SMALL EIGENVALUES OF THE LAPLACIAN FOR ALGEBRAIC MEASURES IN MODULI SPACE, AND MIXING PROPERTIES OF THE TEICHMÜLLER FLOW

ARTUR AVILA AND SÉBASTIEN GOUËZEL

Abstract. We consider the $\mathrm{SL}(2,\mathbb{R})$ action on moduli spaces of quadratic differentials. If μ is an $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure, crucial information about the associated representation on $L^2(\mu)$ (and in particular, fine asymptotics for decay of correlations of the diagonal action, the Teichmüller flow) is encoded in the part of the spectrum of the corresponding foliated hyperbolic Laplacian that lies in $(0,1/4)$ (which controls the contribution of the complementary series). Here we prove that the essential spectrum of an invariant algebraic measure is contained in $[1/4,\infty)$, i.e., for every $\delta>0$, there are only finitely many eigenvalues (counted with multiplicity) in $(0,1/4-\delta)$. In particular, all algebraic invariant measures have a spectral gap.

1. Introduction

For any lattice $\Gamma\subset\mathrm{SL}(2,\mathbb{R})$, the irreducible decomposition of the unitary representation of $\mathrm{SL}(2,\mathbb{R})$ on $L^2(\mathrm{SL}(2,\mathbb{R})/\Gamma)$ consists almost entirely of tempered representations (with fast decay of matrix coefficients): only finitely many non-tempered representations may appear, each with finite multiplicity. This corresponds to the well known result of Selberg (see, e.g., [Iwa95]) that in an hyperbolic surface of finite volume, the Laplacian has only finitely many eigenvalues, with finite multiplicity, in $(0,1/4)$. This has several remarkable consequences, for instance, on the asymptotics of the number of closed geodesics, the main error terms of which come from the small eigenvalues of the Laplacian (by Selberg's trace formula, see [Hej83]), or for the asymptotics of the correlations of smooth functions under the diagonal flow [Rat87].

For a more general ergodic action of $\mathrm{SL}(2,\mathbb{R})$, the situation can be much more complicated: in general, one may even not have a spectral gap ($\mathrm{SL}(2,\mathbb{R})$ does not have Kazhdan's property (T)). Even in the particularly nice situation of the $\mathrm{SL}(2,\mathbb{R})$ action on a homogeneous space G/Γ with G a semi-simple Lie group containing $\mathrm{SL}(2,\mathbb{R})$ and Γ an irreducible lattice in G (a most natural generalization the case $G=\mathrm{SL}(2,\mathbb{R})$ above), non-tempered representations may have a much heavier contribution: for instance, [KS09, Theorem 1] constructs examples (with $G=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)$) where the spectrum of the foliated (along $\mathrm{SO}(2,\mathbb{R})\backslash\mathrm{SL}(2,\mathbb{R})$ orbits) Laplacian on $\mathrm{SO}(2,\mathbb{R})\backslash G/\Gamma$ has an accumulation point in $(0,1/4)$. In fact, whether there is always a spectral gap at all remains an open problem for $G=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)$. While one does expect better behavior in the case where G has no compact factor, it too remains far from fully understood.

Moduli spaces of quadratic differentials present yet another natural generalization of $\mathrm{SL}(2,\mathbb{R})/\Gamma$, with different challenges. Let $g,n\geq 0$ with $3g-3+n>0$, let $\mathcal{M}_{g,n}$ be the moduli space of quadratic differentials on a genus one Riemann surface with n punctures, and with at most simple poles at the punctures (alternatively, it is the cotangent bundle of the moduli space of Riemann surfaces), and

Date: November 24, 2010.

let $\mathcal{M}^1_{g,n} \subset \mathcal{M}_{g,n}$ be the subspace of area one quadratic differentials. There is a natural $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{M}^1_{g,n}$, which has been intensively studied, not least because the corresponding diagonal action gives the Teichmüller geodesic flow. If $g = 0$ and $n = 4$ or if $g = 1$ and $n = 1$, $\mathcal{M}^1_{g,n}$ turns out to be of the form $\mathrm{SL}(2,\mathbb{R})/\Gamma$. In higher genus the $\mathcal{M}^1_{g,n}$ are not homogeneous spaces, and it is rather important to understand to which extent they may still behave as such.

Recall that $\mathcal{M}_{g,n}$ is naturally stratified by the "combinatorial data" of the quadratic differential q (order of zeros, number of poles, and whether or not q is a square of an Abelian differential). Each stratum has a natural complex affine structure, though it is not necessarily connected, the (finitely many) connected components having been classified by Kontsevich-Zorich [KZ03] and Lanneau [Lan08]. Each connected component $\mathcal{C}$ carries a unique (up to scaling) finite invariant measure μ which is $\mathrm{SL}(2,\mathbb{R})$ invariant and absolutely continuous with respect to $\mathcal{C} \cap \mathcal{M}^1_{g,n}$ (in case of the largest, "generic", stratum, which is connected, μ coincides with the Liouville measure in $\mathcal{M}^1_{g,n}$). Those measures were constructed, and shown to be ergodic, by Masur [Mas82] and Veech [Vee82]. In [AGY06] and [AR09], it is shown that for such a *Masur-Veech measure* μ the $\mathrm{SL}(2,\mathbb{R})$ action on $L^2(\mu)$ has a spectral gap.

There are many more ergodic $\mathrm{SL}(2,\mathbb{R})$ invariant measures beyond the Masur-Veech measures, which can be expected to play an important role in the analysis of non-typical $\mathrm{SL}(2,\mathbb{R})$ orbits (the consideration of non-typical orbits arises, in particular, when studying billiards in rational polygons). While all such measures have not yet been classified, it has been recently announced by Eskin and Mirzakhani that they are all "algebraic",[1] a result analogue to one of Ratner's Theorems (classifying $\mathrm{SL}(2,\mathbb{R})$ invariant measures in an homogeneous space [Rat92]). (For squares of Abelian differentials in $\mathcal{M}_{2,0}$, a stronger version of this result, including the classification of the algebraic invariant measures, was obtained earlier by McMullen [McM07].)

Let μ be an algebraic $\mathrm{SL}(2,\mathbb{R})$-invariant measure in some $\mathcal{M}^1_{g,n}$. Our goal in this paper is to see to what extent the action of $\mathrm{SL}(2,\mathbb{R})$ on $L^2(\mu)$ looks like an action on an homogeneous space, especially concerning small eigenvalues of the associated Laplacian acting on the subspace of $\mathrm{SO}(2,\mathbb{R})$ invariant functions in $L^2(\mu)$. Our main theorem states that the situation is almost identical to the $\mathrm{SL}(2,\mathbb{R})/\Gamma$ case (the difference being that we are not able to exclude the possibility that the eigenvalues accumulate at $1/4$):

Main Theorem. *Let μ be an $\mathrm{SL}(2,\mathbb{R})$-invariant algebraic probability measure in the moduli space of quadratic differentials. For any $\delta > 0$, the spectrum of the associated Laplacian in $[0, 1/4 - \delta]$ is made of finitely many eigenvalues, of finite multiplicity.*

This theorem can also be formulated as follows: in the decomposition of $L^2(\mu)$ into irreducible components, the representations of the complementary series occur only discretely, with finite multiplicity. More details are given in the next section.

Our result is independent of the above mentioned theorem of Eskin and Mirzakhani. With their theorem, we obtain that our result in fact applies to all $\mathrm{SL}(2,\mathbb{R})$-invariant probability measures.

As mentioned before, the spectral gap (equivalent to the absence of spectrum in $(0, \epsilon)$ for some $\epsilon > 0$) had been previously established in the particular case of Masur-Veech measures ([AGY06], [AR09]), but without any control of the spectrum

[1]Here we use the term algebraic in a rather lax sense. What has actually been shown is that the corresponding $\mathrm{GL}^+(2,\mathbb{R})$ invariant measure is supported on an affine submanifold of some stratum, along which it is absolutely continuous (with locally constant density in affine charts).

beyond a neighborhood of 0 (which moreover degenerates as the genus increases). Here we not only obtain very detailed information of the spectrum up to the $1/4$ barrier (beyond which the statement is already false even for the modular surface $\mathcal{M}_{1,1}$), but manage to address all algebraic measures, even in the absence of a classification. This comes from the implementation of a rather different, geometric approach, in contrast with the combinatorial one used to establish the spectral gap for Masur-Veech measures (heavily dependent on the precise combinatorial description, in terms of Rauzy diagrams, of the Teichmüller flow restricted to connected components of stratum).

An interesting question is whether there are indeed eigenvalues in $(0, 1/4)$. It is well known that there is no such eigenvalue in $\mathrm{SL}(2,\mathbb{R})/\Gamma$ for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and by Selberg's Conjecture [Sel65] the situation should be the same for any congruence subgroup. It is tempting to conjecture that, in our non-homogeneous situation, there is no eigenvalue either, at least when μ is the Masur–Veech measure. We will however refrain from doing so since we have no serious evidence in one direction or the other. Let us note however that, for some measures μ, there are indeed eigenvalues: for any finite index subgroup Γ of the congruence subgroup $\Gamma(2)$ containing $\{\pm 1\}$, the curve $\mathrm{SL}(2,\mathbb{R})/\Gamma$ can be realized as a Teichmüller curve by [EM09]. Suitably choosing Γ and taking for μ the Liouville measure on the resulting Teichmüller curve, we get an example with eigenvalues. Notice that this shows indeed that there can be no uniform spectral gap for all algebraic measures in all moduli spaces (it is unknown whether there is a uniform spectral gap in each fixed moduli space).

A consequence of our main theorem is that the correlations of well behaved functions have a nice asymptotic expansion (given by the spectrum of the Laplacian). For instance, if f_1 and f_2 are square-integrable $\mathrm{SO}(2,\mathbb{R})$-finite functions (i.e., f_1 and f_2 have only finitely many nonzero Fourier coefficients for the action of $\mathrm{SO}(2,\mathbb{R})$), then their correlations $\int f_1 \cdot f_2 \circ g_t \,\mathrm{d}\mu$ with respect to the Teichmüller flow $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ can be written, for every $0 < \delta < 1$, as $\sum_{i=0}^{M-1} c_i(f_1, f_2) e^{-a_i t} + o(e^{-(1-\delta)t})$, where $0 = a_0 < \cdots < a_{M-1} \leq 1 - \delta$ are the numbers $1 - \sqrt{1 - 4\lambda}$ for λ an eigenvalue of Δ in $[0, (1-\delta^2)/4]$. This follows at once from the asymptotic expansion of matrix coefficients of $\mathrm{SO}(2,\mathbb{R})$-finite functions in [CM82, Theorem 5.6]. A similar expansion certainly holds if f_1 and f_2 are only compactly supported C^∞ functions, but its proof would require more detailed estimates on matrix coefficients.

We expect that our techniques will also be useful in the study of the Ruelle zeta function $\zeta_{\mathrm{Ruelle}}(z) = \prod_\tau (1 - e^{-z|\tau|})$ (where τ runs over the prime closed orbits of the flow g, and $|\tau|$ is the length of τ). Recall that $\zeta_{\mathrm{Ruelle}}(z)$ can be expressed as an alternating product $\prod \zeta_k(z)^{(-1)^k}$, where ζ_k is a dynamical zeta function related to the action of g_t on the space of k-forms (see for instance [Fri86]). Along the proof of the main theorem, we obtain considerable information for the action of the Teichmüller flow in suitably defined Banach spaces, which goes in the direction of providing meromorphic extensions of the functions ζ_k (and therefore also of the Ruelle zeta functions), hence opening the way to precise asymptotic formulas (which should include correction terms coming from small eigenvalues of the Laplacian) for the number of closed geodesics in the support of any algebraic invariant measure.

2. Statements of results

Our results will be formulated in moduli spaces of flat surfaces, as follows. Fix a closed surface S of genus $g \geq 1$, a subset $\Sigma = \{\sigma_1, \dots, \sigma_j\}$ of S and multiplicities $\kappa = (\kappa_1, \dots, \kappa_j)$ with $\sum(\kappa_i - 1) = 2g - 2$. We denote by $\mathrm{Teich} = \mathrm{Teich}(S, \Sigma, \kappa)$ the set of translation structures on S such that the cone angle around each σ_i is equal to $2\pi\kappa_i$, modulo isotopy. Equivalently, this is the space of abelian differentials with zeroes of order $\kappa_i - 1$ at σ_i. Let also $\mathrm{Teich}_1 \subset \mathrm{Teich}$ be the set of area one surfaces.

Given a translation surface x, one can develop closed paths (or more generally paths from singularity to singularity) from the surface to $\mathbb{C}$, using the translation charts. This defines an element $\Phi(x) \in H^1(M, \Sigma; \mathbb{C})$. The resulting *period map* $\Phi : \mathrm{Teich} \to H^1(M, \Sigma; \mathbb{C})$ is a local diffeomorphism, and endows Teich with a canonical complex affine structure.

The mapping class group Γ of (S, Σ, κ) is the group of homeomorphisms of S permuting the elements of Σ with the same κ_i. It acts on Teich and on Teich_1. The space Teich is also endowed with an action of $\mathrm{GL}^+(2, \mathbb{R})$, obtained by postcomposing the translation charts by $\mathrm{GL}^+(2, \mathbb{R})$ elements. The action of the subgroup $\mathrm{SL}(2, \mathbb{R})$ of $\mathrm{GL}^+(2, \mathbb{R})$ leaves Teich_1 invariant. Since the actions of $\mathrm{GL}^+(2, \mathbb{R})$ and Γ commute, we may write the former on the left and the latter on the right.

Definition 2.1. *A measure $\tilde{\mu}$ on* Teich_1 *is admissible if it satisfies the following conditions:*

- *The measure $\tilde{\mu}$ is* $\mathrm{SL}(2, \mathbb{R})$ *and Γ-invariant.*
- *There exists a Γ-invariant linear submanifold Y of* Teich *such that $\tilde{\mu}$ is supported on $X = Y \cap \mathrm{Teich}_1$, and the measure $\tilde{\mu} \otimes \mathrm{Leb}$ on $X \times \mathbb{R}_+^* = Y$ is locally a multiple of the linear Lebesgue measure on Y.*
- *The measure μ induced by $\tilde{\mu}$ on X/Γ has finite mass, and is ergodic under the action of* $\mathrm{SL}(2, \mathbb{R})$ *on X/Γ.*

Although this definition may seem quite restrictive, it follows from the above mentioned theorem of Eskin and Mirzakhani that ergodic $\mathrm{SL}(2, \mathbb{R})$-invariant measures are automatically admissible. The following proposition is much weaker, but we nevertheless include it since its proof is elementary, and is needed to obtain further information on admissible measures (in particular on their local product structure, see Proposition 4.1 below).

Proposition 2.2. *Let X be a Γ-equivariant C^1 submanifold of* Teich_1 *such that X/Γ is connected, and let $\tilde{\mu}$ be a* $\mathrm{SL}(2, \mathbb{R})$ *and Γ-invariant measure on X such that $\tilde{\mu}$ is equivalent to Lebesgue measure, and the induced measure μ in X/Γ is a Radon measure, i.e., it gives finite mass to compact subsets of X/Γ. Then $\tilde{\mu}$ is admissible.*

This proposition should be compared to a result of Kontsevich and Möller in [Möl08]: any $\mathrm{GL}^+(2, \mathbb{R})$-invariant algebraic submanifold of Teich is linear. Here, we obtain the same conclusion if X is only C^1, but we additionally assume the existence of an invariant absolutely continuous Radon measure on X.

Let $\tilde{\mu}$ be an admissible measure, supported by a submanifold X of Teich_1. Every $\mathrm{SL}(2, \mathbb{R})$-orbit in X/Γ is isomorphic to a quotient of $\mathrm{SL}(2, \mathbb{R})$. Therefore, the image of every such orbit in $\mathrm{SO}(2, \mathbb{R})\backslash X/\Gamma$ (the set of translations surfaces in X, modulo the mapping class group, and in which the vertical direction is forgotten) is a quotient of the hyperbolic plane, and is canonically endowed with the hyperbolic Laplacian. Gluing those operators together on the different orbits, we get a Laplacian Δ on $\mathrm{SO}(2, \mathbb{R})\backslash X/\Gamma$, which acts (unboundedly) on $L^2(\mathrm{SO}(2, \mathbb{R})\backslash X/\Gamma, \mu)$, where μ is $\tilde{\mu}$ mod Γ. Our main theorem describes the spectrum of this operator:

Theorem 2.3. *Let $\tilde{\mu}$ be an admissible measure, supported by a manifold X. Denote by μ the induced measure on X/Γ. Then, for any $\delta > 0$, the spectrum of the Laplacian Δ on $L^2(\mathrm{SO}(2, \mathbb{R})\backslash X/\Gamma, \mu)$, intersected with $(0, 1/4 - \delta)$, is made of finitely many eigenvalues of finite multiplicity.*

This theorem can also be formulated in terms of the spectrum of the Casimir operator, or in terms of the decomposition of $L^2(X/\Gamma, \mu)$ into irreducible representations under the action of $\mathrm{SL}(2, \mathbb{R})$: for any $\delta > 0$, there is only a finite number of representations in the complementary series with parameter $u \in (\delta, 1)$ appearing in this decomposition, and they have finite multiplicity. See §3.4 for more details on these notions and their relationships.

Remark 2.4. We have formulated the result in the space X/Γ where Γ is the mapping class group. However, if Γ' is a subgroup of Γ of finite index, then the proof still applies in X/Γ' (of course, there may be more eigenvalues in X/Γ' than in X/Γ). This applies for instance to Γ' the set of elements of Γ that fix each singularity σ_i.

Remark 2.5. In compact hyperbolic surfaces, the spectrum of the Laplacian is discrete. Therefore, the essential spectrum of the Laplacian in $[1/4, \infty)$ in finite volume hyperbolic surfaces comes from infinity, i.e., the cusps. Since the geometry at infinity of moduli spaces of flat surfaces is much more complicated than cusps, one might expect more essential spectrum to show up, and Theorem 2.3 may come as a surprise. However, from the point of view of measure, infinity has the same weight in hyperbolic surfaces and in moduli spaces: the set of points at distance at least H in a cusp has measure $\sim cH^{-2}$, while its analogue in a moduli space is the set of surfaces with systole at most H^{-1}, which also has measure of order $c'H^{-2}$ by the Siegel-Veech formula [EM01]. This analogy (which also holds for recurrence speed to compact sets) justifies heuristically Theorem 2.3.

Quadratic differentials. Let $g, n \geq 0$ be integers such that $3g-3+n > 0$ and let $\mathcal{T}_{g,n}$ be the Teichmüller space of Riemann surfaces of genus g with n punctures. Its cotangent space is the space $\mathcal{Q}_{g,n}$ of quadratic differentials with at most simple poles at the punctures. It is stratified by fixing some appropriate combinatorial data (the number of poles, the number of zeros of each given order, and whether the quadratic differential is a square of an Abelian differential or not). Much of the theory of quadratic differentials is parallel to the one of Abelian differentials, in particular, each stratum in $\mathcal{Q}_{g,n}$ can be seen as a Teichmüller space $\widetilde{\mathrm{Teich}} = \widetilde{\mathrm{Teich}}(\tilde{S}, \tilde{\Sigma}, \tilde{\kappa})$ of *half-translation structures*, which allows one to define a natural action of $\mathrm{GL}^+(2, \mathbb{R})$. Moreover, strata are endowed with a natural affine structure, which allows one to define the notion of admissible measure (in particular, the Liouville measure in $\mathcal{Q}_{g,n}$ is admissible). Thus the statement of Theorem 2.3 still makes sense in the setting of quadratic differentials. As it turns out, it can also be easily derived from the result about Abelian differentials.

This is most immediately seen for strata of squares, in which case $\widetilde{\mathrm{Teich}}$ is the quotient of a Teichmüller space of Abelian differentials Teich by an involution (the rotation of angle π). Taking the quotient by $\mathrm{SO}(2, \mathbb{R})$, we see that the spectrum of the Laplacian for some $\mathrm{SL}(2, \mathbb{R})$-invariant measure in $\widetilde{\mathrm{Teich}}$ is the same as the one for its (involution-symmetric) lift to Teich, to which Theorem 2.3 applies.

Even when $\widetilde{\mathrm{Teich}}$ is not a stratum of squares, it can still be analyzed in terms of certain Abelian differentials (the well-known double cover construction also used in [AR09]). Indeed in this case the Riemann surface with a quadratic differential admits a (holomorphic, ramified, canonical) connected double cover (constructed formally using the doubly-valued square-root of the quadratic differential), to which the quadratic differential lifts to the square of an (also canonical) Abelian differential. This double cover carries an extra bit of information, in the form of a canonical involution, so that $\widetilde{\mathrm{Teich}}$ gets identified with a Teichmüller space of "translation surfaces with involution". Forgetting the involution, the latter can be seen as an affine subspace of a Teichmüller space of translation surfaces, allowing us to apply Theorem 2.3.

Notations. Let us introduce notations for convenient elements of $\mathrm{SL}(2,\mathbb{R})$. For $t\in\mathbb{R}$, let $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}$. Its action on $\mathcal{Q}_g$ is the geodesic flow corresponding to the Teichmüller distance on $\mathcal{T}_g$, and its action in different strata (that we still call the Teichmüller flow) will play an essential role in the proof of our main theorem. We also denote $h_r = \left(\begin{smallmatrix} 1 & r \\ 0 & 1\end{smallmatrix}\right)$ and $\tilde h_r = \left(\begin{smallmatrix} 1 & 0 \\ r & 1\end{smallmatrix}\right)$ the horocycle actions, and $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$ the circle action. Throughout this article, the letter C denotes a constant whose value is irrelevant and can change from line to line.

Sketch of the proof. The usual strategy to prove that the spectrum of the Laplacian is finite in $[0,1/4]$ in a finite volume surface $S=\mathrm{SO}(2,\mathbb{R})\backslash\mathrm{SL}(2,\mathbb{R})/\Gamma$ is the following: one decomposes $L^2(S)$ as $L^2_{\mathrm{cusp}}(S)\oplus L^2_{\mathrm{eis}}(S)$ where $L^2_{\mathrm{cusp}}(S)$ is made of the functions whose average on all closed horocycles vanishes, and $L^2_{\mathrm{eis}}(S)$ is its orthogonal complement. One then proves that the spectrum in $L^2_{\mathrm{eis}}(S)$ is $[1/4,\infty)$ by constructing a basis of eigenfunctions using Eisenstein series, and that the spectrum in $L^2_{\mathrm{cusp}}(S)$ is discrete since convolution with smooth compactly supported functions in $\mathrm{SL}(2,\mathbb{R})$ is a compact operator.

There are two difficulties when trying to implement this strategy in nonhomogeneous situations. Firstly, since the geometry at infinity is very complicated, it is not clear what the good analogue of $L^2_{\mathrm{eis}}(S)$ and Eisenstein series would be. Secondly, the convolution with smooth functions in $\mathrm{SL}(2,\mathbb{R})$ only has a smoothing effect in the direction of the $\mathrm{SL}(2,\mathbb{R})$ orbits, and not in the transverse direction (and this would also be the case if one directly tried to study the Laplacian); therefore, it is very unlikely to be compact.

To solve the first difficulty, we avoid completely the decomposition into Eisenstein and cuspidal components and work in the whole L^2 space. This means that we will not be able to exhibit compact operators (since this would only yield discrete spectrum), but we will rather construct quasi-compact operators, i.e., operators with finitely many large eigenvalues and the rest of the spectrum contained in a small disk. The first part will correspond to the spectrum of the Laplacian in $[0,1/4-\delta]$ and the second part to the non-controlled rest of the spectrum.

Concerning the second difficulty, we will not study the Laplacian nor convolution operators, but another element of the enveloping algebra: the differentiation L_ω in the direction ω of the flow g_t. Of course, its behavior on the space $L^2(X/\Gamma,\mu)$ is very bad, but we will construct a suitable Banach space $\mathcal{B}$ of distributions on which it is quasi-compact. To relate the spectral properties of g_t on $\mathcal{B}$ and of Δ on L^2, we will rely on fine asymptotics of spherical functions in irreducible representations of $\mathrm{SL}(2,\mathbb{R})$ (this part is completely general and does not rely on anything specific to moduli spaces of flat surfaces).

