Asymptotic multivariate finite-time ruin probabilities with heavy-tailed claim amounts: Impact of dependence and optimal reserve allocation

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Abstract. In ruin theory, the univariate model may be found too restrictive to describe accurately the complex evolution of the reserves of an insurance company. In the case where the company is composed of multiple lines of business, we compute asymptotics of finite-time ruin probabilities. Capital transfers between lines are partially allowed. When claim amounts are regularly varying distributed, several forms of dependence between the lines are considered. We also study the optimal allocation of a large global initial reserve in order to minimize the asymptotic ruin probability.

Keywords: Multivariate finite-time ruin probabilities; multivariate regular variation; capital transfer; optimal allocation.

1 Introduction

This paper deals with an insurance company with multiple lines of business. Each line is assumed to be exposed to catastrophic risks like earthquakes, floods or terrorist attacks. Such risks may affect several lines of the company at the same time, so dependence between the lines is considered. Each line may correspond to a business in a specific country or to a type of policy offered by the company. Capital transfers between the lines are strictly regulated. Nevertheless, we assume here that a piece of the amount of each line is allowed to recover losses of an another one.
Our study is done in a finite-time framework. Actually, insurance regulation is based on 1-year time horizon in the standard formula, and on finite-time horizon usually comprised between 1 and 10 years in internal models.

Suppose now that the company owns a global initial reserve to share between the lines. Due to the specific risk exposition of each line, the choice of the allocation may have a huge impact of its solvency. In Loisel (2005) and Biard et al. (2010), this optimal allocation problem is concerned with the minimization of the expected time-integrated negative part of a risk process. In this paper, we focus on the finite-time multivariate ruin probability for our minimization problem.

In risk theory, multivariate context has been studied scarcely compared to the univariate one. For the univariate setting, the reader is referred e.g. to the comprehensive books by Rolski et al. (1999), Asmussen and Albrecher (2010), Goovaerts et al. (2001) and the references therein. Concerning the multivariate setting, light-tailed case is studied in Collamore (1996, 2002) and a discrete approach is investigated in Picard et al. (2003). In this paper, we are concerned by the heavy-tailed case, studied previously by Hult and Lindskog (2006b,a).

The paper is organized as follows. In Section 2, we present the framework of the paper and we define the quantities under interest. Section 3 deals with the computation of the asymptotics of the multivariate finite-time ruin probability in context of dependence and Section 4 investigates optimal allocation problems.

2 Framework

Throughout the paper, vectors are denoted by bold letters. For example, $\mathbf{x} = \left(x^{(1)}, \dots, x^{(d)}\right) \in \mathbb{R}^d$. Moreover, we define $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$, $\mathbf{e}_i$ the unit vector whose ith component is equal to 1 and for $1 \le k \le d-1$, $\mathbf{1}_k = \sum_{i=1}^k \mathbf{e}_i$.

2.1 Multivariate Regular Variation

In order to describe losses of catastrophic risks, we choose the heavy-tailed class of regularly varying random variables. The typical example of these kinds of random variables is the Pareto distribution. This random variable class well describes catastrophic risks in the sense that, in the case of large initial reserve, the ruin of the company may be only caused by one big loss.

Definition 2.1 *A function L on $(0, \infty)$ is slowly varying at ∞ if*

$$\lim_{u \to \infty} \frac{L(tu)}{L(u)} = 1, \quad \text{for every } t > 0.$$

We write $L \in \mathcal{R}_0$.

Definition 2.2 (Univariate Regular Variation) *A $\mathbb{R}$-random variable X is regularly varying if there exists $\alpha > 0$, such that*

$$\lim_{u \to \infty} \frac{P(X > tu)}{P(X > u)} = t^{-\alpha}, \quad \text{for every } t > 0,$$

or equivalently if,

$$P(X > u) = u^{-\alpha} L(u),$$

for some $L \in \mathcal{R}_0$.
We write $X \in \mathcal{R}_{-\alpha}$.

Definition 2.3 (Multivariate Regular Variation) *An $\mathbb{R}^d$-valued random vector*

$$\mathbf{X} = (X^{(1)}, \dots, X^{(d)})$$

with unbounded support is regularly varying if there exists a nonzero Radon measure μ defined on $\mathcal{B}(\overline{\mathbb{R}}^d)$ with $\mu(\overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}) = 0$ such that

$$\lim_{u \to \infty} \frac{P(\mathbf{X} \in uA)}{P(|\mathbf{X}| > u)} = \mu(A), \tag{2.1}$$

for every Borel set $A \in \mathbb{R}^d$ *bounded away from* $\mathbf{0}$ *(i.e.* $\mathbf{0} \notin \bar{A}$*) with* $\mu(\partial A) = 0$.

