(i-j)!(n-i)!}\, P(V_1^* > v, \ldots, V_i^* > v), \end{aligned} \tag{2.14}$$

with $v = \tau/t$. $\diamond$

2.3 Consecutive gauge-loading effects

In this section, we want to incorporate in the risk model the observation that several consecutive claims with large, or small, interoccurrence times are more susceptible to be followed by severe damages. By comparison with the previous model, it will be necessary this time to take (part of) the history of the claim arrival process into account. So, a gauge will be used to register the large, or small, claim interarrival times; initially, the gauge is empty.

2.3.1 Earthquake-type phenomenon

Roughly speaking, for a risk of earthquake-type, the severity of a catastrophe is expected to be more important if the latest catastrophes occur a long time ago. In this sense, let us assume that if k consecutive claim interarrival times are larger than τ, then a more dangerous catastrophe may arise. Moreover, just after, the gauge is put at level 0. Damages caused by standard earthquakes give multivariate regular varying vectors $\mathbf{Z}^{(j)}$ of index $-\beta$, while the more severe form regular varying vectors $\mathbf{Y}^{(j)}$ with index $-\alpha$ and common d.f. F.

Let $M_+(n,k,t,\tau)$ be the random variable that counts the number of sequences, during $(0,t)$, of k consecutive Poisson spacings which are larger than τ, given that $N(t)=n$ (≥ 1).

Proposition 2.3.1 *If $\alpha<\beta$, for $t>0$ and large u, the ruin probability*

$$\psi_k^+(u,t) \sim \left\{ \sum_{n=1}^{\infty} P[N(t)=n] \sum_{j=1}^{\min(\lfloor n/k \rfloor, \lfloor t/k\tau \rfloor)} P\left[M_+(n,k,t,\tau)=j\right]\; q_{j,\alpha} \right\} \overline{F}(u+ct), \qquad (2.15)$$

with, for $0 \leq j \leq \min\left(\lfloor n/k \rfloor, \lfloor t/k\tau \rfloor\right)$,

$$P\left[M_+(n,k,t,\tau)=j\right] = \sum_{i=0}^{k-1} \sum_{x_1,\ldots,x_k \in D_k} \binom{x_1+\cdots+x_k+j}{x_1,\ldots,x_k,j}$$
$$P[M(n,t,\tau)=n-x_1-\cdots-x_k] \Big/ \binom{n}{x_1+\cdots+x_k}, \qquad (2.16)$$

where D_k is the set of all nonnegative integers $x_1,\ldots,x_k$ such that $\sum_{r=1}^{k} r x_r = n-i-kj$ and $n-\sum_{r=1}^{k} x_r \leq \lfloor t/\tau \rfloor$.

Proof. As in the proof of Proposition 2.2.1, we first condition by the value of $N(t)$ and then, by the value of $M_+(n,k,t,\tau)$ to get

$$P\left[S(t)>x\right] = \sum_{n=1}^{\infty} P[N(t)=n] \sum_{j=1}^{n} P[M_+(n,k,t,\tau)=j]$$
$$P[X_1+\ldots+X_n>x|N(t)=n, M_+(n,k,t,\tau)=j].$$

By the model assumptions and using the same arguments as in the proof of Proposition 2.2.1, we have

$$P[X_1+\ldots+X_n>x|N(t)=n, M_+(n,k,t,\tau)=j] = P[Y_1+\ldots+Y_j+Z_1+\ldots+Z_{n-j}>x]$$
$$\sim q_{j,\alpha}\overline{F}(x), \quad 1\leq j\leq n.$$

Thus,

$$P[S(t)>x] = \sum_{n=1}^{\infty} P[N(t)=n] \sum_{j=1}^{n} P[M_+(n,k,t,\tau)=j]\; q_{j,\alpha}\; \overline{F}(x),$$

which leads again to the desired approximation (2.15).

One has now to determine the p.m.f. of the random variable $M_+(n,k,t,\tau)$. This is closely related to the evaluation of the so-called binomial distributions of order k (see e.g. the book by Balakrishnan and Koutras (2002)). Remember that the p.m.f. of $M(n,t,\tau)$ is given in Proposition 2.2. Denote by S the event [a Poisson spacing until time t is larger than τ, given that $N(t)=n$], and let F be the complementary event. Following Philippou and Makri (1986), we observe that a typical element of the event $[M_+(n,k,t,\tau)=j]$ consists in an arrangement of the form

$$a_1a_2\ldots a_{x_1+\ldots+x_k+j}\underbrace{SS\ldots S}_{i}, \text{ with } 0\leq i\leq k-1,$$

where x_1 of the a's are F, x_2 of the a's are SF, …, x_k of the a's are $\underbrace{SS\ldots S}_{k-1}F$ and j of the a's are $\underbrace{SS\ldots S}_{k}$. Note that these x_r are subject to the constraint $x_1+2x_2+\ldots+kx_k+kj+i=n$.

Moreover, by construction of the model, the total number of S's has to be smaller than $\lfloor t/\tau \rfloor$, so that $x_2 + 2x_3 + (k-1)x_k + kj + i \leq \lfloor t/\tau \rfloor$. Now, such arrangements are in number

$$\binom{x_1 + ... + x_k + j}{x_1, \ldots, x_k, j},$$

and each of them has probability

$$P[M(n,t,\tau) = n - x_1 - \cdots - x_k] / \binom{n}{x_1 + \cdots + x_k},$$

hence the formula (2.16). $\diamond$

2.3.2 Flooding-type phenomenon

For a risk of flooding-type, a close succession of claims is expected to be followed by a more important catastrophe. So, we here assume that if k consecutive claim interarrival times are smaller than τ, then the next claim may be more severe. Moreover, the gauge becomes empty once an interarrival time is larger than τ (it corresponds to a reconstruction time). Note that in the model of Section 2.3.1, success runs of length k are counted with a new counter just after each sequence, while in the model here, success runs of length at least k are considered with a new counter only at the first failure. In case of standard floods, damages form multivariate regular varying vectors $\mathbf{Y}^{(j)}$ of index $-\alpha$, while the more severe ones give regular varying vectors $\mathbf{Z}^{(j)}$ with index $-\beta$ and common d.f. G.

Let $M_-(n,k,t,\tau)$ be the random variable that counts the number of sequences, during $(0,t)$, of at least k consecutive Poisson spacings which are smaller than τ, given that $N(t) = n$ (≥ 1).

Proposition 2.3.2 *If $\alpha > \beta$, for $t > 0$ and large u, the ruin probability*

$$\psi_k^-(u,t) \sim \left\{ \sum_{n=1}^{\infty} P[N(t)=n] \sum_{j=1}^{\min(\lfloor (n+1)/(k+1) \rfloor, \lfloor t/(\tau-1) \rfloor)} P\left[M_-(n,k,t,\tau) = j\right] \; q_{j,\beta} \right\} \overline{G}(u+ct), \tag{2.17}$$

with, for $0 \leq j \leq \min\left(\lfloor (n+1)/(k+1) \rfloor, \lfloor t/(\tau-1) \rfloor\right)$,

$$P[M_-(n,k,t,\tau) = j] = \sum_{i=0}^{n} \sum_{x_1,\ldots,x_n \in E_k} \binom{x_1 + \cdots + x_n}{x_1, \ldots, x_n} P[M(n,t,\tau) = x_1 + \cdots + x_n] / \binom{n}{x_1 + \cdots + x_n}, \tag{2.18}$$

where E_k is the set of all nonnegative integers $x_1, \ldots, x_n$ such that $\sum_{r=1}^{n} r x_r = n - i$, $\sum_{r=k+1}^{n} x_r = j - \mathbb{1}_{\{i \geq k\}}$ and $\sum_{r=1}^{n} x_r \leq \lfloor t/\tau \rfloor$.

Proof. Substituting $M_-(n,k,t,\tau)$ for $M_+(n,k,t,\tau)$ and using the same way as in the proof of

Proposition 2.3.1 we get, with the above assumptions, that

$$\begin{aligned} P\left[S(t)>x\right] &= \sum_{n=1}^{\infty} P[N(t)=n]\sum_{j=1}^{n} P[M_-(n,k,t,\tau)=j] \\ &\qquad P[X_1+...+X_n>x|N(t)=n,M_-(n,k,t,\tau)=j] \\ &= \sum_{n=1}^{\infty} P[N(t)=n]\sum_{j=1}^{n} P[M_-(n,k,t,\tau)=j]P[Y_1+\ldots+Y_{n-j}+Z_1+\ldots+Z_j>x] \\ &\sim \sum_{n=1}^{\infty} P[N(t)=n]\sum_{j=1}^{n} P[M_-(n,k,t,\tau)=j]q_{j,\beta}\overline{G}(x), \end{aligned}$$

which leads again to the desired approximation (2.17).

To determine the p.m.f. of the random variable $M_-(n,k,t,\tau)$, one still may argue as for $M_+(n,k,t,\tau)$. Let S be here the event [a Poisson spacing until time t is smaller than τ, given that $N(t)=n$], and let F be the complementary event. An element of the event $[M_-(n,k,t,\tau)=j]$ is an arrangement of the form

$$a_1a_2\ldots a_{x_1+\ldots+x_n}\underbrace{SS\ldots S}_{i},\quad \text{with}\quad 0\le i\le n,$$

where x_1 of the a's are F, x_2 of the a's are SF, ..., x_n of the a's are $\underbrace{SS\ldots S}_{n-1}F$, with the constraints $x_1+2x_2+\ldots+nx_n+i=n$ and $x_{k+1}+\ldots+x_n+\mathbb{1}_{\{i\ge k\}}=j$. In addition, as the number of spacings larger than τ is smaller than $\lfloor t/\tau\rfloor$, one has $x_1+\ldots+x_n\le\lfloor t/\tau\rfloor$. Now, the number of such arrangements is

$$\binom{x_1+...+x_n}{x_1,\ldots,x_n},$$

each of them having probability

$$P[M(n,t,\tau)=x_1+...+x_n]/\binom{n}{x_1+\cdots+x_n},$$

so that the formula (2.18) then follows. $\diamond$

2.4 Numerical analysis

A central step of the numerical analysis is the computation of the p.m.f. of the random variables $M(n,t,\tau)$, $M_+(n,k,t,\tau)$ and $M_-(n,k,t,\tau)$. For that, one can use the exact formulas obtained before. Another possible way is to proceed by recursion. A recursive method is provided in the Appendix for the variables M_+ and M_- (it is not simpler for M). Both methods have their own advantages and drawbacks.

- The exact formulas are easier to implement but the computation is longer.
- The recursive method works faster but implementation is fastidious.

Hereafter, we have chosen to work with the exact formulas because we aim to see the impact of various dependence parameters on the asymptotic ruin probabilities. Someone who wants to investigate larger time horizons for example, might prefer to follow the recursive method.

2.4.1 Impact of a dependence between claim amounts

In this first part, we consider asymptotic ruin probabilities $\psi(u,t)$ for different values of $q_{k,\alpha}$ (see Table 2.1). In order to have a spectrum of dependence, the parameter p (called "dependence parameter") in the BLL copula model is allowed to vary from 0 to 1 with step 0.01. When the dependence parameter equals 0 we have the independence case, and when it equals 1 we obtain the Fréchet upper bound case. The other parameters are :

u	c	λ	Distribution of the riskier claim amount
$1,000,000$	10	0.1	Pareto law with parameter α

For the model of Section 2.2, we examine the case where $\alpha < \beta$. For $t = 10$ and for both $\alpha = 0.5$ and $\alpha = 3$, two cases are investigated, one with $\tau = 1$ and the other with $\tau = 2$. Each case is plotted first separately and then in a same graph. When $\alpha = 0.5$ (Figure 2.1) the asymptotic ruin probability is a decreasing function of the dependence parameter p. It is the opposite when $\alpha = 3$ (Figure 2.2).

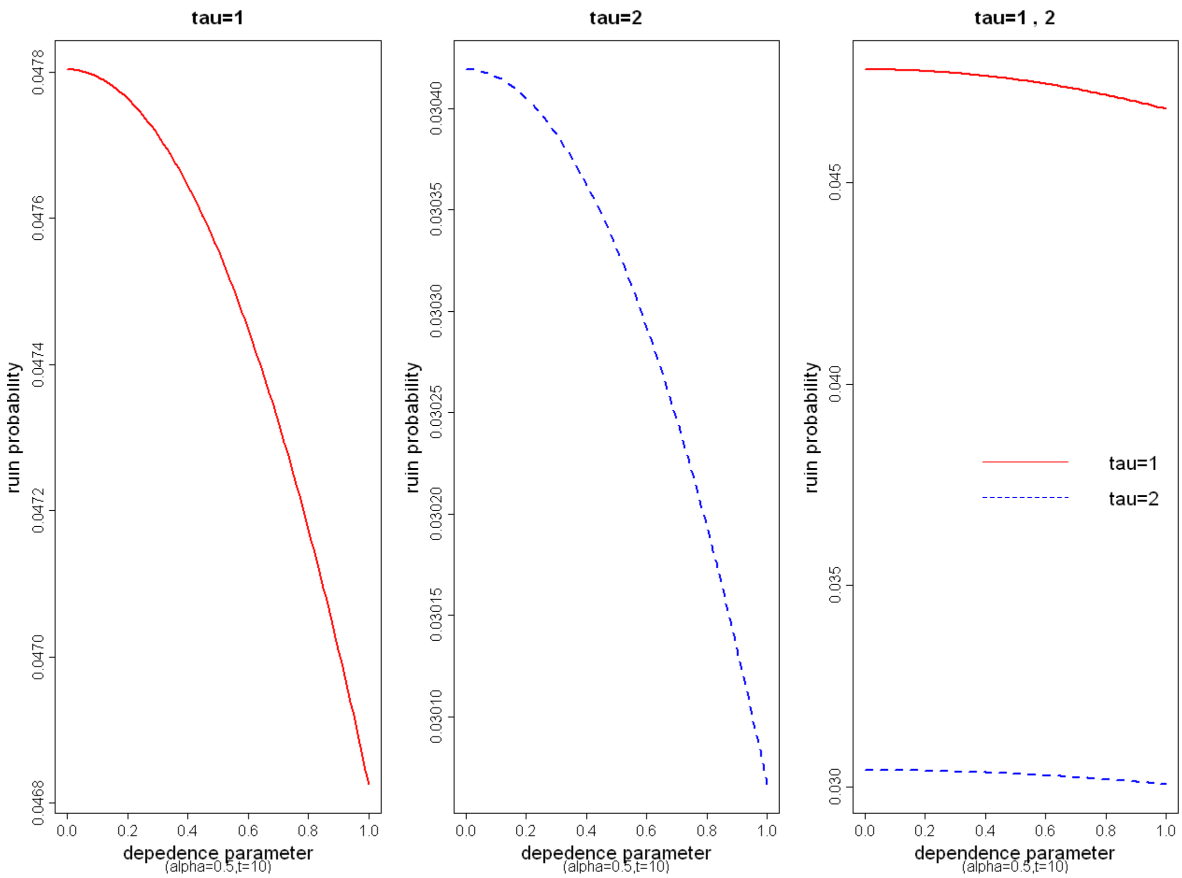

Figure 2.1: Asymptotic ruin probability $\psi(u,t)$ for the model of Section 2.2 as a function of the dependence parameter p in the BLL copula when $\alpha = 0.5$ and $t = 10$.

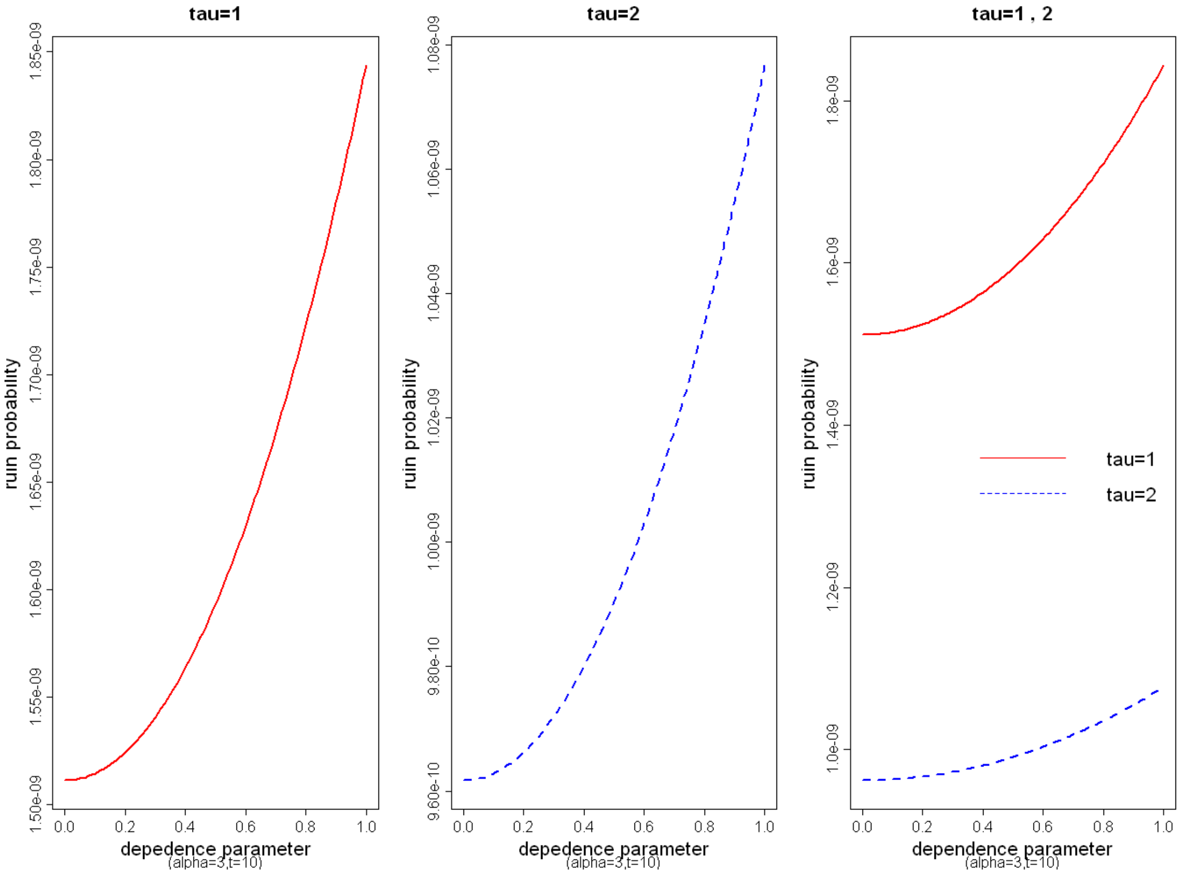

Figure 2.2: Asymptotic ruin probability $\psi(u,t)$ for the model of Section 2.2 as a function of the dependence parameter p in the BLL copula when $\alpha = 3$ and $t = 10$.

For the model of Subsection 2.3.1, with $k = 3$, the same phenomenon is observed (see Figure 2.3 for $\alpha = 0.5$ and Figure 2.4 for $\alpha = 3$).

2.4.2 Impact of a dependence between claim interarrival times and claim amounts

In this second part, we examine the effect of two parameters on the asymptotic ruin probabilities: τ for the model of Section 2.2 and τ and k for the model of Subsection 2.3.1. Let us choose $\alpha = 3$, for instance. For each case, we consider two dependence parameters $p = 0.4$ and $p = 0.8$; these are plotted separately and the relative difference is then plotted in a third graph.

For the model of Section 2.2 with $t = 10$, as expected, the asymptotic ruin probability is a decreasing function of τ (see Figure 2.5).

For the model of Subsection 2.3.1 with $t = 20$, the asymptotic ruin probability is a decreasing function of both τ and k (see Figure 2.6 and Table 2.2).