The main difficulty of the article is the construction of $\mathcal{B}$ and the study of L_ω on $\mathcal{B}$. We rely in a crucial way on the hyperbolicity of g_t, that describes what happens in all the directions of the space under the iteration of the flow. If $\mathcal{B}$ is carefully tuned (its elements should be smooth in the stable direction of the flow, and dual of smooth in the unstable direction), then one can hope to get smoothing effects in every direction, and therefore some compactness. This kind of arguments has been developed in recent years for Anosov maps or flows in compact manifolds and has proved very fruitful (see among others [Liv04, GL06, BT07, BL07]). We use in an essential way the insights of these papers. However, the main difficulty for us is the non-compactness of moduli space: since we can not rely on an abstract compactness argument close to infinity, we have to get explicit estimates there (using a quantitative recurrence estimate of Eskin-Masur [EM01]). We should also make sure that the estimates do not diverge at infinity. Technically, this is done using the Finsler metric of Avila-Gouëzel-Yoccoz [AGY06] (that has good regularity properties uniformly in the Teichmüller space) to define the Banach space $\mathcal{B}$, and plugging the Eskin-Masur function V_δ into the definition of $\mathcal{B}$. On the other hand, special features of the flow under study are very helpful: it is affine (hence no distortion appears), and its stable and unstable manifolds depend smoothly on the base point and are affine. Moreover, it is endowed in a $\mathrm{SL}(2,\mathbb{R})$ action, which implies that its spectrum can not be arbitrary: contrary to [Liv04], we will not need to investigate spectral values with large imaginary part.

Let us quickly describe a central step of the proof. At some point, we need to study the iterates $\mathcal{L}_{T_0}^n$ of the operator $\mathcal{L}_{T_0} f = f \circ g_{T_0}$, for a suitably chosen T_0. Using a partition of unity, we decompose $\mathcal{L}_{T_0}$ as $\tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_2$ where $\tilde{\mathcal{L}}_1$ corresponds to what is going on in a very large compact set K, and $\tilde{\mathcal{L}}_2$ takes what happens outside K into account. We expand $\mathcal{L}_{T_0}^n = \sum_{\gamma_i \in \{1,2\}} \tilde{\mathcal{L}}_{\gamma_1} \cdots \tilde{\mathcal{L}}_{\gamma_n}$. In this sum, if most γ_is are equal to 2, we are spending a lot of time outside K, and the Eskin-Masur function gives us a definite gain. Otherwise, a definite amount of time is spent inside K, where the flow is hyperbolic, and we get a gain λ given by the hyperbolicity constant of the flow inside K. Unfortunately, we only know that λ is strictly less than 1 (and K is very large, so it is likely to be very close to 1). This would be sufficient to get a spectral gap, but not to reach $1/4$ in the spectrum of the Laplacian. A key remark is that, if we define our Banach space $\mathcal{B}$ using C^k regularity, then the gain is better, of order λ^k. Choosing k large enough (at the complete end of the proof), we get estimates as precise as we want, getting arbitrarily close to $1/4$.

In view of this argument, two remarks can be made. Firstly, since we need to use very high regularity, our proof can not be done using a symbolic model since the discontinuities at the boundaries would spoil the previous argument. Secondly, since k is chosen at the very end of the proof, we have to make sure that all our bounds, which already have to be uniform in the non-compact space X/Γ, are also uniform in k.

The paper is organized as follows. In Section 3, we introduce necessary background on irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$, and show that Theorem 2.3 follows from a statement on spectral properties of the differentiation L_ω in the flow direction (Theorem 3.2). In Section 4, we get a precise description of admissible measures, showing that they have a nice local product structure. Along the way, we prove Proposition 2.2. In Section 5, we establish several technical properties of the C^k norm with respect to the Finsler metric of [AGY06] that will be instrumental when defining our Banach space $\mathcal{B}$. In Section 6, we reformulate the recurrence estimates of Eskin-Masur [EM01] in a form that is convenient for us. Finally, we define the Banach space $\mathcal{B}$ in Section 7, and prove Theorem 3.2 in Section 8.

3. Proof of the main theorem: the general part

3.1. **Functional analytic prerequisites.** Let $\overline{\mathcal{L}}$ be a bounded operator on a complex Banach space $(\overline{B}, \|\cdot\|)$. A complex number z belongs to the spectrum $\sigma(\overline{\mathcal{L}})$ of $\overline{\mathcal{L}}$ if $zI - \overline{\mathcal{L}}$ is not invertible. If z is an isolated point in the spectrum of $\overline{\mathcal{L}}$, we can define the corresponding spectral projection $\Pi_z := \frac{1}{2i\pi} \int_C (wI - \overline{\mathcal{L}})^{-1} \,\mathrm{d}w$, where C is a small circle around z (this definition is independent of the choice of C). Then Π_z is a projection, its image and kernel are invariant under $\overline{\mathcal{L}}$, and the spectrum of the restriction of $\overline{\mathcal{L}}$ to the image is $\{z\}$, while the spectrum of the restriction of $\overline{\mathcal{L}}$ to the kernel is $\sigma(\overline{\mathcal{L}}) - \{z\}$. We say that z is an isolated eigenvalue of finite multiplicity of $\overline{\mathcal{L}}$ if the image of Π_z is finite-dimensional, and we denote by $\sigma_{\mathrm{ess}}(\overline{\mathcal{L}})$ the essential spectrum of $\overline{\mathcal{L}}$, i.e., the set of elements of $\sigma(\overline{\mathcal{L}})$ that are not isolated eigenvalues of finite multiplicity.

The spectral radius of $\overline{\mathcal{L}}$ is $r(\overline{\mathcal{L}}) := \sup\{|z| \ : \ z \in \sigma(\overline{\mathcal{L}})\}$, and its essential spectral radius is $r_{\mathrm{ess}}(\overline{\mathcal{L}}) := \sup\{|z| \ : \ z \in \sigma_{\mathrm{ess}}(\overline{\mathcal{L}})\}$. These quantities can also be computed as follows: $r(\overline{\mathcal{L}}) = \inf_{n\in\mathbb{N}} \left\|\overline{\mathcal{L}}^n\right\|^{1/n}$, and $r_{\mathrm{ess}}(\overline{\mathcal{L}}) = \inf \left\|\overline{\mathcal{L}}^n - K\right\|^{1/n}$, where the infimum is over all integers n and all compact operators K. In particular, we get that the essential spectral radius of a compact operator is 0, i.e., the spectrum of a compact operator is made of a sequence of isolated eigenvalues of finite multiplicity tending to 0, as is well known.

So-called Lasota-Yorke inequalities can also be used to estimate the essential spectral radius:

Lemma 3.1. *Assume that, for some $n > 0$ and for all $x \in \overline{B}$, we have*

$$\left\|\overline{\mathcal{L}}^n x\right\| \leq M^n \|x\| + \|x\|',$$

where $\|\cdot\|'$ is a seminorm on $\overline{B}$ such that the unit ball of $\overline{B}$ (for $\|\cdot\|$) is relatively compact for $\|\cdot\|'$. Then $r_{\mathrm{ess}}(\overline{\mathcal{L}}) \leq M$.

This has essentially been proved by Hennion in [Hen93], the statement in this precise form can be found in [BGK07, Lemma 2.2].

Assume now that $\mathcal{L}$ is a bounded operator on a complex normed vector space $(B, \|\cdot\|)$, but that B is not necessarily complete. Then $\mathcal{L}$ extends uniquely to a bounded operator $\overline{\mathcal{L}}$ on the completion $\overline{B}$ of B for the norm $\|\cdot\|$. We will abusively talk about the spectrum, essential spectrum or essential spectral radius of $\mathcal{L}$, thinking of the same data for $\overline{\mathcal{L}}$.

3.2. **Main spectral result.** Let $\tilde{\mu}$ be an admissible measure supported on a manifold X, and let μ be its projection in X/Γ.

We want to study the spectral properties of the differentiation operator L_ω in the direction ω of the flow g_t. As in [Liv04], it turns out to be easier to study directly the resolvent of this operator, given by $R(z)f = \int_{t=0}^{\infty} e^{-zt} f \circ g_t \,\mathrm{d}t$.

Given $\delta > 0$, we will study the operator $\mathcal{M} = R(4\delta)$ on the space $\mathcal{D}^\Gamma$ of C^∞ functions on X, Γ–invariant and compactly supported in X/Γ. Of course, $\mathcal{M}f$ is not any more compactly supported, so we should be more precise.

We want to define a norm $\|\cdot\|$ on $\mathcal{D}^\Gamma$ such that, for any $f \in \mathcal{D}^\Gamma$, the function $f \circ g_t$ (which still belongs to $\mathcal{D}^\Gamma$) satisfies $\|f \circ g_t\| \leq C \|f\|$, for some constant C independent of t. Denoting by $\overline{\mathcal{D}^\Gamma}$ the completion of $\mathcal{D}^\Gamma$ for the norm $\|\cdot\|$, the operator $\mathcal{L}_t : f \mapsto f \circ g_t$ extends continuously to an operator on $\overline{\mathcal{D}^\Gamma}$, whose norm is bounded by C. Therefore, the operator $\mathcal{M} := \int_{t=0}^{\infty} e^{-4\delta t} \mathcal{L}_t$ acts continuously on the Banach space $\overline{\mathcal{D}^\Gamma}$, and it is meaningful to consider its essential spectral radius. We would like this essential spectral radius to be quite small. Since $\|f \circ g_t\| \leq C \|f\|$, the trivial estimate on the spectral radius of $\mathcal{M}$ is $C \int_{t=0}^{\infty} e^{-4\delta t} \,\mathrm{d}t = C/(4\delta)$. This blows up when δ tends to 0. We will get a significantly better bound on the essential spectral radius in the following theorem.

Theorem 3.2. *There exists a norm on $\mathcal{D}^\Gamma$ satisfying the requirement $\|f \circ g_t\| \leq C \|f\|$ (uniformly in $f \in \mathcal{D}^\Gamma$ and $t \geq 0$), such that the essential spectral radius of $\mathcal{M}$ for this norm is at most $1 + \delta$.*

Moreover, for any $f_1 \in \mathcal{D}^\Gamma$, the linear form $f \mapsto \int_{X/\Gamma} f_1 f \,\mathrm{d}\mu$ extends continuously from $\mathcal{D}^\Gamma$ to its closure $\overline{\mathcal{D}^\Gamma}$.

This theorem is proved in Section 8. The main point is of course the assertion on the essential spectral radius, the last one is a technicality that we will need later on.

Let us admit this result for the moment, and see how it implies our main result, Theorem 2.3. Since Theorem 3.2 deals with the spectrum of L_ω, it is not surprising that it implies a description of the spectrum of the action of $\mathrm{SL}(2,\mathbb{R})$. However, we only control the spectrum of L_ω on a quite exotic Banach space of distributions. To obtain information on the action of $\mathrm{SL}(2,\mathbb{R})$, we will therefore follow an indirect path, through meromorphic extensions of Laplace transforms of correlation functions. (It seems desirable to find a more direct and more natural route.)

3.3. **Meromorphic extensions of Laplace transforms.** From Theorem 3.2, we will obtain in this section a meromorphic extension of the Laplace transform of the correlations of smooth functions, to a suitable domain described as follows. For $\delta,\epsilon>0$, define $D_{\delta,\epsilon}\subset\mathbb{C}$ as the set of points $z=x+\mathbf{i}y$ such that either $x>0$, or $(x,y)\in[-1+6\delta,0]\times[-\epsilon,\epsilon]$.

Proposition 3.3. *Let $\delta>0$. Let $f_1,f_2\in\mathcal{D}^\Gamma$, define for $\Re z>0$ a function $F(z)=F_{f_1,f_2}(z)=\int_{t=0}^\infty e^{-zt}\left(\int_{X/\Gamma} f_1\cdot f_2\circ g_t \,\mathrm{d}\mu\right)\,\mathrm{d}t$. Then, for some $\epsilon>0$, the function F admits a meromorphic extension to (a neighborhood of) $D_{\delta,\epsilon}$.*

Moreover, the poles of F in $D_{\delta,\epsilon}$ are located in the set $\{4\delta-1/\lambda_1,\dots,4\delta-1/\lambda_I\}$, where the λ_i are the finitely many eigenvalues of modulus at least $1+2\delta$ of the operator $\mathcal{M}=R(4\delta)$ acting on the space constructed in Theorem 3.2. The residue of F at such a point $4\delta-1/\lambda_i$ is equal to $\int_{X/\Gamma} f_1\cdot\Pi_{\lambda_i}f_2\,\mathrm{d}\mu$, where Π_{λ_i} is the spectral projection of $\mathcal{M}$ associated to $\lambda_i\in\sigma(\mathcal{M})$.

Proof. Heuristically, we have $F(z)=\int_{X/\Gamma} f_1R(z)f_2\,\mathrm{d}\mu$ where $R(z)=\int_{t=0}^\infty e^{-zt}f\circ g_t$, and moreover $R(z)=(z-L_\omega)^{-1}$ where L_ω is the differentiation in the direction ω. Let us fix $z_0=4\delta$. The spectral properties of $\mathcal{M}=R(z_0)=(z_0-L_\omega)^{-1}$ are well controlled by Theorem 3.2. In view of the formal identity

$$(z-L_\omega)^{-1}=(z_0-z)^{-1}(z_0-L_\omega)^{-1}((z_0-z)^{-1}-(z_0-L_\omega)^{-1})^{-1},$$

we are led to define an operator

$$S(z)=\frac{1}{z_0-z}\mathcal{M}\left(\frac{1}{z_0-z}-\mathcal{M}\right)^{-1}, \tag{3.1}$$

which should coincide with $R(z)$. In particular, we should have the equality $F(z)=\int_{X/\Gamma} f_1S(z)f_2\,\mathrm{d}\mu$. Since $S(z)$ is defined for a large set of values of z, this should define the requested meromorphic extension of F to a larger domain.

Let us start the rigorous argument. Let $\overline{\mathcal{D}^\Gamma}$ be the Banach space constructed in Theorem 3.2, and let $\lambda_1,\dots,\lambda_I$ be the finitely many eigenvalues of modulus $\geq 1+2\delta$ of $\mathcal{M}$ acting on $\overline{\mathcal{D}^\Gamma}$. For z with $1/|z_0-z|\geq 1+2\delta$ and $1/(z_0-z)\notin\{\lambda_1,\dots,\lambda_I\}$, we can define on $\overline{\mathcal{D}^\Gamma}$ an operator $S(z)$ by the formula (3.1). It is holomorphic on $D_{\delta,\epsilon}\backslash\{4\delta-1/\lambda_1,\dots,4\delta-1/\lambda_I\}$. Since the points $4\delta-1/\lambda_i$ are poles of finite order (see e.g. [Kat66, III.6.5]), $S(z)$ is even meromorphic on $D_{\delta,\epsilon}$. Let us finally set $G(z)=\int_{X/\Gamma} f_1S(z)f_2\,\mathrm{d}\mu\in\mathbb{C}$, this is well defined by the last statement in Theorem 3.2. The function G is meromorphic and defined on the set $D_{\delta,\epsilon}$, with possible poles at the points $z_0-1/\lambda_1,\dots,z_0-1/\lambda_I$. To conclude, we just have to check that F and G coincide in a neighborhood of z_0.

If z is very close to z_0, $1/(z_0-z)$ is very large so that all series expansions are valid. Then the formula (3.1) gives

$$S(z)f_2=\mathcal{M}(1-(z_0-z)\mathcal{M})^{-1}f_2=\sum_{k=0}^\infty (z_0-z)^k\mathcal{M}^{k+1}f_2.$$

Since $\mathcal{M}^{k+1}f=\int_{t=0}^\infty \frac{t^k}{k!}e^{-z_0t}f\circ g_t\,\mathrm{d}t$, we obtain

$$S(z)f_2=\int_{t=0}^\infty\sum_{k=0}^\infty (z_0-z)^k\frac{t^k}{k!}e^{-z_0t}f\circ g_t\,\mathrm{d}t=\int_{t=0}^\infty e^{(z_0-z)t}e^{-z_0t}f_2\circ g_t\,\mathrm{d}t.$$

This gives the desired result after multiplying by f_1 and integrating.

Let us now compute the residue of $S(z)f_2$ around a point $z_0 - 1/\lambda_i$. We have

$$\begin{aligned} S(z) &= \frac{1}{z_0 - z}\left(\frac{1}{z_0 - z} + \mathcal{M} - \frac{1}{z_0 - z}\right)\left(\frac{1}{z_0 - z} - \mathcal{M}\right)^{-1} \\ &= \frac{1}{(z_0 - z)^2}\left(\frac{1}{z_0 - z} - \mathcal{M}\right)^{-1} - \frac{1}{z_0 - z}. \end{aligned}$$

The term $-(z_0 - z)^{-1}$ is holomorphic around $z_0 - 1/\lambda_i$. Therefore, the residue of S around this point is given by

$$\begin{aligned} &\frac{1}{2\mathbf{i}\pi}\int_{C(z_0 - 1/\lambda_i)} \frac{1}{(z_0 - z)^2}\left(\frac{1}{z_0 - z} - \mathcal{M}\right)^{-1} \mathrm{d}z \\ &\qquad = \frac{1}{2\mathbf{i}\pi}\int_{C(\lambda_i)} w^2 (w - \mathcal{M})^{-1} \frac{\mathrm{d}w}{w^2} = \Pi_{\lambda_i}, \end{aligned}$$

where $C(u)$ denotes a positively oriented path around the point u and we have written $w = 1/(z_0 - z)$. This concludes the proof. □

3.4. **Background on unitary representations of** $\mathrm{SL}(2,\mathbb{R})$**.** Let us describe (somewhat informally) the notion of direct decomposition of a representation. See e.g. [Dix69] for all the details.

Let H_ξ be a family of representations of $\mathrm{SL}(2,\mathbb{R})$, depending on a parameter ξ in a space Ξ, and assume that this family of representations is measurable (in a suitable sense). If m is a measure on Ξ, one can define the direct integral $\int H_\xi \,\mathrm{d}m(\xi)$: an element of this space is a function f defined on Ξ such that $f(\xi) \in H_\xi$ for all ξ, with $\|f\|^2 := \int \|f(\xi)\|^2_{H_\xi} \,\mathrm{d}m(\xi) < \infty$. The group $\mathrm{SL}(2,\mathbb{R})$ acts unitarily on this direct integral, by $(g \cdot f)(\xi) = g(f(\xi))$. If m' is another measure equivalent to m, then the representations $\int H_\xi \,\mathrm{d}m(\xi)$ and $\int H_\xi \,\mathrm{d}m'(\xi)$ are isomorphic.

From now on, let Ξ be the space of all irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$, with its canonical Borel structure (that we will describe below). Any unitary representation H of $\mathrm{SL}(2,\mathbb{R})$ is isomorphic to a direct integral $\int H_\xi \,\mathrm{d}m(\xi)$, where the space H_ξ is a (finite or countable) direct sum of one or several copies of the same representation ξ (we say that H_ξ is quasi-irreducible). Moreover, the measure class of the measure m, and the multiplicity of ξ in H_ξ, are uniquely defined ([Dix69, Théorème 8.6.6]), and the representation H is characterized by these data.

Let us now describe Ξ more precisely. The irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$ have been classified by Bargmann, as follows. An irreducible unitary representation of $\mathrm{SL}(2,\mathbb{R})$ belongs to one of the following families:

- Representations $\mathcal{D}^+_{m+1}$ and $\mathcal{D}^-_{m+1}$, for $m \in \mathbb{N}$. This is the discrete series (except for $m = 0$, where the situation is slightly different: these representations form the "mock discrete series").
- Representations $\mathcal{P}^{+,\mathbf{i}v}$ for $v \in [0, +\infty)$ and $\mathcal{P}^{-,\mathbf{i}v}$ for $v \in (0, \infty)$. This is the principal series (these representations can also be defined for $v < 0$, but they are isomorphic to the same representations with parameter $-v > 0$).
- Representations $\mathcal{C}^u$ for $0 < u < 1$. This is the complementary series.
- The trivial representation.

These representations are described with more details in [Kna01, II.5]. They are all irreducible, no two of them are isomorphic, and any irreducible unitary representation of $\mathrm{SL}(2,\mathbb{R})$ appears in this list. In particular, to any irreducible representation ξ of $\mathrm{SL}(2,\mathbb{R})$ is canonically attached a complex parameter $s(\xi)$ (equal to m in the first case, $\mathbf{i}v$ in the second, u in the third and 1 in the fourth), and the Borel structure of $\mathrm{SL}(2,\mathbb{R})$ is given by this parameter (and the discrete data $\pm$ in the first two cases).

The Casimir operator Ω is a generator of the center of the enveloping algebra of $\mathrm{SL}(2,\mathbb{R})$, i.e., it is a differential operator on $\mathrm{SL}(2,\mathbb{R})$, commuting with every translation, and of minimal degree. It is unique up to scalar multiplication, and we will normalize it as

$$\Omega = (L_W^2 - L_\omega^2 - L_V^2)/4, \tag{3.2}$$

where $W = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $\omega = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are elements of the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$, and L_Z denotes the Lie derivative on $\mathrm{SL}(2,\mathbb{R})$ with respect to the left invariant vector field equal to Z at the identity.

The Casimir operator extends to an unbounded operator in every unitary representation of $\mathrm{SL}(2,\mathbb{R})$. Since it commutes with translations, it has to be scalar on irreducible representations. With the notations we have set up earlier, it is equal to $(1-s(\xi)^2)/4 \in \mathbb{R}$ on an irreducible unitary representation ξ of parameter $s(\xi)$.

An irreducible unitary representation ξ of $\mathrm{SL}(2,\mathbb{R})$ is *spherical* if it contains an $\mathrm{SO}(2,\mathbb{R})$-invariant non-trivial vector. In this case, the $\mathrm{SO}(2,\mathbb{R})$-invariant vectors have dimension 1, let v be an element of unit norm in this set. The spherical function ϕ_ξ is defined on $\mathrm{SL}(2,\mathbb{R})$ by

$$\phi_\xi(g) = \langle g\cdot v, v\rangle, \tag{3.3}$$

it is independent of the choice of v. Taking $g = g_t$, the spherical function is simply the correlations of v under the diagonal flow.

The spherical unitary irreducible representations are the representations $\mathcal{P}^{+,\mathbf{i}v}$ and $\mathcal{C}^u$ (and the trivial one, of course).

Assume now that $\mathrm{SL}(2,\mathbb{R})$ acts on a space Y and preserves a probability measure μ. Then $\mathrm{SL}(2,\mathbb{R})$ acts unitarily on $L^2(Y,\mu)$ by $g\cdot f(x) = f(g^{-1}x)$. Therefore, the Casimir operator also acts $L^2(Y,\mu)$ (as an unbounded operator). Since it commutes with translations, it leaves invariant the space $L^2(\mathrm{SO}(2,\mathbb{R})\backslash Y,\mu)$ (i.e., the space of functions on Y that are $\mathrm{SO}(2,\mathbb{R})$-invariant and square-integrable with respect to μ). On this space, Ω can also be described geometrically as a foliated Laplacian, as follows.

For $x \in Y$, its orbit mod $\mathrm{SO}(2,\mathbb{R})$ is identified with $\mathbb{H} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$, by the map $g\mathrm{SO}(2,\mathbb{R}) \mapsto \mathrm{SO}(2,\mathbb{R})g^{-1}x$ (and changing the basepoint x in the orbit changes the parametrization by an $\mathrm{SL}(2,\mathbb{R})$ element). Therefore, any structure on $\mathbb{H}$ which is $\mathrm{SL}(2,\mathbb{R})$ invariant can be transferred to $\mathrm{SO}(2,\mathbb{R})\backslash Y$. This is in particular the case of the hyperbolic metric of curvature -1, and of the corresponding hyperbolic Laplacian Δ given in coordinates $(x_\mathbb{H}, y_\mathbb{H}) \in \mathbb{H}$ by $-y_\mathbb{H}\left(\frac{\partial^2}{\partial x_\mathbb{H}^2} + \frac{\partial^2}{\partial y_\mathbb{H}^2}\right)$.

Let f_K be a function on $\mathrm{SO}(2,\mathbb{R})\backslash Y$ belonging to the domain of Δ, and let f be its canonical lift to Y. Then Ωf is $\mathrm{SO}(2,\mathbb{R})$-invariant, and is the lift of the function Δf_K on $\mathrm{SO}(2,\mathbb{R})\backslash Y$. This follows at once from the definitions (and our choice of normalization in (3.2)).

