Coxeter orbits and Brauer trees II

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Abstract

The purpose of this paper is to discuss the validity of the assumptions (W) and (S) stated in [13], about the torsion in the modular ℓ-adic cohomology of Deligne-Lusztig varieties associated to Coxeter elements. We prove that both (W) and (S) hold whenever the characteristic of the ground field is a good prime number except for groups of type E_7 or E_8.

Introduction

Let $\mathbf{G}$ be a quasi-simple algebraic group defined over an algebraic closure of a finite field of characteristic p. Let F be the Frobenius endomorphism of $\mathbf{G}$ associated to a split rational $\mathbb{F}_q$-structure. The finite group $G = \mathbf{G}^F$ of fixed points under F is called a (split) finite reductive group.

Let ℓ be a prime number different from p and Λ be a finite extension of $\mathbb{Z}_\ell$. There is strong evidence that the structure of the principal ℓ-block of G is encoded in the cohomology over Λ of some Deligne-Lusztig variety. Precise conjectures have been stated in [5] and [7], and much numerical evidence has been collected. The representation theory of ΛG is highly dependent on the prime number ℓ. In [13], we have studied a special case referred to as the *Coxeter case*. The corresponding primes ℓ are those which divide the cyclotomic polynomial $\Phi_h(q)$ where h is the Coxeter number of W. In that situation, it is to be expected that the cohomology of the Deligne-Lusztig variety $\mathrm{Y}(\dot{c})$ associated to a Coxeter element c describes the principal ℓ-block $b\Lambda G$. More precisely,

- *Hiss-Lübeck-Malle conjecture*: the Brauer tree of $b\Lambda G$ (which has a cyclic defect group) can be recovered from the action of G and F on the cohomology groups $\mathrm{H}_c^i(\mathrm{Y}(\dot{c}), \overline{\mathbb{Q}}_\ell)$ [19];
- *Geometric version of Broué's conjecture*: the complex $b\mathrm{R}\Gamma_c(\mathrm{Y}(\dot{c}),\Lambda)$ induces a derived equivalence between the principal ℓ-blocks of G and the normalizer $N_G(T_c)$ of a torus of type c [5].

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In [13] the author has given a general proof of both of these conjectures, but under some assumptions on the torsion in the cohomology of $\mathrm{Y}(\dot{c})$. The weaker assumption concerns only some eigenspaces of the Frobenius:

(W) For all minimal eigenvalues λ of F, the generalized (λ)-eigenspace of F on $b\mathrm{H}_c^\bullet(\mathrm{Y}(\dot{c}),\Lambda)$ is torsion-free.

We call here "minimal" the eigenvalues of F on the cohomology group in middle degree. If this assumption holds, then we proved in [13] that the Brauer tree of the principal block has the expected shape. However, a stronger assumption is needed to obtain the planar embedding of the tree and Broué's conjecture:

(S) The Λ-modules $b\mathrm{H}_c^i(\mathrm{Y}(\dot{c}),\Lambda)$ are torsion-free.

The purpose of this paper is to discuss these assumptions for the Deligne-Lusztig variety $\mathrm{Y}(c)$ and some interesting quotients, and to prove that they are valid in the majority of cases. The main result in this direction is the following:

Theorem. *Assume that p is a good prime number for* $\mathbf{G}$*. Let b' be the idempotent associated to the principal ℓ-block of T_c. If the type of G is not E_7 or E_8, then the Λ-modules $b\mathrm{H}_c^i(\mathrm{Y}(\dot{c}),\Lambda)b'$ are torsion-free.*

A more general statement is given in Theorem 3.9, with more precise bounds on p in some types. Furthermore, we can also include the groups of type E_7 and E_8 if we assume that we already know the shape of the Brauer tree. Note that the results of [13] remain valid if we consider $b\mathrm{H}_c^i(\mathrm{Y}(\dot{c}),\Lambda)b'$ instead of $b\mathrm{H}_c^i(\mathrm{Y}(\dot{c}),\Lambda)$ in the assumptions (W) and (S). In particular, we obtain a significant number of new cases of the geometric version of Broué's conjecture (see Theorem 3.11). We also deduce new planar embeddings of Brauer trees for the groups of type ${}^2\mathrm{G}_2$ and F_4 with $p \neq 2,3$ (see Theorem 3.12).

Our proof relies on Lusztig's work on the geometry of Deligne-Lusztig varieties associated to Coxeter elements [22]. Many constructions that are derived from $\mathrm{X}(c)$, such as remarkable quotients and smooth compactifications, can be expressed in terms of varieties associated to smaller Coxeter elements. This provides an inductive method for finding the torsion in the cohomology of $\mathrm{X}(c)$. A further refinement adapted from [2] is then used to lift the method up to $\mathrm{Y}(\dot{c})$ and to show that the torsion part of $b\mathrm{H}_c^i(\mathrm{Y}(\dot{c}),\Lambda)b'$ is necessarily a cuspidal module. This reduces the problem of finding the torsion to the problem of finding where cuspidal composition factors can occur in the cohomology. Under some restrictions on p and G, we prove that these cannot occur outside the middle degree. To this end, we determine the generalized Gelfand-Graev modules which have a simple cuspidal module in their head and compute their contribution to the cohomology of $\mathrm{X}(c)$ again using smaller varieties. When no reduction is possible, we can argue as in [22] to deduce the concentration property.

This paper is organized as follows: the first section presents some preliminaries. We have compiled the basic techniques that are used in the modular Deligne-Lusztig theory, together with standard results on the generalized Gelfand-Graev representations. In the following section, we use the geometric results of [22] to rephrase the assumption (S) into a more representation-theoretical condition involving cuspidal modules. The last section is devoted to this problem. We prove that under some restrictions on p and G, the cohomology of $\mathrm{X}(c)$ has cuspidal composition factors in the middle degree only.

1 Preliminaries

In this preliminary section we set up the notation and introduce the main techniques that we will use throughout this paper. These fall into three categories, depending on whether they come from homological algebra, algebraic geometry or Lie theory.

1.1 Homological methods

1.1.1. **Module categories and usual functors.** If $\mathscr{A}$ is an abelian category, we will denote by $C(\mathscr{A})$ the category of cochain complexes, by $K(\mathscr{A})$ the corresponding homotopy category and by $D(\mathscr{A})$ the derived category. We shall use the superscript notation $-$, $+$ and b to denote the full subcategories of bounded above, bounded below or bounded complexes. We will always consider the case where $\mathscr{A} = A$-Mod is the module category over any ring A or the full subcategory A-mod of finitely generated modules. This is actually not a strong restriction, since any small category can be embedded into some module category [26]. Since the categories A-Mod and A-mod have enough projective objects, one can define the usual derived bifunctors $\mathrm{RHom}_A^\bullet$ and $\overset{\mathrm{L}}{\otimes}_A$.

Let H be a finite group and ℓ be a prime number. We fix an ℓ-modular system (K,Λ,k) consisting of a finite extension K of the field of ℓ-adic numbers $\mathbb{Q}_\ell$, the integral closure Λ of the ring of ℓ-adic integers in K and the residue field k of the local ring Λ. We assume moreover that the field K is big enough for H, so that it contains the e-th roots of unity, where e is the exponent of H. In that case, the algebra KH is split semi-simple.

From now on, we shall focus on the case where $A = \mathcal{O}H$, with $\mathcal{O}$ being any ring between (K,Λ,k). By studying the modular representation theory of H we mean studying the module categories $\mathcal{O}H$-mod for various $\mathcal{O}$, and also the different connections between them. In this paper, most of the representations will arise in the cohomology of some complexes and we need to know how to pass from one coefficient ring to another. The scalar extension and ℓ-reduction have a derived counterpart: if C is any bounded complex of ΛH-modules we can form $KC = C \otimes_\Lambda K$ and $\overline{C} = kC = C \overset{\mathrm{L}}{\otimes}_\Lambda k$. Since K is a flat Λ-module, the cohomology of the complex KC is exactly the scalar extension of the cohomology of C. However this does not apply to ℓ-reduction, but the obstruction can be related to the torsion:

Theorem 1.1 (Universal coefficient theorem)**.** *Let C be a bounded complex of ΛH-modules. Assume that the terms of C are free over Λ. Then for all $n \geq 1$ and $i \in \mathbb{Z}$, there exists a short exact sequence of ΛH-modules*

$$0 \longrightarrow \mathrm{H}^i(C) \otimes_\Lambda \Lambda/\ell^n\Lambda \longrightarrow \mathrm{H}^i\big(C \overset{\mathrm{L}}{\otimes}_\Lambda \Lambda/\ell^n\Lambda\big) \longrightarrow \mathrm{Tor}_1^\Lambda(\mathrm{H}^{i+1}(C), \Lambda/\ell^n\Lambda) \longrightarrow 0.$$

In particular, whenever there is no torsion in both C and $\mathrm{H}^\bullet(C)$ then the cohomology of $\overline{C}$ is exactly the ℓ-reduction of the cohomology of C.

1.1.2. **Composition factors in the cohomology.** Let C be a complex of kH-modules and L be a simple kG-module. We denote by P_L the projective cover of L in kG-mod. We can determine the cohomology groups of C in which L occurs as a composition factor using the following standard result:

Lemma 1.2. *Given $i \in \mathbb{Z}$, the following assertions are equivalent:*

(i) *the i-th cohomology group of the complex* $\mathrm{RHom}^\bullet_{kH}(P_L, C)$ *is non-zero;*

(ii) $\mathrm{Hom}_{K^b(kH)}(P_L, C[i])$ *is non-zero;*

(iii) *L is a composition factor of* $\mathrm{H}^i(C)$*.*

Proof. See for example [14, Section 1.1.2]. ☐

The formulation (i) is particularly adapted to our framework and will be extensively used in Section 3.

1.1.3. **Generalized eigenspaces over Λ.** Let M be a finitely generated Λ-module and $f \in \mathrm{End}_\Lambda(M)$. Assume that the eigenvalues of f are in the ring Λ. For $\lambda \in \Lambda$, we have defined in [13, Section 1.2.2] the generalized (λ)-eigenspace $M_{(\lambda)}$ of f on M. Here are the principal properties that we shall use:

- $M_{(\lambda)}$ is a direct summand of M and M is the direct sum of the generalized (λ)-eigenspaces for various $\lambda \in \Lambda$;
- if λ and μ are congruent modulo ℓ then $M_{(\lambda)} = M_{(\mu)}$;
- $(kM)_{(\lambda)} := M_{(\lambda)} \otimes_\Lambda k$ is the usual generalized $\bar{\lambda}$-eigenspace of $\bar{f}$;
- $(KM)_{(\lambda)} := M_{(\lambda)} \otimes_\Lambda K$ is the sum of the usual generalized μ-eigenspace of f where μ runs over the elements of Λ that are congruent to λ modulo ℓ.

1.2 Geometric methods

To any quasi-projective variety X defined over $\overline{\mathbb{F}}_p$ and acted on by H, one can associate a classical object in the derived category $D^b(\mathcal{O}H\text{-Mod})$, namely the cohomology with compact support of X, denoted by $\mathrm{R}\Gamma_c(\mathrm{X}, \mathcal{O})$. It is quasi-isomorphic to a bounded complex of modules that have finite rank over $\mathcal{O}$. Moreover, the cohomology complex behaves well with respect to scalar extension and ℓ-reduction. We have indeed in $D^b(\mathcal{O}H\text{-Mod})$:

$$\mathrm{R}\Gamma_c(\mathrm{X}, \Lambda) \overset{\mathrm{L}}{\otimes}_\Lambda \mathcal{O} \simeq \mathrm{R}\Gamma_c(\mathrm{X}, \mathcal{O}).$$

In particular, the universal coefficient theorem (Theorem 1.1) will hold for ℓ-adic cohomology with compact support.

We give here some quasi-isomorphisms we shall use in Sections 2 and 3. The reader will find references or proofs of these properties in [1, Section 3].