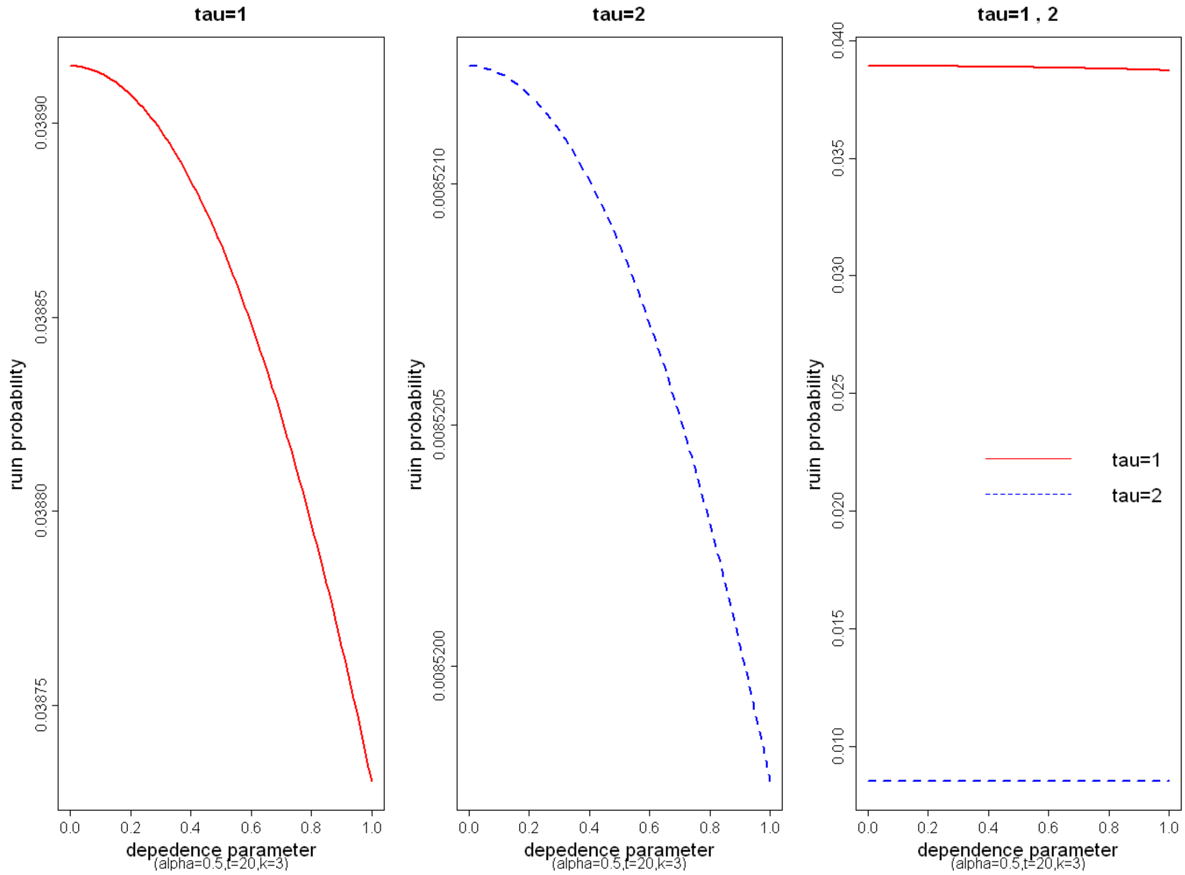

Figure 2.3: Asymptotic ruin probability $\psi_k^+(u,t)$ for the model of Subsection 2.3.1 as a function of the dependence parameter p in the BLL copula when $\alpha = 0.5$ and $t = 20$.

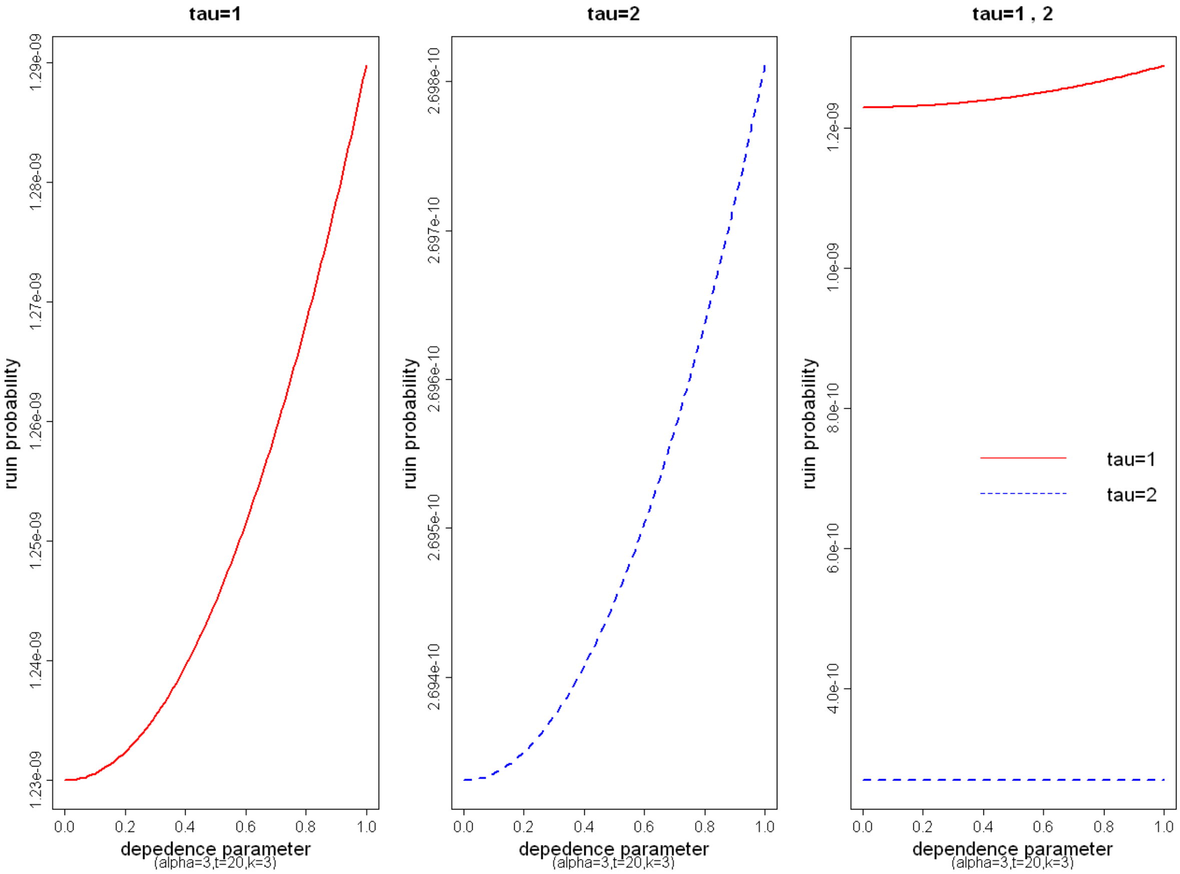

Figure 2.4: Asymptotic ruin probability $\psi_k^+(u,t)$ for the model of Subsection 2.3.1 as a function of the dependence parameter p in the BLL copula when $\alpha = 3$ and $t = 20$.

$\alpha = 3$, $t = 20$ and $\tau = 2$				
k	1	2	3	4
Ruin probability $(R1)$ Dependence parameter $= 0.4$	1.56×10^{-09}	3.08×10^{-10}	2.69×10^{-10}	6.19×10^{-11}
Ruin probability $(R2)$ Dependence parameter $= 0.8$	1.72×10^{-09}	3.21×10^{-10}	2.70×10^{-10}	6.19×10^{-11}
Relative difference $\left(\frac{R2-R1}{R2}\right)$	9.24×10^{-2}	4.01×10^{-2}	8.54×10^{-4}	1.05×10^{-6}

Table 2.2: Asymptotic ruin probability $\psi_k^+(u,t)$ for the model of Subsection 2.3.1 as a function of k when $\alpha = 3$ and $t = 20$.

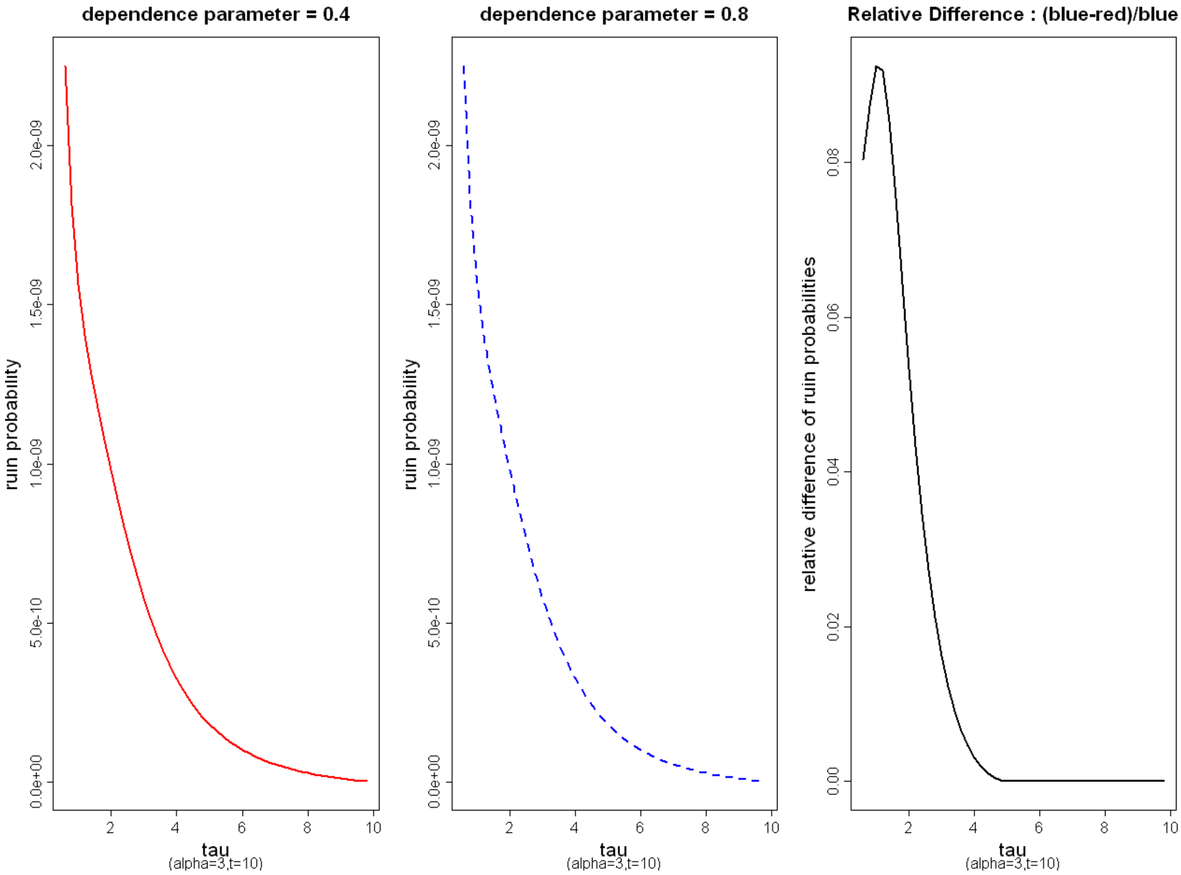

Figure 2.5: Asymptotic ruin probability $\psi(u,t)$ for the model of Section 2.2 as a function of τ when $\alpha = 3$ and $t = 10$.

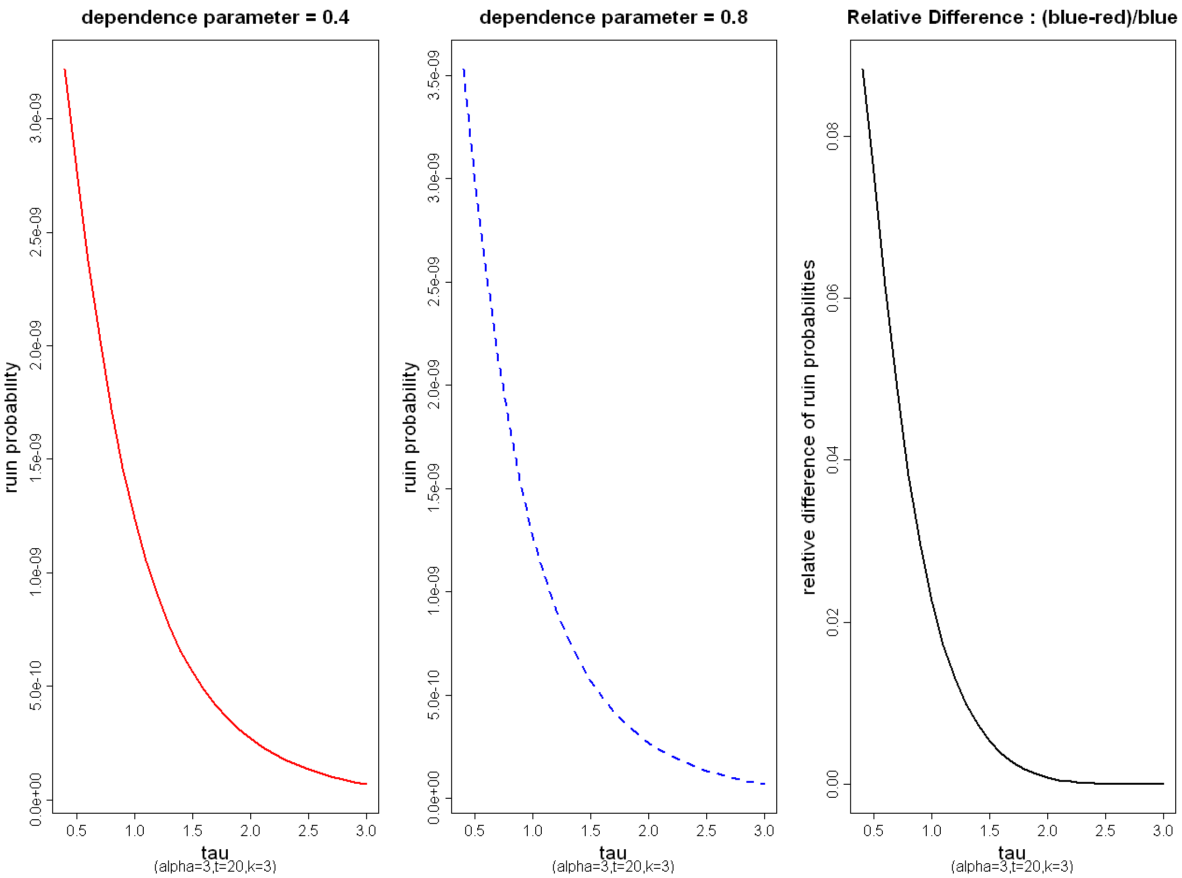

Figure 2.6: Asymptotic ruin probability $\psi_k^+(u,t)$ for the model of Subsection 2.3.1 as a function of τ when $\alpha = 3$ and $t = 20$.

2.5 Appendix

We present in this section a recursive method to compute the p.m.f. of $M_+(n,k,t,\tau)$ and $M_-(n,k,t,\tau)$. The method of proof is directly inspired from Makri and Philippou (2005) (Theorem 4.1).

Proposition 2.5.1 *Let*

$$p_{n,n}(t) = \begin{cases} 0 & \text{if } n\tau > t, \\ \left(1-\frac{n\tau}{t}\right)^n & \text{if } n\tau \leq t, \end{cases}$$

$$p_{0,n}(t) = 1 - p_{n,n}(t),$$

and for $1 \leq l \leq n-1$,

$$p_{l,n}(t) = \begin{cases} 0 & \text{if } l\tau > t, \\ \left(1-\frac{l\tau}{t}\right)^n - \left(1-\frac{(l+1)\tau}{t}\right)^n & \text{if } (l+1)\tau \leq t, \\ \left(1-\frac{l\tau}{t}\right)^n & \text{if } l\tau \leq t < (l+1)\tau. \end{cases}$$

(a) For $n \geq k+1$ *and* $1 \leq j \leq \min(\lfloor n/k \rfloor, \lfloor t/(k\tau) \rfloor)$,

$$\begin{aligned} P\left[M_+(n,k,t,\tau) = j\right] &= p_{n,n}(t)\delta_{j,\lfloor n/k \rfloor} \\ &+ \frac{1}{t}\frac{p_{0,n}(t)}{B(1,n)}\int_0^\tau P\left[M_+(n-1,k,t-s,\tau)=j\right]\left(1-\frac{s}{t}\right)^n ds \\ &+ \frac{1}{t}\sum_{l=1}^{k-1}\frac{p_{l,n}(t)}{B(l+1,n-l)}\int_{l\tau}^t \left(\frac{s}{t}\right)^l \left(1-\frac{s}{t}\right)^{n-l-1} P\left[M_+(n-1-l,k,t-s,\tau)=j\right] ds \\ &+ \frac{1}{t}\sum_{l=k}^{n-1}\frac{p_{l,n}(t)}{B(l+1,n-l)}\int_{l\tau}^t \left(\frac{s}{t}\right)^l \left(1-\frac{s}{t}\right)^{n-l-1} P\left[M_+(n-1-l,k,t-s,\tau)=j-\lfloor l/k \rfloor\right] ds, \end{aligned}$$

(b) for $n \geq k+1$,

$$\begin{aligned} P\left[M_+(n,k,t,\tau) = 0\right] &= \frac{1}{t}\frac{p_{0,n}(t)}{B(1,n)}\int_0^\tau P\left[M_+(n-1,k,t-s,\tau)=0\right]\left(1-\frac{s}{t}\right)^n ds \\ &+ \frac{1}{t}\sum_{l=1}^{k-1}\frac{p_{l,n}(t)}{B(l+1,n-l)}\int_{l\tau}^t \left(\frac{s}{t}\right)^l \left(1-\frac{s}{t}\right)^{n-l-1} P\left[M_+(n-1-l,k,t-s,\tau)=0\right] ds, \end{aligned}$$

(c) for $n \geq k+1$ *and* $j > \min(\lfloor n/k \rfloor, \lfloor t/(k\tau) \rfloor)$,

$$P\left[M_+(n,k,t,\tau) = j\right] = 0,$$

(d) for $n = k$,

$$P\left[M_+(n,k,t,\tau) = 0\right] = 1 - p_{k,k}(t),$$

$$P\left[M_+(n,k,t,\tau) = 1\right] = p_{k,k}(t),$$

(e) for $0 \leq n \leq k-1$,

$$P\left[M_+(n,k,t,\tau) = 0\right] = 1.$$

Proof.

(a) Let $n \geq k+1$ and $1 \leq j \leq \min(\lfloor n/k \rfloor, \lfloor t/(k\tau) \rfloor)$. Define, for $l = 0, \ldots, n-1$, the event $A_l =$ [l interarrival times larger than τ precede the first interarrival time smaller than τ in the sequence of n interarrival times] and the event $B =$ [there is no interarrival time smaller than τ in the sequence of n interarrival times]. Then, we can write that

$$[M_+(n,k,t,\tau) = j] = \left(\bigcup_{l=0}^{n-1} \{[M_+(n,k,t,\tau) = j] \cap A_l\} \right) \cup \{[M_+(n,k,t,\tau) = j] \cap B\},$$

so that we have

$$P[M_+(n,k,t,\tau) = j] = \sum_{l=0}^{n-1} P[M_+(n,k,t,\tau) = j | A_l] \, P(A_l) \; + P[M_+(n,k,t,\tau) = j | B] \, P(B).$$

For $l = 0$,

$$P[M_+(n,k,t,\tau) = j | A_0] = \int_0^{\tau} P[M_+(n-1,k,t-s,\tau) = j] \, f_{U_1}(s) ds.$$

For $1 \leq l \leq k-1$,

$$P[M_+(n,k,t,\tau) = j | A_l] = \int_{l\tau}^{t} P[M_+(n-1-l,k,t-s,\tau) = j] \, f_{U_{l+1}}(s) ds.$$

For $k \leq l \leq n-1$,

$$P[M_+(n,k,t,\tau) = j | A_l] = \int_{l\tau}^{t} P[M_+(n-1-l,k,t-s,\tau) = j - \lfloor l/k \rfloor] \, f_{U_{l+1}}(s) ds.$$

And

$$P[M_+(n,k,t,\tau) = j | B] = \delta_{j, \lfloor n/k \rfloor}.$$

From (2.9), we get, for $1 \leq l \leq n-1$,

$$\begin{aligned}
P(A_l) &= \begin{cases} 0 & \text{if } l\tau > t, \\ P(V_1 > \tau, \ldots, V_l > \tau) - P(V_1 > \tau, \ldots, V_{l+1} > \tau) & \text{if } (l+1)\tau \leq t, \\ P(V_1 > \tau, \ldots, V_l > \tau) & \text{if } \tau \leq t < (l+1)\tau, \end{cases} \\
&= \begin{cases} 0 & \text{if } l\tau > t, \\ \left(1 - \frac{l\tau}{t}\right)^n - \left(1 - \frac{(l+1)\tau}{t}\right)^n & \text{if } (l+1)\tau \leq t, \\ \left(1 - \frac{l\tau}{t}\right)^n & \text{if } l\tau \leq t < (l+1)\tau, \end{cases}
\end{aligned}$$

$$P(A_0) = 1 - P(V_1 > \tau) = 1 - \left(1 - \frac{\tau}{t}\right)^n,$$

and

$$\begin{aligned}
P(B) &= \begin{cases} 0 & \text{if } n\tau > t, \\ P(V_1 > \tau, ...V_n > \tau) & \text{if } n\tau \leq t, \end{cases} \\
&= \begin{cases} 0 & \text{if } n\tau > t, \\ \left(1 - \frac{n\tau}{t}\right)^n & \text{if } n\tau \leq t. \end{cases}
\end{aligned}$$

From David and Nagaraja (2003) p. 134-135, we have, for $l = 1, \ldots, n$,

$$f_{U_l}(u) = \frac{1}{t}\frac{1}{B(l, n+1-l)}\left(\frac{u}{t}\right)^{l-1}\left(1-\frac{u}{t}\right)^{n-l},$$

where $B(.,.)$ is the Beta function. So, we deduce part (a) of the proposition.