Consider the decomposition $L^2(Y,\mu) \simeq \int_\Xi H_\xi \,\mathrm{d}m(\xi)$ of the representation of $\mathrm{SL}(2,\mathbb{R})$ on $L^2(Y,\mu)$ into an integral of quasi-irreducible representations. Denoting by $H_\xi^{\mathrm{SO}(2,\mathbb{R})}$ the $\mathrm{SO}(2,\mathbb{R})$-invariant vectors in H_ξ, we have $L^2(\mathrm{SO}(2,\mathbb{R})\backslash Y,\mu) \simeq \int_\Xi H_\xi^{\mathrm{SO}(2,\mathbb{R})} \,\mathrm{d}m(\xi)$. Therefore, the spectrum of Δ on $L^2(\mathrm{SO}(2,\mathbb{R})\backslash Y,\mu)$ is equal to the set $\{(1-s(\xi)^2)/4\}$, for ξ a spherical representation in the support of m (moreover, the spectral measure of Δ is the image of m under this map). Since the spectrum of the Casimir operator in the interval $(0,1/4)$ only comes from the complementary series representations, which are all spherical, it follows that $\sigma(\Delta)\cap$

$(0,1/4) = \sigma(\Omega) \cap (0,1/4)$, and that the spectral measures coincide. Therefore, it is equivalent to understand $\sigma(\Delta) \cap (0,1/4)$ or to understand representations in the complementary series arising in $L^2(Y,\mu)$. While the former point of view is more elementary, the latter puts it in a larger (and, perhaps, more significant) perspective.

3.5. **Meromorphic extensions of Laplace transforms in abstract** $\mathrm{SL}(2,\mathbb{R})$ **representations.** Given H a unitary representation of $\mathrm{SL}(2,\mathbb{R})$, let us decompose it as $\int_\Xi H_\xi \,\mathrm{d}m(\xi)$ where Ξ is the set of unitary irreducible representations of $\mathrm{SL}(2,\mathbb{R})$, and H_ξ is a direct sum of copies of the irreducible representation ξ. For $f \in H$, we will denote by f_ξ its component in H_ξ.

Let us denote by $\Xi^{\mathrm{SO}(2,\mathbb{R})}$ the set of spherical irreducible unitary representations. Using the parameter s of an irreducible representation described in the previous section, $\Xi^{\mathrm{SO}(2,\mathbb{R})}$ is canonically in bijection with $(0,1] \cup \mathbf{i}[0,+\infty)$. We will denote by ξ_s the representation corresponding to a parameter s.

Proposition 3.4. *Let $f_1, f_2 \in H$ be invariant under* $\mathrm{SO}(2,\mathbb{R})$. *Let us define the Laplace transform of the correlations of f_1, f_2 by*

$$F(z) = F_{f_1,f_2}(z) = \int_{t=0}^{\infty} e^{-zt} \langle g_t \cdot f_1, f_2\rangle,$$

for $\Re(z) > 0$.

The function F admits an holomorphic extension to $\{\Re(z) > -1, z \notin (-1,0]\}$. Moreover, for every $\delta > 0$, the function F can be written on the half-space $\{\Re(z) > -1+2\delta\}$ as the sum of a bounded holomorphic function A_δ, and the function

$$B_\delta(z) = \frac{1}{\sqrt{\pi}} \int_{s\in[\delta,1]} \frac{\Gamma(s/2)}{\Gamma((s+1)/2)} \langle (f_1)_{\xi_s}, (f_2)_{\xi_s}\rangle \frac{\mathrm{d}m(\xi_s)}{z-s+1}.$$

Proof. We fix a decomposition of H_ξ as an orthogonal sum $\bigoplus_{0\leq i<n} \xi_i$, where $n = n(\xi) \in \mathbb{N} \cup \{+\infty\}$ is the multiplicity of ξ in H_ξ, and $\xi_0, \dots, \xi_{n-1}$ are copies of the representation ξ. This decomposition is not canonical, but it can be chosen to depend measurably on ξ (see [Dix69]). If the decomposition ξ is spherical, we fix in every ξ_j a vector $h(\xi,j) \in \xi_j$ of unit norm invariant under $\mathrm{SO}(2,\mathbb{R})$.

Let f be a $\mathrm{SO}(2,\mathbb{R})$-invariant element of H. For $\xi \in \Xi^{\mathrm{SO}(2,\mathbb{R})}$, the element f_ξ of H_ξ can uniquely be decomposed as $\sum_{j<n(\xi)} \hat{f}(\xi,j) h(\xi,j)$, where the coefficients $\hat{f}(\xi,j) \in \mathbb{C}$ depend measurably on ξ, j.

We use this decomposition for f_1 and f_2. Let us recall that we defined the spherical function ϕ_ξ of a representation ξ in (3.3). Since the functions $g_t \cdot h(\xi,j)$ and $h(\xi,j')$ are orthogonal for $j \neq j'$, we have

$$\begin{aligned}
\langle g_t \cdot f_1, f_2\rangle
&= \int_{\Xi^{\mathrm{SO}(2,\mathbb{R})}} \left\langle g_t \left(\sum_{j<n(\xi)} \hat{f}_1(\xi,j) h(\xi,j)\right), \left(\sum_{j'<n(\xi)} \hat{f}_2(\xi,j') h(\xi,'j)\right)\right\rangle \mathrm{d}m(\xi) \\
&= \int_{\Xi^{\mathrm{SO}(2,\mathbb{R})}} \sum_{j<n(\xi)} \hat{f}_1(\xi,j)\overline{\hat{f}_2(\xi,j)} \langle g_t h(\xi,j), h(\xi,j)\rangle \,\mathrm{d}m(\xi) \\
&= \int_{\Xi^{\mathrm{SO}(2,\mathbb{R})}} \left(\sum_{j<n(\xi)} \hat{f}_1(\xi,j)\overline{\hat{f}_2(\xi,j)} \right) \phi_\xi(g_t) \,\mathrm{d}m(\xi) \\
&= \int_{\Xi^{\mathrm{SO}(2,\mathbb{R})}} \langle (f_1)_\xi, (f_2)_\xi\rangle \phi_\xi(g_t) \,\mathrm{d}m(\xi).
\end{aligned}$$

To proceed, we will need fine asymptotics of the spherical functions ϕ_ξ. The first one is due to Ratner [Rat87, Theorem 1]: for all $\delta > 0$, there exists a constant C such that, for any $\xi \in \Xi^{\mathrm{SO}(2,\mathbb{R})}$ with $s(\xi) \notin [\delta, 1]$, and for any $t \geq 0$,

$$|\phi_\xi(g_t)| \leq Ce^{-(1-\delta)t}. \tag{3.4}$$

An important point in this estimate is that the constant C is uniform in ξ, even though ξ varies in a non-compact domain.

For representations in the complementary series, we will use a more precise estimate, as follows. Define a function

$$c(s) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(s/2)}{\Gamma((s+1)/2)}, \tag{3.5}$$

for $s \in (0, 1]$. This function is known as Harish-Chandra's function. For all $\delta > 0$, there exists a constant $C > 0$ such that, for all $s \in [\delta, 1]$ and all $t \geq 0$,

$$\left| \phi_{\xi_s}(g_t) - c(s)e^{(s-1)t} \right| \leq Ce^{-t}. \tag{3.6}$$

This estimate is proved in Appendix A.1.

We will now conclude, using (3.4) and (3.6). Let us decompose $\Xi^{\mathrm{SO}(2,\mathbb{R})}$ (identified through the parameter s with a subset of $\mathbb{C}$) as the union of $[\delta, 1]$ and its complement. Then

$$\begin{aligned}
F(z) &= \int_{\xi_s \in \Xi^{\mathrm{SO}(2,\mathbb{R})}} \int_{t=0}^{\infty} e^{-zt} \langle (f_1)_{\xi_s}, (f_2)_{\xi_s} \rangle \phi_{\xi_s}(g_t) \,\mathrm{d}t \,\mathrm{d}m(\xi_s) \\
&= \int_{s \in \Xi^{\mathrm{SO}(2,\mathbb{R})} \setminus [\delta,1]} \int_{t=0}^{\infty} e^{-zt} \langle (f_1)_{\xi_s}, (f_2)_{\xi_s} \rangle \phi_{\xi_s}(g_t) \,\mathrm{d}t \,\mathrm{d}m(\xi_s) \\
&\quad + \int_{s \in [\delta,1]} \int_{t=0}^{\infty} e^{-zt} \langle (f_1)_{\xi_s}, (f_2)_{\xi_s} \rangle (\phi_{\xi_s}(g_t) - c(s)e^{(s-1)t}) \,\mathrm{d}t \,\mathrm{d}m(\xi_s) \\
&\quad + \int_{s \in [\delta,1]} \int_{t=0}^{\infty} e^{-zt} \langle (f_1)_{\xi_s}, (f_2)_{\xi_s} \rangle c(s)e^{(s-1)t} \,\mathrm{d}t \,\mathrm{d}m(\xi_s).
\end{aligned}$$

Let $B_\delta(z)$ be the last term in this expression, and $A_\delta(z)$ the sum of the two other ones. In A_δ, the factors $\phi_{\xi_s}(g_t)$ and $\phi_{\xi_s}(g_t) - c(s)e^{(s-1)t}$ are bounded, respectively, by $Ce^{-(1-\delta)t}$ and Ce^{-t} (by (3.4) and (3.6)). Therefore, $A_\delta(z)$ extends to an holomorphic function on $\{\Re(z) > -1+\delta\}$, which is bounded on the half-plane $\{\Re(z) \geq -1+2\delta\}$. Since $\int_0^\infty e^{-at} \,\mathrm{d}t = 1/a$ for $\Re(a) > 0$, the function $B_\delta(z)$ is equal to

$$\int_{s \in [\delta,1]} \langle (f_1)_{\xi_s}, (f_2)_{\xi_s} \rangle \frac{c(s)}{z-s+1} \,\mathrm{d}m(\xi_s),$$

for $\Re(z) > 0$. This function can be holomorphically extended to $z \notin [-1+\delta, 0]$, by the same formula. This proves the proposition. □

3.6. **Proof of Theorem 2.3.** We decompose the representation $H = L^2(X/\Gamma, \mu)$ of $\mathrm{SL}(2,\mathbb{R})$ as a direct integral $\int_\Xi H_\xi \,\mathrm{d}m(\xi)$, where the representation H_ξ is the direct sum of one or several copies of the irreducible representation $\xi \in \Xi$. We should prove that, for any $\delta > 0$, the restriction of the measure m to $(\delta, 1)$ (identified with the corresponding set of representations in the complementary series) is made of finitely many Dirac masses, and that at those points the multiplicity of ξ in H_ξ is finite.

Let $\delta > 0$ be small. Consider the eigenvalues λ_i constructed in Proposition 3.3. We claim that, on the interval $(6\delta, 1)$, the measure m only gives mass to the points $4\delta - 1/\lambda_i + 1$, and that at such a point the multiplicity of ξ in H_ξ is bounded by the dimension of the image of the spectral projection Π_{λ_i} described in Proposition 3.3. This will conclude the proof of the theorem.

To proceed, we will use the fact that we have two different expressions for the meromorphic extensions of Laplace transforms, one related to the geometry of Teichmüller space coming from Proposition 3.3, and one given by the abstract theory of representations of $\mathrm{SL}(2,\mathbb{R})$ in Proposition 3.4. Identifying these two expressions gives the results, as follows.

First step: m only gives weight to the points $4\delta - 1/\lambda_i + 1$. Assume by contradiction that m gives positive weight to an interval $[a,b]$ containing no such point. There exists a function $f^{(0)} \in H$ invariant under $\mathrm{SO}(2,\mathbb{R})$ such that the corresponding components $f_\xi^{(0)}$ in H_ξ satisfy $\int_{[a,b]} \left\| f_{\xi_s}^{(0)} \right\|^2_{H_{\xi_s}} \,\mathrm{d}m(\xi_s) > 0$. Consider $\tilde f_{(n)}^{(0)} \in \mathcal{D}^\Gamma$ a sequence of smooth compactly supported functions converging to $f^{(0)}$ in H. The functions $f_{(n)}^{(0)} = \int_{\theta\in\mathbb{S}^1} k_\theta \tilde f_{(n)}^{(0)} \,\mathrm{d}\theta$ also belong to $\mathcal{D}^\Gamma$, are $\mathrm{SO}(2,\mathbb{R})$-invariant, and converge to $f^{(0)}$ in H. In particular, if n is large enough, $\int_{[a,b]} \left\| (f_{(n)}^{(0)})_{\xi_s} \right\|^2_{H_{\xi_s}} \,\mathrm{d}m(\xi_s) > 0$. Let us fix such a function $f = f_{(n)}^{(0)}$.

Consider the function $F_{f,f}(z) = \int_{t=0}^\infty e^{-zt} \langle f, f\circ g_t\rangle \,\mathrm{d}t$, initially defined for $\Re(z) > 0$. By Proposition 3.3, it admits a meromorphic extension to the domain $\mathcal{D}_{\delta,\epsilon}$ for some $\epsilon > 0$, with possible poles only at the points $4\delta - 1/\lambda_i$. Moreover, Proposition 3.4 shows that the same function can be written, on the set $\{\Re(z) > -1+2\delta\}$, as the sum of a bounded holomorphic function and the function

$$B_\delta(z) = \frac{1}{\sqrt{\pi}} \int_{s\in[\delta,1]} \frac{\Gamma(s/2)}{\Gamma((s+1)/2)} \left\| f_{\xi_s} \right\|^2_{H_{\xi_s}} \frac{\mathrm{d}m(\xi_s)}{z-s+1}.$$

It follows that this function B_δ can only have poles at the points $4\delta - 1/\lambda_i$. In particular, it is continuous on the interval $[a-1, b-1]$. Lemma A.1 implies that the measure $\mathrm{d}\nu(s) = \frac{\Gamma(s/2)}{\Gamma((s+1)/2)} \left\| f_{\xi_s} \right\|^2_{H_{\xi_s}} \,\mathrm{d}m(\xi_s)$ gives zero mass to $[a,b]$. In particular, $\int_{[a,b]} \left\| f_{\xi_s} \right\|^2_{H_{\xi_s}} \,\mathrm{d}m(\xi_s) = 0$. This is a contradiction, and concludes the first step.

Second step: at a point $s = 4\delta - 1/\lambda_i + 1$, the multiplicity of ξ_s in H_{ξ_s} is at most the dimension of $\operatorname{Im}\Pi_{\lambda_i}$ *in the Banach space of Theorem 3.2.* We argue again by contradiction. Let $d = \dim \operatorname{Im}\Pi_{\lambda_i}$, assume that the multiplicity of ξ_s in H_{ξ_s} is at least $d+1$. Then one can find in H_{ξ_s} $d+1$ orthogonal functions $f^{(1)}, \dots, f^{(d+1)}$ which are $\mathrm{SO}(2,\mathbb{R})$-invariant. Since m has an atom at ξ_s, these functions are elements of $H = L^2(X/\Gamma, \mu)$. As above, we consider sequences $f_{(n)}^{(k)} \in \mathcal{D}^\Gamma$ of $\mathrm{SO}(2,\mathbb{R})$-invariant functions that converge to $f^{(k)}$ in H.

Let $F_{f_{(n)}^{(k)}, f_{(n)}^{(\ell)}}(z)$ be the meromorphic extension of the Laplace transform of the correlations of $f_{(n)}^{(k)}$ and $f_{(n)}^{(\ell)} \circ g_t$, and let $M_n^{k,\ell}$ denote its residue around the point $4\delta - 1/\lambda_i$. For each n, the residue $M_n^{k,\ell}$ is described by Proposition 3.3. Since the operator Π_{λ_i} has a d-dimensional image, it follows that the rank of the matrix M_n is at most d. On the other hand, Proposition 3.4 shows that $M_n^{k,\ell} = C\langle (f_{(n)}^{(k)})_{\xi_s}, (f_{(n)}^{(\ell)})_{\xi_s}\rangle$ (where $C > 0$ depends only on s and m). When n tends to infinity, the functions $f_{(n)}^{(k)}$ converge to $f^{(k)}$, hence M_n converges to a diagonal matrix. In particular, M_n is of rank $d+1$ for large enough n, a contradiction. □

4. Measures with a local product structure on Teich_1

To construct the Banach space of Theorem 3.2, we will need more geometric information on admissible measures, given by the following proposition.

Proposition 4.1. *Let $\tilde\mu$ be an admissible measure, supported on a submanifold X of* Teich_1. *Then*

(1) *For every $x \in X$, there is a decomposition of the tangent space $\mathcal{T}_x X = \mathbb{R}\omega(x) \oplus E^u(x) \oplus E^s(x)$, where $\omega(x)$ is the direction of the g_t-flow,*

$$E^u(x) = \mathcal{T}_x X \cap D\Phi(x)^{-1}(H^1(M,\Sigma;\mathbb{R})), \quad E^s(x) = \mathcal{T}_x X \cap D\Phi(x)^{-1}(H^1(M,\Sigma;\mathbf{i}\mathbb{R})).$$

(2) *The subspaces $E^s(x)$ and $E^u(x)$ depend in a C^∞ way on $x \in X$, are integrable, and the integral leaves $W^u(x), W^s(x)$ are affine submanifolds of* Teich.

(3) *For every $x \in X$, there is a volume form μ_u on $E^u(x)$ (defined up to sign), such that $x \mapsto \mu_u(x)$ is C^∞. Moreover, $x \mapsto \mu_u(x)$ is constant along the unstable manifolds W^u. Additionally, there exists a scalar $d \geq 0$ such that $(g_t)_*\mu_u = e^{-dt}\mu_u$.*

(4) *For every $x \in X$, there is a volume form μ_s on $E^s(x)$ (defined up to sign), such that $x \mapsto \mu_s(x)$ is C^∞. Moreover, $x \mapsto \mu_s(x)$ is constant along the stable manifolds W^s. Additionally, $(g_t)_*\mu_s = e^{dt}\mu_s$.*

(5) *For every $x \in X$, the volume form $\mathrm{d}\tilde{\mu}(x)$ on $\mathcal{T}_x X$ is equal to the product of* dLeb, *$\mu_u(x)$ and $\mu_s(x)$ respectively in the directions $\omega(x)$, $E^u(x)$ and $E^s(x)$.*

All these data are Γ-equivariant. We say that the decomposition $\mathrm{d}\tilde{\mu} = \mathrm{dLeb} \otimes \mathrm{d}\mu_u \otimes \mathrm{d}\mu_s$ is the affine local product structure of μ.

Note that, since $W^u(x)$ is an affine submanifold, the tangent spaces of $W^u(x)$ at two different points $y_1, y_2 \in W^u(x)$ are canonically identified (i.e., their images under $D\Phi(y_1)$ and $D\Phi(y_2)$ coincide), hence it is meaningful to say in item 3 of the above definition that $y \mapsto \mu_u(y)$ is constant along $W^u(x)$. The same holds for μ_s along W^s.

Note also that E^u and E^s are really the strong stable and unstable manifolds. Indeed, $D\Phi(x)^{-1}(H^1(M,\Sigma;\mathbb{R}))$ is the weak unstable manifold for the flow on Teich, but since we are restricting to $\mathcal{T}_x X$ we are excluding the neutral directions (see the example of area-one surfaces below).

If $x = a + \mathbf{i}b$ in the chart Φ, then for small r we have $h_r(x) = a + rb + \mathbf{i}b$. In particular, the tangent vector to this curve is always $b \in H^1(M,\Sigma;\mathbb{R})$, hence $h_r(x)$ is in the unstable manifold $W^u(x)$. Moreover, the differential of h_r sends $E^u(x)$ to $E^u(h_r x)$, and it is equal to the identity in the chart Φ. In particular, since μ_u is constant along $W^u(x)$, this implies that h_r leaves μ_u invariant, i.e., $(h_r)_*\mu_u = \mu_u$.

The family of volume forms $\mu_u(x)$ on $E^u(x)$ induces a positive measure on each leaf W^u of the unstable foliation, that we also denote by μ_u. In the same way, we get a measure μ_s on each stable manifolds. Let us note that, although the volume forms $\mu_u(x)$ are only defined up to sign, the induced positive measures μ_u are canonical. If the manifolds W^u and W^s were canonically oriented (or at least had a Γ invariant orientation), then $\mu_u(x)$ and $\mu_s(x)$ themselves would not be defined only up to sign, but we do not know if this is always the case.

The scalar d in the above proposition can be identified, see Remark 4.4.

See [BL98] for the notion of local product structure in more complicated non-smooth settings.

Example 4.2. Consider in Teich the subset $X = \mathrm{Teich}_1$ of area one surfaces, with its canonical invariant Lebesgue measure $\tilde{\mu}$. We will describe its affine local product structure. A similar construction is given in [ABEM06, Section 2], in more geometric terms.

First, assume $x \in X$ and $\Phi(x) = a + \mathbf{i}b$. Around x, we identify Teich and $H^1(M,\Sigma;\mathbb{C})$ using Φ. Then the area of $a + a' + \mathbf{i}b$ is $1 + [a', b]$, where $[a', b]$ is the intersection product of a' and b (this is initially defined for elements of $H^1(M;\mathbb{R})$, but since $H^1(M,\Sigma;\mathbb{R})$ projects to $H^1(M;\mathbb{R})$ it extends trivially to $H^1(M,\Sigma;\mathbb{R})$). Therefore, $E^u(x) = \{a' \in H^1(M,\Sigma;\mathbb{R}) \ : \ [a', b] = 0\}$. This depends smoothly on x, and the integral leaves of this distribution are locally the sets $\{(a+a',b) : [a',b] = 0\}$. These are indeed affine submanifolds of Teich.

Let us now define μ_u at the point $a+\mathbf{i}b$. The set $H^1(M,\Sigma;\mathbb{R})$ is endowed with a canonical volume form vol (giving covolume 1 to $H^1(M,\Sigma;\mathbb{Z})$), we let μ_u be the interior product of a and vol, i.e., if $v_1,\dots,v_k$ is a basis of $E^u(x)$, then $\mu_u(x) = \mathrm{vol}(a, v_1,\dots,v_k)$. At a nearby point $x' = a+a'+\mathbf{i}b$ on the same unstable manifold, $\mu_u(x')(v_1,\dots,v_k) = \mathrm{vol}(a+a', v_1,\dots,v_k) = \mathrm{vol}(a,v_1,\dots,v_k)$ since a' belongs to $E^u(x)$. Therefore, $\mu_u(x) = \mu_u(x')$ as claimed.

Let $d = k+1$ be the dimension of $H^1(M,\Sigma;\mathbb{R})$. The differential of g_t, mapping $E^u(x)$ to $E^u(g_t x)$, is simply the multiplication by e^t, therefore $(g_t)_*\mu_u(x) = e^{-(d-1)t}\mu_u(x)$. Since $\mu_u(g_t x) = e^t\mu_u(x)$, we get $(g_t)_*\mu_u(x) = e^{-dt}\mu_u(g_t x)$.

In the same way, we define a volume form $\mu_s(x)$ on $E^s(x)$. It satisfies $(g_t)_*\mu_s = e^{dt}\mu_s$.

Let us finally define a volume form $\tilde\mu'$ on $\mathcal{T}_x X$ as the product of Lebesgue in the flow direction, μ_u and μ_s. It satisfies $(g_t)_*\tilde\mu' = \tilde\mu'$, since the factors e^{-dt} and e^{dt} (coming respectively from μ_u and μ_s) cancel out.

All those data are intrinsically defined, and therefore Γ-invariant. By ergodicity of μ in the quotient X/Γ, we have $\tilde\mu' = c\tilde\mu$ for some $c\in(0,+\infty)$.

We will prove simultaneously Proposition 4.1 (the fact that an admissible measure has a local product structure) and Proposition 2.2 (the fact that an absolutely continuous measure on a smooth submanifold is automatically admissible): indeed, we will start from an absolutely continuous measure and prove simultaneously that it is admissible and that it has an affine local product structure. For this proof, we will use the non-uniform hyperbolicity of the Teichmüller flow. This property is well-known, but we will need it later on in the following precise form. Let us fix on Teich a Γ-invariant Finsler metric. In later arguments, we will use a specific metric which is well behaved at infinity (constructed in Subsection 5.1), but the following statement is valid for any metric.