Proposition 1.3. *Let* X *and* Y *be two quasi-projective varieties acted on by* H*. Then one has the following isomorphisms in the derived category* $D^b(\mathcal{O}H\text{-Mod})$*:*

(i) *The Künneth formula:*

$$\mathrm{R}\Gamma_c(\mathrm{X}\times\mathrm{Y},\mathcal{O}) \simeq \mathrm{R}\Gamma_c(\mathrm{X},\mathcal{O}) \overset{\mathrm{L}}{\otimes} \mathrm{R}\Gamma_c(\mathrm{Y},\mathcal{O}).$$

(ii) *The quotient variety* $H\backslash\mathrm{X}$ *exists. Moreover, if the order of the stabilizer of any point of* X *is prime to* ℓ*, then*

$$\mathrm{R}\Gamma_c(H\backslash\mathrm{X},\mathcal{O}) \simeq \mathcal{O} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}H} \mathrm{R}\Gamma_c(\mathrm{X},\mathcal{O}).$$

If N is a finite group acting on X on the right and on Y on the left, we can form the amalgamated product $\mathrm{X}\times_N\mathrm{Y}$, as the quotient of $\mathrm{X}\times\mathrm{Y}$ by the diagonal action of N. Assume that the actions of H and N commute and that the order of the stabilizer of any point for the diagonal action of N is prime to ℓ. Then $\mathrm{X}\times_N\mathrm{Y}$ is an H-variety and we deduce from the above properties that

$$\mathrm{R}\Gamma_c(\mathrm{X}\times_N\mathrm{Y},\mathcal{O}) \simeq \mathrm{R}\Gamma_c(\mathrm{X},\mathcal{O}) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}N} \mathrm{R}\Gamma_c(\mathrm{Y},\mathcal{O})$$

in the derived category $D^b(\mathcal{O}H\text{-Mod})$.

Proposition 1.4. *Assume that* Y *is an open subvariety of* X*, stable by the action of* H*. Denote by* $\mathrm{Z}=\mathrm{X}\smallsetminus\mathrm{Y}$ *its complement. Then there exists a distinguished triangle in* $D^b(\mathcal{O}H\text{-Mod})$*:*

$$\mathrm{R}\Gamma_c(\mathrm{Y},\mathcal{O}) \longrightarrow \mathrm{R}\Gamma_c(\mathrm{X},\mathcal{O}) \longrightarrow \mathrm{R}\Gamma_c(\mathrm{Z},\mathcal{O}) \rightsquigarrow$$

Moreover, if Y *is both open and closed, then this triangle splits.*

Finally, for a smooth quasi-projective variety, Poincaré-Verdier duality [9] establishes a remarkable relation between the cohomology complexes $\mathrm{R}\Gamma_c(\mathrm{X},\mathcal{O})$ and $\mathrm{R}\Gamma(\mathrm{X},\mathcal{O})$. We shall only need the weaker version for cohomology groups:

Theorem 1.5 (Poincaré duality)**.** *Let* X *be a smooth quasi-projective variety of pure dimension* d*. Then if* $\mathcal{O}$ *is the field* K *or* k*, there exists a non-canonical isomorphism of* $\mathcal{O}H$*-modules*

$$\mathrm{H}_c^i(\mathrm{X},\mathcal{O})^* = \mathrm{Hom}_{\mathcal{O}}\big(\mathrm{H}_c^i(\mathrm{X},\mathcal{O}),\mathcal{O}\big) \simeq \mathrm{H}^{2d-i}(\mathrm{X},\mathcal{O}).$$

1.3 Lie-theoretic methods

Following [21], we recall the construction of the generalized Gelfand-Graev representations for finite reductive groups $\mathbf{G}^F$. The standard results on nilpotent orbits that we need here require that p – the characteristic of the ground field – is a good prime number for $\mathbf{G}$. We shall always assume that it is the case when working with unipotent classes or nilpotent orbits.

1.3.1. **Finite reductive groups.** We keep the basic assumptions of the introduction, with some slight modification: $\mathbf{G}$ is a quasi-simple algebraic group, together with an isogeny F, some power of which is a Frobenius endomorphism. In other words, there exists a positive integer δ such that F^δ defines an $\mathbb{F}_{q^\delta}$-structure on $\mathbf{G}$ for a certain power q^δ of the characteristic p (note that q might not be an integer). For all F-stable algebraic subgroup $\mathbf{H}$ of $\mathbf{G}$, we will denote by H the finite group of fixed points $\mathbf{H}^F$.

We fix a Borel subgroup $\mathbf{B}$ containing a maximal torus $\mathbf{T}$ of $\mathbf{G}$ such that both $\mathbf{B}$ and $\mathbf{T}$ are F-stable. They define a root sytem Φ with basis Δ, and a set of positive (resp. negative) roots Φ^+ (resp. Φ^-). Note that the corresponding Weyl group W is endowed with a action of F, compatible with the isomorphism $W \simeq N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Therefore, the image by F of a root is a positive multiple of some other root, which will be denoted by $\phi^{-1}(\alpha)$, defining thus a bijection $\phi : \Phi \longrightarrow \Phi$. Since $\mathbf{B}$ is also F-stable, this map preserves Δ and Φ^+. We will also use the notation $[\Delta/\phi]$ for a set of representatives of the orbits of ϕ on Δ.

Let $\mathbf{U}$ (resp. $\mathbf{U}^-$) be the unipotent radical of $\mathbf{B}$ (resp. the opposite Borel subgroup $\mathbf{B}^-$). Each root α defines a one-parameter subgroup $\mathbf{U}_\alpha$, and we will denote by $u_\alpha : \mathbb{F} \longrightarrow \mathbf{U}_\alpha$ an isomorphism of algebraic group. The groups $\mathbf{U}_\alpha$ might not be F-stable in general although the groups $\mathbf{U}$ and $\mathbf{U}^-$ are. However we may, and we will, choose the family $(u_\alpha)_{\alpha\in\Phi}$ such that the action of F satisfies $F(u_\alpha(\zeta)) = u_{\phi(\alpha)}(\zeta^{q_\alpha})$ for some power q_α of p.

1.3.2. **Generalized Gelfand-Graev representations.** Let $\mathfrak{g}$ (resp. $\mathfrak{t}$) be the Lie algebra of $\mathbf{G}$ (resp. $\mathbf{T}$) over $\overline{\mathbb{F}}_p$. The action of F on $\mathbf{G}$ induces a morphism $F : \mathfrak{g} \longrightarrow \mathfrak{g}$, which is compatible with the adjoint action of $\mathbf{G}$. Moreover, one can choose a family $(e_\alpha)_{\alpha\in\Phi}$ of tangent vectors to the unipotent groups $(\mathbf{U}_\alpha)_{\alpha\in\Phi}$ such that $F(e_\alpha) = e_{\phi(\alpha)}$. Finally, we fix a non-degenerate $\mathbf{G}$-equivariant associative bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{F}$ and a representative $\dot{w}_0$ of w_0 in $N_G(\mathbf{T})$ that will play the rôle of the opposition automorphism used in [21].

We assume now that the characteristic p of $\overline{\mathbb{F}}_p$ is good for G. Then it is known (see for example [8, Chapter 5]) that to each unipotent class of $\mathbf{G}$ can be naturally attached a *weighted Dynkin diagram*, that is, an additive map $d : \Phi \longrightarrow \mathbb{Z}$ such that $d(\alpha) \in \{0,1,2\}$ for all $\alpha \in \Delta$. Given such a Dynkin diagram, we define the subgroups

$$\mathbf{L}_d = \langle \mathbf{T}, \mathbf{U}_\alpha \mid d(\alpha) = 0 \rangle \qquad \text{and} \qquad \mathbf{U}_i = \prod_{\substack{\alpha\in\Phi^+ \\ d(\alpha)\geq i}} \mathbf{U}_\alpha$$

for all positive integer $i \geq 1$. The corresponding Lie subalgebras will be denoted by $\mathfrak{l}_d = \mathrm{Lie}(\mathbf{L}_d)$ and $\mathfrak{u}(i) = \mathrm{Lie}(\mathbf{U}_i)$. The connection between unipotent and nilpotent elements will be restricted to the map $\phi : \mathbf{U}_2 \longrightarrow \mathfrak{u}(2)$ defined by:

$$\phi\Big(\prod_{\substack{\alpha\in\Phi^+\\ d(\alpha)\geq 2}} u_\alpha(\zeta_\alpha)\Big) = \sum_{\substack{\alpha\in\Phi^+\\ d(\alpha)=2}} \zeta_\alpha e_\alpha$$

Note that this definition does not depend on any order on Φ, since the group $\mathbf{U}_2/\mathbf{U}_3$ is abelian. Then, by [27] the nilpotent class corresponding to d is the unique class $\mathscr{O}$ such that $\mathscr{O}\cap\mathfrak{u}(2)$ is dense in $\mathfrak{u}(2)$. Moreover, if we set $\mathbf{P}_d = \mathbf{L}_d\mathbf{U}_1$, then $\mathscr{O}\cap\mathfrak{u}(2)$ is a single $\mathbf{P}_d$-conjugacy class and for any nilpotent element $N \in \mathscr{O}\cap\mathfrak{u}(2)$, we have $C_{\mathbf{G}}(N) = C_{\mathbf{P}_d}(N)$. Therefore, a complete set of representatives for the G-orbits in $\mathscr{O}^F$ can be found inside $\mathscr{O}^F\cap\mathfrak{u}(2)$.

We fix a non-trivial linear character $\chi : \mathbb{F}_q^\delta \longrightarrow \Lambda^\times$ of $\mathbb{F}_q^\delta$. Following [21], we define for all $N\in\mathscr{O}^F\cap\mathfrak{u}(2)$, a linear character $\varphi_N : U_2 \longrightarrow \Lambda^\times$ of U_2 by

$$\forall x\in U_2 \qquad \varphi_N(x) = \chi\big(\kappa({}^{\dot{w}_0}N, \phi(x))\big).$$

Remark 1.6. If α is a positive root of weight 2, the restriction of φ_N to the finite group $V_\alpha = \big(\mathbf{U}_\alpha\cdot\mathbf{U}_{\phi(\alpha)}\cdots\mathbf{U}_{\phi^{-1}(\alpha)}\big)^F \subset U_2/U_3$ is non-trivial if and only if the coordinate of N on e_α is non-zero. Note also that φ_N depends only on the image of N in $\mathfrak{u}(2)/\mathfrak{u}(3)$.

Finally, Kawanaka showed that there exist an F-stable unipotent subgroup $\mathbf{U}_{1.5}$ of $\mathbf{U}_1$ containing $\mathbf{U}_2$ such that $[U_{1.5} : U_2] = [U_1 : U_{1.5}]$ and an extension $\tilde{\varphi}_N$ of φ_N to a linear character of $U_{1.5}$. The *generalized Gelfand-Graev representation* of G associated to N is then defined to be

$$\Gamma_N = \mathrm{Ind}_{U_{1.5}}^G \Lambda_N$$

where Λ_N denotes the module Λ on which $U_{1.5}$ acts by $\tilde{\varphi}_N$. It is a projective ΛG-module, which depends only on the G-conjugacy class of N. The character of this module will be denoted by γ_N.

1.3.3. **Unipotent support and wave front set.** In [23, Section 13.4], Lusztig has defined a function from the set of unipotent characters to the set of unipotent classes. If ρ is a unipotent character, there exists a unique special character $E\in(\mathrm{Irr}\,W)^F$ such that the Alvis-Curtis dual $D_G(\rho)$ of ρ has a non-trivial scalar product with the almost character R_E associated to E. Under the Springer correspondence, E correspond to the trivial local system on a special unipotent class (note that we use Lusztig's convention: the trivial character of W correspond to the regular orbit of $\mathbf{G}$). When p is assumed to be good, this unipotent class coincides with the *unipotent support* of ρ, defined as the unipotent class of maximal dimension on which the average value of ρ is non-zero (see [24] and [15]). Using the bijection between unipotent classes and nilpotent orbits in good characteristic, we will denote by $\mathscr{O}(\rho)$ the nilpotent orbit corresponding to the unipotent support of ρ.

By [21] and [24, Theorem 11.2] there exists a precise relation between the nilpotent orbit associated to a generalized Gelfand-Graev representation Γ_N and the orbits $\mathscr{O}(\rho)$ of the unipotent components ρ of its character $\gamma_N = [\Gamma_N]$.