(b) When $j = 0$ and $n \geq k$, we observe that

$$[M_+(n,k,t,\tau) = 0] = \bigcup_{l=0}^{k-1}\{[M_+(n,k,t,\tau) = 0] \cap A_l\},$$

so that

$$P[M_+(n,k,t,\tau) = 0] = \sum_{l=0}^{k-1} P[M_+(n,k,t,\tau) = 0|A_l].$$

For $l = 0$,

$$P[M_+(n,k,t,\tau) = 0|A_0] = \int_0^\tau P[M_+(n-1,k,t-s,\tau) = 0] f_{U_1}(s)ds.$$

For $1 \leq l \leq k-1$,

$$P[M_+(n,k,t,\tau) = 0|A_l] = \int_{l\tau}^t P[M_+(n-1-l,k,t-s,\tau) = 0] f_{U_{l+1}}(s)ds.$$

For $l \geq k$,

$$P[M_+(n,k,t,\tau) = 0|A_l] = 0.$$

Thus, we get part (b) of the proposition.

(c) Obvious fact.

(d) For $n = k$,

$$P[M_+(n,k,t,\tau) = 1] = P(V_1 > \tau, V_2 > \tau, \ldots, V_k > \tau) = \left(1 - \frac{k\tau}{t}\right)^k,$$

and

$$P[M_+(n,k,t,\tau) = 0] = 1 - P[M_+(n,k,t,\tau) = 1].$$

(e) Obvious fact. $\diamond$

Proposition 2.5.2 *Let*

$$q_{n,n}(t) = 1 - \left[n\left(1-\frac{\tau}{t}\right)^n - \binom{n}{2}\left(1-\frac{2\tau}{t}\right)^n + \ldots + (-1)^{n-1}\left(1-\frac{n\tau}{t}\right)^n\right],$$

$$q_{0,n}(t) = \left(1-\frac{\tau}{t}\right)^n,$$

and for $1 \leq l \leq n-1$,

$$\begin{aligned} q_{l,n}(t) = & \left[(l+1)\left(1-\frac{\tau}{t}\right)^n - \binom{l+1}{2}\left(1-\frac{2\tau}{t}\right)^n + \ldots + (-1)^l\left(1-\frac{(l+1)\tau}{t}\right)^n\right] \\ & - \left[l\left(1-\frac{\tau}{t}\right)^n - \binom{l}{2}\left(1-\frac{2\tau}{t}\right)^n + \ldots + (-1)^{l-1}\left(1-\frac{l\tau}{t}\right)^n\right]. \end{aligned}$$

(a) *For $n \geq k+1$ and $1 \leq j \leq \min(\lfloor (n+1)/(k+1) \rfloor, \lfloor t/(\tau-1) \rfloor)$,*

$$P\left[M_-(n,k,t,\tau)=j\right] = \delta_{j,1} q_{n,n}(t) + \frac{1}{t}\sum_{l=0}^{k-1} \frac{q_{l,n}(t)}{B(l+1,n-l)} \int_\tau^t \left(\frac{s}{t}\right)^l \left(1-\frac{s}{t}\right)^{n-l-1} P\left[M_-(n-1,k,t-s,\tau)=j\right] ds + \frac{1}{t}\sum_{l=k}^{n-1} \frac{q_{l,n}(t)}{B(l+1,n-l)} \int_\tau^t \left(\frac{s}{t}\right)^l \left(1-\frac{s}{t}\right)^{n-l-1} P\left[M_-(n-1-l,k,t-s,\tau)=j-1\right] ds,$$

(b) *for $n \geq k+1$,*

$$P\left[M_-(n,k,t,\tau)=0\right] = \frac{1}{t}\sum_{l=0}^{k-1} \frac{q_{l,n}(t)}{B(l+1,n-l)} \int_{l\tau}^t \left(\frac{s}{t}\right)^l \left(1-\frac{s}{t}\right)^{n-l-1} P\left[M_-(n-1-l,k,t-s,\tau)=0\right] ds,$$

(c) *for $n \geq k+1$ and $j > \min(\lfloor (n+1)/(k+1) \rfloor, \lfloor t/(\tau-1) \rfloor)$,*

$$P\left[M_-(n,k,t,\tau)=j\right] = 0,$$

(d) *for $n = k$,*

$$P\left[M_-(n,k,t,\tau)=0\right] = 1 - q_{k,k}(t),$$

$$P\left[M_-(n,k,t,\tau)=1\right] = q_{k,k}(t),$$

(e) *for $0 \leq n \leq k-1$,*

$$P\left[M_-(n,k,t,\tau)=0\right] = 1.$$

Proof. We will follow an analogous argument.

(a) Let $n \geq k-1$ and $1 \leq j \leq \min(\lfloor (n+1)/(k+1) \rfloor, \lfloor t/(\tau-1) \rfloor)$. Define this time, for $l = 0, \ldots, n-1$, A_l = [l interarrival times smaller than τ precede the first interarrival time larger than τ in the sequence of n interarrival times] and B = [there is no interarrival time larger than τ in the sequence of n interarrival times]. Then,

$$[M_-(n,k,t,\tau)=j] = \left(\bigcup_{l=0}^{n-1} \}[M_-(n,k,t,\tau)=j] \cap A_l\}\right) \cup \{[M_-(n,k,t,\tau)=j] \cap B\},$$

so that

$$P\left[M_-(n,k,t,\tau)=j\right] = \sum_{l=0}^{n-1} P\left[M_-(n,k,t,\tau)=j|A_l\right] P(A_l) + P\left[M_-(n,k,t,\tau)=j|B\right] P(B).$$

For $0 \leq l \leq k-1$,

$$P\left[M_-(n,k,t,\tau)=j|A_l\right] = \int_\tau^{t-j\tau+1} P\left[M_-(n-1,k,t-s,\tau)=j\right] f_{U_{l+1}}(s) ds.$$

For $k \leq l \leq n-1$,

$$P\left[M_-(n,k,t,\tau)=j|A_l\right]=\int_\tau^{t-j\tau} P\left[M_-(n-1-l,k,t-s,\tau)=j-1\right] f_{U_{l+1}}(s)ds.$$

And

$$P\left[M_-(n,k,t,\tau)=j|B\right]=\delta_{j,1}.$$

From David and Nagaraja (2003) p. 126, 134-135, we have, for $1 \leq l \leq n-1$,

$$\begin{aligned}
P(A_l) &= P(V_1 \leq \tau,\ldots,V_l \leq \tau) - P(V_1 \leq \tau,\ldots,V_{l+1} \leq \tau) \\
&= P(V_1 \leq \tau \cup \ldots \cup V_{l+1} \leq \tau) - P(V_1 \leq \tau \cup \ldots \cup V_l \leq \tau) \\
&= \left[(l+1)\left(1-\frac{\tau}{t}\right)^n - \binom{l+1}{2}\left(1-\frac{2\tau}{t}\right)^n + \ldots + (-1)^l\left(1-\frac{(l+1)\tau}{t}\right)^n\right] \\
&\quad - \left[l\left(1-\frac{\tau}{t}\right)^n - \binom{l}{2}\left(1-\frac{2\tau}{t}\right)^n + \ldots + (-1)^{l-1}\left(1-\frac{l\tau}{t}\right)^n\right],
\end{aligned}$$

$$P(A_0)=P(V_1>\tau)=\left(1-\frac{\tau}{t}\right)^n,$$

and

$$\begin{aligned}
P(B) &= P(V_{(n)} \leq \tau) = 1 - P(V_{(n)} > \tau) \\
&= 1 - \left[n\left(1-\frac{\tau}{t}\right)^n - \binom{n}{2}\left(1-\frac{2\tau}{t}\right)^n + \ldots + (-1)^{n-1}\left(1-\frac{n\tau}{t}\right)^n\right].
\end{aligned}$$

Using again the expression of $f_{U_l}(u)$, $l=1,\ldots,n$, we then obtain part (a) of the proposition.

(b) As for part (b) of Proposition 5.1.

(c) Obvious fact.

(d) For $n=k$,

$$\begin{aligned}
P\left[M_-(n,k,t,\tau)=0\right] &= P\left(V_{(n)}>\tau\right), \\
&= n\left(1-\frac{\tau}{t}\right)^n - \binom{n}{2}\left(1-\frac{2\tau}{t}\right)^n + \ldots + (-1)^{n-1}\left(1-\frac{n\tau}{t}\right)^n,
\end{aligned}$$

and

$$P\left[M_-(n,k,t,\tau)=1\right]=1-P\left(V_{(n)}>\tau\right).$$

(e) Obvious fact. $\diamond$

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Deuxième partie

Théorie de la ruine multivariée : critères de risque et problèmes d'allocation optimale

Chapitre 3

Mesurer le risque avec la partie négative du processus
Asymptotic behavior of the finite-time expected time-integrated negative part of some risk processes and optimal reserve allocation

In the renewal risk model, we study the asymptotic behavior of the expected time-integrated negative part of the process. This risk measure has been introduced by Loisel (2005). Both heavy-tailed and light-tailed claim amount distributions are investigated. The time horizon may be finite or infinite. We apply the results to an optimal allocation problem with two lines of business of an insurance company. The asymptotic behavior of the two optimal initial reserves are computed.

3.1 Introduction

The current change of regulation leads the insurance industry to address new questions regarding solvency. In Europe, insurance groups will have to comply with the new rules, namely Solvency II, by 2011. In comparison to the previous regulation system, Solvency II aims at defining solvency margins that are better adjusted to the underlying risks. Solvency requirements may either be computed thanks to a standard formula, or with internal models that companies are encouraged to develop. While a bottom-up approach is used in the standard formula (one first studies each small risk separately and then aggregates them thanks to a kind of correlation matrix), a top-down approach may be used in some internal or partial internal models: once the main risk drivers for the overall company have been identified and the global solvency capital requirement has been computed, it is necessary to split this overall buffer capital into marginal solvency capitals for each line of business, in order to avoid as far as possible that some lines of business become insolvent too often. Capital fungibility between lines of business or between entities of a large insurance group that lie in different countries is indeed limited by different entity-specific or country-specific solvency constraints.
One possible way to define optimality of the global reserve allocation is to minimize the expected sum of the penalties that each line of business would have to pay due to its temporary potential insolvency. If one neglects discounting factors, a first approximation of this penalty is given by the time-integrated expected negative part of the surplus process. In Loisel (2004, 2005), the author studies this penalty function in infinite time and furnishes a criterion for optimal reserve allocation with different lines of business. Closed-form formulas were available in the classical risk model for exponentially distributed claim amounts, which led to a semi-explicit optimal reserve allocation.
Unfortunately, the hypotheses used in these papers do not perfectly match real-world constraints for practical applications. The first point is that in practice, one may have very different claim amount distributions depending on the kind of insurance risks that are covered: there may be some heavy tails, some light tails or even some very light tails for some particular risks. The second point is that insurance regulation is based on a 1-year time horizon in the standard formula, and on a finite time horizon usually comprised between 1 and 10 years in internal models. To better take those real-world constraints into account, we address the following questions in this paper: can the results obtained by Loisel (2005) for exponentially distributed claim amounts be adapted to the sub-exponential case using results of Embrechts and Veraverbeke (1982)? For large initial global reserve and two lines of business, *with a finite time horizon,* what is the asymptotic optimal part of the initial reserve that one should allocate to each line of business to minimize the sum of the two penalty functions?
To solve these problems for regularly varying, light-tailed and super-exponential claim size distributions, we first need to compute the asymptotics of the finite-time expected time in red and of the expected time-integrated negative part of the considered risk process. In the regularly varying case, it is often said that everything behaves as if one large claim caused ruin. We

wondered whether this heuristic principle could be adapted to our problem, and we show that it applies here. Our paper is organized as follows: in Section 2 we describe our model and use results of Embrechts and Veraverbeke (1982) to obtain analogous results to those of Loisel (2005) in the regular variation case, for infinite time horizon. In Section 3 we derive the asymptotics of the expected time in red and of the expected time-integrated negative part of the considered risk process for finite time horizon and for different classes of claim size distributions. In Section 4, we use these results to obtain asymptotic optimal reserve allocation in some risk models with two lines of business.

3.2 The model

For a uni-dimensional risk processes $U(t) = u + X_t$ that represents the surplus of an insurance company at time t, with initial reserve u and with $X_t = ct - S(t)$, where $c > 0$ is the premium income rate, and $S(t)$ is in the most classical case a compound Poisson process, many risk measures have been considered. The finite-time probability is one and has been studied for different models of risk process. It has been investigated among others by Picard and Lefèvre (1997), Rullière and Loisel (2004) and Lefèvre and Loisel (2008) for classical models. The dependent case has been studied by Biard et al. (2008) and Lefèvre and Loisel (2009). Sensitivity analysis has been carried out by Loisel et al. (2008), Loisel et al. (2009) and Loisel and Privault (2009). We may consider some others risk measures (see for example Gerber (1988), Dufresne and Gerber (1988) and Picard (1994)): the time to ruin $T_u = \inf\{t > 0, u + X_t < 0\}$, the severity of ruin $u+X_{T_u}$, the couple $(T_u, u+X_{T_u})$, the time in red (below 0) from the first ruin to the first time of recovery $T'_u - T_u$ where $T'_u = \inf\{t > T_u, u + X_t = 0\}$, the maximal ruin severity $(\inf_{t>0} u + X_t)$, the aggregate severity of ruin until recovery $J(u) = \int_{T_u}^{T'_u} |u + X_t| dt$,... dos Reis (1993) studied the total time in red $\tau(u) = \int_0^{+\infty} 1_{\{u+X_t<0\}} dt$ using results of Gerber (1988).
All these random variables are drawn from the infinite time ruin theory, or involve the behavior of the risk process between ruin times and recovery times. It seems interesting to consider risk measures based on some fixed time interval $[0, T]$ (T may be infinite). One other of the simplest

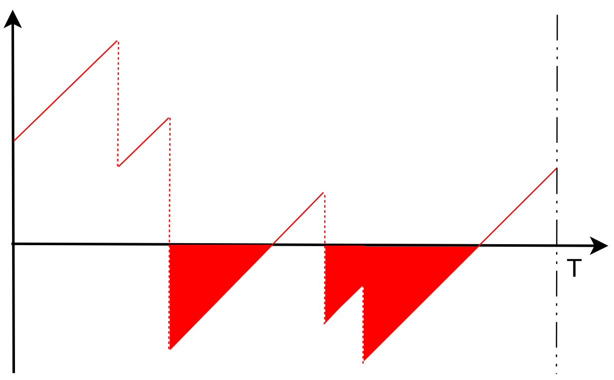

Figure 3.1: Example of a time-aggregated negative part of a risk process.

penalty functions may be the expected value of the time-aggregated negative part of the risk process (see Figure 3.1):

$$E(I_T(u)) = E\left(\int_0^T 1_{\{U(t)<0\}}|U(t)|dt\right).$$

Note that the probability $P(I_T = 0)$ is the probability of non ruin within finite time T. I_T may be seen as the penalty the company will have to pay due to its insolvency until the time horizon T.

These risk measures may be differentiated with respect to the initial reserve u, which makes it possible to compute them quite easily as integrals of other functions of u such as the probability of ruin or the total time in red. Moreover, they have the advantage that the integral over t and the mathematical expectation may be permuted thanks to Fubini's Theorem. Here we recall the two main differentiation theorems (see Loisel (2005)) that are going to be useful for our study:

Theorem 3.2.1 *Assume $T \in \mathbb{R}^+$. Let $(X_t)_{t\in[0,T)}$ be a renewal risk process (possibly modulated by an environment process with finite state space) with time-integrable sample paths. For $u \in \mathbb{R}$, denote by $\tau(u,T)$ the random variable corresponding to the time spent under zero by the process $u + X_t$ between the fixed times 0 and T:*

$$\tau(u,T) = \int_0^T 1_{\{u+X_t<0\}}dt,$$

Let $\tau_0(u,T)$ correspond to the time spent in zero by the process $u + X_t$:

$$\tau_0(u,T) = \int_0^T 1_{\{u+X_t=0\}}dt.$$

Let $I_T(u)$ represent the time-integrated negative part of the process $u + X_t$ between 0 and T:

$$I_T(u) = \int_0^T 1_{\{u+X_t<0\}}|u + X_t|dt$$

and $f(u) = E(I_T(u))$.
For $u \in \mathbb{R}$, if $E(\tau_0(u,T)) = 0$, then f is differentiable at u, and $f'(u) = -E(\tau(u,T))$.

Theorem 3.2.2 *Let $X_t = ct - S(t)$, where $S(t)$ is a compound Poisson process. Consider $T < +\infty$ and define h by $h(u) = E(\tau(u))$ for $u \in \mathbb{R}$. h is differentiable on $\mathbb{R}_*^+ = (0,\infty)$, and for $u > 0$,*

$$h'(u) = -\frac{1}{c}E\left(N^0(u,T)\right),$$

where $N^0(u,T) = Card(\{t \in [0,T], \quad u + ct - S(t) = 0\})$.

We introduce here more notations in the classical compound Poisson model:
An insurance company has an initial surplus $u \geq 0$ and receives premiums continuously at a constant rate $c > 0$. Claims arise according to a homogeneous Poisson process $\{N(t)\}$ with mean λ per unit of time, and, independently of this process, the successive claim amounts $\{W_i\}$ are non-negative independent and identically distributed random variables, with common

distribution function $F_W(x)$ and mean μ. So, the aggregate claims constitute a compound Poisson process $\{S(t)\}$ where $S(t) = \sum_{i=1}^{N(t)} W_i$. The surplus at time t is then given by

$$U(t) = u + ct - S(t), \tag{3.1}$$

and ruin occurs as soon as the surplus becomes negative. One assumes that the net profit condition holds:

$$c > \lambda\mu.$$

Let $\phi(u,T)$ be the probability of non-ruin until time T:

$$\phi(u,T) = P[U(t) = u + ct - S(t) > 0 \;\text{ for }\; 0 < t \leq T], \tag{3.2}$$

and let $\psi(u,T) = 1-\phi(u,T)$ be the probability of ruin before time T. As $T \to \infty$, (3.2) becomes the ultimate non-ruin probability $\phi(u)$, the ultimate ruin probability being $\psi(u) = 1 - \phi(u)$. For $T = \infty$, from Loisel (2005), if $\tau(u)$ is integrable for all $u > 0$, we have

$$E\left(N^0(u,\infty)\right) = \frac{\psi(u)}{1-\psi(0)}, \tag{3.3}$$

for the compound Poisson case and $u > 0$.

We end this section by introducing the mean excess function.

Definition 3.2.3 (Mean excess function) *For a random variable X, the mean excess function $e_X(u)$ is defined by*

$$e_X(u) = E(X - u | X > u).$$

Remark 3.2.4 *A continuous c.d.f. is uniquely determined by its mean excess function since we have*

$$\begin{aligned} e_X(u) &= \int_0^\infty (x-u)dF_X(x)/\overline{F_X}(u) \\ &= \frac{1}{\overline{F_X}(u)} \int_u^\infty \overline{F_X}(x)dx, \quad 0 < u < \infty, \end{aligned}$$

and

$$\overline{F_X}(x) = \frac{e_X(0)}{e_X(x)} \exp\left\{-\int_0^x \frac{1}{e_X(u)}du\right\}, \quad x > 0.$$

3.3 Asymptotics of $E(I_T(u))$ and $E(\tau(u,T))$

This Section gives some results on asymptotics of risk measures we have introduced before. Several cases for the claim size distribution are studied.

3.3.1 A heuristic result with Pareto claim amounts

In the Pareto case, with very large initial reserve u one would expect that one large claim would be responsible for ruin and for the main contribution to the penalty function

$$E(I_T(u)).$$

This is a well-known heuristic result for ruin probabilities, but does it remain true for the expected time-integrated negative part of the risk process? Denote by T_u the time to ruin. Using the decomposition

$$E(I_T(u)) = E\left(I_T(u) \mid T_u \leq T\right) \psi(u,T),$$

the result we expect is that one large claim is likely to cause ruin. Given that this claim occurs, the conditional distribution of this large claim instant is uniform on the interval $[0,T]$ (with average $T/2$), and the average severity at ruin is of the same order as

$$e_W(u) \sim \frac{1}{\alpha-1}u.$$

Consequently, with this approach, it is tempting to say that at the first order, given that ruin occurs before T the risk process stays below zero during an average time $T/2$ at a level equivalent to $-\frac{1}{\alpha-1}u$, which corresponds to an average surface in red

$$\frac{T}{2}\frac{1}{\alpha-1}u.$$

This would lead to the following equivalent:

$$E(I_T(u)) = E\left(I_T(u) \mid T_u \leq T\right) \psi(u,T) \sim \left[\frac{T}{2}\frac{1}{\alpha-1}u\right]\left[\lambda T u^{-\alpha}\right],$$

which may be rewritten as

$$E(I_T(u)) \sim \frac{\lambda T^2}{2(\alpha-1)}u^{-\alpha+1} \tag{3.4}$$

as $u \to +\infty$.
A similar heuristic approach would lead us to guess that the average time spent below zero by the risk process up to time T is

$$E(\tau(u,T)) \sim \frac{\lambda T^2}{2}u^{-\alpha} \tag{3.5}$$

as $u \to +\infty$, as the risk process would remain below zero in average during a time $T/2$ in case of ruin: if ruin occurs, the large claim causing ruin occurs in average at time $T/2$ and the expected severity at ruin is $e_W(u) = u/(\alpha-1)$, so that recovery is almost impossible before time u if u is large enough.