Proposition 4.3. *For any set $K\subset \mathrm{Teich}_1$ which is compact mod Γ, there exists $T = T(K)$ such that, for any point $x\in K$ and any time t such that $g_t x\in K$ and*

$$\mathrm{Leb}\{s\in[0,t] : g_s(x)\in K\}\geq T,$$

then $\|Dg_t(x)v\|_{g_t x}\leq \|v\|_x/2$ for any $v\in E^s(x)$, and $\|Dg_t(x)v\|_{g_t x}\geq 2\|v\|_x$ for any $v\in E^u(x)$.

Proof. The uniform hyperbolicity of the Teichmüller flow in compact subsets of Teich_1/Γ has been proved by Forni in [For02, Lemma 2.1'], for a different norm, the Hodge norm (and for vectors belonging to $H^1(M;\mathbb{C})$ instead of $H^1(M,\Sigma;\mathbb{C})$). To obtain the result for the norm under study, it is sufficient to use the following two facts:

(1) Vectors in $H^1(M,\Sigma;\mathbb{C})$ that vanish in $H^1(M;\mathbb{C})$ are expanded at a constant rate e^t in the unstable direction, and contracted at a constant rate e^{-t} in the stable direction.
(2) In a fixed compact subset of Teich_1/Γ, any two continuous norms are equivalent. □

Proof of Propositions 4.1 and 2.2. Let us fix a measure $\tilde\mu$ as in the assumptions of Proposition 2.2: it is supported on a C^1 submanifold X of Teich_1, equivalent to Lebesgue measure on X, and induces a Radon measure μ in X/Γ. We will prove that $\tilde\mu$ is admissible and that it has an affine local product structure.

For $x\in\mathrm{Teich}_1$, denote by π_ω, π_u and π_s the projections respectively on the flow, unstable and stable direction in the tangent space $\mathcal{T}_x\mathrm{Teich}_1$.

First step: the measure μ has finite mass. In particular, the flow g_t is conservative in the measure space $(X/\Gamma, \mu)$.

Since μ is $\mathrm{SL}(2,\mathbb{R})$-invariant, Athreya's Theorem [Ath06] shows the existence of a compact set K in X/Γ such that, under the iteration of g_t, μ-almost every point spends asymptotically at least half its time in K. It follows from Hopf's ergodic theorem applied to the ergodic components of μ that $\mu(X/\Gamma) \leq 2\mu(K)$. Since μ is a Radon measure, this quantity is finite and the result follows.

Second step: at every point $x \in X$, we have

$$\mathcal{T}_x X = \pi_\omega(\mathcal{T}_x X) \oplus \pi_u(\mathcal{T}_x X) \oplus \pi_s(\mathcal{T}_x X). \tag{4.1}$$

We will prove this property for x in a dense subset of X, since the general case follows by a limiting argument (using the compactness of Grassmannians). The dimensions of $\pi_u(\mathcal{T}_x X)$ and $\pi_s(\mathcal{T}_x X)$ are semi-continuous, hence they are locally constant on a dense subset of X. Moreover, since the flow g_t is conservative and μ has full support, Poincaré's recurrence theorem ensures that almost every point of X comes back close to itself in the quotient X/Γ infinitely often in forward and backward time. We will prove (4.1) for such a point x.

First, since $g_t(x) \in X$ for all $t \geq 0$, we have $\omega(x) = \partial g_t(x)/\partial t|_{t=0} \in \mathcal{T}_x X$. It is therefore sufficient to check that $\pi_u(\mathcal{T}_x X) \subset \mathcal{T}_x X$ and $\pi_s(\mathcal{T}_x X) \subset \mathcal{T}_x X$. By symmetry, it is even sufficient to prove the first inclusion.

Since the dimension of $\pi_u(\mathcal{T}_y X)$ is locally constant around x, there exists a constant C such that, for any y close to x, any vector $w_u \in \pi_u(\mathcal{T}_y X)$ admits a lift w to $\mathcal{T}_y X$ with $\|w\| \leq C\|w_u\|$.

Consider $v \in \mathcal{T}_x X$, and write it as $v = v_\omega + v_u + v_s \in \pi_\omega(\mathcal{T}_x X) \oplus \pi_u(\mathcal{T}_x X) \oplus \pi_s(\mathcal{T}_x X)$. We should prove that $v_u \in \mathcal{T}_x X$. Let $\epsilon > 0$. Consider t very large such that $y = g_{-t}x$ is close to x. By Proposition 4.3, if t is large enough, the norm of $w_u := Dg_{-t}(x) \cdot v_u$ is bounded by ϵ. We may therefore find $w \in \mathcal{T}_y X$ with $\pi_u(w) = w_u$, and $\|w\| \leq C\epsilon$. Write $w = w_\omega + w_u + w_s$. Then $Dg_t(y)w \in \mathcal{T}_x X$, and this vector can be written as $Dg_t(y)(w_\omega + w_s) + v_u$ where $\|Dg_t(y)(w_\omega + w_s)\| \leq C\epsilon$. We have proved that v_u is a limit of points of $\mathcal{T}_x X$, and therefore that $v_u \in \mathcal{T}_x X$. This concludes the proof of the second step.

We can therefore define spaces $E^u(x) = \pi_u(\mathcal{T}_x X) = \mathcal{T}_x X \cap \Phi^{-1}(H^1(M, \Sigma; \mathbb{R}))$ and $E^s(x) = \pi_s(\mathcal{T}_x X) = \mathcal{T}_x X \cap \Phi^{-1}(H^1(M, \Sigma; \mathbf{i}\mathbb{R}))$ such that $\mathcal{T}_x X = \mathbb{R}\omega \oplus E^u(x) \oplus E^s(x)$. Moreover, the dimensions d_u and d_s of those spaces are locally constant. Since the space X/Γ is connected, they are in fact constant. Finally, since the rotation $k_{\pi/2}$ maps $E^u(x)$ to $E^s(k_{\pi/2}x)$, we have $d_u = d_s$.

Let $Y = \mathbb{R}_+^* X$. To simplify notations, we will omit Φ and identify locally Y with a subset of $H^1(M, \Sigma; \mathbb{C})$. Since the tangent vector to the map $t \mapsto tx$ at $x = a + \mathbf{i}b$ is $a + \mathbf{i}b$, we have $\mathcal{T}_x Y = \mathbb{R}(a + \mathbf{i}b) + \mathcal{T}_x X$. With the decomposition of $\mathcal{T}_x X$ given at the first step and the equality $\omega(x) = a - \mathbf{i}b$, we obtain $\mathcal{T}_x Y = \tilde{E}^u(x) \oplus \tilde{E}^s(x)$, where $\tilde{E}^u(x) = \mathcal{T}_x Y \cap H^1(M, \Sigma; \mathbb{R}) = \mathbb{R}a \oplus E^u(x)$, and $\tilde{E}^s(x) = \mathcal{T}_x Y \cap H^1(M, \Sigma; \mathbf{i}\mathbb{R}) = \mathbf{i}\mathbb{R}b \oplus E^s(x)$.

Third step: for any $x \in Y$, we have $\tilde{E}^s(x) = \mathbf{i}\tilde{E}^u(x)$.

Let $e \in \tilde{E}^u(x)$ and $\mathbf{i}f \in \tilde{E}^s(x)$. For small θ, the rotated vector $k_\theta(e + \mathbf{i}f) = (\cos(\theta)e + \sin(\theta)f) + \mathbf{i}(-\sin(\theta)e + \cos(\theta)f)$ belongs to $\mathcal{T}_{k_\theta x} Y$. Taking $f = 0$ and projecting to the real component, we deduce that $\tilde{E}^u(k_\theta x)$ contains $\tilde{E}^u(x)$. Since they have the same dimension, it follows that $\tilde{E}^u(k_\theta x) = \tilde{E}^u(x)$. In the same way, taking $e = 0$, we get $\tilde{E}^u(k_\theta x) = \mathbf{i}^{-1}\tilde{E}^s(x)$. Finally, $\tilde{E}^s(x) = \mathbf{i}\tilde{E}^u(x)$, as desired.

Fourth step: Y is an affine submanifold of Teich.

At every point $x \in Y$, the tangent space $T_x Y = \tilde{E}^u(x) \oplus \tilde{E}^s(x)$ is invariant by complex multiplication, by the third step. This implies that Y is a complex (holomorphic) submanifold of Teich, see e.g. [BER99, Proposition 1.3.14].

Let us show that Y is linear around any point $x_0 \in Y$ (we thank S. Cantat for the following argument). Working in charts and changing coordinates, we can assume that $x_0 = 0$ and that $\mathcal{T}_0 Y = \mathbb{C}^k \subset \mathbb{C}^N$, for $k = d_s + 1 = d_u + 1$. Around 0, the manifold Y can therefore be written as a graph $\{(z, f(z))\}$ for some holomorphic function from $\mathbb{C}^k$ to $\mathbb{C}^{N-k}$. At a point x close to 0, the tangent space $\mathcal{T}_x Y$ is $\{(v, Df(x)v) \ : \ v \in \mathbb{C}^k\}$. In particular, the real part of this tangent space is included in $\{(v, Df(x)v) \ : \ v \in \mathbb{R}^k\}$. Since the dimension of the real part of $\mathcal{T}_x Y$ is exactly k, it follows that $Df(x)v$ is real for any real vector v, i.e., all the matrix coefficients of $Df(x)$ are real. Since a real valued holomorphic function is constant, Df is constant. Therefore, Y is linear around x_0.

Fifth step: the distributions of d_u-dimensional subspaces E^u and E^s are integrable, and the integral leaves are affine submanifolds of Teich.

The strong unstable manifolds form a foliation $\mathcal{F}$ of Teich_1 with affine leaves (see Example 4.2). Moreover, the dimension of $T_x\mathcal{F} \cap T_x X$ is independent of $x \in X$, by the second step. It follows that the restriction of $\mathcal{F}$ to X defines a foliation of X, integrating the distribution of subspaces $T_x\mathcal{F} \cap T_x X = E^u(x)$. In particular, the leaf $W^u(x)$ integrating $E^u(x)$ is locally given by $X \cap \mathcal{F}_x$, which is also equal to $Y \cap \mathcal{F}_x$. Since Y is affine by the fourth step and $\mathcal{F}_x$ is affine, $W^u(x)$ is also affine. The argument is the same for W^s.

Sixth step: the measure $\tilde{\mu}_Y = \tilde{\mu} \otimes \mathrm{Leb}$ on Y is locally a multiple of the linear Lebesgue measure on the linear manifold Y.

Given $x \in Y$, fix a reference Lebesgue measure on Y around x (there is a priori no canonical choice of normalization), and denote by $\tilde{\phi}$ the density of $\tilde{\mu}_Y$ with respect to this Lebesgue measure. We will prove that $\tilde{\phi}$ is constant on strong stable and unstable manifolds in a neighborhood of x. Since the foliations W^s and W^u are smooth and jointly non-integrable, it follows from the classical Hopf argument that $\tilde{\phi}$ is constant. We will work in Y/Γ, around the point $x\Gamma$. Let us denote by ϕ the density of μ around $x\Gamma$.

Since μ has finite mass, we can consider a sequence of compactly supported smooth measures μ_n converging (for the total mass norm) to μ on X/Γ. For any $t \geq 0$,

$$|(g_t)_*\mu_n - \mu| = |(g_t)_*\mu_n - (g_t)_*\mu| = |\mu_n - \mu|.$$

Therefore, for any sequence t_n, the measures $(g_{t_n})_*\mu_n$ converge to μ.

Fix $M > 0$. Let $\phi_{n,t}$ denote the density of $(g_t)_*(\mu_n \otimes \mathrm{Leb})$ in a ball B around $x\Gamma$. Then, for any $n \in \mathbb{N}$ and any $M > 0$, the integral

$$\int_{y \in B} \int_{z \in W^u(y) \cap B} \min(|\phi_{n,t}(z) - \phi_{n,t}(y)|, M) \,\mathrm{dLeb}(z) \,\mathrm{dLeb}(y)$$

converges to 0 when t tends to $+\infty$. Indeed, the integrand is bounded by M, and converges almost everywhere to 0 since the flow is hyperbolic along almost every trajectory and the measure μ_n is smooth. Let us choose t_n such that this integral is at most 2^{-n}. Since $(g_{t_n})_*\mu_n$ converges to μ, the density ϕ_{n,t_n} converges almost everywhere to ϕ along a subsequence. This yields

$$\int_{y \in B} \int_{z \in W^u(y) \cap B} \min(|\phi(z) - \phi(y)|, M) \,\mathrm{dLeb}(z) \,\mathrm{dLeb}(y) = 0.$$

Letting M tend to infinity, we obtain that ϕ is almost everywhere constant along unstable manifolds in B, as desired.

Seventh step: the measure μ is ergodic.

If μ is not ergodic, we can consider an invariant set A for the action of $\mathrm{SL}(2,\mathbb{R})$ on X/Γ, with positive but not total measure. Consider $\tilde\nu$ the restriction of $\tilde\mu$ to the lift of A in Teich_1. The argument in the previous step applies to $\tilde\nu$ and shows that the density of $\tilde\nu \otimes \mathrm{Leb}$ on Y is locally constant. Since Y/Γ is connected, this implies that $\nu \otimes \mathrm{Leb}$ is equivalent to Lebesgue measure on Y/Γ. This is a contradiction, and proves the ergodicity of μ.

Among other things, we have shown that the measure $\tilde\mu$ is admissible. This proves Proposition 2.2. To conclude the proof, we have to complete the construction of the measures μ_u and μ_s forming the affine local product structure of μ.

Eighth step: construction of canonical volume forms $\mu_u(x)$ and $\mu_s(x)$, respectively on $E^u(x)$ and $E^s(x)$, in terms of $\tilde\mu$, which are constant respectively along W^u and W^s. They are only defined up to sign.

Let $x \in X$. Identifying locally Teich with $H^1(M,\Sigma;\mathbb{C})$ thanks to the period map, we write $x = a + \mathbf{i}b$. If $\nu_u(x)$ is any volume form on $\tilde E^u(x) = \mathbb{R}a + E^u(x)$, then it yields a volume form $\nu_s(x)$ on $\tilde E^s(x)$ thanks to the identification of the third step. The product $\nu_u(x) \wedge \nu_s(x)$ is a nonzero volume form on $\tilde E^u(x) \oplus \tilde E^s(x) = \mathcal{T}_x Y$, it is therefore proportional to $\tilde\mu_Y(x)$. Multiplying $\nu_u(x)$ by a unique (up to sign) normalization, we can ensure that $\nu_u(x) \wedge \nu_s(x) = \pm\tilde\mu_Y$. Finally, let $\mu_u(x)$ be the unique volume form on $E^u(x)$ such that $\nu_u(x)$ is the product of $\mu_u(x)$ and Lebesgue measure on $\mathbb{R}a$. Analogously, let $\mu_s(x)$ be the unique volume form on $E^s(x)$ such that $\nu_s(x)$ is the product of $\mu_s(x)$ and Lebesgue measure on $\mathbf{i}\mathbb{R}b$.

This construction is completely canonical up to sign, and $\tilde\mu(x) = \pm\, \mathrm{dLeb} \wedge \mu_u(x) \wedge \mu_s(x)$ by construction, where dLeb denotes Lebesgue measure along $\mathbb{R}\omega(x)$. Since $\tilde\mu$ is Γ-invariant, it follows that μ_u and μ_s are also Γ-invariant (possibly up to sign).

Since μ_u is constructed in a canonical way in terms of $\tilde\mu_Y$ and $\tilde\mu_Y$ is constant along unstable manifolds (by the sixth step), it follows that μ_u is constant along unstable manifolds. In the same way, μ_s is constant along stable manifolds.

Ninth step: there exists $d > 0$ such that $(g_t)_*\mu_u = e^{-dt}\mu_u$ and $(g_t)_*\mu_s = e^{dt}\mu_s$.

Since the action of $\mathrm{SL}(2,\mathbb{R})$ is ergodic, the action of the horocycle flow is also ergodic by Howe-Moore's theorem [HM79]. In particular, we can choose x whose orbit is dense. For $t \geq 0$, the measure $(g_t)_*\mu_u(x)$ is a volume form on $E^u(g_t x)$, and can therefore be written as $e^{d(t)}\mu_u(g_t x)$ for some $d(t) \in \mathbb{R}$. Since the measures μ_u are constant along the unstable manifolds of x and of $g_t x$, it follows that, for any point y in the horocycle through x, we also have $(g_t)_*\mu_u(y) = e^{d(t)}\mu_u(g_t y)$. Since this horocycle is dense, $(g_t)_*\mu_u(z) = e^{d(t)}\mu_u(g_t z)$ for any z.

The function $t \mapsto d(t)$ is continuous and satisfies $d(t+t') = d(t) + d(t')$, we may therefore write $d(t) = -dt$ for some $d \in \mathbb{R}$ (which has to be positive since the flow is expanding along unstable directions). We obtain $(g_t)_*\mu_u = e^{-dt}\mu_u$.

In the same way, we have $(g_t)_*\mu_s = e^{d't}\mu_s$ for some $d' \geq 0$. Since $\tilde\mu = \mathrm{dLeb} \wedge \mu_u \wedge \mu_s$ is g_t-invariant, it follows that $d = d'$.

This concludes the proofs of the ninth step and of Propositions 4.1 and 2.2. □

Remark 4.4. The scalar d constructed in Proposition 4.1 satisfies $d = d_u + 1 = d_s + 1$, where d_u and d_s are the dimensions respectively of E^u and E^s.

To prove this statement, let $\lambda_0, \dots, \lambda_{d_u}$ be the Lyapunov exponents of the Kontsevich-Zorich cocycle (see [For02]) restricted to the bundle $\tilde E^u$, for the measure μ. The Lyapunov exponents of g_t along $\tilde E^u$ are given by $\nu_0 = 1 + \lambda_0, \dots, \nu_{d_u} = 1 + \lambda_{d_u}$, and their sum is equal to d since there is no expansion in the bundle $\tilde E^u/E^u$. Since $\tilde E^s = \mathbf{i}\tilde E^u$, the Lyapunov exponents of the Kontsevich-Zorich cocycle along $\tilde E^s$ are also $\lambda_0, \dots, \lambda_{d_u}$, and it follows that the Lyapunov exponents of g_t along $\tilde E^s$ are $-1 + \lambda_0, \dots, -1 + \lambda_{d_u}$. Since g_t preserves the measure μ which is equivalent to

Lebesgue measure, the sum of its Lyapunov exponents vanishes. Hence, $\sum \lambda_i = 0$. Finally, $d = \sum \nu_i = d_u + 1 + \sum \lambda_i = d_u + 1$.

A suitably generalized Pesin formula also gives that d is the entropy of the measure μ for the flow g_t

5. A good Finsler metric on Teich

5.1. **Construction of the metric.** To define the Banach space satisfying the conclusions of Theorem 3.2, we will need a Finsler metric on Teich, with several good properties:

(1) It should be complete and Γ–invariant.
(2) It should behave in a controlled way close to infinity (technically, it should be slowly varying, see the definition below).
(3) Under the Teichmüller flow, the metric should be non-contracted in the unstable direction, and non-expanded in the stable direction.

It is certainly possible to cook up a metric satisfying these requirements using the Hodge metric of Forni on $H^1(M;\mathbb{R})$ [For02] and extending it first to $H^1(M,\Sigma;\mathbb{R})$ and then to $H^1(M,\Sigma;\mathbb{C})$ (compare for instance [ABEM06]). However, [AGY06] introduced a geometrically defined metric that turns out to satisfy all the above properties. This is the metric we will use for simplicity.

Let us describe this continuous Finsler metric on Teich. Since the tangent space of Teich is everywhere identified with $H^1(M,\Sigma;\mathbb{C})$ through the period map Φ, it is sufficient to define a family of norms on $H^1(M,\Sigma;\mathbb{C})$, depending continuously on the point $x \in$ Teich, as follows:

$$\|v\|_x = \sup \left| \frac{v(\gamma)}{\Phi(x)(\gamma)} \right|,$$

where γ runs over the saddle connections of the surface x. It is proved in [AGY06] that this is indeed a norm, and that the corresponding Finsler metric is complete. Let d denote the distance on Teich coming from this Finsler metric.

The two following straightforward lemmas show that this metric behaves well with respect to the Teichmüller flow.

Lemma 5.1. *The tangent vectors at* 0 *to the families* $t \mapsto g_t(x)$*,* $r \mapsto h_r(x)$*,* $r \mapsto \tilde{h}_r(x)$ *and* $\theta \mapsto k_\theta(x)$ *are all bounded by* 1 *in norm. Therefore,* $d(x, g_t x) \leq |t|$*,* $d(x, h_r x) \leq |r|$*,* $d(x, \tilde{h}_r(x)) \leq |r|$ *and* $d(x, k_\theta x) \leq |\theta|$*.*

Proof. Given x with $\Phi(x) = a + \mathbf{i}b$, we have $\Phi(g_t x) = e^t a + \mathbf{i}e^{-t} b$, hence the tangent vector of the curve $t \mapsto g_t x$ at 0 is $a - \mathbf{i}b$, which is clearly bounded by 1 from the formula. Moreover, $\Phi(h_r x) = a + rb + \mathbf{i}b$, hence the tangent vector to this curve at 0 is b, again bounded by 1. The computations are similar for $\tilde{h}_r$ and k_θ. □

Lemma 5.2. *The Teichmüller flow is non-contracting in the unstable direction and non-expanding in the stable direction, for the above metric. More precisely, for any* $t \geq 0$*, for* $v \in H^1(M,\Sigma;\mathbb{R})$ *and* $w \in H^1(M,\Sigma;\mathbf{i}\mathbb{R})$*, we have* $\|Dg_t(x)v\|_{g_t x} \geq \|v\|_x$ *and* $\|Dg_t(x)w\|_{g_t x} \leq \|w\|_x$*.*

Proof. We have $Dg_t(x)v = e^t v$. Moreover, if $\Phi(x) = a + \mathbf{i}b$, we have $\Phi(g_t x) = e^t a + \mathbf{i}e^{-t} b$. Therefore,

$$\|Dg_t(x)v\|_{g_t x} = \sup_\gamma \frac{e^t |v(\gamma)|}{|e^t a(\gamma) + \mathbf{i}e^{-t} b(\gamma)|} \geq \sup_\gamma \frac{e^t |v(\gamma)|}{|e^t a(\gamma) + \mathbf{i}e^{t} b(\gamma)|} = \|v\|_x .$$

The argument for w is the same. □

The same computation shows that $\|Dg_t(x)v\|_{g_t x} \leq e^{2t} \|v\|_x$ and $\|Dg_t(x)w\|_{g_t x} \geq e^{-2t} \|w\|_x$, which corresponds to the classical fact that the upper and lower Lyapunov exponents of the Teichmüller flow are respectively 2 and -2.

Let $\tilde{\mu}$ be an admissible measure, and let X denote its support. The above Finsler metric can be restricted to every stable or unstable manifold in X, and therefore defines distances d_{W^u}, d_{W^s} on those manifolds. For $r > 0$, we denote by $W^u_r(x)$ the ball of radius r around x in $E^u(x)$ for the distance d_{W^u}.

Fix $x \in X$. Let $\Psi = \Psi_x$ be the canonical local parametrization of the affine manifold $W^u(x)$ by its tangent plane $E^u(x)$. More formally, we define $\Psi(v)$ for $v \in E^u(x)$ as follows. Consider the path κ starting from x with $\kappa'(t) = v$ for all t. For small t, $\kappa(t)$ is well defined and belongs to $W^u(x)$. It is possible that $\kappa(t)$ is not defined for large t, since it could explode to infinity in Teich. If the path κ is well defined for all $t \in [0, 1]$, then we define $\Psi(v) = \kappa(1)$.

Let us denote by $B(0, r)$ the ball of radius r in $E^u(x)$, for the norm $\|\cdot\|_x$. The main result of this section is the following proposition, showing that the norm $\|\cdot\|_x$ varies slowly in fixed size neighborhoods of any point in the non-compact space X. This is a kind of bounded curvature behavior. Note however that this metric depends only in a continuous way on the point, so we can not use true curvature arguments.