We can also use an equivalent definition using the spectral measure.
An $\mathbb{R}^d$*-valued random vector*

$$\mathbf{X} = (X^{(1)}, \dots, X^{(d)})$$

with unbounded support is regularly varying if there exists an $\alpha > 0$ *and a probability measure* σ *on the unit sphere* $\mathbb{S}^{d-1} = \{\mathbf{x} : |\mathbf{x}| = 1\}$ *such that*

$$\lim_{u \to \infty} \frac{P\left(|\mathbf{X}| > xu, \mathbf{X}/|\mathbf{X}| \in S\right)}{P(|\mathbf{X}| > u)} = x^{-\alpha} \sigma(S) , \tag{2.2}$$

for every $x > 0$ *and Borel sets* $S \subset \mathbb{S}^{d-1}$*. The probability measure* σ *is called the spectral measure of* $\mathbf{X}$.

As a consequence, we have for every $u > 0$ *and Borel set* $A \in \mathbb{R}^d$ *bounded away from* $\mathbf{0}$

$$\mu(uA) = u^{-\alpha} \mu(A) .$$

We write $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$.

For a general presentation of heavy-tailed theory, the reader is referred e.g. to the book of Resnick (2007).

2.2 The model

To describe the reserve of an insurance company with d lines of business, we consider a multivariate risk process $(\mathbf{R}_t)_{t \geq 0}$. Denote by $u > 0$ the global initial reserve and by $\mathbf{a} \in (0, 1)^d$ the vector which describes the part of u which is allocated to each branch. As a consequence, we have $a^{(1)} + \ldots + a^{(d)} = 1$. The premium rates are captured in $\mathbf{c} \in (0, \infty)^d$. The aggregate claim amount process $(\mathbf{S}_t)_{t \geq 0}$ is assumed to be a multivariate Poisson process, that is to say

$$\mathbf{S}_t = \sum_{i=1}^{N(t)} \mathbf{X}_i ,$$

where $N(t)$ is a Poisson process with parameter $\lambda > 0$ and $(\mathbf{X}_i)_{i \geq 1}$ is a $\mathbb{R}_+^d$-valued independent and identically distributed sequence. We note by $\mathbf{X}$ their common distribution.
Hence, we have, for $t \geq 0$,

$$\mathbf{R}_t = u\mathbf{a} + \mathbf{c}t - \sum_{i=1}^{N(t)} \mathbf{X}_i . \tag{2.3}$$

Throughout this paper, $\mathbf{X}$ will be regularly varying for some $\alpha > 1$ and measure μ.

2.3 Multivariate finite-time ruin probability

In the univariate setting, the finite-time ruin probability is defined as, for $u, T > 0$,

$$\psi(u, T) = P\left(\exists t \in [0, T] , R_t < 0 | R_0 = u\right) = P\left(\sup_{[0,T]} (S_t - ct) > u\right) .$$

In the multivariate case, there is not a unique definition. For example in Cai and Li (2005, 2007), we can find several definitions, depending on the interest. For $u, T > 0$ let us define

- the probability that the sum of the line reserves becomes negative before T,

$$\psi_{sum}(u,T) = P\left(\sup_{[0,T]}\left\{\sum_{j=1}^{d}(S_t^{(j)} - c^{(j)}t)\right\} > u\right) ; \tag{2.4}$$

- the probability that all the line reserves become negative before T,

$$\psi_{and}(u,T) = P\left(\bigcap_{j=1}^{d}\left\{\sup_{[0,T]}(S_t^{(j)} - c^{(j)}t) > a^{(j)}u\right\}\right) ; \tag{2.5}$$

- the probability that one of the line reserve becomes negative before T,

$$\psi_{or}(u,T) = P\left(\bigcup_{j=1}^{d}\left\{\sup_{[0,T]}(S_t^{(j)} - c^{(j)}(t)) > a^{(j)}u\right\}\right) ; \tag{2.6}$$

- and the probability that all the line reserves are negative at a given time before T,

$$\psi_{sim}(u,T) = P\left(\exists t \in [0,T]\,, \forall j \in [1,d]\,, R_t^{(j)} < 0\right). \tag{2.7}$$