Note that from differentiation theorems in Loisel (2005), Equation (3.5) holds as long as (3.4) holds. We shall now prove that our intuition is correct and that (3.4) holds.

3.3.2 Sub-exponential case

In this Section, we give the asymptotics of $E(I_T(u))$ when u tends to infinity for claim amount distributions that belong to the sub-exponential class.

Definition 3.3.1 *A cdf F with support $(0,\infty)$ is sub-exponential, if for all $n \geq 2$,*

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n.$$

The class of sub-exponential cdfs will be denoted by $\mathcal{S}$.

We first consider the important subclass of regular variation.

Regular variation case

Definition 3.3.2 *A function l on $(0,\infty)$ is slowing varying at ∞ (we write $l \in \mathcal{R}_0$) if*

$$\lim_{x\to\infty} \frac{l(tx)}{l(x)} = 1, \quad t > 0.$$

The convergence is uniform on each compact subset of $t \in (0,\infty)$.

Definition 3.3.3 *A cdf F with support $(0,\infty)$ belongs to the regular variation class if for some $\alpha > 0$*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \quad for\ y > 0.$$

or equivalently if,

$$\overline{F}(x) = x^{-\alpha} l(x),$$

with $l \in \mathcal{R}_0$. We note $F \in \mathcal{R}_{-\alpha}$.
The convergence is uniform on each subset $y \in [y_0,\infty)$ $(0 < y_0 < \infty)$.

Theorem 3.3.4 (Karamata's Theorem) *Let $l \in \mathcal{R}_0$ be locally bounded in $[x_0,\infty]$ for some $x_0 \geq 0$. Then*

- *for $0 < \alpha < 1$,*

$$\int_{x_0}^{x} t^{-\alpha} l(t)dt \sim (1-\alpha)^{-1} x^{-\alpha+1} l(x), \quad x \to \infty,$$

- *for $\alpha > 1$,*

$$\int_{x}^{\infty} t^{-\alpha} l(t)dt \sim (\alpha-1)^{-1} x^{-\alpha+1} l(x), \quad x \to \infty.$$

We first investigate the infinite-time case.
In the sub-exponential case, Embrechts and Veraverbeke (1982) have shown that

$$\psi(u) \sim \frac{\lambda}{c-\lambda\mu} \int_{u}^{+\infty} (1 - F_W(x))dx.$$

In the α-regularly varying case with $\alpha > 1$ (this means that

$$1 - F_W(x) \sim x^{-\alpha} l(x) \text{ as } x \to +\infty,$$

where l is a slowly varying function), this corresponds to

$$\psi(u) \sim \frac{\lambda}{c-\lambda\mu}\frac{1}{\alpha-1}u^{-\alpha+1}l(u).$$

From Theorems 3.2.2 and 3.2.1 and (3.3), we get that

Proposition 3.3.5

$$E\left[\tau(u)\right] \sim \frac{1}{c}\frac{1}{1-\psi(0)}\frac{\lambda}{c-\lambda\mu}\frac{1}{(\alpha-1)(\alpha-2)}u^{-\alpha+2}l(u)$$

for $\alpha > 2$ and

$$E\left[I_\infty(u)\right] \sim \frac{1}{c}\frac{1}{1-\psi(0)}\frac{\lambda}{c-\lambda\mu}\frac{1}{(\alpha-1)(\alpha-2)(\alpha-3)}u^{-\alpha+3}l(u)$$

for $\alpha > 3$.

For real-world applications, finite-time horizon is preferred to infinite-time horizon. This is the reason why we consider a finite-time ruin horizon in the sequel.

Let us begin by a result on the mean-excess function.

Proposition 3.3.6 *For a random variable X with c.d.f. $F_X \in \mathcal{R}_{-\alpha}$ for some $\alpha > 1$, we have for large u,*

$$e_X(u) \sim \frac{u}{\alpha-1}.$$

Proof. For the proof, see for example Embrechts et al. (1997), p 162.
$\diamond$

Theorem 3.3.7 *For a risk process with claim amounts distribution in the regular variation class for some $\alpha > 1$ and c.d.f F_W, we have, for $T > 0$ and large u,*

$$E(I_T(u)) \sim \frac{\lambda T^2}{2(\alpha-1)}u\overline{F_W}(u).$$

Proof. With Proposition 3.3.6 and Remark 3.2.4, we can express $E(I_T(u))$ with the mean-excess function of the compound process $S(t) = \sum_{i=1}^{N(t)} W_i$ which has a c.d.f which belongs to the regular variation class with the same parameter as F_W. Hence, we have

$$\begin{aligned}
E(I_T(u)) &= E\left(\int_{t=0}^{T} \mathbb{1}_{u+X_t<0}\,|u+X_t|\,dt\right)\\
&= \int_{t=0}^{T} E\left(\mathbb{1}_{u+X_t<0}\,|u+X_t|\right)dt && \text{using Fubini's Theorem}\\
&= \int_{t=0}^{T}\int_{x=0}^{\infty} P\left(S(t)>u+ct+x\right)dxdt\\
&= \int_{t=0}^{T}\int_{y=u+ct}^{\infty} P\left(S(t)>y\right)dydt\\
&= \int_{t=0}^{T} \overline{F}_{S(t)}(u+ct)e_{S(t)}(u+ct)dt\\
&\sim \lambda\overline{F}_W(u)\int_{t=0}^{T} te_{S(t)}(u+ct)dt && as\ \ u\to\infty\\
&\sim \frac{\lambda T^2}{2(\alpha-1)}u\overline{F_W}(u).
\end{aligned}$$

◇

The sub-exponential class

Theorem 3.3.8 *For a risk process with claim amounts distribution function, F_W in the sub-exponential class, we have, for $T>0$ and large u,*

$$E(I_T(u)) \sim \frac{\lambda T^2}{2}\left(\int_u^{\infty}\overline{F_W}(v)dv\right).$$

Proof. We have (cf Theorem 3.3.7)

$$E(I_T(u)) = \int_0^T \overline{F_{S(T)}}(u+ct)e_{S(t)}(u+ct)dt.$$

Since $F_W \in \mathcal{S}$, we have from Theorem 3 in Embrechts et al. (1979) that $F_{S(t)} \in \mathcal{S}$ and that

$$\overline{F_{S(t)}}(x) \sim \lambda t\overline{F_W}(x) \ \text{ for } x\to\infty.$$

It follows that

$$e_{S(t)}(x) = \frac{\int_x^{\infty}\overline{F_{S(t)}}(y)dy}{\overline{F_{S(t)}}(x)} \sim \frac{\int_x^{\infty}\overline{F_W}(y)dy}{\overline{F_W}(x)} = e_W(x).$$

Hence, as $u\to\infty$,

$$E(I_T(u)) \sim \frac{\lambda T^2}{2}\overline{F_W}(u)e_W(u) = \frac{\lambda T^2}{2}\int_u^{\infty}\overline{F_W}(x)dx.$$

◇

Remark 3.3.9 *If F_W is regularly varying for some $\alpha>1$, we retrieve $E(I_T(u)) \sim \frac{\lambda T^2}{2}\frac{u\overline{F_W}(u)}{\alpha-1}$.*

3.3.3 Case where the Cramér-Lundberg exponent exists

Definition 3.3.10 (Cramér-Lundberg exponent) *Denote by $\hat{F_W}$ the m.g.f. of F_W. The Cramér-Lundberg exponent R of the risk process $(U(t))_{t\geq 0}$ is defined as the unique positive solution, if it exists, of the equation*

$$\lambda\left(\hat{F_W}(s)-1\right)-cs=0. \tag{3.6}$$

In this Subsection, we assume that the Cramér-Lundberg exponent of the risk process $(U_t)_{t\geq 0}$ exists and is equal to R.

With these assumptions and in the infinite-time case, we have the following well-known result.

Theorem 3.3.11 (The Cramér-Lundberg Approximation) *We have*

$$\psi(u)\sim Ce^{-Ru} \quad \text{as } u\to\infty\ ,$$

where

$$C=\frac{1-\lambda\mu}{\lambda\hat{F_W}'(R)-1}.$$

From Theorems 3.3.11, 3.2.2 and 3.2.1 and (3.3), we get that

Proposition 3.3.12

$$E\left[\tau(u)\right]\sim\frac{1}{c}\frac{1}{1-\psi(0)}\frac{C}{R}e^{-Ru}$$

and

$$E\left[I_\infty(u)\right]\sim\frac{1}{c}\frac{1}{1-\psi(0)}\frac{C}{R^2}e^{-Ru}.$$

In the finite-time case, a convexity argument enables us to show that:

$$C'e^{-Ru}\left(1-e^{-R(c-\lambda\mu)T}\right)\sim E\left[I_{+\infty}(u)\right]-E\left[I_{+\infty}\left(E\left[U(T)\right]\right)\right]\leq E\left[I_T(u)\right]\leq E\left[I_{+\infty}(u)\right]\sim C'e^{-Ru},$$

with $C'=\frac{1}{c}\frac{1}{1-\psi(0)}\frac{C}{R^2}$.

3.3.4 Super-exponential case

In this Section we consider the super-exponential case, i.e. we assume that $E[e^{\theta W_1}]<\infty$ for all $\theta>0$. The aim is to present a large deviation principle (LDP) based on the results in Macci (2008); see Dembo and Zeitouni (1998) for the definition of LDP. We start introducing the function $\Lambda:\mathbb{R}\to\mathbb{R}$ defined by $\Lambda(\theta)=c\theta+\lambda(E[e^{-\theta W_1}]-1)$; moreover let Λ^* be Fenchel-Legendre transform of Λ, i.e. the function $\Lambda^*(x)=\sup_{\theta\in\mathbb{R}}\{\theta x-\Lambda(\theta)\}$. We recall that $\Lambda'(0)=c-\lambda E[W_1]$, and the net profit condition is $\Lambda'(0)>0$.

Proposition 3.3.13 *Assume $\Lambda'(0)\geq -1/T$. Then $\left\{\frac{1}{u^2}I_{Tu}(u):u>0\right\}$ satisfies the LDP with good rate function J defined by*

$$J(z)=\begin{cases} T\Lambda^*\left(\frac{1}{T}\left(-\frac{z}{T}-\sqrt{\left(\frac{z}{T}\right)^2+\frac{2z}{T}}-1\right)\right) & if\ z>0\\ 0 & if\ z=0\\ \infty & if\ z<0.\end{cases}$$

This means that

$$-\inf_{z\in E^\circ} J(z) \le \liminf_{u\to\infty} \frac{1}{u}\log P\left(\frac{1}{u^2} I_{Tu}(u)\in E\right) \le \limsup_{u\to\infty}\frac{1}{u}\log P\left(\frac{1}{u^2} I_{Tu}(u)\in E\right) \le -\inf_{z\in \overline{E}} J(z)$$

for all measurable sets E (E° is the interior of E and $\overline{E}$ is the closure of E).

Proof. We start noting that

$$\begin{aligned}
\frac{1}{u^2} I_{Tu}(u) &= \frac{1}{u^2}\int_0^{Tu} 1_{\{u+ct-\sum_{k=1}^{N(t)} W_k<0\}} \left| u+ct-\sum_{k=1}^{N(t)} W_k\right| dt \\
&= \frac{1}{u^2}\int_0^{T} 1_{\{u+cus-\sum_{k=1}^{N(us)} W_k<0\}} \left| u+cus-\sum_{k=1}^{N(us)} W_k\right| uds \\
&= \int_0^{T} 1_{\{1+cs-\frac{1}{u}\sum_{k=1}^{N(us)} W_k<0\}} \left| 1+cs-\frac{1}{u}\sum_{k=1}^{N(us)} W_k\right| ds.
\end{aligned}$$

Then the LDP holds by Proposition 2.1 in Macci (2008) with $u=1$; indeed here we have $\frac{1}{u}$ in place of ε in Macci (2008). The expression of the rate function is provided by equation (7) in Macci (2008) with $u=1$. $\diamond$

We remark that we could have $\lim_{z\to 0+} J(z) > 0 = J(0)$; see the discussion in Remark 5.1 in Macci (2008).

3.4 Optimal reserve allocation strategy for large initial reserve

In this Section, we consider an insurance company with two lines of business. Two main kinds of phenomena may generate dependence between the two processes.

- Firstly, in some cases, claims for the two lines of business may come from a common event : for example, a car accident may cause a claim for driving insurance, liability and disablement insurance. Hurricanes might cause losses in different countries. This should correspond to simultaneous jumps for the two processes. The most common tool to take this into account is the Poisson common shock model.
- Secondly, there exist other sources of dependence, for example the influence of the weather on health insurance and on agriculture insurance. In this case, claims seem to outcome independently for each line of business, depending on the weather. This seems to correspond rather to models with modulation by a Markov process which describes the evolution of the state of the environment.

The environment state process, denoted by $(J(t))_{t\ge 0}$ is a Markov process with state space $\mathcal{S} = \{1,...,J\}$, initial distribution μ and intensity matrix A.
For $i\in\{1,2\}$, let us define the J independent processes

$$Y_i^j = c_i^j t - \sum_{n=1}^{N_i^j(t)} W_{i,n}^j\ , \quad j=1,...,J,$$

- where $c_i^j > 0$,
- $(W_{i,n}^j)_{n\geq 1}$ is a i.i.d. sequence with common c.d.f. $F_{W_i^j}$ and mean μ_i^j,
- and independent from a Poisson process $(N_i^j(t))_{t\geq 0}$ described below.

Let T_p be the instant of the pth jump of the process $(J(t))_{t\geq 0}$, and define $(U_i(t))_{t\geq 0}$, for $i \in \{1,2\}$ by

$$U_i(t) = u + \sum_{p\geq 1}\sum_{1\leq j\leq J}\left[Y_i^j(T_p) - Y_i^j(T_{p-1})\right]\mathbb{1}_{\{J_{T_{p-1}}=j, T_p\leq t\}} + \sum_{p\geq 1}\sum_{1\leq j\leq J}\left[Y_i^j(t) - Y_i^j(T_{p-1})\right]\mathbb{1}_{\{J_{T_{p-1}}=j, T_{p-1}\leq t\leq T_p\}}.$$

Thus, we have built the two processes modulated by a common process. For an illustration of a single modulated process see Figure 3.2.
To model common shocks, we decompose, for all $j \in \{1,..,J\}$, $(N_1(t))_{t\geq 0}$ and $(N_2(t))_{t\geq 0}$ as follow

$$N_1^j(t) = M_1^j(t) + M^j(t)$$
$$N_2^j(t) = M_2^j(t) + M^j(t)$$

with $(M_1^j(t))_{t\geq 0}$, $(M_2^j(t))_{t\geq 0}$ and $(M^j(t))_{t\geq 0}$ three independent processes with parameter λ_1^j, λ_2^j and λ_j respectively.
For $i = 1,2$ and $u > 0$, we note $\psi_i(u) = P[U_i(t) < 0 \text{ for some } t \geq 0 | U_i(0) = u]$.

The allocation problem is to minimize the risk measure

$$I_T(u_1,u_2) = E\left[I_T^1(u_1)\right] + E\left[I_T^2(u_2)\right],$$

under the constraint $u_1 + u_2 = u$ for large u where

$$I_T^i(u_i) = \int_0^T 1_{\{U_i(t)<0\}}|U_i(t)|dt \quad i = 1,2.$$

For an illustration, see Figure 3.3.

3.4.1 Infinite-time regular variation case

In the Subsection, we assume that the dependence between the two lines of business is only generated by common shocks. There is no environment process. We also assume that the claim amount distribution of the first (resp. second) line of business belongs to the regular variation class with parameter α_1 (resp. α_2) with $3 < \alpha_1 < \alpha_2$. Thus, the second line of business is safer than the first one.
As there are no environment process, the notation in this Subsection is the same but without the state exponent j.

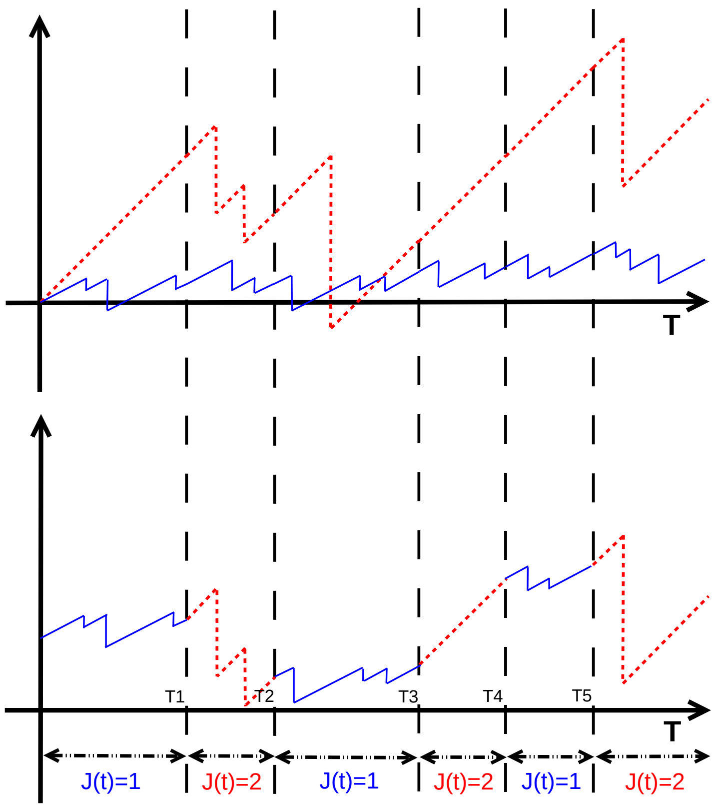

Figure 3.2: A typical modulated risk process with two states (red and blue).

From Proposition 3.3.5, we have for large u and $i = 1, 2$,

$$E\left[I^i_\infty(u)\right] \sim D_i u^3 \overline{F_{W_i}}(u),$$

with

$$D_i = \frac{1}{c_i}\frac{1}{1-\psi_i(0)}\frac{\lambda_i+\lambda}{c-(\lambda_i+\lambda)\mu_i}\frac{1}{(\alpha_i-1)(\alpha_i-2)(\alpha_i-3)}.$$

Lemma 3.4.1 *The couples* $(u, 0)$ *and* $(0, u)$ *do not solve our optimization problem for* u *large enough.*

Proof. Let us choose for example $u/2$ for each line of business. We have

$$I_T(u/2, u/2) \xrightarrow[u\to\infty]{} 0.$$

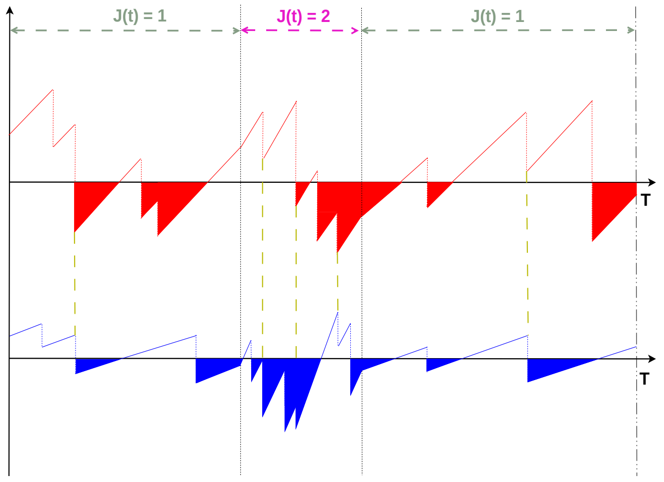

Figure 3.3: Two modulated risk processes with common shocks.

Since $I_T(0,u) = E\left[I_T^1(0)\right] + E\left[I_T^2(u)\right] \geq E\left[I_T^1(0)\right] > 0$ and $I_T(u,0) = E\left[I_T^1(u)\right] + E\left[I_T^2(0)\right] \geq E\left[I_T^2(0)\right] > 0$ for all $u \in \mathbb{R}$, we have the result. $\diamond$

Theorem 3.4.2 *Under the assumptions of this Subsection, the couple (u_1, u_2) which minimizes $I_\infty(u_1, u_2)$ satisfies*

$$\begin{cases} \frac{\partial E\left[I_\infty^1(u_1)\right]}{\partial u_1} = \frac{\partial E\left[I_\infty^1(u_2)\right]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

Moreover, if we denote $u_1 = (1-\beta(u))u$ and $u_2 = \beta(u)u$ with $\beta(u) \in (0,1)$ we have for large u,

$$\beta(u) \sim \left(\frac{D_2^{'}}{D_1^{'}} \frac{\overline{F_{W_2}}(u)}{\overline{F_{W_1}}(u)} \right)^{1/(\alpha_2 - 2)},$$

where

$$D_i^{'} = (\alpha_i - 3)^{-1} D_i \quad i = 1, 2.$$

Note that $\beta(u)$ represents the proportion of the global reserve we allocate to the safer line of business.