Proposition 5.3. *The map Ψ is well defined on $B(0, 1/2)$, and $d_{W^u}(x, \Psi(v)) \leq 2 \|v\|_x$ there. Moreover, for $v \in B(0, 1/2)$, and for every $w \in E^u(x)$,*

$$1/2 \leq \frac{\|w\|_x}{\|w\|_{\Psi(v)}} \leq 2. \tag{5.1}$$

Finally, for $v \in B(0, 1/25)$, we have $d_{W^u}(x, \Psi(v)) \geq \|v\|_x /2$.

Before proving this proposition, let us give a simple consequence for the doubling property of μ_u. Again, the interest of this proposition is that the estimates are uniform, even though X is not compact.

Corollary 5.4. *Let $\tilde{\mu}$ be a measure with an affine local product structure, supported on a submanifold X. There exists $C > 0$ such that, for every $x \in X$ and every $r \leq 1/100$, $\mu_u(W^u_{2r}(x)) \leq C\mu_u(W^u_r(x))$.*

Proof. By Proposition 5.3, $\Psi^{-1}(W^u_{2r}(x)) \subset B(0, 4r)$ and $\Psi^{-1}(W^u_r(x)) \supset B(0, r/2)$. Since $y \mapsto \mu_u(y)$ is constant along $W^u(x)$, we have (denoting by d_u the dimension of $E^u(x)$)

$$\begin{aligned} \mu_u(W^u_{2r}(x)) &= \mu_u(x)(\Psi^{-1}(W^u_{2r}(x))) \leq \mu_u(x)(B(0, 4r)) \\ &= 8^{d_u}\mu_u(x)(B(0, r/2)) \leq 8^{d_u}\mu_u(x)(\Psi^{-1}(W^u_r(x))) = 8^{d_u}\mu_u(W^u_r(x)). \end{aligned}$$ □

The central point in the proof of Proposition 5.3 is the following proposition.

Proposition 5.5. *Let $\kappa : [0, 1] \to \mathrm{Teich}$ be a C^1 path. For each $v \in H^1(M, \Sigma; \mathbb{C})$,*

$$e^{-\operatorname{length}(\kappa)} \leq \frac{\|v\|_{\kappa(0)}}{\|v\|_{\kappa(1)}} \leq e^{\operatorname{length}(\kappa)}.$$

where $\operatorname{length}(\kappa) = \int_{t=0}^1 \|\kappa'(t)\|_{\kappa(t)} \,\mathrm{d}t$.

By symmetry, it is sufficient to prove the upper bound. For the proof, we start with the following lemma. We will write $\kappa(t)(\gamma)$ instead of $\Phi(\kappa(t))(\gamma)$.

Lemma 5.6. *Let γ be a saddle connection surviving in the surface $\kappa(t)$, $t \in [t_1, t_2]$. Then*

$$\frac{|\kappa(t_2)(\gamma)|}{|\kappa(t_1)(\gamma)|} \leq e^{\int_{t_1}^{t_2} \|\kappa'(t)\|_{\kappa(t)} \,\mathrm{d}t}.$$

Proof. Let $t \in [t_1, t_2]$. For small h,

$$\begin{aligned}\log|\kappa(t+h)(\gamma)| &= \log|\kappa(t)(\gamma) + h\kappa'(t)(\gamma) + o(h)| \\ &= \log|\kappa(t)(\gamma)| + \log|1 + h\kappa'(t)(\gamma)/\kappa(t)(\gamma) + o(h)| \\ &= \log|\kappa(t)(\gamma)| + h\Re(\kappa'(t)(\gamma)/\kappa(t)(\gamma)) + o(h).\end{aligned}$$

Hence, $t \mapsto \log|\kappa(t)(\gamma)|$ is differentiable, and its derivative $\Re(\kappa'(t)(\gamma)/\kappa(t)(\gamma))$ is bounded in norm by $\|\kappa'(t)\|_{\kappa(t)}$. The result follows. □

Proof of Proposition 5.5. For $0 \leq t_1' \leq t_2' \leq 1$, let us write

$$I(t_1', t_2') = e^{\int_{t_1'}^{t_2'} \|\kappa'(t)\|_{\kappa(t)} \,\mathrm{d}t}.$$

Let γ be a fixed saddle connection in the surface $\kappa(0)$, we want to show that

$$|v(\gamma)/\kappa(0)(\gamma)| \leq I(0,1)\, \|v\|_{\kappa(1)}. \tag{5.2}$$

We define by induction a sequence of times $t_0 < t_1 < \dots$, and sets Γ_n of saddle connections on the surface $\kappa(t_n)$, as follows.

Let $t_0 = 0$ and $\Gamma_0 = \{\gamma\}$. Assume t_n and Γ_n are defined. If all the saddle connections in Γ_n survive in the surfaces $\kappa(t)$, $t \in [t_n, 1]$, we let $t_{n+1} = 1$ and stop the process here. Otherwise, let $t_{n+1} \in (t_n, 1]$ be the first time one or several saddle connections in Γ_n disappear. If $\tilde\gamma$ is such a saddle connection, it means that other singularity points arrive on $\tilde\gamma$, i.e., $\tilde\gamma$ is split in $\kappa(t_{n+1})$ into a finite set $\{\gamma_1, \dots, \gamma_k\}$ of saddle connections, which are all in the same direction. In particular, in homology, $\tilde\gamma = \sum \gamma_i$, and moreover $|\kappa(t_{n+1})(\tilde\gamma)| = \sum |\kappa(t_{n+1})(\gamma_i)|$. We let Γ_{n+1} be the union of all the saddle connections in Γ_n that survive up to time t_{n+1}, and all the newly created saddle connections γ_i.

We now show that this inductive construction reaches $t = 1$ in a finite number of steps. Let $S_n = \sum_{\tilde\gamma\in\Gamma_n} |\kappa(t_n)(\tilde\gamma)|$. For $\tilde\gamma \in \Gamma_n$, Lemma 5.6 shows that $|\kappa(t_{n+1} - \epsilon)(\tilde\gamma)| \leq I(t_n, t_{n+1} - \epsilon)|\kappa(t_n)(\tilde\gamma)|$. Summing over $\tilde\gamma$ and letting ϵ tend to 0, we get $S_{n+1} \leq I(t_n, t_{n+1})S_n$. In particular, S_n is uniformly bounded, since $S_n \leq I(0, t_n)S_0 \leq I(0,1)S_0$. Moreover, the length of saddle connections in all the surfaces $\kappa(t)$ is bounded from below, since $\kappa([0,1])$ is a compact subset of the Teichmüller space. This implies that the cardinality of Γ_n is uniformly bounded. Since $\#\Gamma_{n+1} \geq \#\Gamma_n + 1$, this would give a contradiction if the inductive process did not stop after finitely many steps.

We claim that, for all n,

$$\sup_{\tilde\gamma\in\Gamma_n} |v(\tilde\gamma)/\kappa(t_n)(\tilde\gamma)| \leq I(t_n, t_{n+1}) \sup_{\tilde\gamma\in\Gamma_{n+1}} |v(\tilde\gamma)/\kappa(t_{n+1})(\tilde\gamma)|. \tag{5.3}$$

Let N be such that $t_N = 1$. Multiplying these inequalities for $n = 0, \dots, N-1$, we obtain (5.2), concluding the proof. We now prove (5.3). Let $\tilde\gamma \in \Gamma_n$. If $\tilde\gamma$ survives up to time t_{n+1}, Lemma 5.6 gives $|v(\tilde\gamma)/\kappa(t_n)(\tilde\gamma)| \leq I(t_n, t_{n+1})|v(\tilde\gamma)/\kappa(t_{n+1})(\tilde\gamma)|$, as desired. Otherwise, $\tilde\gamma$ is split at time t_{n+1} into finitely many saddle connections $\gamma_1, \dots, \gamma_k$. For small $\epsilon > 0$, the saddle connection $\tilde\gamma$ survives from time t_n to time $t_{n+1} - \epsilon$. Therefore, Lemma 5.6 gives $|v(\tilde\gamma)/\kappa(t_n)(\tilde\gamma)| \leq I(t_n, t_{n+1} - \epsilon)|v(\tilde\gamma)/\kappa(t_{n+1} - \epsilon)(\tilde\gamma)|$. When ϵ tends to 0, this tends to

$$\begin{aligned}I(t_n, t_{n+1})\frac{|v(\tilde\gamma)|}{|\kappa(t_{n+1})(\tilde\gamma)|} &= I(t_n, t_{n+1})\frac{|\sum v(\gamma_i)|}{\sum|\kappa(t_{n+1})(\gamma_i)|} \leq I(t_n, t_{n+1})\frac{\sum |v(\gamma_i)|}{\sum|\kappa(t_{n+1})(\gamma_i)|} \\ &\leq I(t_n, t_{n+1}) \sup \frac{|v(\gamma_i)|}{|\kappa(t_{n+1})(\gamma_i)|}.\end{aligned}$$

This proves (5.3). □

Proof of Proposition 5.3. Let κ be the path starting from x with $\kappa' = v$. If κ is well defined on an interval $[0, t_0]$, then for $t \in [0, t_0]$

$$\|\kappa'(t)\|_{\kappa(t)} = \|v\|_{\kappa(t)} \leq \|v\|_x e^{\int_0^t \|\kappa'(r)\|_{\kappa(r)} \, \mathrm{d}r},$$

by Proposition 5.5. Therefore, the function $t \mapsto G(t) = \int_0^t \|\kappa'(r)\|_{\kappa(r)} \, \mathrm{d}r$ satisfies $G'(t) \leq e^{G(t)} \|v\|_x$, i.e., $(-e^{-G(t)})' \leq \|v\|_x$. Integrating this inequality gives $G(t) \leq -\log(1 - t\|v\|_x)$, and therefore

$$\|\kappa'(t)\|_{\kappa(t)} \leq \frac{\|v\|_x}{1 - t\|v\|_x}.$$

If $\|v\|_x < 1$, this quantity remains bounded for $t \in [0, 1]$. Therefore, Ψ is well defined on such vectors v. In particular, Ψ is well defined on the ball $B(0, 1/2)$. Moreover, $d_{W^u}(x, \Psi(v)) \leq \int_0^1 \|\kappa'(t)\|_{\kappa(t)} \leq \|v\|_x / (1 - \|v\|_x)$. For $v \in B(0, 1/2)$, this gives

$$d_{W^u}(x, \Psi(v)) \leq 2\|v\|_x. \tag{5.4}$$

Using the same notation G as above, Proposition 5.5 shows that, for every $v \in B(0, 1/2)$ and every $w \in E^u(x)$, we have $e^{-G(1)} \leq \frac{\|w\|_x}{\|w\|_{\Psi(v)}} \leq e^{G(1)}$. Since $G(1) \leq \log 2$, this proves (5.1).

Let us now prove that, for $v \in B(0, 1/25)$, we also have

$$d_{W^u}(x, \Psi(v)) \geq \|v\|_x / 2. \tag{5.5}$$

Consider $\kappa : [0, 1] \to W^u(x)$ an almost minimizing path for the distance d_{W^u}, between x and $\Psi(v)$. By (5.4), its length is less than $1/10$. Let us lift κ to a path $\tilde{\kappa}$ taking values in $E^u(x)$, starting from 0 and such that $\kappa = \Psi \circ \tilde{\kappa}$, as long as $\tilde{\kappa}$ stays in $B(0, 1/2)$.

While $\tilde{\kappa}(t)$ is defined, we have by (5.1) $\|\tilde{\kappa}'(t)\|_x \leq 2\|\tilde{\kappa}'(t)\|_{\kappa(t)}$. Integrating this inequality from 0 to t, we get

$$\|\tilde{\kappa}(t)\|_x \leq \int_0^t \|\tilde{\kappa}'(r)\|_x \, \mathrm{d}r \leq 2\int_0^t \|\tilde{\kappa}'(r)\|_{\kappa(r)} \, \mathrm{d}r \leq 2\,\mathrm{length}(\kappa) \leq 1/5.$$

Therefore, $\tilde{\kappa}(t)$ stays in $B(0, 1/2)$, and the lifting process may be continued up to $t = 1$, where $\tilde{\kappa}(1) = v$. We get $\|v\|_x \leq 2\,\mathrm{length}(\kappa)$. Hence, $\|v\|_x \leq 2d_{W^u}(x, \Psi(v))$, proving (5.5). $\square$

5.2. **C^k norm and partitions of unity.** When $(E, \|\cdot\|)$ is a normed vector space and f is a C^k function on an open subset of E, let $c_k(f) = \sup |D^k f(x; v_1, \dots, v_k)|$ where the supremum is taken on the points x in the domain of f, and the tangent vectors $v_1, \dots, v_k$ of norm at most 1.

If an affine manifold has a Finsler metric, we can define in the same way the c_k coefficients of a function, using the affine structure to define the k-th differential at every point, and the Finsler metric to measure the tangent vectors. Note that the (possibly non-smooth) variation of the Finsler metric from point to point plays no role in this definition, since it only uses the Finsler metric at a fixed point. Those coefficients behave well under the composition with affine maps.

We can then define the C^k norm of a function by $\|f\|_{C^k} = \sum_{j=0}^k c_j(f)$. When we say that a function is C^k on a non-compact space, we really mean that its C^k norm is finite.

Remark 5.7. There are several more general situations where this definition has a natural extension. Consider for example the following case: W is an affine submanifold of an affine Finsler manifold Z, and v is a vector field defined on W (but pointing in any direction in Z). Then, for $x \in W$ and $v_1, \dots, v_k \in \mathcal{T}_x W$, the

k-th differential $D^k v(x; v_1, \dots, v_k)$ is well defined and belongs to the normed vector space $\mathcal{T}_x Z$. We can therefore define $c_k(v)$ as the supremum of the quantities $\left\|D^k v(x; v_1, \dots, v_k)\right\|_x$, for $x \in W$ and $v_1, \dots, v_k \in \mathcal{T}_x W$ with $\|v_i\|_x \leq 1$. Finally, we set as above $\|v\|_{C^k} = \sum_{j=0}^k c_j(v)$.

Note however that there are several situations where it is not possible to canonically define a C^k norm as above. For instance, on a general Finsler manifold, there is no canonical connection, and therefore $D^k f$ is not well defined. In the same way, in Remark 5.7, if W is not affine or if Z is not affine, then we can not define $\|v\|_{C^k}$. Of course, in a compact subset of W, one could choose charts to define such a norm, but it would depend on the choice of the charts – the equivalence class of the C^k norm is well defined, but the C^k norm itself is not. Further on, we will need to control constants precisely, and it will be very important for us to have a canonical norm.

Consider now an admissible measure $\tilde{\mu}$, supported on a manifold X. Since the local unstable manifolds $W^u(x)$ are affine manifolds, the previous discussion applies to them.

The next proposition constructs good partitions of unity on pieces of such unstable manifolds.

Proposition 5.8. *There exists a constant C with the following property. Let W be a compact subset of an unstable leaf $W^u(x)$. Then there exist finitely many C^∞ functions $(\rho_i)_{i \in I}$ on $W^u(x)$, taking values in $[0,1]$, with $\sum \rho_i = 1$ on W, $\sum \rho_i = 0$ outside of $\{y \in W^u(x) \,:\, d_{W^u}(y, W) \leq 1/200\}$, and each ρ_i is supported in a ball $W^u_{1/200}(x_i)$ for some $x_i \in W$. Moreover, we can ensure that $c_k(\rho_i) \leq C(k!)^2$, and every point of $W^u(x)$ belongs to at most C sets $W^u_{1/200}(x_i)$.*

The precise bound $C(k!)^2$ is not important for the applications we have in mind, what really matters is that we have a bound depending only on k, uniform in x.

Proof. By Proposition 5.3, the norm $\|\cdot\|_x$ is slowly varying in the sense of [Hör03, Definition 1.4.7]. Applying Theorem 1.4.10 there to the sequence $d_k = c/k^{3/2}$ for some $c > 0$, we get a sequence of functions ρ_i satisfying the conclusion of our proposition: they satisfy $c_k(\rho_i) \leq C^k (k!)^{3/2}$ for a constant C depending only on the dimension, so $c_k(\rho_i) \leq C'(k!)^2$, and moreover the assertions on the support are also satisfied. One should only be a little careful since the supports in [Hör03, Theorem 1.4.10] are controlled in terms of fixed norms $\|\cdot\|_x$, while our conclusion deals with the Finsler metric d_{W^u}. Since Proposition 5.3 shows that they are uniformly equivalent in small neighborhoods of the points, this is not an issue. ☐

The next lemma is a particular case of Proposition 5.8 (obtained by letting $W = W^u_{1/200}(x)$), and will be needed later on.

Lemma 5.9. *There exists a constant C with the following property. For any $x \in X$, there exists a function ρ on $W^u(x)$, supported in $W^u_{1/100}(x)$, taking values in $[0,1]$, equal to 1 on $W^u_{1/200}(x)$, with $c_k(\rho) \leq C(k!)^2$.*

The interest of this lemma is, again, that the estimates are uniform in x while this point lives in a noncompact space.

In the next statement, we do not use the distance induced by the Finsler metric on unstable manifolds, but the global distance. Since the previous arguments only rely on Proposition 5.5, which is satisfied in W^u as well as in the whole space, this lemma follows again from the same techniques.

Lemma 5.10. *There exists a constant C with the following property. Let F :* Teich $\to [1, \infty)$ *be a function such that $|\log F(x) - \log F(y)| \leq 2d(x,y)$ for any*

$x, y \in \mathrm{Teich}$. *For any* $V \geq 1$, *there exists a* C^∞ *function* $\rho_V : \mathrm{Teich} \to [0,1]$ *such that* $\rho_V(x) = 1$ *if* $F(x) \leq V$ *and* $\rho_V(x) = 0$ *if* $F(x) \geq 2V$, *satisfying* $c_k(\rho_V) \leq C(k!)^2$.

6. Recurrence estimates

Our goal in this section is to prove the following exponential recurrence estimate. Consider an admissible measure $\tilde\mu$ with its affine local product structure, supported on a submanifold X. If x is a translation surface, let $\mathrm{sys}(x)$ be its systole, i.e., the length of the shortest saddle connection in x.

Proposition 6.1. *Let* $\delta \in (0, 1/4)$. *There exists* $C > 0$ *such that, for any* $x \in X$ *and any* $t \geq 0$,

$$\frac{1}{\mu_u(W^u_{1/100}(x))} \int_{W^u_{1/100}(x)} V_\delta(g_t y) \,\mathrm{d}\mu_u(y) \leq Ce^{-(1-2\delta)t} V_\delta(x) + C,$$

where $V_\delta(x) = \max(1/\mathrm{sys}(x)^{1+\delta}, 1)$. *Moreover, the function* $\log V_\delta$ *is* $(1+\delta)$-*Lipschitz for the Finsler norm of the previous section.*

We will use the following lemma, which is due to Eskin-Masur [EM01] and Athreya [Ath06].

Lemma 6.2. *Fix a neighborhood* $\mathcal{V}$ *of the identity in* $\mathrm{SL}(2,\mathbb{R})$. *For every* $\delta > 0$, *there exists* $C > 0$ *such that, for all* $t > 0$, *there exist a function* $V_\delta^{(t)} : \mathrm{Teich} \to [1, \infty)$ *and a scalar* $b(t) > 0$ *satisfying the following property. For all* $x \in \mathrm{Teich}_1$,

$$\int_0^{2\pi} V_\delta^{(t)}(g_t k_\theta x) \,\mathrm{d}\theta \leq Ce^{-(1-\delta)t} V_\delta^{(t)}(x) + b(t).$$

Moreover,

$$V_\delta^{(t)}(gx) \leq C V_\delta^{(t)}(x) \tag{6.1}$$

for all $x \in \mathrm{Teich}$ *and all* $g \in \mathcal{V}$. *Finally, there exists a constant* $C_{\delta,t}$ *such that* $V_\delta^{(t)}/V_\delta \in [C_{\delta,t}^{-1}, C_{\delta,t}]$.

The order of quantifiers in our statement corrects a mistake in Athreya's Lemma 2.10.

In the next lemma, we transfer the previous estimate on circle averages to estimates on horocycle averages.

Lemma 6.3. *For every* $\delta > 0$, *there exists* C *such that, for any large enough* t, *there exists* $b(t) > 0$ *such that, for any* $x \in \mathrm{Teich}_1$,

$$\int_0^1 V_\delta^{(t)}(g_t h_r x) \,\mathrm{d}r \leq Ce^{-(1-\delta)t} V_\delta^{(t)}(x) + b(t)\,.$$

Proof. Using the decomposition ANK of $\mathrm{SL}(2,\mathbb{R})$, we can write uniquely $h_r = g_{\tau(r)} \tilde h_{\tilde r(r)} k_{\theta(r)}$, where the functions τ, $\tilde r$ and θ depend smoothly on r. One easily checks that $\theta'(0) \neq 0$. In particular, if n is large enough, $r \mapsto \theta(r)$ is a diffeomorphism on $[0, 1/n]$. Using the commutation relation $g_\tau \tilde h_{\tilde r} = \tilde h_{e^{-2\tau}\tilde r} g_\tau$, we get

$$\begin{aligned}\int_0^{1/n} V_\delta^{(t)}(g_t h_r x) \,\mathrm{d}r &= \int_0^{1/n} V_\delta^{(t)}(g_t g_\tau \tilde h_{\tilde r} k_\theta x) \,\mathrm{d}r \\ &= \int_0^{1/n} V_\delta^{(t)}(\tilde h_{\tilde r e^{-2(t+\tau)}} g_\tau g_t k_\theta x) \,\mathrm{d}r.\end{aligned}$$

By (6.1), this is bounded by

$$
\begin{aligned}
C\int_0^{1/n} V_\delta^{(t)}(g_t k_\theta x)\,\mathrm{d}r &= C\int_{\theta([0,1/n])} V_\delta^{(t)}(g_t k_u x)(\theta^{-1})'(u)\,\mathrm{d}u \\
&\leq C\int_0^{2\pi} V_\delta^{(t)}(g_t k_u x)\,\mathrm{d}u \\
&\leq Ce^{-(1-\delta)t}V_\delta^{(t)}(x)+b(t).
\end{aligned}
$$

Therefore,

$$
\int_0^1 V_\delta^{(t)}(g_t h_r x)\,\mathrm{d}r = \sum_{j=0}^{n-1}\int_0^{1/n} V_\delta^{(t)}(g_t h_r h_{j/n} x)\,\mathrm{d}r
$$

$$
\leq \sum_{j=0}^{n-1} Ce^{-(1-\delta)t}V_\delta^{(t)}(h_{j/n}x)+b(t).
$$

With (6.1), this gives the conclusion of the lemma. □

Lemma 6.4. *For every $\delta>0$, there exist C and τ such that, for any $t\geq 0$ and any $x\in \mathrm{Teich}_1$,*

$$
\int_0^1 V_\delta^{(\tau)}(g_t h_r x)\,\mathrm{d}r \leq Ce^{-(1-2\delta)t}V_\delta^{(\tau)}(x)+C\,. \tag{6.2}
$$

The difference with the previous lemma is that we obtain a result valid for all times, with constants independent of the time (while b depends on t in the statement of Lemma 6.3).

Proof. Let us fix τ and b such that, for every $x\in\mathrm{Teich}_1$,

$$
\int_0^1 V_\delta^{(\tau)}(g_\tau h_r x)\,\mathrm{d}r \leq e^{-(1-2\delta)\tau}\int_0^1 V_\delta^{(\tau)}(h_r x)+b. \tag{6.3}
$$

Their existence follows from Lemma 6.3 and (6.1). We can also assume that $e^{2\tau}$ is a (large) integer N.

Let us now prove that, for all $n\in\mathbb{N}$,

$$
\int_0^1 V_\delta^{(\tau)}(g_{(n+1)\tau}h_r x)\,\mathrm{d}r \leq e^{-(1-2\delta)\tau}\int_0^1 V_\delta^{(\tau)}(g_{n\tau}h_r x)\,\mathrm{d}r+b. \tag{6.4}
$$

A geometric series then shows (6.2) for times of the form $n\tau$, and the general result follows from (6.1).