Theorem 1.7 (Kawanaka, Lusztig). *Assume that p is a good prime number for* $\mathbf{G}$. *Let $\mathscr{O}$ be an F-stable nilpotent orbit of* $\mathbf{G}$ *and ρ be a unipotent character of G. Then*

(i) *if* $\dim \mathscr{O}(\rho) \leq \dim \mathscr{O}$ *and* $\mathscr{O} \neq \mathscr{O}(\rho)$ *then for all* $N \in \mathscr{O}^F$, $\langle \rho\,; \gamma_N \rangle = 0$*;*

(ii) *if* $N \in \mathscr{O}(\rho)^F$ *then the multiplicity of ρ in γ_N is bounded by* $|W|^2$*;*

(iii) *there exists* $N \in \mathscr{O}(\rho)^F$ *such that* $\langle \rho\,; \gamma_N \rangle \neq 0$

Remark 1.8. The argument used in [24] is valid only if p is assumed to be large enough. However, Lusztig has communicated to the author that he had a proof of Theorem 1.7 when p is a good prime number, but unfortunately he has not published it. Note that the case of split groups of type A_n, E_n, F_4 or G_2 is already contained in [20, Theorem 2.4.1].

2 General results on the torsion

We present in this section some general results on the torsion in the cohomology of Deligne-Lusztig varieties associated to Coxeter elements. We are motivated by the study of the principal ℓ-block of G in the *Coxeter case*, involving only a specific class of prime numbers ℓ. We will only briefly review the results that we will need about the principal ℓ-block but a fully detailed treatment of the Coxeter case can be found in [13].

The problem of finding the torsion in the cohomology of a given variety is a difficult problem. We shall use here all the specificities of the Deligne-Lusztig varieties associated to Coxeter elements: smooth compactifications, filtrations and remarkable quotients by some finite groups. These will be the principal ingredients to prove that the contribution of the principal ℓ-block to the torsion in the cohomology is necessarily a cuspidal module (see Corollary 2.10). This addresses the problem of finding where cuspidal composition factors can occur in the cohomology. We will discuss this general problem in the next section.

2.1 Review of the Coxeter case

Recall that the p'-part of the order of G is a product of cyclotomic polynomials $\Phi_d(q)$ for various divisors d of the degrees of W (some precautions must be taken for Ree and Suzuki groups [4]). Therefore, if ℓ is any prime number different from p, it should divide at least one of these polynomials. Moreover, if we assume that ℓ is prime to W^F, then there is a unique d such that $\ell \,|\, \Phi_d(q)$.

In this paper we will be interested in the case where d is maximal. Since W is irreducible, it corresponds to the case where $d = h$ is the Coxeter number of the pair (W,F). Explicit values of h can be found in [13]. For a more precise statement − including the Ree and Suzuki groups − recall that a Coxeter element of the pair (W,F) is a product $c = s_{\beta_1} \cdots s_{\beta_r}$ where $\{\beta_1,\ldots,\beta_r\} = [\Delta/\phi]$ is any set of representatives of the orbits of the simple roots under the action of ϕ.

Then the *Coxeter case* corresponds to the situation where ℓ is prime to $|W^F|$ and satisfies:

- "non-twisted" cases: ℓ divides $\Phi_h(q)$;
- "twisted" cases (Ree and Suzuki groups): ℓ divides the order of T_c for some Coxeter element c.

Note that these conditions ensure that the class of q in $k^\times$ is a primitive h-th root of unity.

As in Section 1.1, the modular framework will be given by an ℓ-modular system (K,Λ,k), which we require to be big enough for G. We denote by b an idempotent associated to be the principal block of ΛG. With the assumptions made on ℓ, the ℓ-component of $\mathbf{T}^{cF}$ is a Sylow ℓ-subgroup of G and as such is the defect of b. It will be denoted by T_ℓ.

The structure of the block is closely related to the cohomology of the Deligne-Lusztig varieties associated to c. Fix a representative $\dot{c}$ of c in $N_{\mathbf{G}}(\mathbf{T})$ and define the varieties Y and X by

$$\begin{array}{rl} \mathrm{Y} & = \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g^{-1}F(g) \in \mathbf{U}\dot{c}\mathbf{U}\} \\ {\scriptstyle \pi_c}\big\downarrow {\scriptstyle /\mathbf{T}^{cF}} & \\ \mathrm{X} & = \{g\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid g^{-1}F(g) \in \mathbf{B}c\mathbf{B}\} \end{array}$$

where π_c denotes the restriction to Y of the canonical projection $\mathbf{G}/\mathbf{U} \longrightarrow \mathbf{G}/\mathbf{B}$. They are both quasi-projective varieties endowed with a left action of G by left multiplication. Furthermore, $\mathbf{T}^{cF}$ acts on the right of Y and π_c is isomorphic to the corresponding quotient map, so that it induces a G-equivariant isomorphism of varieties $\mathrm{Y}/\mathbf{T}^{cF} \simeq \mathrm{X}$. Combining the results of [10], [22] and [6] we can parametrize the irreducible characters of the principal ℓ-block:

- the non-unipotent characters of bKG are exactly the θ-isotypic components $\mathrm{H}_c^r(\mathrm{Y},K)_\theta$ where θ runs over the F^δ-orbits in $\mathrm{Irr}\, T_\ell \smallsetminus \{1_{T_\ell}\}$;
- the unipotent characters of bKG are the eigenspaces of F^δ on $\mathrm{H}_c^\bullet(\mathrm{X},K)$. Each eigenvalue is congruent modulo ℓ to $q^{j\delta}$ for a unique integer $j \in \{0,\ldots,h/\delta-1\}$. We denote by χ_j the corresponding irreducible character. Moreover, each Harish-Chandra series that intersects the block corresponds to a root of unity $\zeta \in K$ and the characters in this series are arranged as follows:

$\mathrm{H}_c^r(\mathrm{X},K)$	$\mathrm{H}_c^{r+1}(\mathrm{X},K)$	$\mathrm{H}_c^{r+2}(\mathrm{X},K)$	$\cdots$	$\mathrm{H}_c^{r+M_\zeta-m_\zeta}(\mathrm{X},K)$
χ_{m_ζ}	$\chi_{m_\zeta+1}$	$\chi_{m_\zeta+2}$	$\cdots$	χ_{M_ζ}

Furthermore, the distinction "non-unipotent/unipotent" corresponds to the distinction "exceptional/non-exceptional" in the theory of blocks with cyclic defect groups. The connection is actually much deeper: Hiss, Lübeck and Malle have observed in [19] that the cohomology of the Deligne-Lusztig variety X should not only give the characters of the principal ℓ-block, but also its Brauer tree Γ:

Conjecture HLM (Hiss-Lübeck-Malle). *Let $\Gamma^\bullet$ denote the graph obtained from the Brauer tree of the principal ℓ-block by removing the exceptional node and all edges incident to it. Then the following holds:*

(i) *The connected components of $\Gamma^\bullet$ are labeled by the Harish-Chandra series, hence by some roots of unity $\zeta \in K$.*

(ii) *The connected component corresponding to a root ζ is:*

(iii) *The vertices labeled by χ_{m_ζ} are the only nodes connected to the exceptional node.*

The validity of this conjecture has been checked in all cases where the Brauer tree was known, that is for all quasi-simple groups except the groups of type E_7 and E_8. A general proof has been exposed in [13], but under a precise assumption on the torsion in the cohomology of Y:

(W) For all $\zeta \in K$ corresponding to an Harish-Chandra series in the block, the generalized $(q^{m_\zeta\delta})$-eigenspace of F^δ on $b\mathrm{H}_c^\bullet(\mathrm{Y},\Lambda)$ is torsion-free.

This assumption concerns only the "minimal" eigenvalues, that is the eigenvalues of F^δ on the cohomology of X in middle degree. Unfortunately, we will not be able to prove it in its full generality for the variety Y, but only for the variety X (see Proposition 2.12).

We have investigated in [13] the consequences of a stronger assumption, concerning all the eigenvalues of F^δ:

(S) The Λ-modules $b\mathrm{H}_c^i(\mathrm{Y},\Lambda)$ are torsion-free.

Such an assumption is known to be valid for groups with $\mathbb{F}_q$-rank 1 (since the corresponding Deligne-Lusztig variety is a irreducible affine curve) and for groups of type A_n [2]. The purpose of this paper is to prove that it is actually valid for any quasi-simple group (including E_7 and E_8 if Conjecture (HLM) holds) as soon as p is a good prime number (see Theorem 3.9). As a byproduct, we obtain a proof of the geometric version of Broué's conjecture in the Coxeter case when p is good for $\mathbf{G}$. We also deduce the planar embedding of Brauer trees for groups of type ${}^2\mathrm{G}_2$ and F_4.

2.2 Torsion and cuspidality

Let $I \subset \Delta$ be a ϕ-stable subset of simple roots. We denote by $\mathbf{P}_I$ the standard parabolic subgroup, by $\mathbf{U}_I$ its unipotent radical and by $\mathbf{L}_I$ its standard Levi complement. The corresponding Weyl group is the subgroup W_I of W generated by the simple reflexions in I. All these groups are F-stable. One obtains a Coxeter element c_I of (W_I,F) by removing in c the reflexions associated to the simple roots which are not in I. Written with the Borel subgroup $\mathbf{B}_I = \mathbf{B}\cap\mathbf{L}_I$ of $\mathbf{L}_I$, the Deligne-Lusztig variety X_I associated to c_I is by definition

$$\mathrm{X}_I = \mathrm{X}_{\mathbf{L}_I}(c_I) = \left\{ g\mathbf{B}_I \in \mathbf{L}_I/\mathbf{B}_I \,\middle|\, g^{-1}F(g) \in \mathbf{B}_I c_I \mathbf{B}_I \right\}.$$

Recall that the Harish-Chandra induction and restriction functors are defined over any coefficient ring $\mathcal{O}$ between (K,Λ,k) by

$$\begin{array}{rrcl} \mathrm{R}_{L_I}^G : & \mathcal{O}L_I\text{-mod} & \longrightarrow & \mathcal{O}G\text{-mod} \\ & N & \longmapsto & \mathcal{O}[G/U_I] \otimes_{\mathcal{O}L_I} N \end{array}$$

and

$$\begin{array}{rrcl} {}^*\mathrm{R}_{L_I}^G : & \mathcal{O}G\text{-mod} & \longrightarrow & \mathcal{O}L_I\text{-mod} \\ & M & \longmapsto & M^{U_I}. \end{array}$$

By the results in [22], the restriction of the cohomology of X can be expressed in terms of the cohomology of smaller Coxeter varieties, leading to an inductive approach for studying the torsion. Given the assumptions made on ℓ in Section 2.1, we can prove the following:

Proposition 2.1. *If I is a proper ϕ-stable subset of Δ, then the groups $\mathrm{H}_c^i(\mathrm{X}_I,\Lambda)$ are torsion-free.*

Proof. By induction on the cardinality of I, the case $I=\varnothing$ being trivial. Assume that I is non-empty, and let J be any maximal proper F-stable subset of I. By [22, Corollary 2.10] there exists an isomorphism of varieties $(U_J \cap \mathbf{L}_I)\backslash \mathrm{X}_I \simeq \mathrm{X}_J \times \mathbf{G}_m$ which yields the following isomorphism of Λ-modules

$${}^*\mathrm{R}_{L_J}^{L_I}\big(\mathrm{H}_c^i(\mathrm{X}_I,\Lambda)\big) \simeq \mathrm{H}_c^{i-1}(\mathrm{X}_J,\Lambda) \oplus \mathrm{H}_c^{i-2}(\mathrm{X}_J,\Lambda).$$

Consequently, the restriction to L_J of $\mathrm{H}_c^\bullet(\mathrm{X}_I,\Lambda)_{\mathrm{tor}}$ is a torsion submodule of $\mathrm{H}_c^\bullet(\mathrm{X}_J,\Lambda)$ and hence it is zero by assumption. In other words, any torsion submodule of $\mathrm{H}_c^\bullet(\mathrm{X}_I,\Lambda)_{\mathrm{tor}}$ is a cuspidal ΛL_I-module.