Proof. From Lemma 3.4.1, u_1 and u_2 are not equal to zero, we know from the Lagrange multiplier method, (see Loisel (2005)), that the solution of our problem satisfies

$$\begin{cases} \frac{\partial E\left[I_\infty^1(u_1)\right]}{\partial u_1} = \frac{\partial E\left[I_\infty^1(u_2)\right]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

We know from Proposition 3.3.5 that for $i = 1, 2$ and large u,

$$E\left[I_\infty^i(u)\right] \sim D_i u^3 \overline{F_{W_i}}(u),$$

with

$$D_i = \frac{1}{c_i} \frac{1}{1 - \psi_i(0)} \frac{\lambda_i + \lambda}{c - (\lambda_i + \lambda)\mu} \frac{1}{(\alpha_i - 1)(\alpha_i - 2)(\alpha_i - 3)}.$$

From Theorem 3.2.1, for $i = 1,2$, $u_i \mapsto E\left[I^i_\infty(u_i)\right]$ is differentiable. Moreover since $u_i \mapsto E\left[I^i_\infty(u_i)\right]$ is regularly varying with index $\alpha_i - 3$ and from the Monotone Density Theorem (Theorem A 3.7 on page 568 of Embrechts et al. (1997)), we have for large u,

$$\frac{\partial E\left[I^i_\infty(u_i)\right]}{\partial u_i} \sim D'_i u_i^2 \overline{F_{W_i}}(u_i) \quad i = 1,2,$$

with $D'_i = (\alpha_i - 3)^{-1} D_i$.
Let us denote $u_1 = (1-\beta(u))u$ and $u_2 = \beta(u)u$ with $\beta(u) \in (0,1)$ ($\beta(u)$ represents the proportion of the global reserve u we allocate to the line of business 2). With this notation, we are able to give the asymptotic behavior of u_1 and u_2.
Indeed, we have this following equation to solve, with large u,

$$D'_1\left((1-\beta(u))u\right)^2 \overline{F_{W_1}}((1-\beta(u))u) = D'_2\left(\beta(u)u\right)^2 \overline{F_{W_2}}(\beta(u)u),$$

or equivalently, since $\overline{F_{W_i}}$ is regularly varying with index α_i for $i = 1,2$ and using the uniform convergence property (cf Definition 3.3.3),

$$D'_1(1-\beta(u))^{-\alpha_1+2} u^2 \overline{F_{W_1}}(u) = D'_2 \beta(u)^{-\alpha_2+2} u^2 \overline{F_{W_2}}(u).$$

Thus we have

$$\beta(u)^{\alpha_2-2} = \frac{D'_2}{D'_1}\frac{\overline{F_{W_2}}(u)}{\overline{F_{W_1}}(u)}(1-\beta(u))^{\alpha_1-2} \to 0,$$

since $\alpha_2 > \alpha_1 > 3$ and $1-\beta(u) \in (0,1)$.
Consequently, $\beta(u) \xrightarrow[u\to\infty]{} 0$ and for large u,

$$\frac{\beta(u)^{\alpha_2-2}}{\frac{D'_2}{D'_1}\frac{\overline{F_{W_2}}(u)}{\overline{F_{W_1}}(u)}} = (1-\beta(u))^{\alpha_1-2} \xrightarrow[u\to\infty]{} 1.$$

So, we have

$$\beta(u) \sim \left(\frac{D'_2}{D'_1}\frac{\overline{F_{W_2}}(u)}{\overline{F_{W_1}}(u)}\right)^{1/(\alpha_2-2)}.$$

◇

3.4.2 Finite-time regular variation case

We assume here that claim size distribution is regularly varying with parameter α_i^j for state j and process i with $j \in \{1,...,J\}$ and $i \in \{1,2\}$. We also assume that $\alpha_1^1 < \alpha_1^j$ for all $j \in \{2,...,J\}$ and $\alpha_2^1 < \alpha_2^j$ for all $j \in \{2,...,J\}$ and $1<\alpha_1^1 < \alpha_2^2$. That is to say, state 1 corresponds to a crisis environment with more severe claims and the first line of business is also riskier that the second one.

Proposition 3.4.3 *Under the assumptions of this Section, we have for large u and $i \in \{1,2\}$,*

$$E\left[I^i_T(u)\right] \sim \left(\sum_{j=1}^{J}\mu(j)\left[\int_0^T E(N_i^1(V_j^1(t)))dt\right]\right)\frac{u\overline{F_{W_i^1}}(u)}{\alpha_i^1 - 1},$$

where $V_j^1(t)$ is the time spent by the environment process in state 1 during $[0,t]$ given $J(0) = j$.

Proof. First, rewrite, for $i = 1, 2$, $U_i(t)$ as follows,

$$U_i(t) = u + C_i(t) - S_i(t),$$

where

$$C_i(t) = \sum_{p\geq 1}\sum_{1\leq j\leq J}\left(c_i^j(T_p - T_{p-1})\right)\mathbb{1}_{\{J_{T_{p-1}}=j, T_p\leq t\}} + \sum_{p\geq 1}\sum_{1\leq j\leq J}\left(c_i^j(t - T_{p-1})\right)\mathbb{1}_{\{J_{T_{p-1}}=j, T_{p-1}\leq t\leq T_p\}},$$

and where

$$\begin{aligned} S_i(t) = \sum_{p\geq 1}\sum_{1\leq j\leq J}&\left(\sum_{n=1}^{N_i^j(T_p)} W_{i,n}^j - \sum_{n=1}^{N_i^j(T_{p-1})} W_{i,n}^j\right)\mathbb{1}_{\{J_{T_{p-1}}=j, T_p\leq t\}} \\ &+ \sum_{p\geq 1}\sum_{1\leq j\leq J}\left(\sum_{n=1}^{N_i^j(t)} W_{i,n}^j - \sum_{n=1}^{N_i^j(T_{p-1})} W_{i,n}^j\right)\mathbb{1}_{\{J_{T_{p-1}}=j, T_{p-1}\leq t\leq T_p\}}. \end{aligned}$$

Then, notice that, for all $t > 0$, $S_i(t)$ has the same distribution as

$$\tilde{S}_i(t) = \sum_{j=1}^{J}\sum_{n=1}^{N_i^j(V^j(t))} W_{i,n}^j,$$

where for $j = 1, ..., J$ and $t > 0$, $V^j(t)$ is the time spent by the environment process in state j during $[0, t]$.
In Biard et al. (2008), we have the following result :

$$\begin{aligned} P(U_i(t) < 0) &= \sum_{j=1}^{J}\mu(j)P(\tilde{S}_i(t) > u + C_i(t)|J(0) = j), \\ &\sim \left(\sum_{j=1}^{J}\mu(j)E(N_i^1(V_j^1(t)))\right)\overline{F_{W_i^1}}(u) \quad \text{as } u\to\infty, \end{aligned}$$

for $i = 1, 2$.
Thus, $u \mapsto \sum_{j=1}^{J}\mu(j)P(\tilde{S}_i(t) > u + C_i(t)|J(0) = j)$ is regularly varying with parameter α_i^1 and

from Karamata's Theorem we have for large u,

$$\begin{aligned}
E(I_T^i(u)) &= E\left(\int_{t=0}^{T} \mathbb{1}_{U_i(t)<0} \left|U_i(t)\right| dt\right) \\
&= \int_{t=0}^{T} E\left(\mathbb{1}_{U_i(t)<0} \left|U_i(t)\right|\right) dt \qquad \text{using Fubini's Theorem} \\
&= \int_{t=0}^{T}\int_{x=0}^{\infty}\sum_{j=1}^{J} \mu(j) P(\tilde{S}_i(t) > u + C_i(t) + x | J(0) = j) dx dt \\
&= \int_{t=0}^{T}\int_{y=u}^{\infty}\sum_{j=1}^{J} \mu(j) P(\tilde{S}_i(t) > y + C_i(t) | J(0) = j) dy dt \\
&\sim \int_{t=0}^{T} \frac{u}{\alpha_i^1 - 1}\sum_{j=1}^{J} \mu(j) P(\tilde{S}_i(t) > u | J(0) = j) dt \\
&\sim \left(\sum_{j=1}^{J} \mu(j)\left[\int_0^T E(N_i^1(V_j^1(t)))dt\right]\right)\frac{u\overline{F_{W_i^1}}(u)}{\alpha_i^1 - 1}
\end{aligned}$$

$\diamond$

Lemma 3.4.4 *The couples $(u, 0)$ and $(0, u)$ do not solve our optimization problem for u large enough.*

Proof. The proof is the same as in Lemma 3.4.1. $\diamond$

Theorem 3.4.5 *Under the assumptions of this Subsection, the couple (u_1, u_2) which minimizes $I_T(u_1, u_2)$ satisfies*

$$\begin{cases} \frac{\partial E\left[I_T^1(u_1)\right]}{\partial u_1} = \frac{\partial E\left[I_T^1(u_2)\right]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

Moreover, if we denote $u_1 = (1 - \beta(u))u$ and $u_2 = \beta(u)u$ with $\beta(u) \in (0, 1)$ we have for large u,

$$\beta(u) \sim \left(\frac{K_2}{K_1}\frac{\overline{F_{W_2^1}}(u)}{\overline{F_{W_1^1}}(u)}\right)^{1/\alpha_2},$$

where

$$K_i = \left(\sum_{j=1}^{J} \mu(j)\left[\int_0^T E(N_i^1(V_j^1(t)))dt\right]\right) \qquad i = 1, 2.$$

Note that $\beta(u)$ represents the proportion of the global reserve we allocate to the safer line of business.

Proof. From Lemma 3.4.4, u_1 and u_2 are not equal to zero, we know from the Lagrange multiplier method, (see Loisel (2005)), that the solution of our problem satisfies

$$\begin{cases} \frac{\partial E\left[I_T^1(u_1)\right]}{\partial u_1} = \frac{\partial E\left[I_T^1(u_2)\right]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

We know from Proposition 3.4.3 that for large u,

$$E\left[I_T^i(u_i)\right] \sim K_i u_i \frac{\overline{F_{W_i^1}}(u_i)}{\alpha_i^1 - 1} \quad i = 1, 2,$$

with

$$K_i = \left(\sum_{j=1}^{J} \mu(j)\left[\int_0^T E(N_i^1(V_j^1(t)))dt\right]\right) \quad i = 1, 2.$$

From Theorem 3.2.1, for $i = 1, 2$, $u_i \mapsto E\left[I_T^i(u_i)\right]$ is differentiable. Moreover, since $u_i \mapsto E\left[I_T^i(u_i)\right]$ is regularly varying with index $\alpha_i^1 - 1$ and from the Monotone Density Theorem (Theorem A 3.7 on page 568 of Embrechts et al. (1997)), we have for large u,

$$\frac{\partial E\left[I_T^i(u_i)\right]}{\partial u_i} \sim K_i \overline{F_{W_i^1}}(u_i) \quad i = 1, 2.$$

Let us denote $u_1 = (1-\beta(u))u$ and $u_2 = \beta(u)u$ with $\beta(u) \in (0, 1)$ ($\beta(u)$ represents the proportion of the global reserve u we allocate to the line of business 2). With this notation, we are able to give the asymptotic behavior of u_1 and u_2.
Indeed, we have this following equation to solve, with large u,

$$K_1 \overline{F_{W_1^1}}((1 - \beta(u))u) = K_2 \overline{F_{W_2^1}}(\beta(u)u).$$

or equivalently, since $\overline{F_{W_i^1}}$ is regularly varying with index α_i^1 for $i = 1, 2$ and using the uniform convergence property (cf Definition 3.3.3),

$$K_1(1 - \beta(u))^{-\alpha_1^1} \overline{F_{W_1^1}}(u) = K_2 \beta(u)^{-\alpha_2^1} \overline{F_{W_2^1}}(u).$$

Thus we have

$$\beta(u)^{\alpha_2^1} = \frac{K_2}{K_1} \frac{\overline{F_{W_2^1}}(u)}{\overline{F_{W_1^1}}(u)} (1 - \beta(u))^{\alpha_1^1} \to 0,$$

since $\alpha_2 > \alpha_1 > 1$ and $1 - \beta(u) \in (0, 1)$.
Consequently, $\beta(u) \xrightarrow[u\to\infty]{} 0$ and for large u,

$$\frac{\beta(u)^{\alpha_2^1}}{\frac{K_2}{K_1} \frac{\overline{F_{W_2^1}}(u)}{\overline{F_{W_1^1}}(u)}} = (1 - \beta(u))^{\alpha_1^1} \xrightarrow[u\to\infty]{} 1.$$

So,

$$\beta(u) \sim \left(\frac{K_2}{K_1} \frac{\overline{F_{W_2^1}}(u)}{\overline{F_{W_1^1}}(u)}\right)^{1/\alpha_2^1}.$$

⋄ Note that K_1 and K_2 may be computed from an adaptation of Proposition 5.2 in Biard et al. (2008). For example, if we consider only one state (e.g. state 1), we have $K_1 = \frac{\lambda_1^1 T^2}{2}$ and $K_2 = \frac{\lambda_2^1 T^2}{2}$ and for large u,

$$\beta(u) \sim \left(\frac{\lambda_2^1 \overline{F_{W_2^1}}(u)}{\lambda_1^1 \overline{F_{W_1^1}}(u)}\right)^{1/\alpha_2^1}.$$

3.4.3 Infinite time case where Cramér-Lundberg exponent exists

In the Subsection, we assume that the dependence between the two lines of business is only generated by common shocks. There is no environment process. We also assume that the Cramér-Lundberg exponent of the risk process $(U_1(t))_{t\geq 0}$ (resp. $(U_2(t))_{t\geq 0}$) exists and is equal to R_1 (resp. R_2). Finally, we assume that $R_1 < R_2$, that is to say that the second line of business is safer than the first one.
As there is no environment process, the notations in this Subsection are the same but without the state exponent j.

From Proposition 3.3.12, we have for large u and $i = 1, 2$,

$$E\left[I^i_\infty(u)\right] \sim M_i e^{-R_i u},$$

with

$$M_i = \frac{1}{c_i}\frac{1}{1-\psi_i(0)}\frac{1-(\lambda_i+\lambda)\mu_i}{R_i^2((\lambda_i+\lambda)\hat{F_{W_i}}'(R_i)-1)}.$$

Lemma 3.4.6 *The couples $(u, 0)$ and $(0, u)$ do not solve our optimization problem for u large enough.*

Proof. The proof is the same as in Lemma 3.4.1. $\diamond$

Theorem 3.4.7 *Under the assumptions of this Subsection, the couple (u_1, u_2) which minimizes $I_\infty(u_1, u_2)$ satisfies*

$$\begin{cases} \frac{\partial E\left[I^1_\infty(u_1)\right]}{\partial u_1} = \frac{\partial E\left[I^1_\infty(u_2)\right]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

For large u, the solution is given by

$$\begin{aligned} u_1 &= \frac{R_2}{R_1+R_2}u + \frac{1}{R_1+R_2}\log\left(\frac{M_2'}{M_1'}\right) + o(1), \\ u_2 &= u - u_1 + o(1), \end{aligned}$$

where

$$M_i' = -R_i M_i \quad i = 1, 2.$$

Proof. From Lemma 3.4.6, u_1 and u_2 are not equal to zero, we know from the Lagrange multiplier method, see Loisel (2005), that the solution of our problem satisfies

$$\begin{cases} \frac{\partial E\left[I^1_\infty(u_1)\right]}{\partial u_1} = \frac{\partial E\left[I^1_\infty(u_2)\right]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

We know from Proposition 3.3.12 that for $i = 1, 2$ and large u,

$$E\left[I^i_\infty(u)\right] \sim M_i e^{-Ru},$$

with

$$M_i = \frac{1}{c_i}\frac{1}{1-\psi_i(0)}\frac{1-(\lambda_i+\lambda)\mu_i}{R_i^2((\lambda_i+\lambda)\hat{F_{W_i}}'(R_i)-1)}.$$

From Theorem 3.2.1, for $i = 1, 2$, $u_i \mapsto E\left[I^i_\infty(u_i)\right]$ is differentiable. For $i = 1, 2$, we have for large u,

$$\frac{\partial E\left[I^i_\infty(u_i)\right]}{\partial u_i} \sim M_i' e^{-R_i u_i} \quad i = 1, 2,$$

with $M_i' = -R_i M_i$.
We have this following equation to solve, with large u,

$$M_1' e^{-R_1 u_1} = M_2' e^{-R_2(u-u_1)}.$$

The solution is as in the statement of the theorem. $\diamond$

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Chapitre 4

Probabilité de ruine multivariée
Asymptotic multivariate finite-time ruin probabilities with heavy-tailed claim amounts : Impact of dependence and optimal reserve allocation

In ruin theory, the univariate model may be found too restrictive to describe accurately the complex evolution of the reserves of an insurance company. In the case where the company is composed of multiple lines of business, we compute asymptotics of finite-time ruin probabilities. Capital transfers between lines are partially allowed. When claim amounts are regularly varying distributed, several forms of dependence between the lines are considered. We also study the optimal allocation of a large global initial reserve in order to minimize the asymptotic ruin probability.

4.1 Introduction

This paper deals with an insurance company with multiple lines of business. Each line is assumed to be exposed to catastrophic risks like earthquakes, floods or terrorist attacks. Such risks may affect several lines of the company, so dependence between the lines is considered. Each line may correspond to a business in a specific country or to a type of policy offered by the company. Capital transfers between the lines are strictly regulated. Nevertheless, we assume here that a piece of the amount of each line is allowed to recover losses of an another one.
Since the new solvency rules in Europe conduce to study the insurance company reserve in a finite-time framework, we compute multivariate finite-time ruin probabilities.

Suppose now that the company owns a global initial reserve to share between the lines. Due to the specific risk exposition of each line, the choice of the allocation may have a huge impact of its solvency. In Loisel (2005) and Biard et al. (2010), this optimal allocation problem is concerned with the minimization of the expected time-integrated negative part of a risk process. In this paper, we focus on the finite-time multivariate ruin probability for our minimization problem.

In risk theory, multivariate context has been studied scarcely compared to the univariate one. For the univariate setting, the reader is referred e.g. to the comprehensive books by Rolski et al. (1999), Asmussen (2000), Goovaerts et al. (2001) and the references therein. Concerning the multivariate setting, light-tailed case is studied in Collamore (1996, 2002) and a discrete approach is investigated in Picard et al. (2003). In this paper, we are concerned by the heavy-tailed case and use some results of Hult and Lindskog (2006b,a).

The paper is organized as follows. In Section 4.2, we present the framework of the paper, comprising the regular variation setting, the multivariate risk model and the definition of the investigated multivariate ruin probability. Section 4.3 deals with the computation of the asymptotics of the multivariate finite-time ruin probability in context of dependence and Section 4.4 investigates optimal allocation problems.

4.2 Framework

Throughout the paper, vectors are denoted by bold letters. For example, $\mathbf{x} = \left(x^{(1)}, \ldots, x^{(d)}\right) \in \mathbb{R}^d$. Moreover, we define $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{e}_i$ the unit vector whose ith component is equal to 1 and for $1 \leq k \leq d-1$, $\mathbf{1}_k = \sum_{i=1}^{k} \mathbf{e}_i$.

4.2.1 Multivariate Regular Variation

In order to describe losses of catastrophic risks, we choose the heavy-tailed class of regularly varying random variables. The typical example of these kinds of random variables is the Pareto distribution. This random variable class well describes catastrophic risks in the sense that one big loss may endanger the solvency of the company.

Definition 4.2.1 *A function L on $(0,\infty)$ is slowly varying at ∞ if*

$$\lim_{u\to\infty}\frac{L(tu)}{L(u)}=1,\quad \text{for every } t>0.$$

We write $L\in\mathcal{R}_0$

Definition 4.2.2 (Univariate Regular Variation) *A $\mathbb{R}$-random variable X is regularly varying if there exists $\alpha>0$, such that*

$$\lim_{u\to\infty}\frac{P(X>tu)}{P(X>u)}=t^{-\alpha},\quad \text{for every } t>0,$$

or equivalently if,

$$P(X>u)=u^{-\alpha}L(u),$$

for some $L\in\mathcal{R}_0$.
We write $X\in\mathcal{R}_{-\alpha}$.

Definition 4.2.3 (Multivariate Regular Variation) *An $\mathbb{R}^d$-valued random vector*

$$\mathbf{X}=(X^{(1)},\dots,X^{(d)})$$

with unbounded support is regularly varying if there exists a nonzero Radon measure μ defined on $\mathcal{B}(\overline{\mathbb{R}}^d)$ with $\mu(\overline{\mathbb{R}}^d\setminus\{\mathbf{0}\})=0$ such that

$$\lim_{u\to\infty}\frac{P\left(\mathbf{X}\in uA\right)}{P(|\mathbf{X}|>u)}=\mu(A), \tag{4.1}$$

for every Borel set $A\in\mathbb{R}^d$ bounded away from $\mathbf{0}$ (i.e. $\mathbf{0}\notin\bar{A}$) with $\mu(\partial A)=0$.