To prove (6.4), write $g_{(n+1)\tau}h_r = g_\tau g_{n\tau}h_r = g_\tau h_{e^{2n\tau}r}g_{n\tau}$ with $e^{2n\tau}=N^n=M$. Then, writing $r'=Mr$,

$$
\begin{aligned}
\int_0^1 V_\delta^{(\tau)}(g_{(n+1)\tau}h_r x)\,\mathrm{d}r &= \sum_{j=0}^{M-1}\int_0^{1/M} V_\delta^{(\tau)}(g_{(n+1)\tau}h_r h_{j/M}x)\,\mathrm{d}r \\
&= \sum_{j=0}^{M-1}\int_0^{1/M} V_\delta^{(\tau)}(g_\tau h_{Mr}g_{n\tau}h_{j/M}x)\,\mathrm{d}r \\
&= \frac{1}{M}\sum_{j=0}^{M-1}\int_0^{1} V_\delta^{(\tau)}(g_\tau h_{r'}g_{n\tau}h_{j/M}x)\,\mathrm{d}r' \\
&\leq \frac{1}{M}\sum_{j=0}^{M-1}\left(e^{-(1-2\delta)\tau}\int_0^{1} V_\delta^{(\tau)}(h_{r'}g_{n\tau}h_{j/M}x)\,\mathrm{d}r'+b\right),
\end{aligned}
$$

where the last inequality follows from (6.3) applied to the point $g_{n\tau}h_{j/M}x$. Changing again variables in the opposite direction, we get (6.4). □

Proof of Proposition 6.1. The log-smoothness of V_δ readily follows from the fact that $\log \operatorname{sys}$ is 1-Lipschitz by [AGY06, Lemma 2.12].

Let τ be given by Lemma 6.4. Since V_δ is within a multiplicative constant of $V_\delta^{(\tau)}$, it also satisfies the inequality (6.2) (with a different constant C).

Fix $r \in [0, 1/100]$. Since μ_u is invariant under h_r,

$$\begin{aligned}\int_{W^u_{1/100}(x)} V_\delta(g_t y)\,\mathrm{d}\mu_u(y) &= \int_{W^u_{1/100}(x)} V_\delta(g_t h_r h_{-r} y)\,\mathrm{d}\mu_u(y)\\ &= \int_{h_{-r}W^u_{1/100}(x)} V_\delta(g_t h_r z)\,\mathrm{d}\mu_u(z) \leq \int_{W^u_{1/50}(x)} V_\delta(g_t h_r z)\,\mathrm{d}\mu_u(z).\end{aligned}$$

Averaging over r, we get

$$\begin{aligned}\int_{W^u_{1/100}(x)} V_\delta(g_t y)\,\mathrm{d}\mu_u(y) &\leq 100 \int_{r=0}^{1/100} \int_{W^u_{1/50}(x)} V_\delta(g_t h_r z)\,\mathrm{d}\mu_u(z)\,\mathrm{d}r\\ &\leq 100 \int_{W^u_{1/50}(x)} \left(\int_0^1 V_\delta(g_t h_r z)\,\mathrm{d}r \right) \mathrm{d}\mu_u(z).\end{aligned}$$

This is bounded by $\mu_u(W^u_{1/50}(x))(Ce^{-(1-2\delta)t}V_\delta(x) + C)$, using (6.2) for V_δ and the fact that $V_\delta(z)/V_\delta(x)$ is uniformly bounded for all $z \in W^u_{1/50}(x)$ (since $\log V_\delta$ is Lipschitz). The result follows since the measures of $W^u_{1/50}(x)$ and $W^u_{1/100}(x)$ are comparable by Corollary 5.4. □

7. Distributional coefficients

In this section, we introduce a distributional norm on smooth functions, similar in many respects to the norms introduced in [GL06] (the differences are the control at infinity, and the fact that we only use vector fields pointing in the stable direction or the flow direction – this is simpler than the approach of [GL06], and is made possible here by the smooth structure of the stable foliation). Let us fix $\tilde{\mu}$ an admissible measure with its affine local product structure, supported by a manifold X. Let also $\delta > 0$ be a fixed small number, as in the previous section.

Consider a smooth vector field v^s on a piece of unstable manifold $W^u_{1/100}(x)$, such that for every $y \in W^u_{1/100}(x)$, $v^s(y) \in E^s(y)$. We can define its c_k coefficients as in Remark 5.7. For a vector field $v^\omega(y) = \psi(y)\omega(y)$ defined on $W^u_{1/100}(x)$, we let its c_k coefficient be $c_k(\psi)$. The definitions of $\|v^s\|_{C^k}$ and $\|v^\omega\|_{C^k}$ follow. Let us stress that these definitions only involve base points that are located on an unstable manifold: this implies that these norms behave well under g_{-t}, which is contracting along such an unstable manifold, and is at the heart of the proof of Lemma 8.2 below.

We want to use such vector fields to differentiate functions, several times. However, the Lie derivative $L_{v_1}L_{v_2}f$ of a function f can only be defined if $L_{v_2}f$ is defined on an open set, which means that v_2 has to be defined on an open set. Therefore, we will need to extend the above vector fields to whole open sets, as follows.

Consider first a smooth vector field v^s on $W^u_{1/100}(x)$, pointing everywhere in the stable direction. We will now construct an extension $\overline{v^s}$ of v^s to a neighborhood of $W^u_{1/200}(x)$ in X.

For $y \in W^u_{1/100}(x)$, the stable manifold $W^s(y)$ is affine, its tangent space is everywhere equal to $E^s(y)$, and we may therefore define $\overline{v^s}(z) = v^s(y)$ for $z \in W^s(y)$: this extended vector field is still tangent to the direction E^s. Finally, for small t, we define $\overline{v^s}(g_t z) = Dg_t(z) \cdot \overline{v^s}(z)$, i.e., we push the vector field by g_t.

Since g_t sends stable direction to stable direction, $\overline{v^s}$ is everywhere tangent to the stable direction. Since the unstable direction, the stable direction and the flow direction are transverse at every point, we can uniquely parameterize a point in a neighborhood of $W^u_{1/200}(x)$ as $g_t(z)$ for some $z \in W^s_\epsilon(y)$, $y \in W^u_{1/200+\epsilon}(x)$. This defines the extension of v^s.

If v^ω is a vector field along $W^u_{1/100}(x)$ pointing everywhere in the flow direction, we can also define an extension $\overline{v^\omega}$ as follows. Along W^u, write $v^\omega(y) = \psi(y)\omega(y)$, where the function ψ is smooth. Let $\overline{v^\omega}(g_t z) = \psi(y)\omega(g_t z)$ for $z \in W^s_\epsilon(y)$ as above, this defines a smooth vector field extending v^ω as desired.

For $k, \ell \in \mathbb{N}$, $\alpha \in \{s, \omega\}^\ell$ and $x \in X$, we can now define a distributional coefficient of the C^∞ function f at x, as follows (the function V_δ has been defined in Proposition 6.1):

$$e_{k,\ell,\alpha}(f;x) := \frac{1}{V_\delta(x)} \frac{1}{\mu_u(W^u_{1/200}(x))} \sup \left| \int_{W^u_{1/200}(x)} \phi \cdot L_{\overline{v_1}} \cdots L_{\overline{v_\ell}} f \, \mathrm{d}\mu_u \right|, \tag{7.1}$$

where the supremum is over all compactly supported functions $\phi : W^u_{1/200}(x) \to \mathbb{C}$ with $\|\phi\|_{C^{k+\ell}} \leq 1$, and all vector fields $v_1, \dots, v_\ell$ defined on $W^u_{1/100}(x)$ such that $v_j(y) \in E^s(y)$ if $\alpha_j = s$ and $v_j(y) \in \mathbb{R}\omega(y)$ if $\alpha_j = \omega$, and $\|v_j\|_{C^{k+\ell+1}(W^u_{1/100}(x))} \leq 1$. Note that the domain of definition of the vector fields is *larger* than the domain of integration in (7.1) – this will be useful for extension purposes below. Note also that we use the Lie derivative with respect to the extended vector fields $\overline{v_j}$, but the norm requirements on the vector fields v_j are only along W^u and not in the transverse direction.

Define $e_{k,\ell,\alpha}(f) = \sup_x e_{k,\ell,\alpha}(f;x)$. Let $e_{k,\ell}(f) = \sum_{\alpha \in \{s,\omega\}^\ell} e_{k,\ell,\alpha}(f)$. Finally, let

$$\|f\|_k = \sup_{0 \leq \ell \leq k} e_{k,\ell}(f). \tag{7.2}$$

Remark 7.1. If $f_1 \in \mathcal{D}^\Gamma$ then we have the estimate

$$\int_{X/\Gamma} f_1 f \, \mathrm{d}\mu \leq C(f_1) e_{k,0}(f) \leq C(f_1) \|f\|_k, \quad f \in \mathcal{D}^\Gamma,$$

where $C(f_1)$ depends on the support of f_1 as well as its C^k norm therein. This is readily obtained by decomposing f_1 as a sum of finitely many functions with small support (using partitions of unity), using locally the disintegration of μ along local unstable manifolds, and applying the definition of $e_{k,0}$ to bound the integrals along those.

We will also need a weaker norm, that we denote by $\|\cdot\|'_k$, given by

$$\|f\|'_k = \sup \frac{1}{V_\delta(x)} \frac{1}{\mu_u(W^u_{1/200}(x))} \left| \int_{W^u_{1/200}(x)} \phi \cdot L_{\overline{v_1}} \cdots L_{\overline{v_\ell}} f \, \mathrm{d}\mu_u \right|, \tag{7.3}$$

where the supremum is over $0 \leq \ell \leq k-1$, over all points $x \in X$, all compactly supported functions $\phi : W^u_{1/200}(x) \to \mathbb{C}$ with $\|\phi\|_{C^{k+\ell+1}} \leq 1$, and all vector fields $v_1, \dots, v_\ell$ defined on $W^u_{1/100}(x)$ and pointing either in the stable direction or in the flow direction, such that $\|v_j\|_{C^{k+\ell+1}(W^u_{1/100}(x))} \leq 1$. Apart from constants, the difference with the norm $\|f\|_k$ is that we allow less derivatives (at most $k-1$ instead of k), and that the test function ϕ has one more degree of smoothness (it is in $C^{k+\ell+1}$ instead of $C^{k+\ell}$). Therefore, the norm $\|f\|'_k$ is weaker in all directions than the norm $\|f\|_k$. Hence, the following compactness result is not surprising.

Proposition 7.2. *Let K be a compact set mod Γ, and let $k \in \mathbb{N}$. Let f_n be a sequence of functions in $\mathcal{D}^\Gamma$, supported in K, and with $\|f_n\|_k \leq 1$. Then there exists a subsequence $f_{j(n)}$ which is Cauchy for the norm $\|\cdot\|'_k$.*

In other words, if we work with the completions of the spaces, then the unit ball for the norm $\|\cdot\|_k$ is relatively compact for the norm $\|\cdot\|'_k$ if we consider only functions on X/Γ that are supported in a fixed compact set.

The rest of this subsection is devoted to the proof of this proposition (it is similar to the proof of Lemma 2.1 in [GL06]). We will need a preliminary lemma.

Let us fix for any r a C^r norm on the functions supported in K, such that this norm is Γ-invariant. Such a norm is not canonically defined, but this will not be a problem in the statements or results to follow since multiplicative constants do not matter.

Lemma 7.3. *There exists a constant $C(k, \ell, K)$ such that any smooth function f supported in K satisfies the following property. For any $x \in K$, any $C^{k+\ell}$ vector fields $v_1, \dots, v_\ell$ defined on a neighborhood of $W^u_{1/100}(x)$ with $\|v_j\|_{C^{k+\ell}} \leq 1$, and any $C^{k+\ell}$ function ϕ, compactly supported on $W^u_{1/200}(x)$ with $\|\phi\|_{C^{k+\ell}} \leq 1$,*

$$\left| \int_{W^u_{1/200}(x)} \phi \cdot L_{v_1} \cdots L_{v_\ell} f \, \mathrm{d}\mu_u \right| \leq C \sum_{\ell' \leq \ell} e_{k,\ell'}(f).$$

The interest of this lemma is that the vector fields v_j can be any vector fields, not only canonical extensions of vector fields pointing in the stable direction or in the flow direction. Moreover, we also weaken the smoothness of the vector fields v_j, requiring them only to be $C^{k+\ell}$ instead of $C^{k+\ell+1}$.

Proof. We prove the statement of the lemma by induction on ℓ. For $\ell = 0$, this is clear from the definitions. Let us decompose the vector field v_1 as $v_1^u + v_1^s + v_1^\omega$ where those three components point, respectively, in the unstable direction, in the stable direction and in the flow direction. Along $W^u_{1/100}(x)$, decomposing v_1^s along coordinates vector fields, we can write it as a linear combination of vector fields of the form $\psi_1^s w_1^s$ where ψ_1^s is a function bounded in $C^{k+\ell}$ and w_1^s is a C^∞ vector field with $\|w_1^s\|_{C^{k+\ell+1}} \leq C$. To simplify notations, we will omit a summation and assume that we can write $v_1^s(y) = \psi_1^s(y) w_1^s(y)$. In the same way, we write $v_1^\omega(y) = \psi_1^\omega(y)\omega(y)$ where $\|\psi_1^\omega\|_{C^{k+\ell}} \leq C$. For convenience, we introduce the notation $w_1^\omega = \omega$.

Let $g = L_{v_2} \dots L_{v_\ell} f$. Since $L_{v_1} g$ only depends on the value of the vector field v_1 (and not its derivatives), we have, along $W^u_{1/200}(x)$, $L_{v_1} g = L_{v_1^u} g + \psi_1^s L_{\overline{w_1^s}} g + \psi_1^\omega L_{\overline{w_1^\omega}} g$. Moreover,

$$\int_{W^u_{1/200}(x)} \phi \cdot L_{v_1^u} g \, \mathrm{d}\mu_u = -\int_{W^u_{1/200}(x)} L_{v_1^u} \phi \cdot g \, \mathrm{d}\mu_u,$$

which is bounded by $C \sum_{\ell' \leq \ell - 1} e_{k,\ell'}(f)$ by the induction hypothesis, since the function $L_{v_1^u}\phi$ is $C^{k+\ell-1}$ and is multiplied by $\ell - 1$ derivatives of f against $C^{k+\ell-1}$ vector fields.

It remains to bound $\int_{W^u_{1/200}(x)} \phi \psi_1^{\alpha_1} \cdot L_{\overline{w_1^{\alpha_1}}} L_{v_2} \cdots L_{v_\ell} f \, \mathrm{d}\mu_u$, for some $\alpha_1 \in \{s, \omega\}$. Let us exchange the vector fields to put $L_{\overline{w_1^{\alpha_1}}}$ in the last position. Since $[L_v, L_w] = L_{[v,w]}$, the error we make is bounded by the integral of a $C^{k+\ell}$ function multiplied by $\ell - 1$ derivatives of f against $C^{k+\ell-1}$ vector fields. By the induction hypothesis, this is again bounded by $C \sum_{\ell' \leq \ell - 1} e_{k,\ell'}(f)$.

It remains to bound $\int_{W^u_{1/200}(x)} \phi \psi_1^{\alpha_1} \cdot L_{v_2} \cdots L_{v_\ell} L_{\overline{w_1^{\alpha_1}}} f \, \mathrm{d}\mu_u$. In the same way as above, we decompose v_2 into its unstable, stable and flow part, integrate by parts to get rid of the unstable part, and exchange the vector fields to put the remaining parts of v_2 at the end. Iterating this process ℓ times, we end up with an estimate

$$\left|\int_{W^u_{1/200}(x)} \phi \cdot L_{v_1} \cdots L_{v_\ell} f \,\mathrm{d}\mu_u\right|$$
$$\leq C \sum_{\ell' \leq \ell-1} e_{k,\ell'}(f) + C \sup_{\alpha\in\{s,\omega\}^\ell} \left|\int_{W^u_{1/200}(x)} \phi\psi_1^{\alpha_1}\cdots\psi_\ell^{\alpha_\ell} \cdot L_{\overline{w_1^{\alpha_1}}}\cdots L_{\overline{w_\ell^{\alpha_\ell}}} f \,\mathrm{d}\mu_u\right|.$$

By construction, the vector fields $\overline{w_j^{\alpha_j}}$ are canonical extensions of $C^{k+\ell+1}$ vector fields defined along $W^u_{1/200}(x)$ and pointing in the stable or flow direction. Therefore, the latter integrals are bounded by $Ce_{k,\ell}(f)$ by definition of this coefficient. □

Proof of Proposition 7.2. The first step of the proof is to show that, to estimate $\|f\|'_k$, it is sufficient to work with finitely many unstable manifolds. More precisely, we will show that, for any $\epsilon > 0$, there exist finitely many points $(x_i)_{i\in I}$ such that, for any function f supported in K and Γ-invariant,

$$\|f\|'_k \leq C\epsilon \|f\|_k + C \sup \left|\int_{W^u_{1/200}(x_i)} \phi \cdot L_{v_1}\cdots L_{v_\ell} f \,\mathrm{d}\mu_u\right|, \tag{7.4}$$

where the supremum is taken over all $0 \leq \ell \leq k-1$, all $i \in I$, all functions ϕ compactly supported on $W^u_{1/200}(x_i)$ and all vector fields v_j defined in some fixed neighborhood $\mathcal{U}_i$ of $W^u_{1/100}(x_i)$ with $C^{k+\ell+1}$ norm bounded by 1.

Since K/Γ is compact, it is sufficient to show that integrals along the unstable manifold of a point x_1 can be controlled by similar integrals along the unstable manifold of a nearby point x_0. Let x_0, x_1 be two nearby points in K (so that their unstable spaces $E^u(x_0)$ and $E^u(x_1)$ are also close). Consider a smooth path x_t from x_0 to x_1, and a smooth family of maps sending $E^u(x_0)$ to $E^u(x_t)$. Parameterizing locally the (affine) unstable manifold of the point x_t by its tangent space (by the map Ψ_{x_t} introduced before Proposition 5.3), we obtain a family of affine maps $\Phi_t : W^u_{1/50}(x_0) \to W^u(x_t)$ with $\Phi_0 = \mathrm{id}$, that we extend smoothly to diffeomorphisms defined on a neighborhood of $W^u_{1/50}(x_0)$.

Fix $0 \leq \ell \leq k-1$ and consider a $C^{k+\ell+1}$ function ϕ compactly supported on $W^u_{1/400}(x_1)$, and $C^{k+\ell+1}$ vector fields $v_1, \dots, v_\ell$ along $W^u_{1/50}(x_1)$, each of them pointing either in the stable direction or in the flow direction, with $C^{k+\ell+1}$ norm bounded by 1. We want to bound the integral

$$I_1 = \int_{W^u(x_1)} \phi \cdot L_{\overline{v_1}}\cdots L_{\overline{v_\ell}} f \,\mathrm{d}\mu_u,$$

using data along $W^u(x_0)$.

For each t, we define vector fields v_j^t on a neighborhood of $W^u_{1/75}(x_t)$ by $v_j^0 = (\Phi_1)^*\overline{v_j}$, and $v_j^t = (\Phi_t)_* v_j^0$. Letting $J_t \in (0, +\infty)$ be the jacobian of Φ_t from $W^u(x_0)$ to $W^u(x_t)$, we can rewrite I_1 as a sum of two terms

$$I_1 = \int_{W^u(x_0)} \phi\circ\Phi_1 \cdot L_{v_1^0}\cdots L_{v_\ell^0}(f\circ\Phi_1) J_1 \,\mathrm{d}\mu_u = \int_{W^u(x_0)} \phi\circ\Phi_1 \cdot L_{v_1^0}\cdots L_{v_\ell^0} f \cdot J_1 \,\mathrm{d}\mu_u$$
$$+ \int_{t=0}^1 \frac{\partial}{\partial t}\left(\int_{W^u(x_0)} \phi\circ\Phi_1 \cdot L_{v_1^0}\cdots L_{v_\ell^0}(f\circ\Phi_t)\cdot J_1 \,\mathrm{d}\mu_u\right) \mathrm{d}t.$$

The first term is bounded by the second term in the right hand side of (7.4). Writing $w_t = (\partial\Phi_t/\partial t) \circ \Phi_t^{-1}$, the integrand of the second term at fixed t is

$$\int_{W^u(x_0)} \phi \circ \Phi_1 \cdot L_{v_1^0} \cdots L_{v_\ell^0}((L_{w_t} f) \circ \Phi_t) \cdot J_1 \, \mathrm{d}\mu_u$$
$$= \int_{W^u(x_t)} \phi \circ \Phi_1 \circ \Phi_t^{-1} \cdot L_{v_1^t} \cdots L_{v_\ell^t} L_{w_t} f \cdot J_1 J_t^{-1} \, \mathrm{d}\mu_u.$$

This is an integral along an unstable manifold of a $C^{k+\ell+1}$ function multiplied by $\ell+1$ derivatives of f against $C^{k+\ell+1}$ vector fields. By Lemma 7.3 (applied to $\ell' = \ell+1$, which is licit since $\ell < k$ by assumption), this is bounded in terms of $\|f\|_k$. Moreover, if x_0 and x_1 are close enough, the $C^{k+\ell+1}$ norm of the vector field w_t is arbitrarily small, and we get that this integral is bounded by $C\epsilon \|f\|_k$. Putting together the two terms, we see that I_1 is bounded by the right hand side of (7.4). Up to constants (which do depend on K), the norm $\|f\|'_k$ is defined using integrals similar to I_1, but where ϕ is allowed to have a larger support $W^u_{1/200}(x_1)$ and the v_j may have a smaller domain of definition $W^u_{1/100}(x_1)$. However, this is not a problem, since those more general integrals can be decomposed as sums of a bounded number of integrals like I_1, using partitions of unity. This concludes the proof of (7.4).

It is now easy to conclude the proof. Fix smooth bump functions ρ_i compactly supported in $\mathcal{U}_i$ (the domain of definition of the v_j in (7.4)) and equal to 1 in a neighborhood of $W^u_{1/200}(x_i)$. Since $C^{k+\ell+1}$ is compactly included in $C^{k+\ell}$, for each x_i, $i \in I$, we can choose finitely many functions $\phi_{m,i}$ compactly supported in $W^u_{1/200}(x_i)$ and finitely many vector fields $v_{j,m,i}$ defined in $\mathcal{U}_i$, such that for all functions ϕ and vector fields v_j which are bounded by 1 in $C^{k+\ell+1}$, there exists m such that ϕ and $\rho_i v_j$ are ϵ-close to $\phi_{m,i}$ and $\rho_i v_{j,m,i}$ in $C^{k+\ell}$. By Lemma 7.3, this gives with (7.4)

$$\|f\|'_k \leq C'\epsilon \|f\|_k + \sup_{i,m} \left| \int_{W^u_{1/200}(x_i)} \phi_{m,i} \cdot L_{v_{1,m,i}} \cdots L_{v_{\ell,m,i}} f \, \mathrm{d}\mu_u \right|.$$

Consider now a sequence f_n with $\|f_n\|_k \leq 1$. We extract a subsequence $f_{j(n)}$ along which all the finitely many quantities $\int_{W^u_{1/200}(x_i)} \phi_{m,i} \cdot L_{v_{1,m,i}} \cdots L_{v_{\ell,m,i}} f_{j(n)} \, \mathrm{d}\mu_u$ converge. It follows that $\limsup_{n,n' \to \infty} \left\|f_{j(n)} - f_{j(n')}\right\|'_k \leq 2C'\epsilon$. Letting ϵ tend to 0 and using a standard diagonal argument, we get the required Cauchy sequence. □

8. A good bound on the essential spectral radius of $\mathcal{M}$

Let $\tilde{\mu}$ be an admissible measure with its affine local product structure, supported by a submanifold X of Teich_1. In this section, we prove Theorem 3.2. As in the statement of this theorem, let us write $\mathcal{M}f = \int_{t=0}^{\infty} e^{-4\delta t} \mathcal{L}_t f \, \mathrm{d}t$ (to be interpreted as explained in §3.2), where $\delta > 0$ is fixed and $\mathcal{L}_t f = f \circ g_t$.

To prove Theorem 3.2, we have to construct a good norm on $\mathcal{D}^\Gamma$. It turns out that the norms $\|\cdot\|_k$ that we have constructed in the previous section in (7.2) are suitable for this purpose. The following statement contains Theorem 3.2 (see also Remark 7.1).

Theorem 8.1. *For all k, there exists $C > 0$ such that $\|\mathcal{L}_t f\|_k \leq C \|f\|_k$, uniformly in $t \geq 0$. Therefore, $\mathcal{M}$ acts continuously on the completion of $\mathcal{D}^\Gamma$ for the norm $\|\cdot\|_k$.*

Moreover, if k is large enough, then the essential spectral radius of $\mathcal{M}$ on this space is at most $1 + \delta$.