Let $r_I = |I/\phi|$ be the number of ϕ-orbits in I. Following [22], we can define a smooth compactification $\overline{\mathrm{X}}_I$ of X_I. It has a filtration by closed L_I-subvarieties $\overline{\mathrm{X}}_I = D_{r_I}(I) \supset D_{r_I-1}(I) \supset \cdots \supset D_0(I) = L_I/B_I$ such that each pair $(D_a(I), D_{a-1}(I))$ leads to the following long exact sequences of kL_I-modules:

$$\cdots \longrightarrow \bigoplus_{\substack{J\subset I \ \phi\text{-stable} \\ |J/\phi|=a}} \mathrm{R}_{L_J}^{L_I}\big(\mathrm{H}_c^i(\mathrm{X}_J,k)\big) \longrightarrow \mathrm{H}^i(D_a(I),k) \longrightarrow \mathrm{H}^i(D_{a-1}(I),k) \longrightarrow \cdots$$

Since kL_I is a semi-simple algebra, we are in the following situation:

- any cuspidal composition factor of a kL_I-module M is actually a direct summand of M. Therefore we can naturally define the cuspidal part M_{cusp} of M as the sum of all cuspidal submodules;
- if $J \neq I$ then an induced module $\mathrm{R}_{L_J}^{L_I}(M)$ has a zero cuspidal part;
- by Poincaré duality (see Theorem 1.5) $\mathrm{H}^i(\overline{\mathrm{X}}_I,k)$ is isomorphic to the k-dual of the kL_I-module $\mathrm{H}^{2r_I-i}(\overline{\mathrm{X}}_I,k)$.

Consequently, we can argue as in [22] and use the following isomorphisms :

$$\mathrm{H}_c^i(\mathrm{X}_I,k)_{\mathrm{cusp}} \simeq \mathrm{H}^i(\overline{\mathrm{X}}_I,k)_{\mathrm{cusp}} \simeq \mathrm{H}^{2r_I-i}(\overline{\mathrm{X}}_I,k)^*_{\mathrm{cusp}} \simeq \mathrm{H}_c^{2r_I-i}(\mathrm{X}_I,k)^*_{\mathrm{cusp}}$$

to deduce that $\mathrm{H}_c^i(\mathrm{X}_I,k)$ has a zero cuspidal part whenever $i > r_I$.

Finally, $\mathrm{H}_c^i(\mathrm{X}_I,\Lambda)_{\mathrm{tor}}\otimes_\Lambda k$ is a cuspidal submodule of $\mathrm{H}_c^i(\mathrm{X}_I,\Lambda)\otimes_\Lambda k$ which is also a submodule of $\mathrm{H}_c^i(\mathrm{X}_I,k)$ by the universal coefficient theorem (see Theorem 1.1). Therefore it must be zero if $i\neq r_I$ and $\mathrm{H}_c^i(\mathrm{X}_I,\Lambda)$ is torsion-free. Note that the cohomology group in middle degree is also torsion-free since X_I is an irreducible affine variety: we have indeed $\mathrm{H}_c^{r_I-1}(\mathrm{X}_I,k)=0$ so that $\mathrm{H}_c^{r_I}(\mathrm{X}_I,\Lambda)_{\mathrm{tor}}=0$ by the universal coefficient theorem. □

Remark 2.2. We have actually shown that the cohomology of a Deligne-Lusztig variety associated to a Coxeter element is torsion-free whenever ℓ does not divide the order of the corresponding finite reductive group. One can conjecture that this should hold for any Deligne-Lusztig variety. Note that the assumption on ℓ is crucial, otherwise an induced module can have cuspidal composition factors. For example if ℓ divides $\Phi_h(q)$, then $\mathrm{R}_T^G(k)=k[G/B]$ has at least one cuspidal composition factor − denoted by S_0 in [13].

As an immediate consequence of Proposition 2.1 and the isomorphism in [22, Corollary 2.10], we obtain:

Corollary 2.3. *The torsion part of any group* $\mathrm{H}_c^i(\mathrm{X},\Lambda)$ *is a cuspidal* ΛG*-module.*

We can therefore reduce the problem of finding the torsion in the cohomology of X to the problem of finding where cuspidal composition factors can occur in the kG-modules $\mathrm{H}_c^i(\mathrm{X},k)$. This is the general approach that we shall use throughout this paper. It is justified by the fact that if we assume that the modules $\mathrm{H}_c^i(\mathrm{X},\Lambda)$ are torsion-free and that Conjecture (HLM) holds, then by [13] cuspidal composition factors can occur only in $\mathrm{H}_c^r(\mathrm{X},k)$.

2.3 Reduction: from Y to X

In this section we shall give conditions on X ensuring that the principal part of the cohomology of Y is torsion-free. Unlike Y, the variety X has a smooth compactification (the compactification of Y constructed in [3] is only rationally smooth in general). Therefore we can use duality theorems to study precise concentration properties of cuspidal modules in the cohomology of X (see Sections 2.4 and 3.1). We will not only restrict the cohomology of Y to the principal block b of ΛG but also to the principal block b' of $\Lambda\mathbf{T}^{cF}$. This is not a strong restriction since the complexes $b\mathrm{R}\Gamma_c(\mathrm{Y},\Lambda)$ and $b\mathrm{R}\Gamma_c(\mathrm{Y},\Lambda)b'$ are conjecturally isomorphic in $K^b(\Lambda G\text{-Mod})$. Note that this is already the case when the coefficient ring is K (see [6, Theorem 5.24]). By Proposition 1.3 we have

$$\mathrm{R}\Gamma_c(\mathrm{Y},\Lambda)b'\simeq\mathrm{R}\Gamma_c(\mathrm{Y},\Lambda)\otimes_{\Lambda T_{\ell'}}\Lambda\simeq\mathrm{R}\Gamma_c(\mathrm{Y}/T_{\ell'},\Lambda)$$

where $T_{\ell'}$ is the ℓ'-component of the abelian group $\mathbf{T}^{cF}$. The quotient variety $\mathrm{Y}/T_{\ell'}$ will be denoted by Y_ℓ.

From now on, we shall work exclusively with Y_ℓ instead of Y . This will not have any incidence on the results we have deduced from (W) and (S) in [13]. We start by proving an analog of [22, Corollary 2.10] for the variety Y_ℓ:

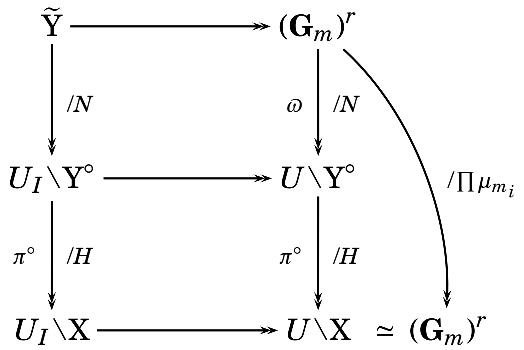
Proposition 2.4. *Let I be a maximal proper ϕ-stable subset of Δ. Then there exists an isomorphism of Λ-modules*

$$\mathrm{H}_c^i(U_I\backslash \mathrm{Y}_\ell,\Lambda) \simeq {}^*\mathrm{R}_{L_I}^G\big(\mathrm{H}_c^i(\mathrm{Y}_\ell,\Lambda)\big) \simeq \mathrm{H}_c^{i-1}(\mathrm{X}_I,\Lambda)\oplus \mathrm{H}_c^{i-2}(\mathrm{X}_I,\Lambda).$$

Proof. Recall that the isomorphism $U_I\backslash \mathrm{X} \simeq \mathrm{X}_I \times \mathbf{G}_m$ constructed in [22] is not G-equivariant. We can check that it is nevertheless V_I-equivariant, where $\mathbf{V}_I = \mathbf{U}\cap \mathbf{L}_I$ is a maximal unipotent subgroup of $\mathbf{L}_I$. We argue as in [2]: the isomorphism cannot be lifted up to Y_ℓ but only to an abelian covering of this variety. Let Y° be the preimage in Y of a connected component of $U\backslash \mathrm{Y}$. It is stable by the action of U but not necessarily by the action of $\mathbf{T}^{cF}$ which permutes transitively the connected components of $U\backslash\mathrm{Y}$ (since $U\backslash \mathrm{Y}/\mathbf{T}^{cF} \simeq U\backslash \mathrm{X}$ is connected). If we denote by H the stabilizer of the component $U\backslash \mathrm{Y}^\circ$ in $\mathbf{T}^{cF}$, we have the following $V_I \times (\mathbf{T}^{cF})^{\mathrm{op}}$-equivariant isomorphism

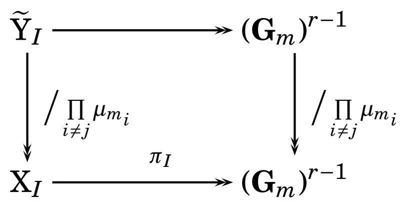
$$U_I\backslash \mathrm{Y}^\circ \times_H \mathbf{T}^{cF} \simeq U_I\backslash \mathrm{Y}. \tag{2.5}$$

By Abhyankar's lemma, we can now lift the Galois covering $U\backslash \mathrm{Y}^\circ \longrightarrow U\backslash \mathrm{X} \simeq (\mathbf{G}_m)^r$ up to a covering $\varpi : (\mathbf{G}_m)^r \longrightarrow (\mathbf{G}_m)^r$ with Galois group $\prod \mu_{m_i}$. Details of this construction are given in [2] and [12]. We define the variety $\widetilde{\mathrm{Y}}$ using the following diagram in which all the squares are cartesian:



(2.6)

Under the isomorphism $U_I\backslash \mathrm{X} \simeq \mathrm{X}_I \times \mathbf{G}_m$ the map $U_I\backslash \mathrm{X} \longrightarrow (\mathbf{G}_m)^r$ decomposes into $\pi_I \times \mathrm{id}_{\mathbf{G}_m}$ where $\pi_I : \mathrm{X}_I \longrightarrow V_I\backslash \mathrm{X}_I \simeq (\mathbf{G}_m)^{r-1}$ is the quotient map associated to X_I. In particular, we can form the following fiber product



(2.7)

in order to decompose the variety $\widetilde{\mathrm{Y}}$ into a product $\widetilde{\mathrm{Y}} \simeq \widetilde{\mathrm{Y}}_I \times \mathbf{G}_m$ which is compatible with the group decomposition $\prod \mu_{m_i} \simeq \big(\prod_{i\neq j}\mu_{m_i}\big)\times \mu_{m_j}$.

Recall from [2] that the cohomology of $\mathbf{G}_m$ endowed with the action of μ_m can be represented by a complex concentrated in degrees 1 and 2:

$$0 \longrightarrow \Lambda\mu_m \xrightarrow{\zeta-1} \Lambda\mu_m \longrightarrow 0$$

where ζ is any primitive m-th root of 1. Such a complex will be denoted by $Z(\mu_m)[-1]$ so that the cohomology of $Z(\mu_m)$ vanishes outside the degrees 0 and 1. As in [2, Section 3.3.2], we shall write $Z_{\mathbf{T}^{cF}}(\mu_m)$ for $Z(\mu_m)\otimes_{\Lambda\mu_m}\Lambda\mathbf{T}^{cF}$.