We can also use an equivalent definition using the spectral measure.
An $\mathbb{R}^d$-valued random vector

$$\mathbf{X}=(X^{(1)},\dots,X^{(d)})$$

with unbounded support is regularly varying if there exists an $\alpha>0$ and a probability measure σ on the unit sphere $\mathbb{S}^{d-1}=\{\mathbf{x}:|\mathbf{x}|=1\}$ such that

$$\lim_{u\to\infty}\frac{P\left(|\mathbf{X}|>xu,\mathbf{X}/|\mathbf{X}|\in S\right)}{P(|\mathbf{X}|>u)}=x^{-\alpha}\sigma(S)\ , \tag{4.2}$$

for every $x>0$ and Borel sets $S\subset\mathbb{S}^{d-1}$. The probability measure σ is called the spectral measure of $\mathbf{X}$.

As a consequence, we have for every $u>0$ and Borel set $A\in\mathbb{R}^d$ bounded away from $\mathbf{0}$

$$\mu(uA)=u^{-\alpha}\mu(A)\ .$$

We write $\mathbf{X}\in\mathcal{MR}_{-\alpha,\mu}$.

For a general presentation of heavy-tailed theory, the reader is referred e.g. to the book of Resnick (2007).

4.2.2 The model

To describe the reserve of an insurance company with d lines of business, we consider a multivariate risk process $(\mathbf{R}_t)_{t\geq 0}$. Denote by $u > 0$ the global initial reserve and by $\mathbf{a} \in (0,1)^d$ the vector which describes the part of u which is allocated to each branch. As a consequence, we have $a^{(1)} + \ldots + a^{(d)} = 1$. The premium rates are captured in $\mathbf{c} \in (0,\infty)^d$. The aggregate claim amount process $(\mathbf{S}_t)_{t\geq 0}$ is assumed to be a multivariate Poisson process, that is to say

$$\mathbf{S}_t = \sum_{i=1}^{N(t)} \mathbf{X}_i \ ,$$

where $N(t)$ is a Poisson process with parameter $\lambda > 0$ and $(\mathbf{X}_i)_{i\geq 1}$ is a $\mathbb{R}_+^d$-valued independent and identically distributed sequence. We note by $\mathbf{X}$ their common distribution.
Hence, we have, for $t \geq 0$,

$$\mathbf{R}_t = u\mathbf{a} + \mathbf{c}t - \sum_{i=1}^{N(t)} \mathbf{X}_i \ . \tag{4.3}$$

Throughout this paper, $\mathbf{X}$ will be regularly varying for some $\alpha > 1$ and measure μ.

4.2.3 Multivariate finite-time ruin probability

In the univariate setting, the finite-time ruin probability is defined as, for $u, T > 0$,

$$\psi(u,T) = P\left(\exists t \in [0,T]\,, R_t < 0 | R_0 = u\right) = P\left(\sup_{[0,T]}(S_t - ct) > u\right) \ .$$

In the multivariate case, there is not a unique definition. For example in Cai and Li (2005, 2007), we can find several definitions, depending on the interest. For $u, T > 0$ let us define

$$\psi_{sum}(u,T) = P\left(\sup_{[0,T]}\left\{\sum_{j=1}^{d}(S_t^{(j)} - c^{(j)}t)\right\} > u\right) \ , \tag{4.4}$$

$$\psi_{and}(u,T) = P\left(\bigcap_{j=1}^{d}\left\{\sup_{[0,T]}(S_t^{(j)} - c^{(j)}t) > a^{(j)}u\right\}\right) \ , \tag{4.5}$$

$$\psi_{or}(u,T) = P\left(\bigcup_{j=1}^{d}\left\{\sup_{[0,T]}(S_t^{(j)} - c^{(j)}(t)) > a^{(j)}u\right\}\right) \ , \tag{4.6}$$

and

$$\psi_{sim}(u,T) = P\left(\exists t \in [0,T]\,, \forall j \in [1,d]\,, R_t^{(j)} < 0\right). \tag{4.7}$$

Here, we investigate the definition proposed by Hult and Lindskog (2006a). For $\beta \in [0,1]$, let

$$F_\beta = \left\{\mathbf{x} : \beta\sum_{k=1}^{d}\left(x^{(k)} \vee 0\right) < -\sum_{k=1}^{d}\left(x^{(k)} \wedge 0\right)\right\} \ , \tag{4.8}$$

where $\vee = \min$ and $\wedge = \max$. For $T > 0$, we define the multivariate finite-time ruin probability $\psi_{d,\beta}(u,T)$ as the probability that the risk reserve process $\mathbf{R}_t$ hits F_β at some time t before T. Explicitly, for $u, T > 0$, we have

$$\psi_{d,\beta}(u,T) = P\left(\exists t \in [0,T], \ \ \mathbf{R}_t \in F_\beta\right). \tag{4.9}$$

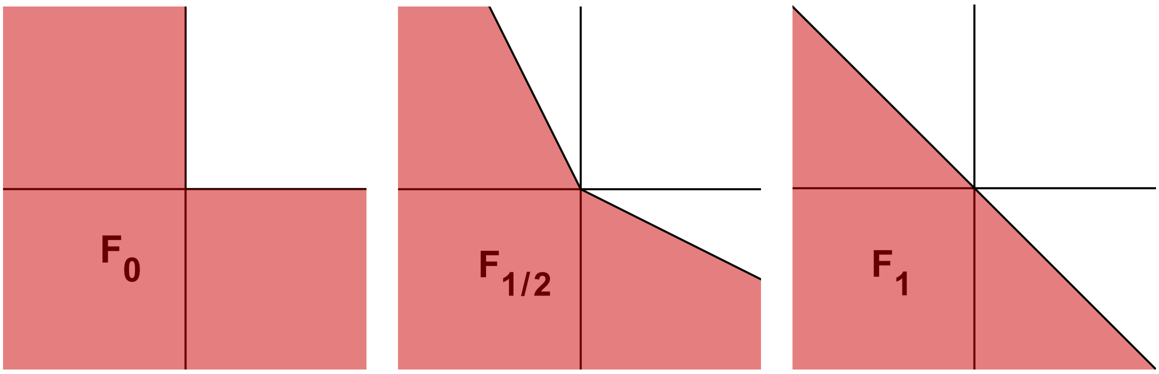

Figure 4.1: F_β for β=0,1/2 and 1 in two dimensions

Remark 4.2.4 *The ruin set F_β corresponds to the possibility to transfer from positive line a fraction $\beta \in [0,1]$ to cover a negative position of another line. For $\beta = 0$, no transfer is allowed and $\psi_{d,0} = \psi_{or}$ and for $\beta = 1$, transfer is allowed without restrictions and $\psi_{d,1} = \psi_{sum}$.*

In Figure 4.1, the set F_β is represented for β =0,1/2 and 1 in the the two-dimensional case.

The following result, from Hult and Lindskog (2006a), gives the asymptotic of the finite-time multivariate ruin probability for a large initial reserve.

Proposition 4.2.5 (Hult and Lindskog (2006a)) *For a risk process $(\mathbf{R}_t)_{t\geq 0}$ given by (4.3) with a common distribution $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some $\alpha > 1$ and measure μ, we have, for $T > 0$ and large u,*

$$\psi_{d,\beta}(u,T) \sim (\lambda T)\mu(\mathbf{a} - F_\beta)P\left(|\mathbf{X}| > u\right). \tag{4.10}$$

This result is the base of our computations. Actually, after giving the assumptions on the dependence structure between claim amount of each line, we can exhibit μ and then get the asymptotic ruin probability. The following lemma gives $\mu(\mathbf{a} - F_\beta)$ for some basic forms of $\mathbf{X}$.

Lemma 4.2.6 *Let X be a positive random variable which is regularly varying with some $\alpha > 0$.*

- **Case 1)** *If for some $1 \leq j \leq d$, $\mathbf{X} = X\mathbf{e}_j$, then $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some measure μ and we have*
$$\mu(\mathbf{a} - F_\beta) = \left(\beta + a^{(j)}(1-\beta)\right)^{-\alpha}. \tag{4.11}$$

- **Case 2)** *If, for some $1 \leq k \leq d$,$\mathbf{X} = X\mathbf{1}_k$, then $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some measure μ and we have*
$$\mu(d^{-1}\mathbf{1} - F_\beta) = \left(d^{-1}\left(\frac{\beta(d-k)}{k} + 1\right)|\mathbf{1}_k|\right)^{-\alpha}. \tag{4.12}$$

- **Case 3)** *If $\mathbf{X} = X\mathbf{1}$, then $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some measure μ and we have*

$$\mu(\mathbf{a} - F_\beta) = \left(\frac{\sum_{i=1}^{k^*} a^{(i:d)} + \beta\sum_{i=k^*+1}^{d} a^{(i:d)}}{k^* + \beta(d-k^*)}|\mathbf{1}|\right)^{-\alpha}, \tag{4.13}$$

where for $1 \leq i \leq d$, $a^{(i:d)}$ is the ith larger component of $\mathbf{a}$ *and*

$$k^* = \inf \left\{ k \in [1, d-1] : a^{(k+1:d)} > \frac{\sum_{i=1}^{k} a^{(i:d)} + \beta \sum_{i=k+1}^{d} a^{(i:d)}}{k + \beta(d-k)} \right\} .$$

Proof. Let $A = \mathbf{a} - F_\beta$. We have

$$A = \left\{ \mathbf{x} : \beta \sum_{i=1}^{d} \left((a^{(i)} - x^{(i)}) \vee 0 \right) < - \sum_{i=1}^{d} \left((a^{(i)} - x^{(i)}) \wedge 0 \right) \right\} .$$

- **Case 1)** Let $\mathbf{X} = X\mathbf{e}_j$ for some $1 \leq j \leq d$. Since $X \in \mathcal{R}_{-\alpha}$, there exists a function $L \in \mathcal{R}_0$ such that $P(X > u) = u^{-\alpha}L(u)$. Moreover, from Karamata's Theorem (see e.g. Embrechts et al. (1997), Theorem A3.6 p 567), we have for large u, $\frac{\partial}{\partial u} P(X > u) \sim -\alpha u^{-\alpha-1} L(u)$. Let $B \subset \mathbb{R}^d$ be a Borel set bounded away from $\mathbf{0}$. We have

$$\begin{aligned}
\lim_{u\to\infty} \left(P(|X\mathbf{e}_j| > u)\right)^{-1} P\left(X\mathbf{e}_j \in uB\right) &= \lim_{u\to\infty} \left(P(X > u)\right)^{-1} P\left(u^{-1}X\mathbf{e}_j \in B\right) \\
&= \lim_{u\to\infty} \left(u^{-\alpha}L(u)\right)^{-1} \int \mathbb{1}_B(r\mathbf{e}_j) dF_{u^{-1}X}(r) \\
&= \lim_{u\to\infty} \left(u^{-\alpha}L(u)\right)^{-1} \int \mathbb{1}_B(r\mathbf{e}_j) dF_X(ru) \\
&= \lim_{u\to\infty} \left(u^{-\alpha}L(u)\right)^{-1} \int \mathbb{1}_B(r\mathbf{e}_j) u\alpha(ur)^{-\alpha-1} L(ur) dr \\
&= \int \mathbb{1}_B(r\mathbf{e}_j) \alpha r^{-\alpha-1} dr .
\end{aligned}$$

Hence, we have $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ with μ defined for all Borel set $B \subset \mathbb{R}^d$ bounded away from $\mathbf{0}$ as

$$\mu(B) = \int \mathbb{1}_B(r\mathbf{e}_j) \alpha r^{-\alpha-1} dr .$$

Since we have

$$\begin{aligned}
r\mathbf{e}_j \in A &\Leftrightarrow \beta \sum_{\substack{1 \leq i \leq d \\ i \neq j}} a^{(i)} < -(a^{(j)} - r) \\
&\Leftrightarrow r > a^{(j)}(1-\beta) + \beta .
\end{aligned}$$

the first result follows.

- **Case 2)** In this case, using the same way, we obtain that $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ with μ defined for all Borel set $B \subset \mathbb{R}^d$ bounded away from $\mathbf{0}$ as

$$\mu(B) = \int \mathbb{1}_B(r\mathbf{1}_k/|\mathbf{1}_k|) \alpha r^{-\alpha-1} dr ,$$

and since $\mathbf{a} = d^{-1}\mathbf{1}$,

$$\begin{aligned}
r\mathbf{1}_k/|\mathbf{1}_k| \in A &\Leftrightarrow \beta(d-k)d^{-1} < -k(d^{-1} - r/|\mathbf{1}_k|) \\
&\Leftrightarrow r > d^{-1} \left(\frac{\beta(d-k)}{k} + 1 \right) |\mathbf{1}_k| ,
\end{aligned}$$

and the second result follows.

• **Case 3)** In this case we get that $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ with μ defined for all Borel set $B \subset \mathbb{R}^d$ bounded away from $\mathbf{0}$ as

$$\mu(B) = \int \mathbb{1}_B(r\mathbf{1}/|\mathbf{1}|)\alpha r^{-\alpha-1}dr \,.$$

It remains to find $\{r : r\mathbf{1}/|\mathbf{1}| \in A\}$.

$$r\mathbf{1}/|\mathbf{1}| \in A \Leftrightarrow \beta \sum_{i=1}^{d} \Big((a^{(i)} - r/|\mathbf{1}|) \vee 0 \Big) < - \sum_{i=1}^{d} \Big((a^{(i)} - |\mathbf{1}|) \wedge 0 \Big) \,.$$

Denote by $a^{(k:d)}$ the kth larger component of $\mathbf{a}$. Let $1 \le k \le d-1$ and assume that $a^{(k:d)} \le r/|\mathbf{1}| < a^{(k+1:d)}$. Then

$$\begin{aligned} r\mathbf{1}/|\mathbf{1}| \in A &\Leftrightarrow \beta \sum_{i=k+1}^{d} a^{(i:d)} - \beta(d-k)r/|\mathbf{1}| < - \sum_{i=1}^{k} a^{(i:d)} + kr/|\mathbf{1}| \\ &\Leftrightarrow r/|\mathbf{1}| > \frac{\sum_{i=1}^{k} a^{(i:d)} + \beta \sum_{i=k+1}^{d} a^{(i:d)}}{k + \beta(d-k)} \,. \end{aligned}$$

Let $K = \left\{ k \in [1, d-1] : a^{(k+1:d)} > \frac{\sum_{i=1}^{k} a^{(i:d)} + \beta \sum_{i=k+1}^{d} a^{(i:d)}}{k+\beta(d-k)} \right\}$. K is lower bounded by 1 and $d-1 \in K$ (since $a^{(d:d)} > 1/d$), so there exists a K-minimal element denoted by k^*. Thus,

$$r\mathbf{1}/|\mathbf{1}| \in A \Leftrightarrow r/|\mathbf{1}| > \frac{\sum_{i=1}^{k^*} a^{(i:d)} + \beta \sum_{i=k^*+1}^{d} a^{(i:d)}}{k^* + \beta(d-k^*)}$$

and the third result follows.

$\diamond$

4.3 Computation of ruin probabilities in the presence of dependence

4.3.1 A simple model of dependence

In this Subsection, we investigate a simple model of dependence between the lines of business. For each claim occurrence, we allow the claim amount of a branch either be independent of the others or equal to a common random variable. Explicitly, the distribution of $\mathbf{X} = (X^1, ..., X^d)$ is such that, for $1 \le j \le d$,

$$X^{(j)} = I^{(j)} W^{(0)} + (1 - I^{(j)}) W^{(j)},$$

where,

- $(W^{(j)})_{0 \le j \le d}$ is an i.i.d. non-negative random vector with common distribution $W \in \mathcal{R}_{-\alpha}$, for some $\alpha > 1$,
- and $(I^{(j)})_{1 \le j \le d}$ is a vector of independent Bernoulli random variables with same parameter $p \in [0, 1]$, and independent from $(W^{(j)})_{0 \le j \le d}$.

Let F be the c.d.f. of W.
Note that dependence is only measured through the parameter p.
Here, we assume that $\mathbf{a} = d^{-1}\mathbf{1}$.

Lemma 4.3.1 *Let $\mathbf{X_1} \in \mathcal{MR}_{-\alpha,\mu_1}$ for some $\alpha > 0$ and some Radon measure μ_1. Let $\mathbf{X_2} \in \mathcal{MR}_{-\alpha,\mu_2}$ for same $\alpha > 0$ and some Radon measure μ_2. Moreover we assume that, for some function $L \in \mathcal{R}_{-\alpha}$, there exists $c_1, c_2 > 0$ such that, for large u*

$$P\left(|\mathbf{X_1}| > u\right) \sim c_1 u^{-\alpha} L(u) \ ,$$

and

$$P\left(|\mathbf{X_2}| > u\right) \sim c_2 u^{-\alpha} L(u) \ .$$

If $\mathbf{X_1}$ and $\mathbf{X_2}$ are independent, then $\mathbf{X_1} + \mathbf{X_2} \in \mathcal{MR}_{\left(-\alpha, \frac{c_1}{c_1+c_2}\mu_1 + \frac{c_2}{c_1+c_2}\mu_2\right)}$.

Proof. Let $A \in \mathbb{R}^d$ be a Borel set bounded away from $\mathbf{0}$ For $i = 1, 2$, $\mathbf{X_i} \in \mathcal{MR}_{-\alpha,\mu_i}$, so from Definition 4.2.3

$$\lim_{u\to\infty} \frac{P\left(\mathbf{X_i} \in uA\right)}{P(|\mathbf{X_i}| > u)} = \mu_i(A) \ . \tag{4.1}$$

Since $P\left(|\mathbf{X_i}| > u\right) \sim c_i u^{-\alpha} L(u)$,

$$\lim_{u\to\infty} u^{\alpha} \tilde{L}(u) P\left(\mathbf{X_i} \in uA\right) = c_i \mu_i(A) \ ,$$

with $\tilde{L} = 1/L \in \mathcal{R}_0$. So from Hult and Lindskog (2006b) Proposition A.1,

$$\lim_{u\to\infty} u^{\alpha} \tilde{L}(u) P\left(\mathbf{X_1} + \mathbf{X_2} \in uA\right) = c_1\mu_1(A) + c_2\mu_2(A) \ .$$

By independence and regular variation,

$$P\left(|\mathbf{X_1} + \mathbf{X_2}| > u\right) \sim P\left(|\mathbf{X_1}| > u\right) + P\left(|\mathbf{X_2}| > u\right) \sim (c_1 + c_2) u^{-\alpha} L(u) \ .$$

Hence,

$$\lim_{u\to\infty} \frac{P\left(\mathbf{X_1} + \mathbf{X_2} \in uA\right)}{P(|\mathbf{X_1} + \mathbf{X_1}| > u)} = \frac{c_1}{c_1 + c_2}\mu_1(A) + \frac{c_2}{c_1 + c_2}\mu_2(A) \ ,$$

and the result follows. $\diamond$

Proposition 4.3.2 *Under the assumptions of this subsection, we have, for $T > 0$ and large u,*

$$\psi_{d,\beta}(u,T) \sim \Bigg\{ (1-p)^d d\left((d-1)\beta + 1\right)^{-\alpha} + \\ \sum_{k=1}^{d} \binom{d}{k} p^k (1-p)^{d-k} \left[\left(\left(\frac{d-k}{k} \right) \beta + 1 \right)^{-\alpha} + (d-k)\left((d-1)\beta + 1\right)^{-\alpha} \right] \Bigg\} d^{\alpha} (\lambda T) \overline{F}(u).$$

Proof. Since $W \in \mathcal{R}_{-\alpha}$, there exists slowly varying function L such that, for $u > 0$, $P(W > u) = u^{-\alpha} L(u)$.
For $k = 1, \ldots, d-1$, let

$$\mathbf{X^{(k)}} = W^{(0)} \mathbf{1}_k + \sum_{i=k+1}^{d} W^{(i)} \mathbf{e}_i.$$

Let $\mathbf{X}^{(\mathbf{0})} = \sum_{i=1}^{d} W^{(i)}\mathbf{e}_i$ and $\mathbf{X}^{(d)} = W^{(0)}\mathbf{1}$.
Since for $k = 1, \ldots, d-1$, $P\left(|W^{(0)}\mathbf{1}_k| > u\right) \sim |\mathbf{1}_k|^{\alpha} P(W > u)$ and for $i = 1, \ldots, d$, $P\left(|W^{(i)}\mathbf{e}_i| > u\right) \sim P(W > u)$, we have, from Lemma 4.3.1, for $k = 1, \ldots, d-1$, $\mathbf{X}^{(\mathbf{k})} \in \mathcal{MR}_{-\alpha,\mu_k}$ with, for all Borel set $B \in \mathbb{R}^d$ bounded away from $\mathbf{0}$

$$\mu_k(B) = \frac{|\mathbf{1}_k|^{\alpha}\tilde{\mu}_{0,k}(B) + (d-k)\tilde{\mu}_1(B)}{|\mathbf{1}_k|^{\alpha} + (d-k)} ,$$

where, from Lemma 4.2.6 Case 2,

$$\tilde{\mu}_{0,k}(B) = \lim_{u\to\infty} \frac{P\left(W\mathbf{1}_k \in uB\right)}{P\left(|W\mathbf{1}_k| > u\right)} = \left(d^{-1}\left(\frac{\beta(d-k)}{k} + 1\right)|\mathbf{1}_k|\right)^{-\alpha} ,$$

and, from Lemma 4.2.6 Case 1 with $a^{(1)} = d^{-1}$,

$$\tilde{\mu}_1(B) = \lim_{u\to\infty} \frac{P\left(W\mathbf{e}_1 \in uB\right)}{P\left(|W\mathbf{e}_1| > u\right)} = \left(\beta + d^{-1}(1-\beta)\right)^{-\alpha} .$$