This section is devoted to the proof of this result. Until the end of its proof, we will always specify if a constant depends on k, by using a subscript as in C_k. Most constants will be independent of k, and this will be very important for the argument, since k will be chosen only at the very end of the proof.

For technical reasons, it is convenient to work with another norm that is equivalent to $\|\cdot\|_k$. For $A \geq 1$, let us first define a norm equivalent to $\|\cdot\|_{C^k}$, by $\|f\|_{C^k_A} = \sum_{j=0}^k c_j(f)/(j!A^j)$. Since $c_j(fg) \leq \sum_{m=0}^j \binom{j}{m} c_m(f)c_{j-m}(g)$, it follows that $\|fg\|_{C^k_A} \leq \|f\|_{C^k_A} \|g\|_{C^k_A}$. Moreover, for any fixed C^k function f and any $\epsilon > 0$, if A is large enough, then $\|f\|_{C^k_A} \leq (1+\epsilon)\sup|f|$. Let us define $e^A_{k,\ell,\alpha}(f;x)$ like $e_{k,\ell,\alpha}(f;x)$, but replacing the requirements $\|\phi\|_{C^{k+\ell}} \leq 1$ and $\|v_j\|_{C^{k+\ell+1}} \leq 1$ (for the supremum taken in (7.1)) by $\|\phi\|_{C^{k+\ell}_A} \leq 1$ and $\|v_j\|_{C^{k+\ell+1}_A} \leq 1$.

We will need to deal separately with the case where all the vector fields in the definition of $e^A_{k,\ell,\alpha}$ point in the stable direction, and the case where at least one vector field points in the flow direction. Let us therefore define $e^A_{k,\ell,s}(f;x) = e^A_{k,\ell,\{s,\dots,s\}}(f;x)$, and $e^A_{k,\ell,\omega}(f;x) = \sup e^A_{k,\ell,\alpha}(f;x)$, where the supremum is over all $\alpha \in \{s,\omega\}^\ell$ different from $\{s,\dots,s\}$. Let $e^A_{k,\ell,s}(f) = \sup_x e^A_{k,\ell,s}(f;x)$, and similarly for $e^A_{k,\ell,\omega}(f)$. For $B \geq 1$, let $\|f\|^{A,B}_{k,s} = \sum_{\ell=0}^k B^{-\ell} e^A_{k,\ell,s}(f)$, and similarly for $\|f\|^{A,B}_{k,\omega}$. Finally, let $\|f\|^{A,B}_k = \|f\|^{A,B}_{k,s} + \|f\|^{A,B}_{k,\omega}$. This norm is equivalent to $\|f\|_k$, but more convenient for a lot of inequalities.

In the statements below, when we say "for all large enough A, B...", we mean: if A is large enough, then, if B is large enough (possibly depending on A), then... The assumption "for all large enough k, A, B" should be interpreted in the same way.

We now start the proof. Some arguments are borrowed from [GL06]. We write $\mathcal{D}$ for the set of C^∞ functions supported in a compact set mod Γ. It contains the previously defined set $\mathcal{D}^\Gamma$ of functions in $\mathcal{D}$ that are Γ-invariant.

Lemma 8.2. *There exists a constant $C_0 \geq 1$ satisfying the following property. For every $k, \ell \in \mathbb{N}$ and every $\alpha \in \{s,\omega\}^\ell$, if A is large enough, then for every $t \geq 0$, every $f \in \mathcal{D}$ and every $x \in X$,*

$$e^A_{k,\ell,\alpha}(f \circ g_t; x) \leq C_0 e^A_{k,\ell,\alpha}(f) \left(e^{-(1-2\delta)t} + 1/V_\delta(x)\right). \tag{8.1}$$

Proof. We first give the proof for $\ell = 0$.

Fix some point x, and some compactly supported function $\phi : W^u_{1/200}(x) \to \mathbb{C}$ with $\|\phi\|_{C^k_A} \leq 1$. We want to estimate $\int_{W^u_{1/200}(x)} \phi(y) \cdot f \circ g_t(y) \,\mathrm{d}\mu_u(y)$. We change variables, letting $z = g_t(y)$. By Proposition 4.1, the resulting jacobian has the form e^{-dt} for some $d > 0$. The integral becomes an integral over $g_t(W^u_{1/200}(x))$. Proposition 5.8 provides a partition of unity $(\rho_i)_{i\in I}$ on this set, with good properties. In particular, ρ_i is supported in a ball $W^u_{1/200}(x_i)$. The integral becomes

$$\sum_i \int_{W^u_{1/200}(x_i)} \rho_i(z)\phi(g_{-t}z) \cdot f(z)\, e^{-dt} \,\mathrm{d}\mu_u(z).$$

Since g_{-t} is affinely contracting along W^u, $\|\phi \circ g_{-t}\|_{C^k_A} \leq \|\phi\|_{C^k_A} \leq 1$. Therefore, the C^k_A norm of $\rho_i \cdot \phi \circ g_{-t}$ is bounded by $\|\rho_i\|_{C^k_A}$. If A is large enough, this is at most 2 (since the coefficients c_m of ρ_i, for $1 \leq m \leq k$, are uniformly bounded by

Proposition 5.8). Therefore, the above integral is bounded by

$$
\sum_i Ce^A_{k,0}(f)V_\delta(x_i)\mu_u(W^u_{1/200}(x_i))e^{-dt} \leq Ce^A_{k,0}(f)\sum_i \int_{W^u_{1/200}(x_i)} V_\delta(z)\, e^{-dt}\, \mathrm{d}\mu_u(z), \tag{8.2}
$$

since $\log V_\delta$ is Lipschitz by Proposition 6.1. The covering multiplicity of the sets $W^u_{1/200}(x_i)$ is uniformly bounded, by Proposition 5.8. Moreover, all those sets are included in $\{z \ :\ d(z, g_t(W^u_{1/200}(x))) \leq 1/200\}$, which is itself included in $g_t(W^u_{1/100}(x))$ since g_{-t} contracts the distance along W^u. Therefore, (8.2) is bounded by

$$
Ce^A_{k,0}(f)\int_{g_t(W^u_{1/100}(x))} V_\delta(z)\, e^{-dt}\, \mathrm{d}\mu_u(z) = Ce^A_{k,0}(f)\int_{W^u_{1/100}(x)} V_\delta(g_t y)\, \mathrm{d}\mu_u(y).
$$

By Proposition 6.1, this is bounded by $Ce^A_{k,0}(f)\mu_u(W^u_{1/100}(x))(e^{-(1-2\delta)t}V_\delta(x)+1)$. Finally,

$$
\frac{1}{V_\delta(x)}\frac{1}{\mu_u(W^u_{1/200}(x))}\left|\int_{W^u_{1/200}(x)} \phi\cdot f\circ g_t\, \mathrm{d}\mu_u(y)\right| \leq Ce^A_{k,0}(f)\frac{\mu_u(W^u_{1/100}(x))}{\mu_u(W^u_{1/200}(x))}\left(e^{-(1-2\delta)t}+1/V_\delta(x)\right).
$$

The ratio of the measures is bounded, by Corollary 5.4. This proves (8.1) when $\ell=0$.

Assume now $\ell>0$, we have to estimate

$$
\int_{W^u_{1/200}(x)} \phi\cdot L_{\overline{v_1}}\cdots L_{\overline{v_\ell}}(f\circ g_t)\, \mathrm{d}\mu_u, \tag{8.3}
$$

where the vector fields v_j are defined on $W^u_{1/100}(x)$, satisfy $\|v_j\|_{C^{k+\ell+1}_A}\leq 1$, and point in the direction E^s or $\mathbb{R}\omega$. Consider a function ρ equal to 1 in $W^u_{1/200}(x)$ and compactly supported in $W^u_{1/100}(x)$ (as constructed in Lemma 5.9), and define a new vector field $v_{j,1}=\rho\cdot v_j$. It coincides with v_j on $W^u_{1/200}(x)$, therefore the integral (8.3) can also be written using $v_{j,1}$ instead of v_j. Moreover, if A is large enough, the definition of the $C^{k+\ell+1}_A$ norm ensures that

$$
\|v_{j,1}\|_{C^{k+\ell+1}_A} = \|\rho\cdot v_j\|_{C^{k+\ell+1}_A} \leq \|\rho\|_{C^{k+\ell+1}_A}\|v_j\|_{C^{k+\ell+1}_A} \leq 2^{1/\ell}.
$$

Let $w_j=(g_t)_* v_{j,1}$. Since the extension $\overline{w_j}$ is defined using the affine structure and the flow direction, which are invariant under the affine flow g_t, it follows that $\overline{w_j}=(g_t)_*\overline{v_{j,1}}$. Therefore,

$$
L_{\overline{v_{1,1}}}\cdots L_{\overline{v_{\ell,1}}}(f\circ g_t)(y) = L_{\overline{w_1}}\cdots L_{\overline{w_\ell}}f(g_t y).
$$

We claim that the vector fields w_j are bounded by $2^{1/\ell}$ in $C^{k+\ell+1}_A$ (even better, $c_m(w_j)\leq c_m(v_{j,1})$ for all m). We can then proceed as in the $\ell=0$ case, getting simply an additional error factor equal to $\prod_{j=1}^{\ell}\|w_j\|_{C^{k+\ell+1}_A}\leq 2$. One should pay attention to the fact that, with the above definition, the vector fields w_j are not always defined on all the balls $W_{1/100}(x_i)$, for those x_i that are close to the boundary of $g_t(W_{1/200}(x))$. This is not a problem since w_j is compactly supported in $g_t(W_{1/100}(x))$ by construction: one may therefore extend it by 0 wherever it is not defined (this is why we had to use $v_{j,1}$ and not v_j in this construction).

It remains to check the formula $c_m(w_j)\leq c_m(v_{j,1})$. It comes from the fact that the definition of c_m only involves differentiation along directions in W^u, and that

g_{-t} is contracting along this manifold. If $\alpha_j = \omega$, i.e., v_j points in the flow direction, this estimate is straightforward. Let us therefore assume that $\alpha_j = s$, i.e., v_j points in the stable direction. Consider a point z in the domain of definition of w_j, and m vectors $u_1, \dots, u_m$ at that point which are tangent to $W^u(x)$, with $\|u_m\|_z \leq 1$. Write $y = g_{-t}z$. We get

$$D^m w_j(z; u_1, \dots, u_m) = e^{-t} D^m v_{j,1}(g_{-t}z; Dg_{-t}(z) \cdot u_1, \dots, Dg_{-t}(z) \cdot u_m).$$

Therefore,

$$\begin{aligned}\|D^m w_j(z; u_1, \dots, u_m)\|_y &= e^{-t} \|D^m v_{j,1}(g_{-t}z; Dg_{-t}(z) \cdot u_1, \dots, Dg_{-t}(z) \cdot u_m)\|_y \\ &\leq e^{-t} c_m(v_{j,1}) \|Dg_{-t}(z)u_1\|_y \cdots \|Dg_{-t}(z)u_m\|_y .\end{aligned}$$

Since the differential $Dg_{-t}(z)$ contracts in the direction of W^u by Lemma 5.2, we have $\|Dg_{-t}(z)u_n\|_y \leq \|u_n\|_z \leq 1$. This yields

$$\|D^m w_j(z; u_1, \dots, u_m)\|_y \leq e^{-t} c_m(v_{j,1}). \tag{8.4}$$

We are interested in bounding $\|D^m w_j(z; u_1, \dots, u_m)\|_z$. Since $d(y, z) \leq |t|$ by Lemma 5.1, Proposition 5.5 shows that the ratio between $\|\cdot\|_y$ and $\|\cdot\|_z$ is at most e^t. This cancels the factor e^{-t} in (8.4), and we get the conclusion. □

Corollary 8.3. *For every $k \in \mathbb{N}$, for every large enough A and B, for every $t \geq 0$ and every $f \in \mathcal{D}$, we have $\|f \circ g_t\|_k^{A,B} \leq 2C_0 \|f\|_k^{A,B}$.*

Proof. The function V_δ is bounded from below by 1. Taking the supremum over x in (8.1), we get $e^A_{k,\ell,\alpha}(f \circ g_t) \leq 2C_0 e^A_{k,\ell,\alpha}(f)$. The result follows from the definition of the $\|\cdot\|_k^{A,B}$ norm. □

It follows from this corollary that we can define the operator $\mathcal{M}$ on $\overline{\mathcal{D}^\Gamma}$. Let $N \in \mathbb{N}$, we will study the norm of $\mathcal{M}^N$. We have

$$\mathcal{M}^N f = \int_{t=0}^{\infty} \frac{t^{N-1}}{(N-1)!} e^{-4\delta t} \mathcal{L}_t f \, \mathrm{d}t. \tag{8.5}$$

We will estimate differently the contributions $\|\mathcal{M}^n f\|_{k,\omega}^{A,B}$ and $\|\mathcal{M}^n f\|_{k,s}^{A,B}$ to $\|\mathcal{M}^n f\|_k^{A,B}$. Let us first deal with the former.

Lemma 8.4. *For any $N \in \mathbb{N}$, for any k, if A and B are large enough, we have*

$$\left\|\mathcal{M}^N f\right\|_{k,\omega}^{A,B} \leq 5C_0 \|f\|_k^{A,B} .$$

Proof. We will prove that, for any N, k, ℓ and A sufficiently large, there exists a constant $C_{N,k,\ell,A}$ such that

$$e^A_{k,\ell,\omega}(\mathcal{M}^N f) \leq C_{N,k,\ell,A} \sum_{\ell' < \ell} e^A_{k,\ell'}(f) + 4C_0 e^A_{k,\ell}(f). \tag{8.6}$$

Taking B much larger than all $C_{N,k,\ell,A}$ for $0 \leq \ell \leq k$, this implies directly the statement of the lemma.

Let us fix N, k, ℓ, A. We split $\mathcal{M}^N$ as the sum of $\mathcal{M}_1 := \int_0^D \frac{t^{N-1}}{(N-1)!} e^{-4\delta t} \mathcal{L}_t \, \mathrm{d}t$ and $\mathcal{M}_2 := \int_D^\infty \frac{t^{N-1}}{(N-1)!} e^{-4\delta t} \mathcal{L}_t \, \mathrm{d}t$, where D is suitably large.

Lemma 8.2 shows that $e^A_{k,\ell,\omega}(\mathcal{L}_t f) \leq 2C_0 e^A_{k,\ell,\omega}(f)$. Hence, if D is large enough (depending on N), we have $e^A_{k,\ell,\omega}(\mathcal{M}_2 f) \leq C_0 e^A_{k,\ell,\omega}(f)$. The term $\mathcal{M}_2$ is therefore not a problem to prove (8.6).

Let us handle $\mathcal{M}_1$. Consider first a point x such that $V_\delta(x) \geq e^{(1-2\delta)D}$. For such a point x, Lemma 8.2 gives $e^A_{k,\ell,\omega}(\mathcal{L}_t f; x) \leq 2C_0 e^A_{k,\ell,\omega}(f) e^{-(1-2\delta)t}$ for $t \leq D$.

In particular,

$$\begin{aligned} e^A_{k,\ell,\omega}(\mathcal{M}_1 f; x) &\leq \int_{t=0}^{D} \frac{t^{N-1}}{(N-1)!} e^{-4\delta t} e^A_{k,\ell,\omega}(\mathcal{L}_t f; x) \,\mathrm{d}t \\ &\leq 2C_0 \int_{t=0}^{D} \frac{t^{N-1}}{(N-1)!} e^{-4\delta t} e^A_{k,\ell,\omega}(f) e^{-(1-2\delta)t} \,\mathrm{d}t \leq 2C_0 e^A_{k,\ell,\omega}(f) \end{aligned}$$

since $\int_{t=0}^{\infty} \frac{t^{N-1}}{(N-1)!} e^{-(1+2\delta)t} \,\mathrm{d}t \leq \int_{t=0}^{\infty} \frac{t^{N-1}}{(N-1)!} e^{-t} \,\mathrm{d}t = 1$. This concludes the proof for such points x.

It remains to consider points x with $V_\delta(x) \leq e^{(1-2\delta)D}$. This set is very large if D is large, but it is compact mod Γ. Fix such a point x, we have to estimate integrals of the form $\int_{W^u_{1/200}(x)} \phi \cdot L_{\overline{v_1}} \cdots L_{\overline{v_\ell}}(\mathcal{M}_1 f) \,\mathrm{d}\mu_u$, where $\|\phi\|_{C_A^{k+\ell}} \leq 1$ and $\|v_j\|_{C_A^{k+\ell+1}} \leq 1$, and at least one of the vector fields v_j points in the flow direction.

To begin, assume that the last vector field v_ℓ points in the flow direction, i.e., $v_\ell(y) = \psi(y)\omega(y)$ for some function ψ with $\|\psi\|_{C_A^{k+\ell+1}} \leq 1$. In the expression $L_{\overline{v_1}} \cdots L_{\overline{v_{\ell-1}}}(\psi L_\omega(\mathcal{M}_1 f))$, if we use at least one of the Lie derivatives to differentiate ψ, we obtain a term bounded by $C_{k,\ell,A} e^A_{k,\ell'}(\mathcal{M}_1 f)$ for some $\ell' < \ell$. This is bounded by $C_{N,k,\ell,A,D} e^A_{k,\ell'}(f)$ by Lemma 8.2. This error term is compatible with (8.6). The remaining term is $\psi L_{\overline{v_1}} \cdots L_{\overline{v_{\ell-1}}}(L_\omega(\mathcal{M}_1 f))$. Since $\mathcal{M}_1 f = \int_{t=0}^{D} h(t)\mathcal{L}_t f \,\mathrm{d}t$ for some smooth function h, we have $L_\omega(\mathcal{M}_1 f) = h(D)\mathcal{L}_D f - h(0) f - \int_{t=0}^{D} h'(t)\mathcal{L}_t f \,\mathrm{d}t$. Therefore, the integral we are studying can be bounded in terms of $\ell - 1$ derivatives of f (or images of f under operators $\mathcal{L}_t$), and this is bounded by $C_{N,k,\ell,A,D} e^A_{k,\ell-1}(f)$. This error term is again compatible with (8.6).

Assume now that one of the vector fields v_j points in the flow direction, but that it is not necessarily the last one. We can exchange the vector fields to put the vector field $\overline{v_j}$ in the last position and conclude as above. Since $[L_{w_1}, L_{w_2}] = L_{[w_1,w_2]}$, the additional error corresponds to the integration of $\ell - 1$ derivatives of $\mathcal{M}_1 f$ against a $C^{k+\ell}$ function, but one of the vector fields is not the canonical extension of a vector field defined on $W^u_{1/100}(x)$. Since we work in the set $\{V_\delta \leq e^{(1-2\delta)D}\}$ which is compact mod Γ, Lemma 7.3 shows that this error is bounded in terms of $\sup_{\ell' < \ell} e_{k,\ell'}(f)$, and is again compatible with (8.6). $\square$

It remains to study $\left\|\mathcal{M}^N f\right\|^{A,B}_{k,s}$. We will rather estimate $\|\mathcal{L}_t f\|^{A,B}_{k,s}$ if t is large enough, this will readily gives estimates for $\left\|\mathcal{M}^N f\right\|^{A,B}_{k,s}$ by (8.5).

Let us fix some constants. First, we recall that C_0 has been defined in Lemma 8.2. Let $T_0 > 0$ be large enough so that $40C_0 \leq e^{\delta T_0}$. Let $V = 2e^{(3-2\delta)T_0}$, and define

$$K = \{x \in \mathrm{Teich}_1 \;:\; V_\delta(x) \leq 4V e^{2T_0}\}. \tag{8.7}$$

This set is compact mod Γ. Finally, applying Proposition 4.3 to K, we get a time $T = T(K)$.

We will study the operator $\mathcal{L}_{nT_0}$, for all n large enough so that $nT_0 \geq T/\delta$. By Lemma 5.10, we can define a C^∞ function ρ_V such that $\rho_V(x) = 1$ if $V_\delta(x) \leq V$ and $\rho_V(x) = 0$ if $V_\delta(x) \geq 2V$. Write $\psi_1 = \rho_V$ and $\psi_2 = 1 - \rho_V$ so that $\psi_1 + \psi_2 = 1$. We decompose $\mathcal{L}_{T_0}(f) = \mathcal{L}_{T_0}(\psi_1 f) + \mathcal{L}_{T_0}(\psi_2 f) = \tilde{\mathcal{L}}_1 f + \tilde{\mathcal{L}}_2 f$. Therefore, $\mathcal{L}_{nT_0} = \sum_{\gamma \in \{1,2\}^n} \tilde{\mathcal{L}}_{\gamma_1} \cdots \tilde{\mathcal{L}}_{\gamma_n}$.

We first give a lemma ensuring that the multiplication by ρ_V or $1 - \rho_V$ in the definition of $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ is not harmful, and then we will turn to the study of $\tilde{\mathcal{L}}_{\gamma_1} \cdots \tilde{\mathcal{L}}_{\gamma_n}$ for $\gamma \in \{1,2\}^n$. We will handle in Lemma 8.6 the case where most γ_i are equal to 2 (i.e., most time is spent close to infinity, and we can use the good recurrence estimates of Proposition 6.1), and in Lemma 8.7 the case where a definite proportion of the γ_i is equal to 1 (i.e., some time is spent in the compact set K, and we can take advantage of the hyperbolicity of the flow there).

Lemma 8.5. *Let $k \in \mathbb{N}$, and let $\psi : X \to [0,1]$ be a C^{2k} function supported in a compact set mod Γ. If A and B are large enough, then for any $f \in \mathcal{D}$, we have $\|\psi f\|_{k,s}^{A,B} \leq 3 \|f\|_{k,s}^{A,B}$.*

Proof. Let us prove that, for every k, $\ell \leq k$ and every large enough A, there exists a constant $C_{k,\ell,A}$ such that, for any $f \in \mathcal{D}$,

$$e_{k,\ell,s}^A(\psi f) \leq 2e_{k,\ell,s}^A(f) + C_{k,\ell,A} \sum_{\ell'<\ell} e_{k,\ell',s}^A(f). \tag{8.8}$$

The statement of the lemma follows directly from this estimate if B is much larger than any of the $C_{k,\ell,A}$.

To estimate $e_{k,\ell,s}^A(\psi f)$, we have to compute integrals of the form

$$\int_{W^u_{1/200}(x)} \phi \cdot L_{\overline{v_1}} \cdots L_{\overline{v_\ell}}(\psi f) \, \mathrm{d}\mu_u,$$

where $\|\phi\|_{C_A^{k+\ell}} \leq 1$ and $v_1, \dots, v_\ell$ have a $C_A^{k+\ell+1}$–norm along $W^u_{1/100}(x)$ bounded by 1. We can use each $L_{\overline{v_i}}$ to differentiate either ψ or f. If we differentiate ψ m times for some $m > 0$, we obtain an integral of $\ell - m$ derivatives of f against a $C^{k+\ell-m}$ function, hence this is bounded by $Ce_{k,\ell',s}^A(f)$ for $\ell' = \ell - m$ (note that we are working in the lift of a compact subset of Teich_1/Γ, hence the $C^{k+\ell}$ norm of the extended vector fields $\overline{v_j}$ is bounded). The remaining term is $\int \phi\psi \cdot L_{\overline{v_1}} \cdots L_{\overline{v_\ell}} f$. If A is large enough, $\|\phi\psi\|_{C_A^{k+\ell}} \leq \|\phi\|_{C_A^{k+\ell}} \|\psi\|_{C_A^{k+\ell}} \leq 2$, hence this integral is bounded by $2e_{k,\ell,s}^A(f)$. We have proved (8.8). □

Lemma 8.6. *For every $k, n \in \mathbb{N}$, for every $\gamma \in \{1,2\}^n$, for every large enough A, B, we have for every $f \in \mathcal{D}$*

$$\left\| \tilde{\mathcal{L}}_{\gamma_1} \cdots \tilde{\mathcal{L}}_{\gamma_n} f \right\|_{k,s}^{A,B} \leq (10C_0)^n e^{-(1-2\delta)T_0 \#\{i \,:\, \gamma_i = 2\}} \|f\|_{k,s}^{A,B}.$$

Proof. It is sufficient to prove that

$$\left\| \tilde{\mathcal{L}}_1 f \right\|_{k,s}^{A,B} \leq 10C_0 \|f\|_{k,s}^{A,B} \quad \text{and} \quad \left\| \tilde{\mathcal{L}}_2 f \right\|_{k,s}^{A,B} \leq 10C_0 e^{-(1-2\delta)T_0} \|f\|_{k,s}^{A,B}.$$

Since V_δ is bounded from below by 1, Lemma 8.2 shows that $\|\mathcal{L}_{T_0} f\|_{k,s}^{A,B} \leq 2C_0 \|f\|_{k,s}^{A,B}$ if A is large enough. Therefore,

$$\left\| \tilde{\mathcal{L}}_1 f \right\|_{k,s}^{A,B} = \|\mathcal{L}_{T_0}(\rho_V f)\|_{k,s}^{A,B} \leq 2C_0 \|\rho_V f\|_{k,s}^{A,B} \leq 6C_0 \|f\|_{k,s}^{A,B},$$

by Lemma 8.5, if A and B are large enough.