Using Proposition 1.3, we can now express the previous constructions in cohomological terms. Equation 2.5 together with Diagram 2.6 leads to

$$\begin{aligned}\mathrm{R}\Gamma_c(U_I\backslash \mathrm{Y},\Lambda) &\simeq \mathrm{R}\Gamma_c(U_I\backslash \mathrm{Y}^\circ,\Lambda)\overset{\mathrm{L}}{\otimes}_{\Lambda H}\Lambda\mathbf{T}^{cF}\\ &\simeq \mathrm{R}\Gamma_c(\widetilde{\mathrm{Y}},\Lambda)\overset{\mathrm{L}}{\otimes}_{\Lambda\prod\mu_{m_i}}\Lambda\mathbf{T}^{cF}\end{aligned}$$

Using the isomorphism $\widetilde{\mathrm{Y}}\simeq\widetilde{\mathrm{Y}}_I\times\mathbf{G}_m$ and the notation that we have introduced, it can be written as

$$\begin{aligned}\mathrm{R}\Gamma_c(U_I\backslash \mathrm{Y},\Lambda) &\simeq \Big(\mathrm{R}\Gamma_c(\widetilde{\mathrm{Y}}_I,\Lambda)\overset{\mathrm{L}}{\otimes}_{\Lambda\prod\limits_{i\neq j}\mu_{m_i}}\Lambda\mathbf{T}^{cF}\Big)\overset{\mathrm{L}}{\otimes}_{\Lambda\mathbf{T}^{cF}}\Big(\mathrm{R}\Gamma_c(\mathbf{G}_m,\Lambda)\overset{\mathrm{L}}{\otimes}_{\Lambda\mu_{m_j}}\Lambda\mathbf{T}^{cF}\Big)\\ &\simeq \Big(\mathrm{R}\Gamma_c(\widetilde{\mathrm{Y}}_I,\Lambda)\overset{\mathrm{L}}{\otimes}_{\Lambda\prod\limits_{i\neq j}\mu_{m_i}}\Lambda\mathbf{T}^{cF}\Big)\overset{\mathrm{L}}{\otimes}_{\Lambda\mathbf{T}^{cF}}Z_{\mathbf{T}^{cF}}(\mu_{m_j})[-1]\end{aligned}\tag{2.8}$$

in the derived category $D^b(\text{Mod-}\Lambda\mathbf{T}^{cF})$.

In general, the cohomology groups of the complex $Z_{\mathbf{T}^{cF}}(\mu_{m_j})$ are endowed with a non-trivial action of $\mathbf{T}^{cF}$ since the map $\mu_{m_j}\longrightarrow\mathbf{T}^{cF}$ is not surjective. One can actually compute explicitly the image of this map: by [2, Proposition 3.5] it is precisely the group $N_c(Y_{c,c_I})$ (see [1, Section 4.4.2] for the definition). Therefore we obtain

$$\mathrm{H}^0(Z_{\mathbf{T}^{cF}}(\mu_{m_j}))=\mathrm{H}^1(Z_{\mathbf{T}^{cF}}(\mu_{m_j}))\simeq\Lambda\big[\mathbf{T}^{cF}/N_c(Y_{c,c_I})\big].$$

By [1, Proposition 4.4], the quotient $\mathbf{T}^{cF}/N_c(Y_{c,c_I})$ is isomorphic to $\mathbf{T}^{c_IF}/N_{c_I}(Y_{c,c_I})$ which has order prime to ℓ. In particular, the image of the map $\mu_{m_j}\longrightarrow\mathbf{T}^{cF}$ contains the ℓ-Sylow subgroup of $\mathbf{T}^{cF}$ and the map $\mu_{m_j}\longrightarrow\mathbf{T}^{cF}/T_{\ell'}$ is definitely onto. Since $Z_{\mathbf{T}^{cF}}(\mu_{m_j})$ fits into the following distinguished triangle in $D^b(\Lambda\mathbf{T}^{cF}\text{-Mod})$

$$\Lambda\big[\mathbf{T}^{cF}/N_c(Y_{c,c_I})\big]\longrightarrow Z_{\mathbf{T}^{cF}}(\mu_{m_j})\longrightarrow\Lambda\big[\mathbf{T}^{cF}/N_c(Y_{c,c_I})\big][-1]\rightsquigarrow$$

we deduce that the coinvariants under T_ℓ have a relatively simple shape

$$\Lambda\longrightarrow Z_{\mathbf{T}^{cF}}(\mu_{m_j})\overset{\mathrm{L}}{\otimes}_{\Lambda T_{\ell'}}\Lambda\longrightarrow\Lambda[-1]\rightsquigarrow\tag{2.9}$$

Together with the expression of $\mathrm{R}\Gamma_c(\widetilde{\mathrm{Y}}_I,\Lambda)\overset{\mathrm{L}}{\otimes}_{\Lambda\prod\limits_{i\neq j}\mu_{m_i}}\Lambda\mathbf{T}^{cF}$ given in Formula 2.8, this triangle yields

$$\mathrm{R}\Gamma_c(\widetilde{\mathrm{Y}}_I,\Lambda)\overset{\mathrm{L}}{\otimes}_{\Lambda\prod\limits_{i\neq j}\mu_{m_i}}\Lambda[-1]\longrightarrow\mathrm{R}\Gamma_c(U_I\backslash\mathrm{Y}_\ell,\Lambda)\longrightarrow\mathrm{R}\Gamma_c(\widetilde{\mathrm{Y}}_I,\Lambda)\overset{\mathrm{L}}{\otimes}_{\Lambda\prod\limits_{i\neq j}\mu_{m_i}}\Lambda[-2]\rightsquigarrow$$

and then simply

$$\mathrm{R}\Gamma_c(\mathrm{X}_I,\Lambda)[-1]\longrightarrow\mathrm{R}\Gamma_c(U_I\backslash\mathrm{Y}_\ell,\Lambda)\longrightarrow\mathrm{R}\Gamma_c(\mathrm{X}_I,\Lambda)[-2]\rightsquigarrow$$

by definition of $\widetilde{\mathrm{Y}}_I$.

We claim that the connecting maps $\mathrm{H}_c^{i-2}(\mathrm{X}_I,\Lambda) \longrightarrow \mathrm{H}_c^i(\mathrm{X}_I,\Lambda)$ coming form the long exact sequence associated to the previous triangle are actually zero. Since the triangle 2.9 splits over K, the scalar extension of these morphisms are zero. But by Proposition 2.1 the modules $\mathrm{H}_c^i(\mathrm{X}_I,\Lambda)$ are torsion-free, so that the connecting morphisms are indeed zero over Λ. Consequently, we obtain short exact sequences

$$0 \longrightarrow \mathrm{H}_c^{i-1}(\mathrm{X}_I,\Lambda) \longrightarrow \mathrm{H}_c^i(\mathrm{Y}_\ell,\Lambda)^{U_I} \longrightarrow \mathrm{H}_c^{i-2}(\mathrm{X}_I,\Lambda) \longrightarrow 0$$

which finishes the proof. □

Corollary 2.10. *The torsion part of any cohomology group* $\mathrm{H}_c^i(\mathrm{Y}_\ell,\Lambda)$ *is a cuspidal* ΛG*-module.*

As a consequence, the assumption (S) holds for Y_ℓ if every cuspidal module in the block occurs in the cohomology group in middle degree only. By construction of Y_ℓ, it is actually sufficient to consider the cohomology of X:

Corollary 2.11. *Assume that one of the following holds:*

(1) *The cohomology of* X *over* Λ *is torsion-free and Conjecture* (HLM) *holds.*

(2) *Cuspidal composition factors occur in* $\mathrm{H}_c^i(\mathrm{X},k)$ *for* $i = r$ *only.*

Then the cohomology of Y_ℓ *over* Λ *is torsion-free.*

Proof. We have already mentioned at the end of Section 2.2 that assertions (1) and (2) are equivalent. Denote by $\pi_\ell : \mathrm{Y}_\ell \longrightarrow \mathrm{X}$ the quotient map by the ℓ-Sylow subgroup of $\mathbf{T}^{cF}$. The push-forward of the constant sheaf $\underline{k}_{\mathrm{Y}_\ell}$ on Y_ℓ is obtained from successive extensions of constant sheaves $\underline{k}_{\mathrm{X}}$:

$$(\pi_\ell)_*(\underline{k}_{\mathrm{Y}_\ell}) \simeq \begin{array}{c} \underline{k}_{\mathrm{X}} \\ \underline{k}_{\mathrm{X}} \\ \vdots \\ \underline{k}_{\mathrm{X}} \end{array} .$$

In the derived category, the complex $\mathrm{R}\Gamma_c(\mathrm{Y}_\ell,k) = \mathrm{R}\Gamma_c(\mathrm{X},(\pi_\ell)_*(\underline{k}_{\mathrm{Y}_\ell}))$ can also be obtained from extensions of $\mathrm{R}\Gamma_c(\mathrm{X},k)$'s. In other words, there exists a family of complexes $\mathrm{R}\Gamma_c(\mathrm{Y}_\ell,k) = C_0, C_1, \ldots, C_n = \mathrm{R}\Gamma_c(\mathrm{X},k)$ that fit into the following distinguished triangles in $D^b(kG\text{-Mod-}kT_\ell)$:

$$C_{i+1} \longrightarrow C_i \longrightarrow \mathrm{R}\Gamma_c(\mathrm{X},k) \rightsquigarrow$$

If we apply the functor $\mathrm{RHom}_{kG}^\bullet(P,-)$ to each of these triangles, we can deduce that the complexes $\mathrm{RHom}_{kG}^\bullet(P,C_i)$ are concentrated in degree r whenever $\mathrm{RHom}_{kG}^\bullet(P,C_n)$ is. Taking P to be the projective cover of any cuspidal kG-module, we conclude that cuspidal composition factors of the cohomology of Y_ℓ can only occur in middle degree. □

2.4 On the assumption (W) for X

We give here a proof of the analog of the assumption (W) for the Deligne-Lusztig variety X, using the smooth compactification $\overline{\mathrm{X}}$ constructed in [10].

Proposition 2.12. *If λ is an eigenvalue of F^δ on $\mathrm{H}_c^r(\mathrm{X},K)$, then the Λ-modules $\mathrm{H}_c^i(\mathrm{X},\Lambda)_{(\lambda)}$ are all torsion-free.*

Proof. In the line with [22], we shall denote by $D = D_{r-1}(\Delta)$ the complement of X in $\overline{\mathrm{X}}$. Let I be a ϕ-stable subset of Δ. By [22, Section 7], the generalized (λ)-eigenspaces of F^δ on $\mathrm{H}_c^i(\mathrm{X}_I,K)$ are zero whenever $i \neq |I/\phi|$. By Proposition 2.1, this remains true for cohomology groups with coefficients in k. From the long exact sequences

$$\cdots \longrightarrow \bigoplus_{\substack{I\ \phi\text{-stable}\\ |I/\phi|=a}} \mathrm{R}_{L_I}^G\big(\mathrm{H}_c^i(\mathrm{X}_I,k)\big) \longrightarrow \mathrm{H}_c^i(D_a(\Delta),k) \longrightarrow \mathrm{H}_c^i(D_{a-1}(\Delta),k) \longrightarrow \cdots \tag{2.13}$$

we deduce that $\mathrm{H}_c^i(D,k)_{(\lambda)}$ is zero if $i \geq r$. Consequently, for all $i > r$ we have

$$\mathrm{H}_c^i(\mathrm{X},k)_{(\lambda)} \simeq \mathrm{H}^i(\overline{\mathrm{X}},k)_{(\lambda)}.$$

The eigenvalue $\mu = q^{2r}\lambda^{-1}$ is a "maximal" eigenvalue of F^δ and as such does not occur in the cohomology of the varieties X_I for proper subsets I. From the previous long exact sequences we deduce that the generalized $\bar{\mu}$-eigenspace of F^δ on $\mathrm{H}_c^i(D,k)$ is always zero. Since $\overline{\mathrm{X}}$ is a smooth variety, we can apply Poincaré duality (see Thereom 1.5) in order to obtain the following isomorphisms:

$$\mathrm{H}_c^i(\mathrm{X},k)_{(\lambda)} \simeq \mathrm{H}^i(\overline{\mathrm{X}},k)_{(\lambda)} \simeq \big(\mathrm{H}^{2r-i}(\overline{\mathrm{X}},k)_{(\mu)}\big)^* \simeq \big(\mathrm{H}_c^{2r-i}(\mathrm{X},k)_{(\mu)}\big)^*.$$

Since X is an irreducible affine variety, the cohomology of X vanishes outside the degrees $r,\ldots,2r$. This proves that $\mathrm{H}_c^i(\mathrm{X},k)_{(\lambda)} = 0$ whenever $i > r$, and the result follows from the universal coefficient theorem. ☐

Remark 2.14. One cannot deduce that the assumption (W) holds for Y_ℓ using this result. The method that we used in the proof of Corollary 2.11 does not preserve the generalized eigenspaces of F^δ.