Here, we investigate the definition proposed by Hult and Lindskog (2006a). For $\beta \in [0,1]$, let

$$F_\beta = \left\{\mathbf{x} : \beta \sum_{k=1}^{d}\left(x^{(k)} \vee 0\right) < -\sum_{k=1}^{d}\left(x^{(k)} \wedge 0\right)\right\} , \tag{2.8}$$

where $\vee$ = min and $\wedge$ = max. For $T > 0$, we define the multivariate finite-time ruin probability $\psi_{d,\beta}(u,T)$ as the probability that the risk reserve process $\mathbf{R}_t$ hits F_β at some time t before T. Explicitly, for $u, T > 0$, we have

$$\psi_{d,\beta}(u,T) = P\left(\exists t \in [0,T],\quad \mathbf{R}_t \in F_\beta\right). \tag{2.9}$$

Remark 2.4 *The ruin set F_β corresponds to the possibility to transfer from positive line a fraction $\beta \in [0,1]$ to cover a negative position of another line. For $\beta = 0$, no transfer is allowed and $\psi_{d,0} = \psi_{or}$ and for $\beta = 1$, transfer is allowed without restrictions and $\psi_{d,1} = \psi_{sum}$.*

In Figure 1, the set F_β is represented for β =0, 1/2 and 1 in the the two-dimensional case.

The following result, from Hult and Lindskog (2006a), gives the asymptotic of the finite-time multivariate ruin probability for a large initial reserve.

Proposition 2.5 (Hult and Lindskog (2006a)) *For a risk process $(\mathbf{R}_t)_{t\geq 0}$ given by (2.3) with a common distribution $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some $\alpha > 1$ and measure μ, we have, for $T > 0$ and large u,*

$$\psi_{d,\beta}(u,T) \sim (\lambda T)\mu(\mathbf{a} - F_\beta)P\left(|\mathbf{X}| > u\right). \tag{2.10}$$

This result is the base of our computations. Actually, after giving the assumptions on the dependence structure between claim amount of each line, we can exhibit μ and then get the asymptotic ruin probability. The following lemma gives $\mu(\mathbf{a} - F_\beta)$ for some basic forms of $\mathbf{X}$.

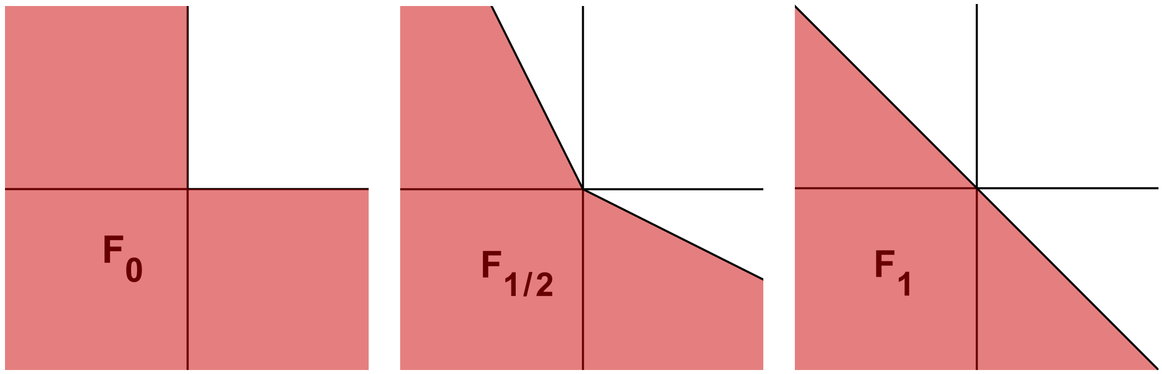

Figure 1: F_β for β=0, 1/2 and 1 in two dimensions

Lemma 2.6 *Let X be a positive random variable which is regularly varying with some $\alpha > 1$.*

- **Case 1)** *If for some $1 \le j \le d$, $\mathbf{X} = X\mathbf{e}_j$, then $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some measure μ and we have*

$$\mu(\mathbf{a} - F_\beta) = \left(\beta + a^{(j)}(1-\beta)\right)^{-\alpha} . \tag{2.11}$$

- **Case 2)** *If, for some $1 \le k \le d$,$\mathbf{X} = X\mathbf{1}_k$, then $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some measure μ and we have*

$$\mu(d^{-1}\mathbf{1} - F_\beta) = \left(d^{-1}\left(\frac{\beta(d-k)}{k} + 1\right)|\mathbf{1}_k|\right)^{-\alpha} . \tag{2.12}$$