We turn to $\tilde{\mathcal{L}}_2 f = \mathcal{L}_{T_0}((1-\rho_V)f)$. Let $x \in X$. Since $\log V_\delta$ is 2-Lipschitz, $V_\delta(g_{T_0} y) \leq e^{2T_0} V_\delta(y)$ for all y. If $V_\delta(x) \leq e^{-2T_0} V/2$, it follows that $V_\delta(y) \leq e^{-2T_0} V$ on $W^u_{1/100}(x)$, and therefore that $V_\delta(g_{T_0} y) \leq V$ on $g_{T_0}(W^u_{1/100}(x))$. Hence, $1 - \rho_V = 0$ on this set. The definition of $e_{k,\ell,s}^A$ gives $e_{k,\ell,s}^A(\mathcal{L}_{T_0}((1-\rho_V)f); x) = 0$. We therefore obtain by Lemma 8.2

$$\begin{aligned}
e_{k,\ell,s}^A(\mathcal{L}_{T_0}((1-\rho_V)f) &= \sup_{V_\delta(x) \geq e^{-2T_0} V/2} e_{k,\ell,s}^A(\mathcal{L}_{T_0}((1-\rho_V)f; x) \\
&\leq \sup_{V_\delta(x) \geq e^{-2T_0} V/2} C_0 e_{k,\ell,s}^A((1-\rho_V)f) \left(e^{-(1-2\delta)T_0} + 1/V_\delta(x) \right) \\
&\leq C_0 e_{k,\ell,s}^A((1-\rho_V)f) \left(e^{-(1-2\delta)T_0} + 2e^{2T_0}/V \right).
\end{aligned}$$

Taking into account the definition of $\|\cdot\|_{k,s}^{A,B}$ and the equality $2e^{2T_0}/V = e^{-(1-2\delta)T_0}$, we obtain

$$\|\mathcal{L}_{T_0}((1-\rho_V)f)\|_{k,s}^{A,B} \leq 2C_0 \|(1-\rho_V)f\|_{k,s}^{A,B} e^{-(1-2\delta)T_0}.$$

By Lemma 8.5, $\|(1-\rho_V)f\|_{k,s}^{A,B} \leq \|f\|_{k,s}^{A,B} + \|\rho_V f\|_{k,s}^{A,B} \leq 4\|f\|_{k,s}^{A,B}$ if A, B are large enough. We obtain $\left\|\tilde{\mathcal{L}}_2 f\right\|_{k,s}^{A,B} \leq 8C_0 e^{-(1-2\delta)T_0} \|f\|_{k,s}^{A,B}$ as desired. ☐

We defined an auxiliary norm $\|\cdot\|_k'$ in (7.3).

Lemma 8.7. *Consider $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\#\{i \ : \ \gamma_i = 1\} \geq T/T_0$. Then, for all k, if A and B are large enough,*

$$\left\|\tilde{\mathcal{L}}_{\gamma_1} \cdots \tilde{\mathcal{L}}_{\gamma_n} f\right\|_{k,s}^{A,B} \leq 2^{-k/2} \cdot 12C_0 \|f\|_k^{A,B} + C_{n,k,A,B} \|\psi_\gamma f\|_k', \tag{8.9}$$

where the function ψ_γ is C^∞ and supported in a compact set mod Γ.

The point of this lemma is that, if γ is fixed, we can choose k very large to make the first term in (8.9) arbitrarily small, while the second term gives a compact contribution (thanks to Proposition 7.2), and will therefore not be an issue to control the essential spectral radius.

Proof. We can write $\tilde{\mathcal{L}}_{\gamma_1} \dots \tilde{\mathcal{L}}_{\gamma_n} f = \mathcal{L}_{nT_0}(\psi f)$, where $\psi = \psi_\gamma = \prod_{j=1}^n \psi_{\gamma_j} \circ g_{-(n-j)T_0}$ is C^∞ and compactly supported.

To estimate $e_{k,\ell,s}^A(\tilde{\mathcal{L}}_{\gamma_1} \dots \tilde{\mathcal{L}}_{\gamma_n} f)$ for some $0 \leq \ell \leq k$, we should estimate integrals of the form

$$\int_{W_{1/200}^u(x)} \phi \cdot L_{\overline{v_1}} \cdots L_{\overline{v_\ell}}(\mathcal{L}_{nT_0}(\psi f)) \, \mathrm{d}\mu_u, \tag{8.10}$$

where $\|\phi\|_{C_A^{k+\ell}} \leq 1$, the vector fields v_j all point in the stable direction and $\|v_j\|_{C_A^{k+\ell+1}} \leq 1$.

As in the proof of Lemma 8.2, we first replace v_j by a compactly supported vector field $v_{j,1}$ on $W_{1/100}^u(x)$, with $\|v_{j,1}\|_{C_A^{k+\ell+1}} \leq 2^{1/2}$ (assuming A is large enough). Let w_j be the push-forward of $v_{j,1}$ under g_{nT_0}, and let (ρ_i) be a partition of unity on $g_{nT_0}(W_{1/200}^u(x))$ (c.f. Proposition 5.8). The integral (8.10) becomes

$$\sum_{i \in I} \int_{W_{1/200}^u(x_i)} \rho_i(z)\phi(g_{-nT_0} z) \cdot L_{\overline{w_1}} \cdots L_{\overline{w_\ell}}(\psi f)(z) \, e^{-dnT_0} \, \mathrm{d}\mu_u(z).$$

Let $I' \subset I$ be the set of is such that ψ is not identically zero on $W_{1/200}^u(x_i)$. We claim that, for $i \in I'$, for all $y \in W_{1/200}^u(x_i)$,

$$\mathrm{Leb}\{s \in [0, nT_0] \ : \ g_{-s}(y) \in K\} \geq T, \tag{8.11}$$

where K is defined in (8.7). Indeed, let $z \in W_{1/200}^u(x_i)$ satisfy $\psi(z) \neq 0$. For all j with $\gamma_j = 1$, we have $\psi_1(g_{-(n-j)T_0} z) \neq 0$, therefore $V_\delta(g_{-(n-j)T_0} z) \leq 2V$. Since $g_{-(n-j)T_0}$ is a contraction along W^u, we obtain $V_\delta(g_{-(n-j)T_0} y) \leq 4V$ for any $y \in W_{1/200}^u(x_i)$. For any $s \in [0, T_0]$, $V_\delta(g_{-s} g_{-(n-j)T_0} y) \leq e^{2s} V_\delta(g_{-(n-j)T_0} y) \leq e^{2T_0} 4V$, i.e., $g_{-s} g_{-(n-j)T_0} y \in K$. This implies that

$$\mathrm{Leb}\{s \in [0, nT_0] \ : \ g_{-s}(y) \in K\} \geq T_0 \#\{j \ : \ \gamma_j = 1\},$$

which is greater than or equal to T, by the assumptions of the lemma. This proves (8.11).

Fix now $i \in I'$, we work along $W_{1/200}^u(x_i)$. Since g_t is uniformly hyperbolic along trajectories that spend a time at least T in K (by Proposition 4.3), we have $c_m(\phi \circ g_{-nT_0}) \leq 2^{-m} c_m(\phi)$, and $c_m(w_j) \leq 2^{-m-1} c_m(v_{j,1})$ (note that we have a gain even for $m = 0$ since the vector itself is contracted by the differential of g_{nT_0}). This gives $\|\phi \circ g_{-nT_0}\|_{C_A^{k+\ell}} \leq \|\phi\|_{C_A^{k+\ell}}$ (there is no gain here at level $m = 0$, so no gain overall) and $\|w_j\|_{C_A^{k+\ell+1}} \leq 2^{-1} \|v_{j,1}\|_{C_A^{k+\ell+1}} \leq 2^{-1/2}$. This gives a gain of $2^{-1/2}$ with respect to the non-contracting situation of Lemma 8.2, and we end up after the same computations with

$$e^A_{k,\ell,s}(\mathcal{L}_{nT_0}(\psi f)) \leq 2C_0 \cdot 2^{-\ell/2} e^A_{k,\ell,s}(\psi f). \tag{8.12}$$

This gives a certain gain if ℓ is large. In particular, for $\ell = k$, we obtain a gain of $2^{-k/2}$, as in the estimate (8.9) we are trying to prove. However, this is not sufficient for smaller ℓ. Assume now $\ell < k$, we will regularize the function ϕ by convolution in this case.

For $\epsilon > 0$, we consider a function $\tilde{\phi}$ on $W^u_{1/100}(x_i)$ such that $c_m(\phi - \tilde{\phi}) \leq \epsilon$ for $m < k + \ell$, $c_{k+\ell}(\tilde{\phi}) \leq 2c_{k+\ell}(\phi)$ and $c_{k+\ell+1}(\tilde{\phi}) \leq C_{k,A}/\epsilon$. Note that, since $\tilde{\phi}$ is obtained by convolution between ϕ and a kernel of support of size ϵ, the support of $\tilde{\phi}$ is larger than that of ϕ. Since all the functions we are considering are multiplied by the partition of unity ρ_i, this is not a problem.

Along $W^u_{1/200}(x_i)$, the function $\phi' = (\phi - \tilde{\phi}) \circ g_{-nT_0}$ satisfies $c_m(\phi') \leq \epsilon$ for $m < k + \ell$ and $c_{k+\ell}(\phi') \leq 2 \cdot 2^{-(k+\ell)} c_{k+\ell}(\phi)$. Choosing $\epsilon = 2^{-(4+k+\ell)}$, we have finally $\|\phi'\|_{C_A^{k+\ell}} \leq e^{1/A} 2^{-4-k-\ell} + 2^{1-k-\ell} \|\phi\|_{C_A^{k+\ell}} \leq 2^{(3/2)-k-\ell}$ for any $A \geq 1$. Let us decompose in (8.10) the function ϕ as $\phi' + \tilde{\phi}$. The resulting term coming from ϕ' is similar to (8.12) but with an additional factor $2^{(3/2)-k-\ell}$, while the term coming from $\tilde{\phi}$ is bounded in terms of $\|f\|'_k$, since there are at most $\ell < k$ derivatives of f integrated against a function in $C^{k+\ell+1}$. In the end, we get

$$e^A_{k,\ell,s}(\mathcal{L}_{nT_0}(\psi f)) \leq 2C_0 \cdot 2^{-\ell/2} \cdot 2^{(3/2)-k-\ell} e^A_{k,\ell,s}(\psi f) + C_{n,\gamma,A,k} \|\psi f\|'_k .$$

Summing the last equation for $\ell = 0, \dots, k-1$ and (8.12) for $\ell = k$, we obtain

$$\|\mathcal{L}_{nT_0}(\psi f)\|^{A,B}_{k,s} \leq 4C_0 2^{-k/2} \|\psi f\|^{A,B}_{k,s} + C_{n,\gamma,A,k} \|\psi f\|'_k .$$

Since the function ψ is C^∞ and compactly supported, Lemma 8.5 applies if B is large enough. This concludes the proof. □

To simplify notations, we write $O^{comp}(f)$ for terms bounded by $\|\psi f\|'_k$, for some C^∞ function ψ in Teich_1 that is supported in a compact set mod Γ. This notation is invariant under $\mathcal{L}_t$ for fixed t (since this operator acts continuously for $\|\cdot\|'_k$), and under addition (if ψ_1 and ψ_2 are two C^∞ functions whose support is compact mod Γ, consider a function ψ with the same properties which is equal to 1 on $\mathrm{supp}(\psi_1) \cup \mathrm{supp}(\psi_2)$, then $\|\psi_1 f\|'_k = \|\psi_1 \psi f\|'_k \leq C(\psi_1) \|\psi f\|'_k$, and a similar inequality holds for ψ_2).

Corollary 8.8. *For every $n \in \mathbb{N}$ with $n \geq T/(\delta T_0)$, if k, A, B are large enough, we have*

$$\|\mathcal{L}_{nT_0} f\|^{A,B}_{k,s} \leq e^{-(1-4\delta)nT_0} \|f\|^{A,B}_k + O^{comp}(f).$$

Proof. We write $\mathcal{L}_{nT_0} f = \sum_{\gamma \in \{1,2\}^n} \tilde{\mathcal{L}}_{\gamma_1} \cdots \tilde{\mathcal{L}}_{\gamma_n} f$, and estimate the terms coming from each γ.

If $\#\{j : \gamma_j = 1\} \geq \delta n$, then the resulting term is bounded by Lemma 8.7. Otherwise, $\#\{j : \gamma_j = 2\} \geq (1-\delta)n$, and Lemma 8.6 gives an upper bound of the form $(10C_0)^n e^{-(1-2\delta)T_0(1-\delta)n} \|f\|^{A,B}_k$. Since $(1-2\delta)(1-\delta) \geq 1 - 3\delta$, we obtain after summing over the 2^n possible values of γ

$$\begin{aligned}\|\mathcal{L}_{nT_0} f\|^{A,B}_{k,s} \leq 2^n (10C_0)^n e^{-(1-3\delta)T_0 n} \|f\|^{A,B}_k + 2^n \cdot 12C_0 2^{-k/2} \|f\|^{A,B}_k \\ + C_{n,k,A,B} \sum \|\psi_{n,\gamma} f\|'_k .\end{aligned}$$

Choosing k large enough, we can make sure that $12C_0 2^{-k/2} \leq (10C_0)^n e^{-(1-3\delta)T_0 n}$, and we obtain a bound of the form

$$(40C_0)^n e^{-(1-3\delta)T_0 n} \|f\|_k^{A,B} + C_{n,k,A,B} \sum \|\psi_{n,\gamma} f\|'_k .$$

Since $40C_0 \leq e^{\delta T_0}$, this implies the statement of the corollary. □

Corollary 8.9. *For any large enough N, if k, A, B are large enough, we have*

$$\left\|\mathcal{M}^N f\right\|_{k,s}^{A,B} \leq 2C_0(e^{(1-4\delta)T_0} + 2) \|f\|_k^{A,B} + O^{comp}(f).$$

Proof. We start from the formula

$$\mathcal{M}^N f = \int_{t=0}^{\infty} \frac{t^{N-1}}{(N-1)!} e^{-4\delta t} \mathcal{L}_t f \ \mathrm{d}t = \sum_{n=0}^{\infty} \int_{nT_0}^{(n+1)T_0} \frac{t^{N-1}}{(N-1)!} e^{-4\delta t} \mathcal{L}_t f \ \mathrm{d}t.$$

On an interval $[nT_0, (n+1)T_0]$ with small n (i.e., $n < T/(\delta T_0)$), we use the simple bound $\|\mathcal{L}_t f\|_{k,s}^{A,B} \leq 2C_0 \|f\|_k^{A,B}$ coming from Lemma 8.2. Since for any fixed $T_* > 0$, $\int_0^{T_*} \frac{t^{N-1}}{(N-1)!} \ \mathrm{d}t$ tends to zero when $N \to \infty$, the contribution of those intervals is bounded, say, by $2C_0 \|f\|_k^{A,B}$ if N is large enough.

We use the same trivial bound on the intervals $[nT_0, (n+1)T_0]$ with very large n ($n \geq n_0(N)$ to be chosen later). The contribution of these intervals is then bounded by

$$\int_{n_0(N)T_0}^{\infty} \frac{t^{N-1}}{(N-1)!} e^{-4\delta t} 2C_0 \|f\|_k^{A,B} \ \mathrm{d}t.$$

Choosing $n_0(N)$ large enough, we can ensure that this is bounded by $2C_0 \|f\|_k^{A,B}$.

Consider now n in between. For $t \in [nT_0, (n+1)T_0]$, we have

$$\begin{aligned} \|\mathcal{L}_t f\|_{k,s}^{A,B} &\leq 2C_0 \|\mathcal{L}_{nT_0} f\|_{k,s}^{A,B} \leq 2C_0 e^{-(1-4\delta)nT_0} \|f\|_k^{A,B} + O^{comp}(f) \\ &\leq 2C_0 e^{(1-4\delta)T_0} e^{-(1-4\delta)t} \|f\|_k^{A,B} + O^{comp}(f). \end{aligned}$$

Integrating over t and then summing over n, we get a contribution bounded by

$$2C_0 e^{(1-4\delta)T_0} \int_{t=0}^{\infty} \frac{t^{N-1}}{(N-1)!} e^{-4\delta t} e^{-(1-4\delta)t} \|f\|_k^{A,B} \ \mathrm{d}t + O^{comp}(f),$$

which is bounded by $2C_0 e^{(1-4\delta)T_0} \|f\|_k^{A,B} + O^{comp}(f)$ since $\int_{t=0}^{\infty} \frac{t^{N-1}}{(N-1)!} e^{-t} \ \mathrm{d}t = 1$. □

Proof of Theorem 8.1. The first part of the statement is contained in Lemma 8.2. It remains to estimate the essential spectral radius of $\mathcal{M}$. Adding the estimates of Lemma 8.4 and of Corollary 8.9, we have for large enough N, k, A, B

$$\left\|\mathcal{M}^N f\right\|_k^{A,B} \leq 2C_0(e^{(1-4\delta)T_0} + 5) \|f\|_k^{A,B} + O^{comp}(f).$$

Let us fix once and for all N large enough so that $2C_0(e^{(1-4\delta)T_0} + 5) \leq (1+\delta)^N$, and k, A, B such that the previous estimate holds. This estimate translates into the following: there exists a C^∞ function ψ supported in a compact set mod Γ such that, for any function f in $\mathcal{D}$,

$$\left\|\mathcal{M}^N f\right\|_k^{A,B} \leq (1+\delta)^N \|f\|_k^{A,B} + \|\psi f\|'_k .$$

The unit ball of $\mathcal{D}^\Gamma$ for the norm $\|\cdot\|_k$ is relatively compact for the semi-norm $\|f\|' := \|\psi f\|'_k$, by Proposition 7.2. By Hennion's Theorem (Lemma 3.1), it follows that the essential spectral radius of $\mathcal{M}$ for the norm $\|\cdot\|_k^{A,B}$ on the space $\mathcal{D}^\Gamma$ is at most $1+\delta$. Since this norm is equivalent to $\|\cdot\|_k$, this concludes the proof of Theorem 8.1. □

Appendix A.

A.1. **Spherical functions.** In this section, we prove the estimate (3.6) on the behavior of the spherical function ϕ_{ξ_s} when ξ_s is a representation of $\mathrm{SL}(2,\mathbb{R})$ in the complementary series. It is a consequence of classical estimates on spherical functions, let us for instance follow the computations in [Hel00]. For $\Re s \in [-1,1]$, let us define coefficients $\Gamma_n(s)$ by $\Gamma_0 = 1$, $\Gamma_n = 0$ if n is odd, and $n(n-s)\Gamma_n(s) = \sum_{0<k\leq n/2} \Gamma_{n-2k}(s)(2n-4k-s+1)$ if n is even. It is easy to check by induction that these coefficients grow more slowly than any exponential. In particular (see, e.g., [Hel00, Lemma 4.13]), for every $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\forall s \in [-1,1],\ \forall n \in \mathbb{N}, \quad |\Gamma_n(s)| \leq C e^{\epsilon n}. \tag{A.1}$$

These coefficients are chosen so that $t \mapsto e^{(s-1)t} \sum \Gamma_n(s) e^{-2nt}$ satisfies an explicit differential equation of order 2 which is also satisfied by ϕ_{ξ_s}. Another solution of the same equation is $t \mapsto e^{(-s-1)t} \sum \Gamma_n(-s) e^{-2nt}$. It follows that ϕ_{ξ_s} is a linear combination of those two functions. One can identify the coefficients in this linear combination (they are given by the c function (3.5)), to obtain the following formula for ϕ_{ξ_s}: for every $s \in (0,1] \cup \mathbf{i}(0,+\infty)$,

$$\phi_{\xi_s}(g_t) = c(s) e^{(s-1)t} \sum_{n\geq 0} \Gamma_n(s) e^{-2nt} + c(-s) e^{(-s-1)t} \sum_{n \geq 0} \Gamma_n(-s) e^{-2nt}.$$

This is [Hel00, Theorem IV.5.5] in the case of $\mathrm{SL}(2,\mathbb{R})$ (the formula for c is given in [Hel00, Theorem IV.6.4]).

For $s \in [\delta, 1]$, the dominating term in this formula is $c(s)e^{(s-1)t}$, and the sum of the other terms is bounded by Ce^{-t} if $t \geq 1$, by (A.1). Since $\phi_{\xi_s}(g_t) - c(s)e^{(s-1)t}$ is uniformly bounded for $t \in [0,1]$ and $s \in [\delta, 1]$, the estimate (3.6) follows.

A.2. **Boundary behavior of Cauchy transforms.**

Lemma A.1. *Let ν be a nonnegative measure on $[0,1]$, with finite mass. Assume that the function $F(z) = \int_{s\in[0,1]} \frac{\mathrm{d}\nu(s)}{z-s+1}$, defined for $z \in \mathbb{C} - [-1,0]$, admits a continuous extension to an interval $[a-1, b-1] \subset [-1,0]$. Then $\nu[a,b] = 0$.*

Proof. Let us first show that, if F is continuous at a point $x-1$, with $x \in [0,1]$, then

$$\nu[x-\epsilon, x+\epsilon] = o(\epsilon). \tag{A.2}$$

We have

$$F(x-1+\mathbf{i}y) = \int \frac{\mathrm{d}\nu(s)}{x+\mathbf{i}y-s} = \int \frac{x-s-\mathbf{i}y}{(x-s)^2+y^2} \,\mathrm{d}\nu(s).$$

As a consequence,

$$\mathrm{Im}(F(x-1+\mathbf{i}y) - F(x-1-\mathbf{i}y)) = -2 \int \frac{y}{(x-s)^2+y^2} \,\mathrm{d}\nu(s).$$

If F can be extended continuously to $x-1$, this quantity tends to 0. For $s \in [x-y, x+y]$, the integrand is at least $y/(2y^2)$, therefore

$$\begin{aligned} \nu[x-y, x+y]/y &\leq 2 \int_{s=x-y}^{x+y} \frac{y}{(x-s)^2+y^2} \,\mathrm{d}\nu(s) \\ &\leq |\,\mathrm{Im}(F(x-1+\mathbf{i}y) - F(x-1-\mathbf{i}y))| \to 0. \end{aligned}$$

This proves (A.2).

Assume now that F can be continuously extended to a whole interval $[a-1, b-1]$. For any $x \in [a,b]$, we have $\nu[x-\epsilon, x+\epsilon] = o(\epsilon)$. By [Mat95, Theorem 2.12], for any $\rho > 0$, we can cover $[a,b]$ with intervals I_n with bounded overlap, with $\nu(I_n) \leq \rho |I_n|$.

Therefore, $\nu[a,b] \leq \sum \nu(I_n) \leq \rho \sum |I_n| \leq \rho C(b-a)$. Letting ρ tend to 0, we obtain $\nu[a,b] = 0$. $\square$

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CNRS UMR 7586, Institut de Mathématiques de Jussieu, 175 rue du Chevaleret, 75013, Paris, France & IMPA, Estrada Dona Castorina 110, 22460-320, Rio de Janeiro, Brazil
E-mail address: artur@math.jussieu.fr

IRMAR, CNRS UMR 6625, Université de Rennes 1, 35042 Rennes, France
E-mail address: sebastien.gouezel@univ-rennes1.fr