Note however that Bonnafé and Rouquier have constructed in [3] a compactification $\overline{\mathrm{Y}}_\ell$ of Y_ℓ such that $\overline{\mathrm{Y}}_\ell \smallsetminus \mathrm{Y}_\ell \simeq \overline{\mathrm{X}} \smallsetminus \mathrm{X}$. This compactification is only rationally smooth in general, but if it happened to be k-smooth, the previous method would extend to Y_ℓ and finish the proof of the conjecture of Hiss-Lübeck-Malle.

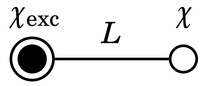
3 Cuspidal composition factors in $\mathrm{H}_c^\bullet(\mathrm{X},k)$

We have reduced the proof of the assumption (S) to showing that the kG-modules $\mathrm{H}_c^i(\mathrm{X},k)$ have no cuspidal composition factors unless $i = r$. In other words, the cohomology of $\mathrm{RHom}_{kG}^\bullet\big(\overline{P}_L,\mathrm{R}\Gamma_c(\mathrm{X},k)\big)$ should vanish outside the degree r whenever L is cuspidal.

Throughout this section and unless otherwise specified, we shall alway assume that Conjecture (HLM) holds for $(\mathbf{G},F)$ (which is known to be true except for the groups of type E_7 or E_8). In that case, the above result can be deduced from the characteristic zero case if we can show that the cohomology of $\mathrm{RHom}^\bullet_{\Lambda G}\big(P_L, \mathrm{R}\Gamma_c(\mathrm{X},\Lambda)\big)$ is torsion-free. Indeed, the irreducible unipotent components of $[P_L]$ are the characters χ_{m_ζ}'s, and they occur in the cohomology of X in middle degree only. We shall divide the proof according to the depth of χ_{m_ζ}: if χ_{m_ζ} is cuspidal, we prove that the contribution of P_L to the cohomology of X and its compactification $\overline{\mathrm{X}}$ are the same. In the case where χ_{m_ζ} is not cuspidal, it is induced from the cohomology of a smaller Coxeter variety and we show that P_L can be computed in terms of this cohomology, which we know to be torsion-free.

3.1 Harish-Chandra series of length 1

We start by dealing with the case of cuspidal kG-modules that occur as ℓ-reductions of cuspidal unipotent characters. By Conjecture (HLM), these correspond to subtrees of the Brauer tree of the form



where χ is a cuspidal unipotent character and L is the unique composition factor of the ℓ-reduction of χ. By [22, Proposition 4.3], the character χ occurs only in the cohomology of X in middle degree. The following is a modular analog of this result.

Proposition 3.1. *Assume that Conjecture (HLM) holds for $(\mathbf{G},F)$. Let χ be a cuspidal character of G occurring in $\mathrm{H}_c^r(\mathrm{X},K)$. The ℓ-reduction of χ gives a unique simple cuspidal kG-module and it occurs as a composition factor of $\mathrm{H}_c^i(\mathrm{X},k)$ for $i=r$ only.*

Proof. We keep the notation used in the course of the proof of Proposition 2.12: D is the complement of X in the compactification $\overline{\mathrm{X}}$. Recall that the cohomology of D can be computed in terms of induced characters afforded by the cohomology of smaller Coxeter varieties. Since χ is cuspidal, we can therefore use the long exact sequences 2.13 to show that χ does not occur in $\mathrm{H}_c^\bullet(D,K)$.

Denote by L the unique composition factor of the ℓ-reduction of χ. By Conjecture (HLM), L does not appear in the ℓ-reduction of any other unipotent character of the block. Therefore, using Proposition 2.1 and the long exact sequences 2.13 again, we deduce that L cannot occur as a composition factor of $\mathrm{H}_c^\bullet(D,k)$. In particular, if we denote by $\overline{P}_L \in kG$-mod the projective cover of L then

$$\mathrm{Hom}_{kG}\big(\overline{P}_L, \mathrm{H}_c^i(\mathrm{X},k)\big) \simeq \mathrm{Hom}_{kG}\big(\overline{P}_L, \mathrm{H}_c^i(\overline{\mathrm{X}},k)\big).$$

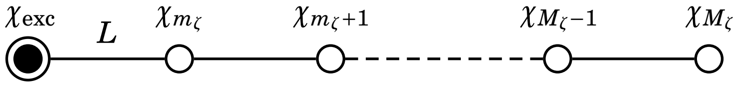
Moreover, since duality preserve cuspidality, the same argument applies for the dual of the cohomology of D. Subsequently, using Poincaré duality, we obtain

$$\mathrm{Hom}_{kG}\big(\overline{P}_L,\mathrm{H}_c^i(\overline{\mathrm{X}},k)\big) \simeq \mathrm{Hom}_{kG}\big(\overline{P}_L,\mathrm{H}_c^{2r-i}(\overline{\mathrm{X}},k)^*\big) \simeq \mathrm{Hom}_{kG}\big(\overline{P}_L,\mathrm{H}_c^{2r-i}(\mathrm{X},k)^*\big).$$

Now the cohomology of the irreducible affine variety X vanishes outside the degrees $r,\ldots,2r$. It follows that $\mathrm{Hom}_{kG}\big(\overline{P}_L,\mathrm{H}_c^i(\mathrm{X},k)\big)=0$ whenever $i>r$. □

3.2 Inducing nilpotent orbits

We now turn to the case of cuspidal kG-modules occurring as composition factors of the ℓ-reduction of a non-cuspidal unipotent character. If we assume that Conjecture (HLM) holds, then by the results of [13] the situation corresponds to branches of the following form:



where $m_\zeta < M_\zeta$.

At the level of characters, χ_{m_ζ} is obtain by inducing a cuspidal unipotent character occurring in the cohomology of a smaller Coxeter variety X_I. Unfortunately, the projective cover P_L of L cannot be constructed the same way. However, it turns out to be a direct summand of a generalized Gelfand-Graev module attached to some unipotent class induced from $\mathbf{L}_I$. The purpose of this section is to show that for these specific classes, the contribution of the generalized Gelfand-Graev module in the cohomology of X can be obtained from the cohomology of X_I.

Throughout this section and unless otherwise specified, we shall always assume that p is a good prime number. This restriction is necessary for using the results in Section 1.3 on nilpotent orbits.

3.2.1. **Induction of orbits.** Recall that for any nilpotent $\mathbf{L}_I$-orbit $\mathscr{O}_I$ in $\mathfrak{l}_I$, there exists a unique nilpotent $\mathbf{G}$-orbit $\mathscr{O}$ such that $\mathscr{O}\cap(\mathscr{O}_I+\mathfrak{u}_I)$ is dense in $\mathscr{O}_I+\mathfrak{u}_I$ where $\mathfrak{u}_I=\mathrm{Lie}\,\mathbf{U}_I$ [25]. The orbit $\mathscr{O}$ is called the *induction of* $\mathscr{O}_I$ from $\mathbf{L}_I$ to $\mathbf{G}$ and will be denoted by $\mathrm{Ind}_{\mathbf{L}_I}^{\mathbf{G}}\,\mathscr{O}_I$. Note that it is clearly F-stable whenever $\mathscr{O}_I$ is. In our situation, the special nilpotent orbit associated to χ_{m_ζ} turns out to be the induction of the orbit associated to the corresponding cuspidal character:

Lemma 3.2. *Let $I\subset\Delta$ be a ϕ-stable subset of simple roots and χ_I be a cuspidal character occurring in $\mathrm{H}_c^{r_I}(\mathrm{X}_I,K)$. If χ is the (unique) irreducible component of $\mathrm{R}_{L_I}^G(\chi_I)$ occuring in $\mathrm{H}_c^r(\mathrm{X},K)$ then*

$$\mathscr{O}(\chi)=\mathrm{Ind}_{\mathbf{L}_I}^{\mathbf{G}}\,\mathscr{O}(\chi_I).$$

Proof. Let $E\in\mathrm{Irr}\,W$ (resp. $E_I\in\mathrm{Irr}\,W_I$) be the unique special representation such that the Alvis-Curtis dual $D_G(\chi)$ (resp. $D_{L_I}(\chi_I)$) has a non-zero scalar product with the almost character R_E (resp. R_{E_I}). Then by definition E (resp. E_I) is the image of $(\mathscr{O}(\chi),1)$ (resp. $(\mathscr{O}(\chi_I),1)$) under the Springer correspondence (see [24, Section 11] or Section 1.3). We claim that E is the unique special component of

$\mathrm{J}_{W_I}^W E_I$. Using [11, Theorem 8.11] and [], it is easy to check that E is a component of $\mathrm{Ind}_{W_I}^W E_I$ since

$$\langle R_{\mathrm{Res}_{W_I}^W E}\,;\chi_I\rangle_{L_I} = \langle R_E\,;\mathrm{R}_{L_I}^G(\chi_I)\rangle_G \geq \langle R_E\,;\chi^*\rangle_G > 0.$$

where χ^* is the irreducible character of G such that $D_G(\chi) = \pm\chi^*$ (note that $\chi_I^* = \chi_I$ since χ_I is cuspidal).

Recall that the irreducible components of $\mathrm{R}_{L_I}^G(\chi_I)$ are labelled by the irreducible characters of the normalizer of χ_I in W_I. Now by the observation made in [22, Section 7.8], the character χ^* correspond to the trivial character of the normalizer of χ_I in $N_W(W_I)/W_I$ (whereas χ corresponds to the signature). In particular, $a_{\chi^*} = a_{\chi_I}$. This forces E to be a component of $\mathrm{J}_{W_I}^W E_I$ and the result follows from [25, Theorem 3.5]. □

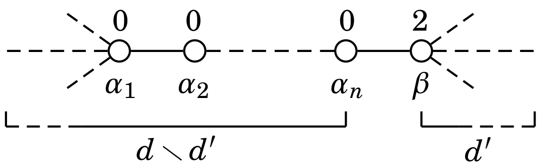
3.2.2. **Good representatives.** In general, there is no easy way to deduce the weighted Dynkin diagram of $\mathrm{Ind}_{\mathbf{L}_I}^{\mathbf{G}}\,\mathscr{O}_I$ from the diagram of $\mathscr{O}_I$. However, for the specific orbits we are interested in, one can observe (see Table 1) that they are obtained by adding the weight 2 to any simple root in $\Delta\smallsetminus I$. Using the particular shape of these diagrams we can find nice representatives of the induced orbit:

Lemma 3.3. *Let $\mathscr{O}$ be an F-stable nilpotent $\mathbf{G}$-orbit with weighted Dynkin diagram d. Let I be a ϕ-stable subset of Δ and denote by J the complement set. Assume that*

(i) *the restriction of d to I is the weighted Dynkin diagram of a nilpotent $\mathbf{L}_I$-orbit $\mathscr{O}_I$;*

(ii) *for all $\gamma\in J$, $d(\gamma)=2$. In other words, the restriction of d to J is the weighted Dynkin diagram of the regular nilpotent $\mathbf{L}_J$-orbit;*

(iii) *if d' is any connected component of the restriction of d to J, then d' is connected to $d_{|I}$ as follows:*



where all the roots adjacent to α_1 are in the set $\{\gamma\in I\,|\,d(\gamma)>0\}\cup\{\alpha_2,\beta\}$. In particular, s_{α_i} acts trivially on any other connected component of $d_{|J}$.

Then $\mathscr{O}$ is the induction of $\mathscr{O}_I$. Moreover, any nilpotent element $N\in\mathscr{O}^F$ is conjugated by G to an element N_I+N_J where $N_I\in\mathscr{O}_I\cap\mathfrak{u}(2)$ and $N_J\in\mathfrak{u}_I$ is such that the projection of N_J to $\mathfrak{g}(2)$ is a regular nilpotent element of $\mathfrak{l}_J$.