Moreover, we have ,

$$\mu_0(B) = \tilde{\mu}_1(B) ,$$

and from Lemma 4.2.6 Case 2 with $k = d$,

$$\mu_d(B) = \tilde{\mu}_{0,d}(B) = \lim_{u\to\infty} \frac{P\left(W\mathbf{1} \in uB\right)}{P\left(|W\mathbf{1}| > u\right)} = d^{\alpha}|1|^{-\alpha} .$$

By construction $\mathbf{X}$ is composed of a sum of random variables of the form

$$\mathbf{X}_{\Delta} = \left\{\left[W^{(0)}\sum_{i\in\Delta}\mathbf{e}_i\right] + \sum_{i\in\{1,\ldots,d\}\setminus\Delta}\left[W^{(i)}\mathbf{e}_i\right]\right\}\mathbb{1}_{\left\{\cap_{i\in\Delta}\{I^{(i)}=1\}\bigcup\cup_{i\in\{1,\ldots,d\}\setminus\Delta}\{I^{(i)}=0\}\right\}}$$

for some subset Δ of $\{1,\ldots,d\}$. For all subset Δ of $\{1,\ldots,d\}$, $P(|\mathbf{X}_{\Delta}| > u) \sim c_{\Delta}u^{-\alpha}L(u)$ for some constant c_{Δ}.
Thus, from Lemma 4.3.1 $\mathbf{X} \in MR_{-\alpha,\mu}$ with , for all Borel set $B \in \mathbb{R}^d$ bounded away from $\mathbf{0}$,

$$\mu(B) = \lim_{u\to\infty} \frac{P(\mathbf{X} \in uB)}{P(|\mathbf{X}| > u)} .$$

Let $A = d^{-1}\mathbf{1} - F_{\beta}$. Note that A is symmetric in each direction.
Denote by M the random variable which counts the number of random variables equal to $W^{(0)}$ in $\mathbf{X}$. We have, for large u,

$$\begin{aligned}
\mu(A)P\left(|\mathbf{X}| > u\right) &\sim P\left(\mathbf{X} \in uA\right) \\
&\sim \sum_{k=0}^{d} P(M=k)P\left(\mathbf{X} \in uA | M=k\right) \\
&\sim \sum_{k=0}^{d}\binom{d}{k}p^k(1-p)^{d-k}P\left(\mathbf{X}^{(\mathbf{k})} \in uA\right) \text{ (A is symmetric in each direction)}, \\
&\sim \sum_{k=0}^{d}\binom{d}{k}p^k(1-p)^{d-k}\frac{P\left(\mathbf{X}^{(\mathbf{k})} \in uA\right)}{P\left(|\mathbf{X}^{(k)}| > u\right)}P\left(|\mathbf{X}^{(k)}| > u\right) , \\
&\sim \sum_{k=0}^{d}\binom{d}{k}p^k(1-p)^{d-k}\mu_k(A)P\left(|\mathbf{X}^{(k)}| > u\right) .
\end{aligned}$$

Since from Proposition 4.2.5, we have for $T > 0$ and large u

$$\psi_{d,\beta}(u,T) \sim (\lambda T)\mu(d^{-1}\mathbf{1} - F_\beta)P\left(|\mathbf{X}| > u\right),$$

we get the result. $\diamond$

Corollary 4.3.3 *When no transfer is allowed ($\beta = 0$), we get, for large u and $T > 0$,*

$$\psi_{d,0}(u,T) \sim \left\{ d(1-p)^d + \sum_{k=1}^{d} \binom{d}{k} p^k (1-p)^{d-k}(d-k+1) \right\} d^\alpha (\lambda T)\overline{F}(u)\,.$$

This result corresponds to ψ_{or} (4.6).

Corollary 4.3.4 *When transfer is allowed without restriction ($\beta = 1$), we get, for large u and $T > 0$,*

$$\psi_{d,1}(u,T) \sim \left\{ \sum_{k=0}^{d} \binom{d}{k} p^k (1-p)^{d-k} \left[k^\alpha + (d-k)\right] \right\} (\lambda T)\overline{F}(u)\,.$$

This result corresponds to ψ_{sum} (4.4).

4.3.2 A Poisson shock model

In the Subsection, we study a classical Poisson shock model; when a claim occurs, it may affect either one specific line of business or all the lines. Explicitly, let X be a non-negative random variable which is regularly varying with some $\alpha > 1$. For $1 \le j \le d$, we assume that the specific claims of the business line j arrive at the jump times of a Poisson process $(N_t^j)_{t\ge 0}$ with intensity $\lambda^{(j)}$. Let X_k^j be the kth specific claim amount of line j. Assume that, for $1 \le j \le d$ and $k \ge 1$, $(X_k^j)_{k\ge 1}$ is an i.i.d. sequence with common distribution X. Thus, the specific aggregate claim amount process of the line j is, for $t \ge 0$,

$$S_t^j = \sum_{k=1}^{N_t^j} X_k^j\,.$$

We assume that the claims which affect all the lines of business arrive at the jump times of a Poisson process $(N_t^0)_{t\ge 0}$ with intensity $\lambda^{(0)}$. Let $\mathbf{X}_k^0 = \mathbf{1}X_k^0$ be the vector of the kth claim amounts of this kind. We assume again that $(X_k^0)_{k\ge 1}$ is an i.i.d. sequence with common distribution X. Here, for simplification, we have assumed that all the lines of business pay the same amount for a common claim. Thus, the common aggregate claim amount process is, for $t \ge 0$,

$$\mathbf{S}_t^0 = \sum_{k=1}^{N_t^0} \mathbf{X}_k^0\,.$$

We also assume that all $(N_t^j)_{t\ge 0}$ and $(X_k^j)_{k\ge 1}$, $j \in \{0,\ldots,d\}$ are independent. From the compound Poisson process properties, we are able to get a risk process of the type of (4.3), since we can write

$$\mathbf{S}_t = \sum_{k=1}^{N_t^0} \mathbf{X}_k^0 + \sum_{j=1}^{d} \sum_{k=1}^{N_t^j} X_k^j \mathbf{e}_j = \sum_{k=1}^{N_t} \mathbf{X}_k\,,$$

where $N(t) = \sum_{j=0}^d N_t^j$ is a Poisson process with intensity $\overline{\lambda} = \lambda^{(0)} + \lambda^{(1)} + \ldots + \lambda^{(d)}$, and $\mathbf{X}_k = \mathbf{X}_k^0 \delta_0(\xi_k) + \sum_{j=1}^d X_k^j \mathbf{e}_j \delta_j(\xi_k)$ with, $(\xi_k)_{k\ge 1}$ an i.i.d. sequence of random variables independent of all others random variables and with $P(\xi_k = j) = \lambda^{(j)}/\overline{\lambda}$ for $k \ge 1$ and $0 \le j \le d$.

Proposition 4.3.5 *Under the above assumptions, we have, for $T > 0$ and large u,*

$$\psi_{d,\beta}(u,T) \sim \left\{ \lambda^{(0)} \left[\frac{\sum_{j=1}^{k^*} a^{(j:d)} + \beta \sum_{j=k^*+1}^{d} a^{(j:d)}}{k^* + \beta(d-k^*)} \right]^{-\alpha} + \sum_{j=1}^{d} \lambda^{(j)} \left[\beta + a^{(j)}(1-\beta) \right]^{-\alpha} \right\} T\overline{F}(u) ,$$

where for $1 \le j \le d$, $a^{(j:d)}$ is jth larger component of $\mathbf{a}$ and

$$k^* = \inf \left\{ k \in [1, d-1] : a^{(k+1:d)} > \frac{\sum_{j=1}^{k} a^{(j:d)} + \beta \sum_{j=k+1}^{d} a^{(j:d)}}{k + \beta(d-k)} \right\} .$$

Proof. From Proposition 4.2.5, we have for $T > 0$ and large u

$$\psi_{d,\beta}(u,T) \sim (\overline{\lambda} T)\mu(\mathbf{a} - F_\beta) P\left(|\mathbf{X}| > u\right) .$$

Let $A = \mathbf{a} - F_\beta$.
From Hult and Lindskog (2006a), Section 4, we have

$$P\left(|\mathbf{X}| > u\right) \sim \left(\frac{\lambda^{(0)}}{\overline{\lambda}} (|\mathbf{1}_2|^\alpha - 1) + 1 \right) P(X > u) ,$$

and

$$\mu(A) = \frac{\lambda^{(0)} |\mathbf{1}|^\alpha \mu_0(A) + \sum_{j=1}^{d} \lambda^{(j)} \mu_j(A)}{\lambda^{(0)} \left(|\mathbf{1}|^\alpha - 1\right) + \overline{\lambda}}$$

where

$$\mu_0(A) = \lim_{u \to \infty} \frac{P(X\mathbf{1} \in uA)}{P(|X\mathbf{1}| > u)} ,$$

and, for $1 \le j \le d$,

$$\mu_j(A) = \lim_{u \to \infty} \frac{P(X\mathbf{e}_j \in uA)}{P(|X\mathbf{e}_j| > u)} .$$

We get the result using 4.2.6, Case 1 and Case 3. $\diamond$

Corollary 4.3.6 *When no transfer is allowed ($\beta = 0$), we get, for large u and $T > 0$,*

$$\psi_{d,0}(u,T) \sim \left\{ \lambda^{(0)} \left[\min_{1 \le j \le d} \{a^{(j)}\} \right]^{-\alpha} + \sum_{j=1}^{d} \lambda^{(j)} \left[a^{(j)} \right]^{-\alpha} \right\} T\overline{F}(u) .$$

This result corresponds to ψ_{or} (4.6).

Corollary 4.3.7 *When transfer is allowed without restriction ($\beta = 1$), we get, for large u and $T > 0$,*

$$\psi_{d,1}(u,T) \sim \left\{ \lambda^{(0)} (d^\alpha - 1) + 1 \right\} T\overline{F}(u) .$$

This result corresponds to ψ_{sum} (4.4).

4.4 Optimal allocation problems

Throughout this Section, we assume $\beta = 0$. So we write $\psi_{d,\beta=0} = \psi_d$. This case corresponds to the probability that at least one of the line business becomes negative before T without money transfer.

In this section, we suppose that the company owns a global initial reserve u to allocate to the d lines of business in order to minimize its finite-time ruin probability. Explicitly, we have the following optimal problem :

$$\begin{cases} \min\limits_{\mathbf{a}\in(0,1)^d} \psi_d(u,T) \ , \\ \text{under the constraint } a^{(1)} + \ldots + a^{(d)} = 1 \ . \end{cases} \tag{4.1}$$

Here, we are going to minimize the asymptotics of $\psi_d(u,T)$ we denote by $\tilde{\psi}_d(u,T)$. Thus the problem (4.1) becomes :

$$\begin{cases} \min\limits_{\mathbf{a}\in(0,1)^d} \tilde{\psi}_d(u,T) \ , \\ \text{under the constraint } a^{(1)} + \ldots + a^{(d)} = 1 \ . \end{cases} \tag{4.2}$$

We investigate the following cases.

- **Case 1 :** the company is composed of d lines of business and they are mutually independent; explicitly, the model is the Subsection 4.3.2 one with $\lambda^{(0)} = 0$.
- **Case 2 :** the company is composed of two lines of business and their dependence structure is described by the Poisson shock model of Subsection 4.3.2.
- **Case 3 :** the company is composed of three lines of business, one is independent from the others and the two others are dependent via the Poisson shock model of Subsection 4.3.2.

4.4.1 Case 1

In this Subsection, we start with the model of Subsection 4.3.2 wherein $\lambda^{(0)}$ is assumed to be equal to zero. That is to say that $\mathbf{X}$ is composed with d mutually independent random variables, so the business lines are mutually independent too.

Proposition 4.4.1 *Under the above assumptions, we have for $T > 0$ and large u,*

$$\psi_d(u,T) \sim \left\{ \sum_{j=1}^{d} \lambda^{(j)} \left[a^{(j)}\right]^{-\alpha} \right\} T\overline{F}(u) \ .$$

Proof. Take $\lambda^{(0)} = 0$ in Corollary 4.3.6. $\diamond$

The following proposition gives the optimal allocation of our optimization problem (4.2).

Proposition 4.4.2 *Under the assumptions of the Subsection 4.4.1, the solution of (4.2) is, for all $1 \leq i \leq d$,*

$$a^{(i)*} = \left\{ \frac{\lambda^{(i)\,\frac{1}{\alpha+1}}}{\sum_{j=1}^{d} \lambda^{(j)\,\frac{1}{\alpha+1}}} \right\} \ .$$

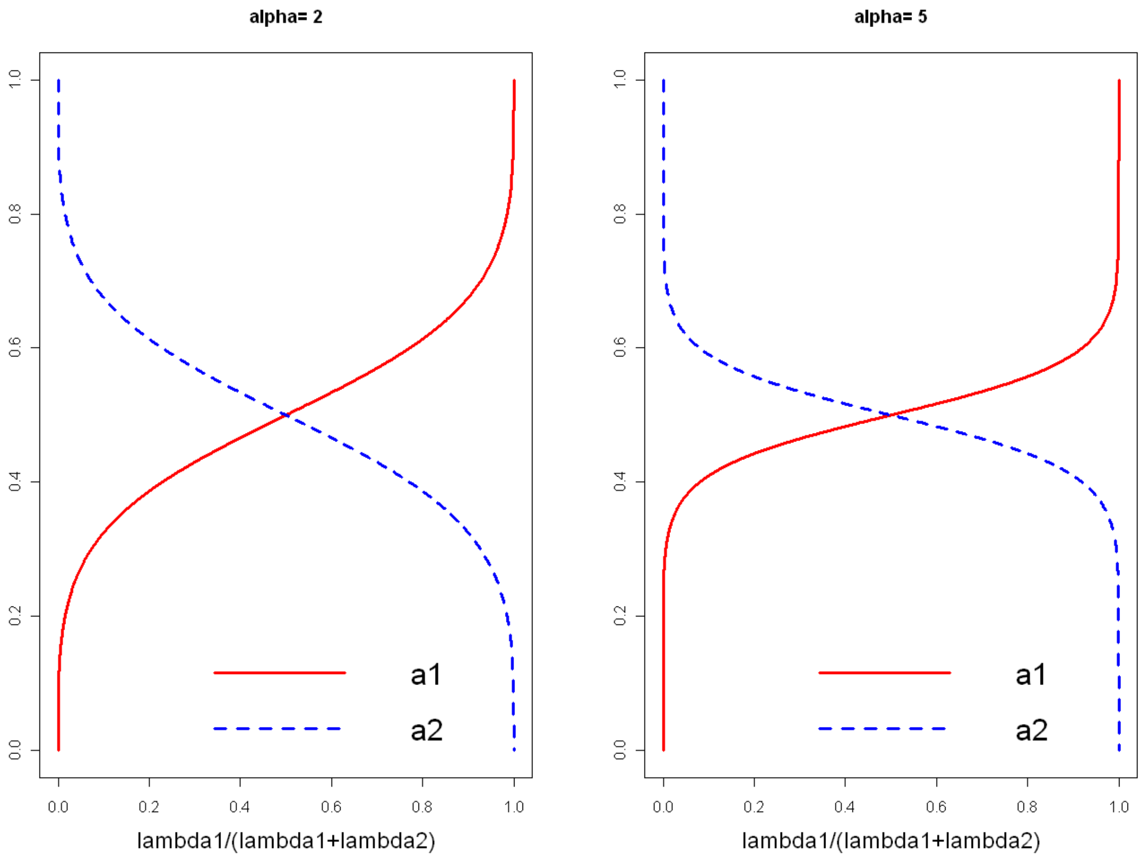

Figure 4.2: Optimal solution of the Case 1 for $d = 2$

Proof. Let $g : \mathbf{a} \in (0,1)^d \mapsto g(\mathbf{a}) = \sum_{i=1}^{d} \lambda^{(i)} \left[a^{(i)}\right]^{-\alpha}$. g is a continuous, differentiable and strictly convex function on $(0,1)^d$. Using the method of Lagrange multipliers, we find a $\mathbf{a}^*$ which minimizes g on $\{(a^{(1)}, \ldots, a^{(d)}) \in (0,1)^d \ , \ a^{(1)} + \ldots + a^{(d)} = 1\}$. Since g is strictly convex, on the non empty open convex set $\Omega = \{a^{(1)} + \ldots + a^{(d)} = 1\}$ this minimum is unique. $\diamond$

In Figure 4.2, for $d = 2$, we represent, $a^{(1)}$ and $a^{(2)}$ as a function of $\lambda^{(1)}/\overline{\lambda}$ ($\lambda^{(1)}/\overline{\lambda}$ varies from 0 to 1). Both cases $\alpha = 2$ and $\alpha = 5$ are plotted.

4.4.2 Case 2

In this Subsection, we investigate the two dimensional case wherein the dependence structure is described by the model of Subsection 4.3.2. Let $a \in (0,1)$ such that $a^{(1)} = a$ and $a^{(2)} = 1 - a$.

Proposition 4.4.3 *Under the above assumptions, we have for $T > 0$ and large u,*

$$\psi_2(u,T) \sim \left\{\lambda^{(0)} \left[\min(a; 1-a)\right]^{-\alpha} + \lambda^{(1)} a^{-\alpha} + \lambda^{(2)} (1-a)^{-\alpha}\right\} T\overline{F}(u) \ .$$

Proof. Take $d = 2$ in Corollary 4.3.6. $\diamond$

Proposition 4.4.4 *Under the assumptions of the Subsection 4.4.2, the solution of (4.2) is*

$$a* = \begin{cases} \frac{1}{2} & \text{if } \lambda^{(0)} > |\lambda^{(1)} - \lambda^{(2)}| \,, \\ \frac{\lambda^{(1)}\frac{1}{\alpha+1}}{\lambda^{(1)}\frac{1}{\alpha+1} + \left(\lambda^{(0)}+\lambda^{(2)}\right)\frac{1}{\alpha+1}} & \text{if } \lambda^{(0)} \leq \lambda^{(1)} - \lambda^{(2)} \,, \\ \frac{\left(\lambda^{(0)}+\lambda^{(1)}\right)\frac{1}{\alpha+1}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)\frac{1}{\alpha+1} + \lambda^{(2)}\frac{1}{\alpha+1}} & \text{if } \lambda^{(0)} \leq \lambda^{(2)} - \lambda^{(1)} \,. \end{cases}$$

Proof. Let, for $0 < a \leq 1/2$

$$g_1(a) = (\lambda^{(0)} + \lambda^{(1)})a^{-\alpha} + \lambda^{(2)}[1-a]^{-\alpha} \,,$$

and for $1/2 \geq a < 1$

$$g_2(a) = (\lambda^{(0)} + \lambda^{(2)})\,[1-a]^{-\alpha} + \lambda^{(1)}a^{-\alpha} \,.$$

g_1 (resp. g_2) is differentiable and strictly convex on $(0, 1/2)$ (resp. $(1/2, 1)$). Moreover $g_1(1/2) = g_2(1/2)$ for all $a_1 \in (0, 1/2)$ and $a_2 \in (1/2, 1)$; $g'(a_1) < g'(a_2)$. Let

$$g(a) = \begin{cases} g_1(a) & 0 < x \leq 1/2 \\ g_2(a) & 1/2 < a < 1 \end{cases} .$$

Thus, g is continuous on $(0, 1)$ and g' is strictly increasing on $(0, 1/2) \cup (1/2, 1)$. So, g is strictly convex on $(0, 1)$. As a consequence, on the non empty open convex set $(0, 1)$, there exists a unique a^* which minimizes g. Since $g(0) = g(1) = +\infty$, we have

$$a^* = \begin{cases} \underset{(0,1/2)}{\arg\min}\, g_1 & \text{if } g_1'(1/2) > 0 \,, \\ \underset{(1/2,1)}{\arg\min}\, g_2 & \text{if } g_2'(1/2) < 0 \,, \\ 1/2 & \text{if } g_1'(1/2) < 0 \text{ and } g_2'(1/2) > 0 \,, \end{cases}$$

that is to say

$$a^* = \begin{cases} a_1^* \;/\; g_1'(a_1^*) = 0 & \text{if } g_1'(1/2) > 0 \,, \\ a_2^* \;/\; g_2'(a_2^*) = 0 & \text{if } g_2'(1/2) < 0 \,, \\ 1/2 & \text{if } g_1'(1/2) < 0 \text{ and } g_2'(1/2) > 0 \,. \end{cases}$$

Since $\arg\min g = \arg\min \tilde{\psi}_{2,\beta}$, we get the result. $\diamond$

Remark 4.4.5 *There are three different forms of the optimal allocation in Proposition 4.4.4. When $\lambda^{(0)}$ is large, it conduces to minimize the comonotonic part, so we allocate half of the reserve to each line. When $\lambda^{(1)}$ is large, the optimal solution is the same as in the case where the two lines of business are independent and where $\lambda^{(2)}$ is switched with $\lambda^{(2)} + \lambda^{(0)}$. We have also the symetric case.*

In Figures 4.3, 4.4 and 4.5, for $d = 2$, we represent, $a^{(1)}$ and $a^{(2)}$ as a function of $\lambda^{(1)}/\overline{\lambda}$ (we fix $\lambda^{(0)}$ and $\lambda^{(1)}/\overline{\lambda}$ varies from 0 to $1 - \lambda^{(0)}$). in the three figures, both cases $\alpha = 2$ and $\alpha = 5$ are plotted. Figures 4.3, 4.4, and 4.5 represent respectively cases $\lambda^{(0)} = 0.1$, $\lambda^{(0)} = 0.3$ and $\lambda^{(0)} = 0.5$. In Figure 4.2, for $d = 2$, we represent, $a^{(1)}$ and $a^{(2)}$ as a function of $\lambda^{(1)}/\overline{\lambda}$ ($\lambda^{(1)}/\overline{\lambda}$ varies from 0 to 1). In Figure 4.2, for $d = 2$, we represent, $a^{(1)}$ and $a^{(2)}$ as a function of $\lambda^{(1)}/\overline{\lambda}$ ($\lambda^{(1)}/\overline{\lambda}$ varies from 0 to 1).