- **Case 3)** *If $\mathbf{X} = X\mathbf{1}$, then $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ for some measure μ and we have*

$$\mu(\mathbf{a} - F_\beta) = \left(\frac{\sum_{i=1}^{k^*} a^{(i:d)} + \beta \sum_{i=k^*+1}^{d} a^{(i:d)}}{k^* + \beta(d-k^*)}|\mathbf{1}|\right)^{-\alpha} , \tag{2.13}$$

where for $1 \le i \le d$, $a^{(i:d)}$ is the ith larger component of $\mathbf{a}$ and

$$k^* = \inf\left\{k \in [1, d-1] : a^{(k+1:d)} > \frac{\sum_{i=1}^{k} a^{(i:d)} + \beta \sum_{i=k+1}^{d} a^{(i:d)}}{k + \beta(d-k)}\right\} .$$

Proof. Let $A = \mathbf{a} - F_\beta$. We have

$$A = \left\{\mathbf{x} : \beta \sum_{i=1}^{d}\left((a^{(i)} - x^{(i)}) \vee 0\right) < -\sum_{i=1}^{d}\left((a^{(i)} - x^{(i)}) \wedge 0\right)\right\} .$$

- **Case 1)** Let $\mathbf{X} = X\mathbf{e}_j$ for some $1 \le j \le d$. Since $X \in \mathcal{R}_{-\alpha}$, there exists a function $L \in \mathcal{R}_0$ such that $P(X > u) = u^{-\alpha}L(u)$. Moreover, from Karamata's Theorem (see e.g. Embrechts et al. (1997), Theorem A3.6 p 567), we have for large u, $\frac{\partial}{\partial u}P(X > u) \sim -\alpha u^{-\alpha-1}L(u)$.

Let $B \subset \mathbb{R}^d$ be a Borel set bounded away from $\mathbf{0}$. We have

$$\begin{aligned}\lim_{u\to\infty}\left(P(|X\mathbf{e}_j|>u)\right)^{-1}P\left(X\mathbf{e}_j\in uB\right) &= \lim_{u\to\infty}(P(X>u))^{-1}P\left(u^{-1}X\mathbf{e}_j\in B\right)\\ &= \lim_{u\to\infty}(u^{-\alpha}L(u))^{-1}\int \mathbb{1}_B(r\mathbf{e}_j)dF_{u^{-1}X}(r)\\ &= \lim_{u\to\infty}(u^{-\alpha}L(u))^{-1}\int \mathbb{1}_B(r\mathbf{e}_j)dF_X(ru)\\ &= \lim_{u\to\infty}(u^{-\alpha}L(u))^{-1}\int \mathbb{1}_B(r\mathbf{e}_j)u\alpha(ur)^{-\alpha-1}L(ur)dr\\ &= \int \mathbb{1}_B(r\mathbf{e}_j)\alpha r^{-\alpha-1}dr\ .\end{aligned}$$

Hence, we have $\mathbf{X}\in\mathcal{MR}_{-\alpha,\mu}$ with μ defined for all Borel set $B\subset\mathbb{R}^d$ bounded away from $\mathbf{0}$ as

$$\mu(B)=\int \mathbb{1}_B(r\mathbf{e}_j)\alpha r^{-\alpha-1}dr\ .$$

Since we have

$$\begin{aligned}r\mathbf{e}_j\in A &\Leftrightarrow \beta\sum_{\substack{1\le i\le d\\ i\neq j}}a^{(i)}<-(a^{(j)}-r)\\ &\Leftrightarrow r>a^{(j)}(1-\beta)+\beta\ .\end{aligned}$$

the first result follows.

- **Case 2)** In this case, using the same way, we obtain that $\mathbf{X}\in\mathcal{MR}_{-\alpha,\mu}$ with μ defined for all Borel set $B\subset\mathbb{R}^d$ bounded away from $\mathbf{0}$ as

$$\mu(B)=\int \mathbb{1}_B(r\mathbf{1}_k/|\mathbf{1}_k|)\alpha r^{-\alpha-1}dr\ ,$$

and since $\mathbf{a}=d^{-1}\mathbf{1}$,

$$\begin{aligned}r\mathbf{1}_k/|\mathbf{1}_k|\in A &\Leftrightarrow \beta(d-k)d^{-1}<-k(d^{-1}-r/|\mathbf{1}_k|)\\ &\Leftrightarrow r>d^{-1}\left(\frac{\beta(d-k)}{k}+1\right)|\mathbf{1}_k|\ ,\end{aligned}$$

and the second result follows.