Proof. We keep the notation introduced in Section 1.3. The Lie algebra $\mathfrak{u}(2)$ is the subalgebra of $\mathfrak{u}=\mathrm{Lie}\,\mathbf{U}$ generated by the vectors e_γ for $d(\gamma)\geq 2$. Note that the assumption (ii) forces $\mathfrak{u}_I\subset\mathfrak{u}(2)$. We can therefore consider the subvariety $Z=\mathscr{O}_I\cap\mathfrak{u}(2)+\mathfrak{u}_I$ of $\mathfrak{u}(2)$. We claim that Z is stable by the action of the parabolic group $\mathbf{P}_d=\langle\mathbf{T},\mathbf{U}_\gamma\,|\,d(\gamma)\geq 0\rangle$. Indeed,

– $\mathfrak{u}_I$ is stable by the action of $\mathbf{P}_I$ and $\mathbf{P}_d \subset \mathbf{P}_I$ by (ii);
– $\mathscr{O}_I \cap \mathfrak{u}(2)$ is stable by $\mathbf{L}_d \subset \mathbf{L}_I$ and $\mathbf{L}_I \cap \mathbf{U}_1$;
– for all $x \in \mathfrak{l}_I \cap \mathfrak{u}$, we have $\mathbf{U}_I \cdot x \in x + \mathfrak{u}_I$ and therefore $\mathbf{U}_I \cdot (\mathscr{O}_I \cap \mathfrak{u}(2)) \subset Z$.

Now, by (i) the variety Z is dense in $\mathfrak{u}(2)$. In particular, it must contain the unique $\mathbf{P}_d$-orbit which is dense in $\mathfrak{u}(2)$ so that we have

$$\mathscr{O} \cap \mathfrak{u}(2) \subset \mathscr{O}_I \cap \mathfrak{u}(2) + \mathfrak{u}_I.$$

We deduce that any element of $\mathscr{O}^F$ is conjugated by G to an element $N_I + N_J$ with $N_I \in \mathscr{O}_I \cap \mathfrak{u}(2)$ and $N_J \in \mathfrak{u}_I$.

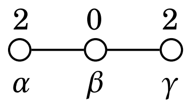
For simplicity, we assume that F acts trivially on d, although this assumption is actually unnecessary. Let $\pi : \mathfrak{u}(2) \longrightarrow \mathfrak{g}(2)$ be the canonical projection. The image of N_J by π involves the vectors e_β for $\beta \in J$ but also some of the $e_{\alpha+\beta}$ for $\alpha \in I$ such that $d(\alpha) = 0$. By the assumption (iii), the contribution of a connected component d' of $d_{|J}$ to the element $\pi(N_J)$ can be written

$$\sum_{i=1}^{n} \lambda_i e_{\alpha_i+\cdots+\alpha_n+\beta} + \mu e_\beta + \text{ linear combination of some } e_\gamma\text{'s with } \gamma \in J$$

If one the λ_i's is non-zero, then by applying $s_{\alpha_n} \cdots s_{\alpha_i}$ we can assume that $\mu \neq 0$. In that case, we can conjugate $\pi(N_J)$ by $\prod_i u_{\alpha_i+\cdots+\alpha_n}(\pm\lambda_i/\mu)$ to get rid of the λ_i's. Moreover, the assumption (iii) ensures that none of these operations affects the contribution to $\pi(N_J)$ of any other connected component of $d_{|J}$. Consequently, N_J is conjugated by $L_d \subset L_I$ to a element N'_J such that $\pi(N'_J) \in \mathfrak{l}_J$.

It remains to show that $\pi(N'_J)$ is regular as an element of $\mathfrak{l}_I$. Let $\gamma \in J$. If e_γ has a zero coefficient in $\pi(N'_J)$ then so has any element in $\mathbf{L}_d \cdot \pi(N'_J)$ since $\pi(N'_J)$ involves only simple roots. Now, the property of $\mathbf{P}_d \cdot (N_I + N'_J)$ to be dense in $\mathfrak{u}(2)$ forces $\mathbf{L}_d \cdot \pi(N'_J)$ to be dense in $\mathfrak{u}_I \cap \mathfrak{g}(2)$. As a consequence, the coefficient of all the e_γ's for $\gamma \in J$ must be non-zero and therefore $\pi(N'_J)$ is regular. □

Remark 3.4. The properties (i) and (ii) are not sufficient for finding such representatives. For example, the subregular orbit in type A_3 corresponds to the diagram



and hence is the induction of the trivial orbit in $\mathfrak{l}_\beta$. However, any element of the form $xe_\alpha + ye_\gamma + ze_{\alpha+\beta+\gamma}$ is contained in an orbit strictly smaller than the subregular.

Table 1 lists the special orbits $\mathscr{O}(\chi_I)$ associated to non-trivial cuspidal unipotent characters χ_I occuring in the cohomology of X_I (first three columns), together with the orbits that are induced from it (last three columns). We have omitted the cuspidal characters of groups of maximal semi-simple rank for obvious reasons, but also the cuspidal character of type ${}^2\mathrm{B}_2$ since it corresponds to the case of a bad prime. Following [8, Chapter 13], we have labelled the orbits by appropriate combinatorial objects:

- Type A_n: partitions λ of $n+1$.
- Type B_n with $p \neq 2$: pairs of partitions (α,β) with $2|\alpha|+|\beta| = 2n+1$ such that all parts of β are odd and distinct.
- Type C_n with $p \neq 2$: pairs of partitions (α,β) with $|\alpha|+|\beta| = n$ such that all parts of β are distinct.
- Type D_n with $p \neq 2$: pairs of partitions (α,β) with $2|\alpha|+|\beta| = 2n$ such that all parts of β are odd and distinct.

Using the table one can readily check that all of the induced orbits satisfy the assumptions of Lemma 3.3. More generally, it is likely to be the case for any cuspidal orbit (in the sense of [16]).

3.2.3. **Generalized Gelfand-Graev modules in** $\mathrm{R}\Gamma_c(\mathrm{X})$**.** Using the above specific representatives, we shall now compute the contribution of the corresponding generalized Gelfand-Graev module in the cohomology of X.

Proposition 3.5. *Let $\mathscr{O}$ be an F-stable nilpotent* $\mathbf{G}$*-orbit in $\mathfrak{g}$ with weighted Dynkin diagram d. Let $I \subset \Delta$ be a ϕ-stable set of simple roots satisfying the assumptions of Lemma 3.3. Then for all $N \in \mathscr{O}^F \cap \mathfrak{u}(2)$, we have the following isomorphism in $D^b(\Lambda\text{-Mod})$:*

$$\mathrm{RHom}^\bullet_{\Lambda G}\big(\Gamma_N, \mathrm{R}\Gamma_c(\mathrm{X},\Lambda)\big) \simeq \mathrm{RHom}^\bullet_{\Lambda L_I}\big(\Gamma_{\mathrm{pr}_{\mathfrak{l}_I}(N)}, \mathrm{R}\Gamma_c(\mathrm{X}_I,\Lambda)\big)[r_I - r].$$

Proof. We denote by J the complement set of I in Δ. Recall that the generalized Gelfand-Graev representation does not depend on the P_d-orbit of N. Since $P_d \subset P_I$, we can assume using Lemma 3.3 that $N = N_I + N_J$ with $N_I \in \mathscr{O}_I \cap \mathfrak{u}(2)$ and $N_J \in \mathfrak{u}_I$ is such that the image of N_J in $\mathfrak{g}(2)$ is a regular element of $\mathfrak{l}_J$.

The generalized Gelfand-Graev representation is a projective module obtained by inducing a one-dimensional module Λ_N from $U_{1.5}$ to G, where $U_{1.5}$ acts on Λ_N via a linear character $\widetilde{\varphi}_N$. Note that by construction, $\widetilde{\varphi}_{N_I}$ is the restriction of $\widetilde{\varphi}_N$ to the group $U_{1.5} \cap L_I$. The unipotent subgroup $\mathbf{V} = \mathbf{U}_I \cap \mathbf{U}_J$ is a normal subgroup of $\mathbf{U}$, and the quotient $\mathbf{V}\backslash\mathbf{U}$ is isomorphic to $\mathbf{V}_I \times \mathbf{V}_J$ as an algebraic group (recall that $\mathbf{V}_I = \mathbf{U} \cap \mathbf{L}_I$ and $\mathbf{V}_J = \mathbf{U} \cap \mathbf{L}_J$). We shall denote by π_V the composition of this isomorphism with the canonical projection $\mathbf{U} \twoheadrightarrow \mathbf{V}\backslash\mathbf{U}$. By Remark 1.6 and the particular shape of N_J, it is clear that the restriction of $\widetilde{\varphi}_N$ to V is trivial. Moreover, the induced linear character on $V\backslash U_{1.5}$ decomposes into $\widetilde{\varphi}_{N_I} \boxtimes \psi$ where ψ is a regular character of V_J (see [2, Definition 2.1.1]).

Let us study the geometric counterpart of this decomposition. By [22, Theorem 2.6], there exists a U-equivariant isomorphism of varieties

$$\mathrm{X} \simeq \Big\{ u \in \mathbf{U} \;\Big|\; u^{-1}F(u) \in \prod_{\alpha \in I} u_\alpha(\mathbf{G}_m) \prod_{\beta \in J} u_\beta(\mathbf{G}_m) \Big\}.$$

The method used in [12, Proposition 1.2] to describe the quotient of $\mathbf{B}$ by the group $D(\mathbf{U})^F$ extends to any unipotent normal subgroup of $\mathbf{B}$ instead of $D(\mathbf{U})$ (see [14, Section 2.3.2] for more details). Using the isomorphism $\mathbf{V}\backslash\mathbf{U} \simeq \mathbf{V}_I \times \mathbf{V}_J$, we can therefore realize the quotient of the Deligne-Lusztig variety by V as:

$$V\backslash \mathrm{X} \simeq \left\{ (\bar u, v_1, v_2) \in \mathbf{U}\times\mathbf{V}_I\times\mathbf{V}_J \;\middle|\; \begin{array}{l} \pi_{\mathbf{V}}(\bar u) = v_1^{-1}F(v_1)v_2^{-1}F(v_2) \\ \bar u \in \prod_{\alpha\in I} u_\alpha(\mathbf{G}_m) \prod_{\beta\in J} u_\beta(\mathbf{G}_m) \end{array} \right\}.$$

This can be rephrased in the following $V_I \times V_J$-equivariant isomorphism:

$$V\backslash \mathrm{X} \simeq \mathrm{X}_{\mathbf{L}_I}(c_I)\times \mathrm{X}_{\mathbf{L}_J}(c_J) \simeq \mathrm{X}_I \times \mathrm{X}_J. \qquad (3.6)$$

Denote by e_N (resp. e_{N_I}, resp. e_{N_J}) the idempotent of $\Lambda U_{1.5}$ corresponding to $\widetilde\varphi_N$ (resp. to $\widetilde\varphi_{N_I}$, resp. to ψ). By adjunction, we obtain in the derived category $D^b(\Lambda\text{-Mod})$:

$$\begin{aligned} \mathrm{RHom}^\bullet_{\Lambda G}\big(\Gamma_N, \mathrm{R}\Gamma_c(\mathrm{X},\Lambda)\big) &\simeq \mathrm{RHom}^\bullet_{\Lambda G}\big(\mathrm{Ind}_{U_{1.5}}^G \Lambda_N, \mathrm{R}\Gamma_c(\mathrm{X},\Lambda)\big) \\ &\simeq \mathrm{RHom}^\bullet_{\Lambda U_{1.5}}\big(\Lambda_N, \mathrm{R}\Gamma_c(\mathrm{X},\Lambda)\big) \\ \mathrm{RHom}^\bullet_{\Lambda G}\big(\Gamma_N, \mathrm{R}\Gamma_c(\mathrm{X},\Lambda)\big) &\simeq e_N \mathrm{R}\Gamma_c(\mathrm{X},\Lambda). \end{aligned}$$

Now the restriction of $\widetilde\varphi_N$ to V is trivial and hence e_N must act as zero on any non-trivial ΛV-module. Using Proposition 1.3 together with Formula 3.6, we obtain

$$e_N\mathrm{R}\Gamma_c(\mathrm{X},\Lambda) \simeq e_N\mathrm{R}\Gamma_c(V\backslash\mathrm{X},\Lambda) \simeq e_{N_I}\mathrm{R}\Gamma_c(\mathrm{X}_I,\Lambda)\otimes e_{N_J}\mathrm{R}\Gamma_c(\mathrm{X}_J,\Lambda).$$

We use [2, Theorem 3.10] to conclude: since ψ is a regular character of V_J, the complex $e_{N_J}\mathrm{R}\Gamma_c(\mathrm{X}_J,\Lambda)$ is quasi-isomorphic to $\Lambda[-r_J]$ where $r_J = r - r_I$ is the dimension of X_J. ☐

The main consequence of this proposition is the following: if I is a proper subset of Δ then the cohomology of the complex $\mathrm{RHom}^\bullet_{\Lambda G}\big(\Gamma_N, \mathrm{R}\Gamma_c(\mathrm{X},\Lambda)\big)$ is torsion-free. Together with Theorem 1.7 it gives:

Corollary 3.7. *Assume that Conjecture (HLM) holds. Assume moreover that p is a good prime number so that Theorem 1.7 holds. Let L be the unique simple cuspidal kG-module that occur in any ℓ-reduction of χ_{m_ζ}. Then L occurs as a composition factor of $\mathrm{H}^i_c(\mathrm{X},k)$ for $i = r$ only.*

Proof. By [22], there exists a ϕ-stable subset I of Δ and an irreducible cuspidal component χ_I of $\mathrm{H}_c^{r_I}(\mathrm{X}_I,K)$ such that χ_{m_ζ} is a component of $\mathrm{R}_{L_I}^G(\chi_I)$. By Lemma 3.2, the special nilpotent orbit $\mathscr{O}(\chi_{m_\zeta})$ is induced from the orbit associated to χ_I. Moreover, we can check in Table 1 that it satisfies the assumptions of Lemma 3.3. Therefore, for any $N \in \mathscr{O}(\chi_{m_\zeta})^F$ the complex $\mathrm{RHom}^\bullet_{\Lambda G}\big(\Gamma_N, \mathrm{R}\Gamma_c(\mathrm{X},\Lambda)\big)$ is torsion-free by the above proposition.