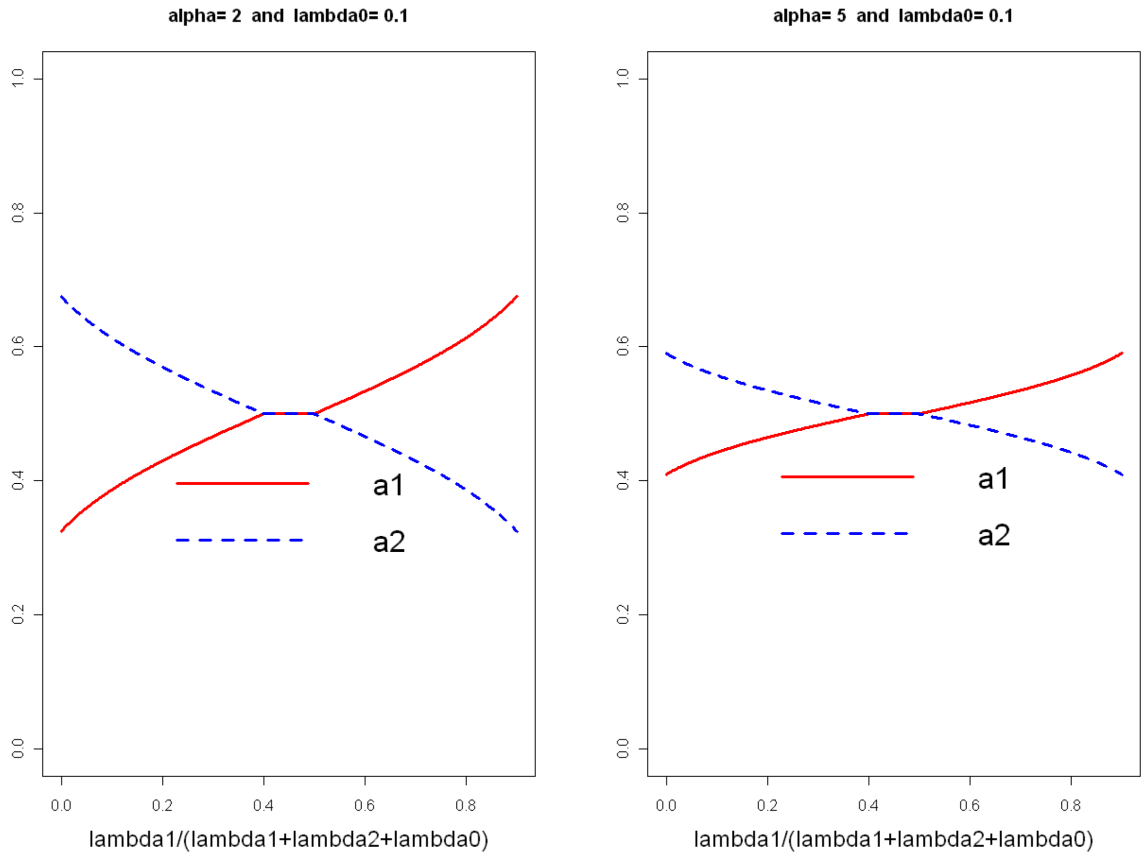

Figure 4.3: Optimal solution of the Case 2 for $d = 2$ and $\lambda^{(0)} = 0.1$

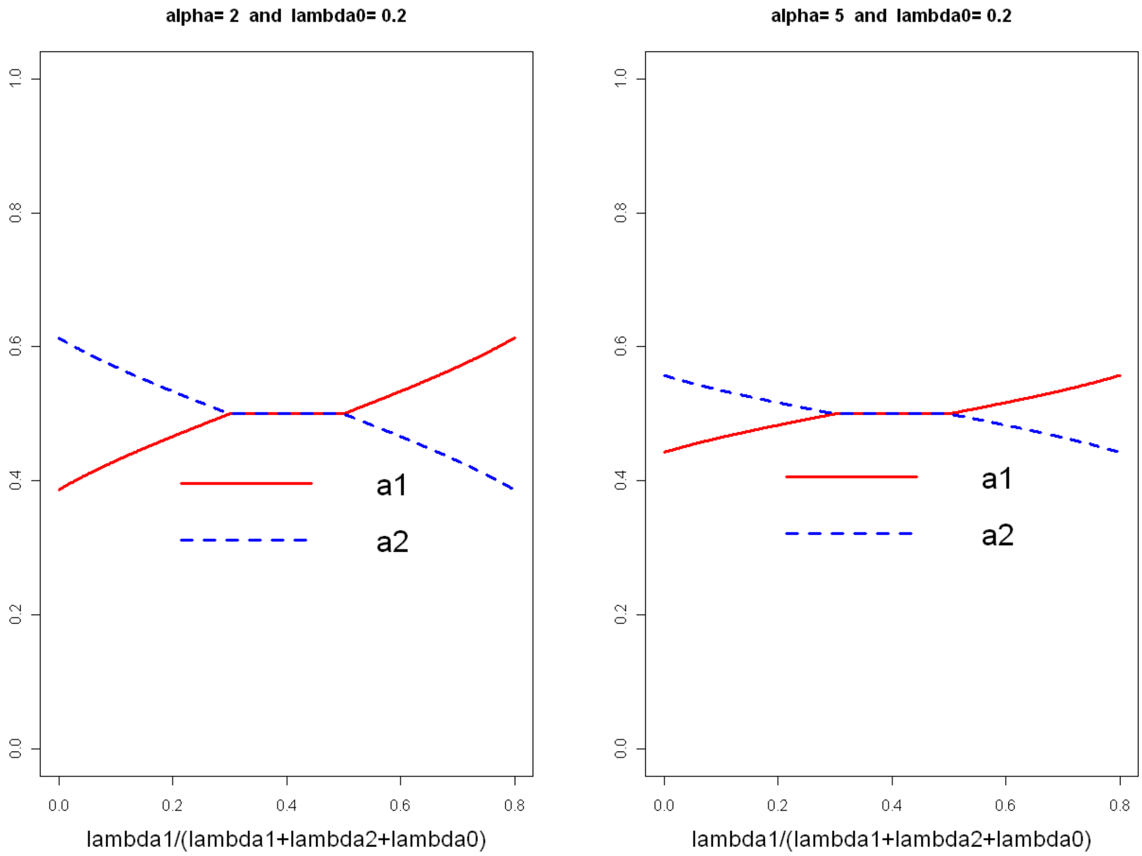

Figure 4.4: Optimal solution of the Case 2 for $d = 2$ and $\lambda^{(0)} = 0.3$

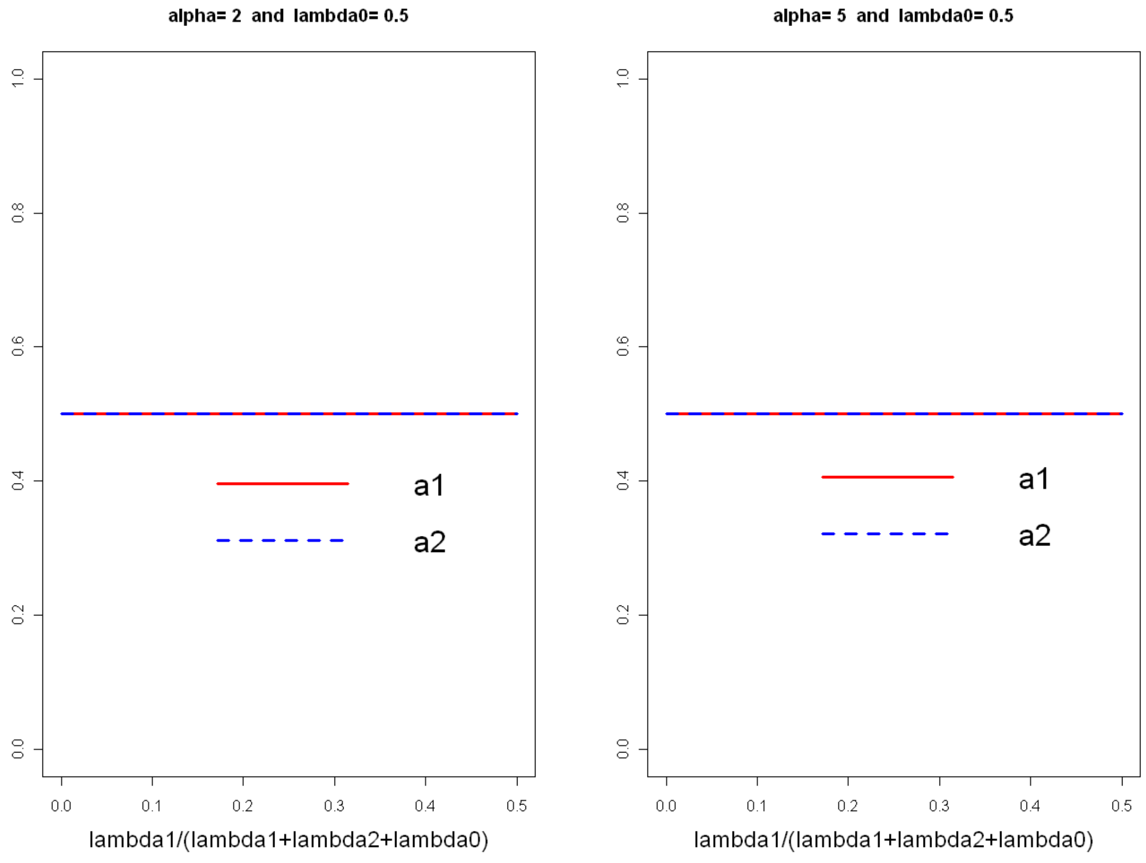

Figure 4.5: Optimal solution of the Case 2 for $d = 2$ and $\lambda^{(0)} = 0.5$

4.4.3 Case 3

In this Subsection, we assume that the insurance company has three lines of business, two dependent through the common shock model of Subsection 4.3.2, and one independent from the two others. Explicitly, we have (with a simple adaptation of the Subsection 4.3.2 model) :

$$\mathbf{S}_t = \sum_{k=1}^{N(t)} \mathbf{X}_k \ ,$$

where $N(t)$ is a Poisson process with intensity $\overline{\lambda} = \lambda^{(0)} + \lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)}$, and $\mathbf{X}_k = X_k^0 \mathbf{1}_2 \delta_0(\xi_k) + \sum_{i=1}^3 X_k^i \mathbf{e}_i \delta_i(\xi_k)$ with,

- $\lambda_i > 0 \ \ , \ 0 \le i \le 3$,
- for $0 \le i \le 3$, $(X_k^i)_{k \ge 1}$ is an i.i.d sequence with common distribution $X \in \mathcal{R}_{-\alpha}$,
- and with all $(X_k^i)_{k \ge 1}$, $0 \le i \le 3$ independent and independent from $N(t)$,
- $(\xi_k)_{k \ge 1}$ an i.i.d. sequence of random variables independent of all others random variables and with $P(\xi_k = i) = \lambda^{(i)} / \overline{\lambda}$ for $k \ge 1$ and $0 \le i \le 3$.

Denote by $\psi_{\tilde{3}}$ the ruin probability associated with the above model.

Proposition 4.4.6 *Under the above assumptions, we have, for $T > 0$ and large u,*

$$\psi_{\tilde{3}}(u,T) \sim \left\{ \lambda^{(0)} \left[\min(a^{(1)}; a^{(2)}) \right]^{-\alpha} + \lambda^{(1)} a^{(1)^{-\alpha}} + \lambda^{(2)} a^{(2)^{-\alpha}} + \lambda^{(3)} a^{(3)^{-\alpha}} \right\} T \overline{F}(u) \ .$$

Proof. Let $A = \mathbf{a} - F_\beta$.
We have, for large u

$$P(|\mathbf{X}| > u) \sim \frac{\lambda^{(0)}}{\overline{\lambda}} P(|X^0 \mathbf{1}_2| > u) + \sum_{i=1}^{3} \frac{\lambda^{(i)}}{\overline{\lambda}} P(|X^i \mathbf{e}_i| > u) \sim \left(\frac{\lambda^{(0)}}{\overline{\lambda}} (|\mathbf{1}_2|^\alpha - 1) + 1 \right) P(X > u) \ .$$

Moreover $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ with

$$\mu(A) = \frac{\lambda^{(0)} |\mathbf{1}_2|^\alpha \mu_{(1,2)}(A) + \sum_{i=1}^3 \lambda^{(i)} \mu_i(A)}{\lambda^{(0)} (|\mathbf{1}_2|^\alpha - 1) + \overline{\lambda}} \ ,$$

where

$$\mu_{(1,2)}(A) = \lim_{u \to \infty} \frac{P(X \mathbf{1}_2 \in uA)}{P(|X \mathbf{1}_2| > u)} \ ,$$

and, for $1 \le j \le d$,

$$\mu_j(A) = \lim_{u \to \infty} \frac{P(X \mathbf{e}_j \in uA)}{P(|X \mathbf{e}_j| > u)} \ .$$

We get the result using 4.2.6, Case 2 and Case 3. $\diamond$

Proposition 4.4.7 *Under the assumptions of the Subsection 4.4.3, the solution of (4.2) is following.*

- *If* $\lambda^{(0)} > |\lambda^{(1)} - \lambda^{(2)}|$, *then*

$$\begin{cases} a^{(1)^*} = a^{(2)^*} = \frac{1}{2} \frac{\left(2^\alpha\left(\lambda^{(0)}+\lambda^{(1)}+\lambda^{(2)}\right)\right)^{\frac{1}{\alpha+1}}}{\left(2^\alpha\left(\lambda^{(0)}+\lambda^{(1)}+\lambda^{(2)}\right)\right)^{\frac{1}{\alpha+1}}+\lambda^{(3)\frac{1}{\alpha+1}}}, \\ a^{(3)^*} = \frac{\lambda^{(3)\frac{1}{\alpha+1}}}{\left(2^\alpha\left(\lambda^{(0)}+\lambda^{(1)}\right)\right)^{\frac{1}{\alpha+1}}+\lambda^{(3)\frac{1}{\alpha+1}}}. \end{cases}$$

- *If* $\lambda^{(0)} \le \lambda^{(1)} - \lambda^{(2)}$, *then*

$$\begin{cases} a^{(1)^*} = \frac{\lambda^{(1)\frac{1}{\alpha+1}}}{\lambda^{(1)\frac{1}{\alpha+1}}+\left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}}+\lambda^{(3)\frac{1}{\alpha+1}}}, \\ a^{(2)^*} = \frac{\left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}}}{\lambda^{(1)\frac{1}{\alpha+1}}+\left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}}+\lambda^{(3)\frac{1}{\alpha+1}}}, \\ a^{(3)^*} = \frac{\lambda^{(3)\frac{1}{\alpha+1}}}{\lambda^{(1)\frac{1}{\alpha+1}}+\left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}}+\lambda^{(3)\frac{1}{\alpha+1}}}. \end{cases}$$

- *If* $\lambda^{(0)} \le \lambda^{(2)} - \lambda^{(1)}$, *then*

$$\begin{cases} a^{(1)^*} = \frac{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}}+\lambda^{(2)\frac{1}{\alpha+1}}+\lambda^{(3)\frac{1}{\alpha+1}}}, \\ a^{(2)^*} = \frac{\lambda^{(2)\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}}+\lambda^{(2)\frac{1}{\alpha+1}}+\lambda^{(3)\frac{1}{\alpha+1}}}, \\ a^{(3)^*} = \frac{\lambda^{(3)\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}}+\lambda^{(2)\frac{1}{\alpha+1}}+\lambda^{(3)\frac{1}{\alpha+1}}}. \end{cases}$$

Proof. Fix $a^{(3)} \in (0,1)$. Let, for $a^{(1)} \in (0, 1-a_3)$,

$$g_1(a^{(1)}) = \lambda^{(0)} \min(a^{(1)}, 1-a^{(3)}-a^{(1)})^{-\alpha} + \lambda^{(1)} a^{(1)^{-\alpha}} + \lambda^{(2)}(1-a^{(3)}-a^{(2)})^{-\alpha} + \lambda^{(3)} a^{(3)^{-\alpha}}.$$

Using the same way as in the proof of Proposition 4.4.4, we get $a^{(1)^*} = g_3(a^3) = \underset{(0,1)}{\arg\min}\, g_1$. Then $a^{(3)^*} = \underset{(0,1)}{\arg\min}\, g_3$ and we get the result. $\diamond$

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Conclusion

Dans cette thèse, nous avons utilisé la théorie développée autour des distributions à variations régulières pour obtenir des équivalents des critères de risque tels que la probabilité de ruine, univariée et multivariée ou l'espérance de l'intégrale temporelle de la partie négative du processus de risque. Ces équivalents sont au premier ordre et il serait intéressant d'obtenir des ordres supérieurs pour vérifier la qualité des approximations. Nous avons aussi présenté des modèles où les hypothèses fortes d'indépendance et de stationnarité sont relâchées. Cette voie là est à développer pour des travaux futurs surtout dans la modélisation de la dépendance dynamique. La théorie de la ruine multivariée est encore peu étudiée dans la littérature et ouvre un champ d'étude assez vaste. Nous pouvons citer comme axes de recherche la modélisation de la dépendance entre les branches, les problèmes d'allocation optimale et la mesure du risque. Ces thèmes ont tous été abordés dans la thèse et pourront faire l'objet de futurs travaux.

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Dépendance et événements extrêmes en théorie de la ruine : étude univariée et multivariée, problèmes d'allocation optimale

Résumé

Cette thèse présente de nouveaux modèles et de nouveaux résultats en théorie de la ruine, lorsque les distributions des montants de sinistres sont à queue épaisse. Les hypothèses classiques d'indépendance et de stationnarité, ainsi que l'analyse univariée sont parfois jugées trop restrictives pour décrire l'évolution complexe des réserves d'une compagnie d'assurance. Dans un contexte de dépendance entre les montants de sinistres, des équivalents de la probabilité de ruine univariée en temps fini sont obtenus. Cette dépendance, ainsi que les autres paramètres du modèle sont modulés par un processus Markovien d'environnement pour prendre en compte des possibles crises de corrélation. Nous introduisons ensuite des modèles de dépendance entre les montants de sinistres et les temps inter-sinistres pour des risques de type tremblements de terre et inondations. Dans un cadre multivarié, nous présentons divers critères de risques tels que la probabilité de ruine multivariée ou l'espérance de l'intégrale temporelle de la partie négative du processus de risque. Nous résolvons des problèmes d'allocation optimale pour ces différentes mesures de risque. Nous étudions alors l'impact de la dangerosité des risques et de la dépendance entre les branches sur cette allocation optimale.

Mots-clés: théorie de la ruine, distribution à queue épaisse, dépendance spatio-temporelle, processus multivariés, allocation optimale.

Dependence and extreme events in ruin theory : univariate and multivariate study, optimal allocation problems

Abstract

This PhD thesis presents new models and new results in ruin theory, in the case where claim amounts are heavy-tailed distributed. Classical assumptions like independence and stationarity and univariate analysis are sometimes too restrictive to describe the complex evolution of the reserves of an insurance company. In a dependence context, asymptotics of univariate finite-time ruin probability are computed. This dependence, and the other model parameters are modulated by a Markovian environment process to take into account possible correlation crisis. Then, we introduce some models which describe dependence between claim amounts and claim interarrival times we can find in earthquake or flooding risks. In multivariate framework, we present some risk criteria like multivariate ruin probability or the expectation of the time integrated negative part of the risk process. We solve some problems of optimal allocation for these risk measures. Then, we study the impact of the risk dangerousness and of the dependence between lines on this optimal allocation.

Keywords: ruin theory, heavy-tailed distribution, spatio-temporal dependence, multivariate process, optimal allocation

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