- **Case 3)** In this case we get that $\mathbf{X}\in\mathcal{MR}_{-\alpha,\mu}$ with μ defined for all Borel set $B\subset\mathbb{R}^d$ bounded away from $\mathbf{0}$ as

$$\mu(B)=\int \mathbb{1}_B(r\mathbf{1}/|\mathbf{1}|)\alpha r^{-\alpha-1}dr\ .$$

It remains to find $\{r : r\mathbf{1}/|\mathbf{1}|\in A\}$.

$$r\mathbf{1}/|\mathbf{1}|\in A\Leftrightarrow \beta\sum_{i=1}^{d}\left((a^{(i)}-r/|\mathbf{1}|)\vee 0\right)<-\sum_{i=1}^{d}\left((a^{(i)}-|\mathbf{1}|)\wedge 0\right)\ .$$

Denote by $a^{(k:d)}$ the kth larger component of $\mathbf{a}$. Let $1 \leq k \leq d-1$ and assume that $a^{(k:d)} \leq r/|\mathbf{1}| < a^{(k+1:d)}$. Then

$$r\mathbf{1}/|\mathbf{1}| \in A \Leftrightarrow \beta \sum_{i=k+1}^{d} a^{(i:d)} - \beta(d-k)r/|\mathbf{1}| < -\sum_{i=1}^{k} a^{(i:d)} + kr/|\mathbf{1}|$$
$$\Leftrightarrow r/|\mathbf{1}| > \frac{\sum_{i=1}^{k} a^{(i:d)} + \beta \sum_{i=k+1}^{d} a^{(i:d)}}{k+\beta(d-k)} .$$

Let $K = \left\{ k \in [1, d-1] : a^{(k+1:d)} > \frac{\sum_{i=1}^{k} a^{(i:d)} + \beta \sum_{i=k+1}^{d} a^{(i:d)}}{k+\beta(d-k)} \right\}$. K is lower bounded by 1 and $d-1 \in K$ (since $a^{(d:d)} > 1/d$), so there exists a K-minimal element denoted by k^*. Thus,

$$r\mathbf{1}/|\mathbf{1}| \in A \Leftrightarrow r/|\mathbf{1}| > \frac{\sum_{i=1}^{k^*} a^{(i:d)} + \beta \sum_{i=k^*+1}^{d} a^{(i:d)}}{k^*+\beta(d-k^*)}$$

and the third result follows.

$\diamond$

3 Computation of ruin probabilities in the presence of dependence

3.1 A simple model of dependence

In this Subsection, we investigate a simple model of dependence between the lines of business. For each claim occurrence, we allow the claim amount of a branch either be independent of the others or equal to a common random variable. This model is inspired by Biard et al. (2008) who have introduced a model of dependence between claim amounts in univariate setting. Explicitly, the distribution of $\mathbf{X} = (X^1, ..., X^d)$ is such that, for $1 \leq j \leq d$,

$$X^{(j)} = I^{(j)} W^{(0)} + (1 - I^{(j)}) W^{(j)},$$

where,

- $(W^{(j)})_{0 \leq j \leq d}$ is an i.i.d. non-negative random vector with common distribution $W \in \mathcal{R}_{-\alpha}$, for some $\alpha > 1$,
- and $(I^{(j)})_{1 \leq j \leq d}$ is a vector of independent Bernoulli random variables with same parameter $p \in [0,1]$, and independent from $(W^{(j)})_{0 \leq j \leq d}$.

Let F be the c.d.f. of W.
Note that dependence is only measured through the parameter p.
Here, we assume that $\mathbf{a} = d^{-1}\mathbf{1}$.

Lemma 3.1 *Let $\mathbf{X_1} \in \mathcal{MR}_{-\alpha,\mu_1}$ for some $\alpha > 0$ and some Radon measure μ_1. Let $\mathbf{X_2} \in \mathcal{MR}_{-\alpha,\mu_2}$ for same $\alpha > 0$ and some Radon measure μ_2. Moreover we assume that, for some function $L \in \mathcal{R}_{-\alpha}$, there exists $c_1, c_2 > 0$ such that, for large u*

$$P(|\mathbf{X_1}| > u) \sim c_1 u^{-\alpha} L(u) ,$$

and

$$P(|\mathbf{X_2}| > u) \sim c_2 u^{-\alpha} L(u) \, .$$

If $\mathbf{X_1}$ *and* $\mathbf{X_2}$ *are independent, then* $\mathbf{X_1} + \mathbf{X_2} \in \mathcal{MR}_{\left(-\alpha, \frac{c_1}{c_1+c_2}\mu_1 + \frac{c_2}{c_1+c_2}\mu_2\right)}$.