Since p is assumed to be good for $\mathbf{G}$, Theorem 1.7 ensures that there exists a nilpotent element $N \in \mathscr{O}(\chi_{m_\zeta})^F$ such that

- χ_{m_ζ} is an irreducible component of $[\Gamma_N]$;
- for all $i = m_\zeta + 1, \ldots, M_\zeta$, the character χ_i does not occur in $[\Gamma_N]$ (using the observation made in [22, Section 7.8] one can check for example that $a_{\chi_i} < a_{\chi_{m_\zeta}}$).

We deduce that the projective cover P_L of L in ΛG-mod is a direct summand of the projective module Γ_N. Indeed, it has character $[P_L] = \chi_{\mathrm{exc}} + \chi_{m_\zeta}$ and any other projective indecomposable module that involves χ_{m_ζ} must also involve $\chi_{m_\zeta+1}$. In particular, the cohomology of $\mathrm{RHom}^\bullet_{\Lambda G}\big(P_L, \mathrm{R}\Gamma_c(\mathrm{X},\Lambda)\big)$ is also torsion-free and hence vanishes outside the degree r by [22, Proposition 4.3]. □

Remark 3.8. The particular case where $\chi_{m_\zeta} = \mathrm{St}_G$ and $I = \varnothing$ has been already solved in [2]. It corresponds to the regular orbit, which is induced from the trivial **T**-orbit. in that case, the generalized Gelfand-Graev representation is an usual Gelfand-Graev module and no restriction on p is needed.

3.3 Main results

We now have all the ingredients for proving that there is no torsion in the cohomology of Y_ℓ. Recall that we have shown that the torsion part of the cohomology is necessarily a cuspidal ΛG-module (see Corollary 2.10). By the universal coefficient theorem, it is therefore sufficient to prove that the module $\mathrm{H}_c^i(\mathrm{Y}_\ell, k)$ has no cuspidal composition factors if $i > r$. By Corollary 2.11, this property holds whenever it holds for the Deligne-Lusztig X. In the framework of derived categories, we are then reduced to show that for any cuspidal kG-module L lying in the block, the complex

$$\mathrm{RHom}^\bullet_{\Lambda G}\big(\overline{P}_L, \mathrm{R}\Gamma_c(\mathrm{X},k)\big)$$

is quasi-isomorphic to a complex concentrated in degree r. If L happens to be a composition factor of the ℓ-reduction of a cuspidal unipotent character, then by Proposition 3.1 it cannot occur outside the cohomology in middle degree. Corollary 3.7 deals with the case where L is a composition factor of the ℓ-reduction of an induced character, but we have to assume that both Theorem 1.7 and Conjecture (HLM) hold. This excludes only the Ree groups of type ${}^2\mathrm{F}_4$ but gives some conditions in other types. Here is the precise result that we obtain:

Theorem 3.9. *Let* **G** *be a quasi-simple group. According to the type of* $(\mathbf{G}, F)$*, we make the following assumptions:*

- A_n, ${}^2A_n$, B_2, ${}^2B_2$, D_4, ${}^2D_n$, ${}^3D_4$, G_2 *and* ${}^2G_2$*: no restriction on p.*
- B_n, C_n, D_n, E_6, ${}^2E_6$ *and* F_4*: p is good.*
- E_7 *and* E_8*: p is good and Conjecture (HLM) holds.*

Then in the set-up of Section 2.1, the Λ*-modules* $b\mathrm{H}_c^i(\mathrm{Y}_\ell, \Lambda)$ *are torsion-free.*

Remark 3.10. As mentioned in Remark 1.8, Lusztig has a general proof of Theorem 1.7 when p is a good prime number but he has not published it. However, the results in [24] and [20] include the case of large characteristic and the case of split groups of type A_n, E_n, F_4 and G_2.

We deduce from [13, Theorem 4.12] that in any of the above cases, the geometric version of Broué's conjecture holds. This extends significantly the previous results of Puig [28] (for $\ell \,|\, q-1$), Rouquier [29] (for $\ell \,|\, \phi_h(q)$ and $r = 1$) and Bonnafé-Rouquier [2] (for $\ell \,|\, \phi_h(q)$ and $(\mathbf{G}, F)$ of type A_n).

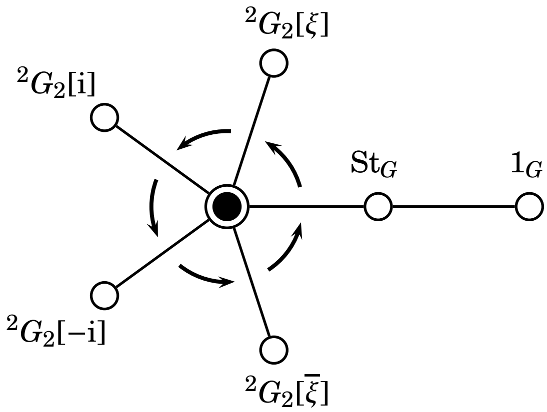
Theorem 3.11. *Let* $(\mathbf{G},F)$ *be a quasi-simple group satisfying the assumptions of Theorem 3.9 and* c *be a Coxeter element of* (W,F). *Let* ℓ *be a prime number not dividing the order of* W^F *and satisfying one of the two following assumptions, depending on the type of* $(\mathbf{G},F)$*:*

- *"non-twisted" cases:* ℓ *divides* $\Phi_h(q)$*;*
- *"twisted" cases:* ℓ *divides the order of* T_c.

Then the complex $b\mathrm{R}\Gamma_c(\mathrm{Y}(\dot{c}),\Lambda)b'$ *induces a splendid and perverse equivalence between the principal* ℓ*-blocks* $b\Lambda G$ *and* $b'\Lambda N_G(T_c)$

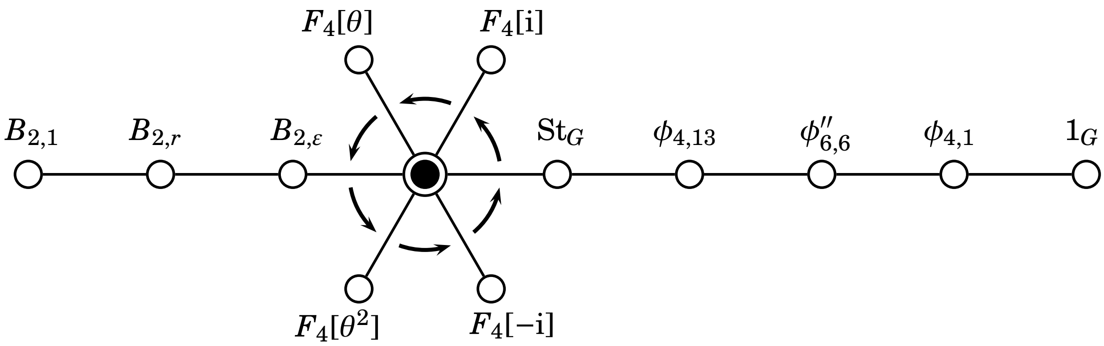
Using [13, Theorem 4.14] we also deduce the planar embedding of the Brauer tree of the principal ℓ-block for groups of type ${}^2\mathrm{G}_2$ and F_4 with $p \neq 2,3$ (compare with [17] and [18]).

Theorem 3.12. (i) *Assume that* ℓ *is odd and divides* $q^2 - q\sqrt{3}+1$. *Then the planar embedded Brauer tree of the principal* ℓ*-block of the Ree group* ${}^2G_2(q^2)$ *is*



where $\mathrm{i} = \xi^3$ *and* ξ *is the unique* 12*th root of unity in* $\Lambda^\times$ *congruent to* q *modulo* ℓ.

(ii) *Assume that* $\ell \neq 2,3$ *and divides* $q^4 - q^2 + 1$. *Assume moreover that* $p \neq 2,3$. *Then the planar embedded Brauer tree of the principal* ℓ*-block of the simple group of type* $F_4(q)$ *is*



where θ *(resp.* i*) is the unique third (resp. fourth) root of unity in* $\Lambda^\times$ *congruent to* q^4 *(resp.* q^3*) modulo* ℓ.

Table 1: Induced orbits

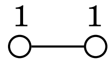
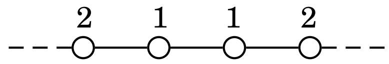
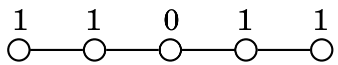
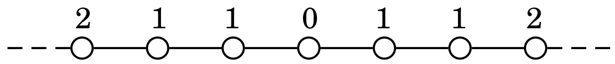
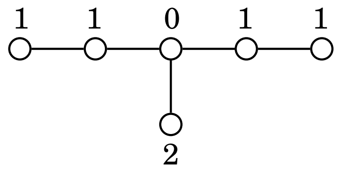
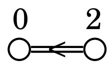
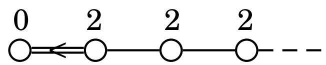
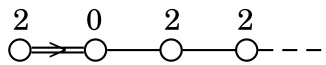
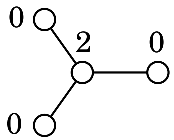
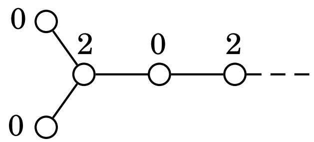
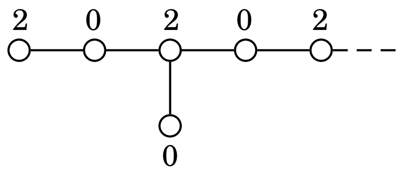
Type	Weighted Dynkin diagram	Label	Type	Weighted Dynkin diagram	Label
${}^2A_2$		[1,2]	${}^2A_{2n}$		$[1,2n]$
${}^2A_5$		$[1,2,3]$	${}^2A_{2n+1}$		$[1,2,2n-1]$
			${}^2E_6$		$D_5(a_1)$
B_2		$[1];[3]$	B_n		$[1];[2n-1]$
			C_n		$[];[1,n-1]$
D_4		$[1,3];[]$	D_n		$[1];[3,2n-5]$
			E_n		$E_n(a_3)$

Table 1: Induced orbits (continued)

Type	Weighted Dynkin diagram	Label	Type	Weighted Dynkin diagram	Label
E_6	0 0 2 0 0 0	$D_4(a_1)$	E_7	0 0 2 0 0 2 2	$E_7(a_5)$
			E_8	0 0 2 0 0 2 2 0	$E_8(b_5)$
E_7	1 0 1 0 1 0 0	A_4+A_1	E_8	1 0 1 0 1 0 2 0	$E_6(a_1)+A_1$

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