Proof. Let $A \in \mathbb{R}^d$ be a Borel set bounded away from $\mathbf{0}$ For $i = 1, 2$, $\mathbf{X_i} \in \mathcal{MR}_{-\alpha,\mu_i}$, so from Definition 2.3

$$\lim_{u\to\infty} \frac{P(\mathbf{X_i} \in uA)}{P(|\mathbf{X_i}| > u)} = \mu_i(A) \, . \tag{3.1}$$

Since $P(|\mathbf{X_i}| > u) \sim c_i u^{-\alpha} L(u)$,

$$\lim_{u\to\infty} u^{\alpha} \tilde{L}(u) P(\mathbf{X_i} \in uA) = c_i \mu_i(A) \, ,$$

with $\tilde{L} = 1/L \in \mathcal{R}_0$. So from Hult and Lindskog (2006b) Proposition A.1,

$$\lim_{u\to\infty} u^{\alpha} \tilde{L}(u) P(\mathbf{X_1} + \mathbf{X_2} \in uA) = c_1 \mu_1(A) + c_2 \mu_2(A) \, .$$

By independence and regular variation,

$$P(|\mathbf{X_1} + \mathbf{X_2}| > u) \sim P(|\mathbf{X_1}| > u) + P(|\mathbf{X_2}| > u) \sim (c_1 + c_2) u^{-\alpha} L(u) \, .$$

Hence,

$$\lim_{u\to\infty} \frac{P(\mathbf{X_1} + \mathbf{X_2} \in uA)}{P(|\mathbf{X_1} + \mathbf{X_1}| > u)} = \frac{c_1}{c_1 + c_2} \mu_1(A) + \frac{c_2}{c_1 + c_2} \mu_2(A) \, ,$$

and the result follows.
⋄

Proposition 3.2 *Under the assumptions of this subsection, we have, for* $T > 0$ *and large* u,

$$\psi_{d,\beta}(u, T) \sim \Big\{ (1-p)^d d \left((d-1)\beta + 1 \right)^{-\alpha} + \sum_{k=1}^{d} \binom{d}{k} p^k (1-p)^{d-k} \left[\left(\left(\frac{d-k}{k} \right) \beta + 1 \right)^{-\alpha} + (d-k) \left((d-1)\beta + 1 \right)^{-\alpha} \right] \Big\} d^{\alpha} (\lambda T) \overline{F}(u).$$

Proof. Since $W \in \mathcal{R}_{-\alpha}$, there exists slowly varying function L such that, for $u > 0$, $P(W > u) = u^{-\alpha} L(u)$.

By construction $\mathbf{X}$ is composed of a sum of random variables of the form

$$\mathbf{X}_\Delta = \left\{ \left[W^{(0)} \sum_{i\in\Delta} \mathbf{e}_i \right] + \sum_{i\in\{1,\ldots,d\}\setminus\Delta} \left[W^{(i)} \mathbf{e}_i \right] \right\} \mathbb{1}_{\{\cap_{i\in\Delta}\{I^{(i)}=1\} \bigcup \cup_{i\in\{1,\ldots,d\}\setminus\Delta}\{I^{(i)}=0\}\}}$$

for some subset Δ of $\{1, \ldots, d\}$. For all subset Δ of $\{1, \ldots, d\}$, $P(|\mathbf{X}_\Delta| > u) \sim c_\Delta u^{-\alpha} L(u)$ for some constant c_Δ.
Thus, from Lemma 3.1 $\mathbf{X} \in MR_{-\alpha,\mu}$ with , for all Borel set $B \in \mathbb{R}^d$ bounded away from $\mathbf{0}$,

$$\mu(B) = \lim_{u\to\infty} \frac{P(\mathbf{X} \in uB)}{P(|\mathbf{X}| > u)} \, .$$

For $k=1,\dots,d-1$, let

$$\mathbf{X}^{(\mathbf{k})} = W^{(0)}\mathbf{1}_k + \sum_{i=k+1}^{d} W^{(i)}\mathbf{e}_i.$$

Let $\mathbf{X}^{(\mathbf{0})} = \sum_{i=1}^{d} W^{(i)}\mathbf{e}_i$ and $\mathbf{X}^{(d)} = W^{(0)}\mathbf{1}$.
Let $A = d^{-1}\mathbf{1} - F_\beta$. Note that the set $A = d^{-1}\mathbf{1} - F_\beta$ is symmetric in each direction.
Denote by M the random variable which counts the number of random variables equal to $W^{(0)}$ in $\mathbf{X}$. We have, for large u,

$$\begin{aligned}
\mu(A)P(|\mathbf{X}|>u) &\sim P(\mathbf{X}\in uA)\\
&\sim \sum_{k=0}^{d} P(M=k)P(\mathbf{X}\in uA|M=k)\\
&\sim \sum_{k=0}^{d} \binom{d}{k} p^k(1-p)^{d-k} P\left(\mathbf{X}^{(\mathbf{k})}\in uA\right) \quad (A \text{ is symmetric in each direction}),\\
&\sim \sum_{k=0}^{d} \binom{d}{k} p^k(1-p)^{d-k} \frac{P\left(\mathbf{X}^{(\mathbf{k})}\in uA\right)}{P(|\mathbf{X}^{(k)}|>u)} P\left(|\mathbf{X}^{(k)}|>u\right) .
\end{aligned}$$

Since for $k=1,\dots,d-1$, $P\left(|W^{(0)}\mathbf{1}_k|>u\right) \sim |\mathbf{1}_k|^\alpha P(W>u)$ and for $i=1,\dots,d$, $P\left(|W^{(i)}\mathbf{e}_i|>u\right) \sim P(W>u)$, we have, from Lemma 3.1, for $k=1,\dots,d-1$, $\mathbf{X}^{(\mathbf{k})} \in \mathcal{MR}_{-\alpha,\mu_k}$ with, for all Borel set $B\in\mathbb{R}^d$ bounded away from $\mathbf{0}$

$$\mu_k(B) = \frac{|\mathbf{1}_k|^\alpha \tilde{\mu}_{0,k}(B) + (d-k)\tilde{\mu}_1(B)}{|\mathbf{1}_k|^\alpha + (d-k)} ,$$

where,

- from Lemma 2.6 Case 2,

$$\tilde{\mu}_{0,k}(B) = \lim_{u\to\infty} \frac{P(W\mathbf{1}_k \in uB)}{P(|W\mathbf{1}_k|>u)} ,$$

and

$$\tilde{\mu}_{0,k}(A) = \left(d^{-1}\left(\frac{\beta(d-k)}{k}+1\right)|\mathbf{1}_k|\right)^{-\alpha} ,$$

- and, from Lemma 2.6 Case 1 with $a^{(1)} = d^{-1}$,

$$\tilde{\mu}_1(B) = \lim_{u\to\infty} \frac{P(W\mathbf{e}_1 \in uB)}{P(|W\mathbf{e}_1|>u)} ,$$

and

$$\tilde{\mu}_1(A) = \left(\beta + d^{-1}(1-\beta)\right)^{-\alpha} .$$

Moreover, we have, for all Borel set $B\in\mathbb{R}^d$ bounded away from $\mathbf{0}$

- $$\mu_0(B) = \tilde{\mu}_1(B) \,,$$
  so
  $$\mu_0(A) = \left(\beta + d^{-1}(1-\beta)\right)^{-\alpha} \,,$$
- and from Lemma 2.6 Case 2 with $k = d$,
  $$\mu_d(B) = \tilde{\mu}_{0,d}(B) = \lim_{u\to\infty} \frac{P(W\mathbf{1} \in uB)}{P(|W\mathbf{1}| > u)} \,,$$
  so
  $$\mu_d(A) = d^{\alpha}|1|^{-\alpha} \,.$$

Moreover, by independence and regular variation, we have for large u, for $1 \le k \le d-1$,

$$P\left(\left|X^{(k)}\right| > u\right) \sim (|\mathbf{1}_k|^{\alpha} + (d-k))\,\overline{F}(u) \,,$$

and

$$P\left(\left|X^{(d)}\right| > u\right) \sim |\mathbf{1}|^{\alpha}\overline{F}(u) \,,$$

and

$$P\left(\left|X^{(0)}\right| > u\right) \sim d\overline{F}(u) \,.$$

Thus, we get

$$\mu(A)P(|\mathbf{X}| > u) \sim \sum_{k=0}^{d} \binom{d}{k} p^k (1-p)^{d-k} \mu_k(A) P\left(|\mathbf{X}^{(k)}| > u\right) \,.$$

Moreover, since from Proposition 2.5, we have for $T > 0$ and large u

$$\psi_{d,\beta}(u,T) \sim (\lambda T)\mu(d^{-1}\mathbf{1} - F_{\beta})P(|\mathbf{X}| > u) \,,$$

we get the result.
◇

Corollary 3.3 *When no transfer is allowed ($\beta = 0$), we get, for large u and $T > 0$,*

$$\psi_{d,0}(u,T) \sim \left\{ d(1-p)^d + \sum_{k=1}^{d} \binom{d}{k} p^k (1-p)^{d-k}(d-k+1) \right\} d^{\alpha}(\lambda T)\overline{F}(u) \,.$$

This result corresponds to ψ_{or} (2.6).

Corollary 3.4 *When transfer is allowed without restriction ($\beta = 1$), we get, for large u and $T > 0$,*

$$\psi_{d,1}(u,T) \sim \left\{ \sum_{k=0}^{d} \binom{d}{k} p^k (1-p)^{d-k} \left[k^{\alpha} + (d-k)\right] \right\} (\lambda T)\overline{F}(u) \,.$$

This result corresponds to ψ_{sum} (2.4).

3.2 A Poisson shock model

In the Subsection, we study a classical Poisson shock model; when a claim occurs, it may affect either one specific line of business or all the lines. Explicitly, let X be a non-negative random variable which is regularly varying with some $\alpha > 1$. For $1 \le j \le d$, we assume that the specific claims of the business line j arrive at the jump times of a Poisson process $(N_t^j)_{t\ge 0}$ with intensity $\lambda^{(j)}$. Let X_k^j be the kth specific claim amount of line j. Assume that, for $1 \le j \le d$ and $k \ge 1$, $(X_k^j)_{k\ge 1}$ is an i.i.d. sequence with common distribution X. Thus, the specific aggregate claim amount process of the line j is, for $t \ge 0$,

$$S_t^j = \sum_{k=1}^{N_t^j} X_k^j .$$

We assume that the claims which affect all the lines of business arrive at the jump times of a Poisson process $(N_t^0)_{t\ge 0}$ with intensity $\lambda^{(0)}$. Let $\mathbf{X}_k^0 = X_k^0 \mathbf{1}$ be the vector of the kth claim amounts of this kind. We assume again that $(X_k^0)_{k\ge 1}$ is an i.i.d. sequence with common distribution X. Here, for simplification, we have assumed that all the lines of business pay the same amount for a common claim. Thus, the common aggregate claim amount process is, for $t \ge 0$,

$$\mathbf{S}_t^0 = \sum_{k=1}^{N_t^0} \mathbf{X}_k^0 .$$

We also assume that all $(N_t^j)_{t\ge 0}$ and $(X_k^j)_{k\ge 1}$, $j \in \{0, \dots, d\}$ are independent. From the compound Poisson process properties, we are able to get a risk process of the type of (2.3), since we can write

$$\mathbf{S}_t = \sum_{k=1}^{N_t^0} \mathbf{X}_k^0 + \sum_{j=1}^{d} \sum_{k=1}^{N_t^j} X_k^j \mathbf{e}_j = \sum_{k=1}^{N_t} \mathbf{X}_k ,$$

where $N(t) = \sum_{j=0}^d N_t^j$ is a Poisson process with intensity $\overline{\lambda} = \lambda^{(0)} + \lambda^{(1)} + \ldots + \lambda^{(d)}$, and $\mathbf{X}_k = \mathbf{X}_k^0 \delta_0(\xi_k) + \sum_{j=1}^d X_k^j \mathbf{e}_j \delta_j(\xi_k)$ with, $(\xi_k)_{k\ge 1}$ an i.i.d. sequence of random variables independent of all others random variables and with $P(\xi_k = j) = \lambda^{(j)}/\overline{\lambda}$ for $k \ge 1$ and $0 \le j \le d$.

Proposition 3.5 *Under the above assumptions, we have, for $T > 0$ and large u,*

$$\psi_{d,\beta}(u,T) \sim \left\{ \lambda^{(0)} \left[\frac{\sum_{j=1}^{k^*} a^{(j:d)} + \beta \sum_{j=k^*+1}^{d} a^{(j:d)}}{k^* + \beta(d - k^*)} \right]^{-\alpha} + \sum_{j=1}^{d} \lambda^{(j)} \left[\beta + a^{(j)}(1-\beta) \right]^{-\alpha} \right\} T\overline{F}(u) ,$$

where for $1 \le j \le d$, $a^{(j:d)}$ is jth larger component of $\mathbf{a}$ and

$$k^* = \inf \left\{ k \in [1, d-1] : a^{(k+1:d)} > \frac{\sum_{j=1}^{k} a^{(j:d)} + \beta \sum_{j=k+1}^{d} a^{(j:d)}}{k + \beta(d-k)} \right\} .$$

Proof. From Proposition 2.5, we have for $T > 0$ and large u

$$\psi_{d,\beta}(u,T) \sim (\overline{\lambda} T) \mu(\mathbf{a} - F_\beta) P(|\mathbf{X}| > u) .$$

Let $A = \mathbf{a} - F_\beta$.
From Hult and Lindskog (2006a), Section 4, we have

$$P(|\mathbf{X}| > u) \sim \left(\frac{\lambda^{(0)}}{\overline{\lambda}}(|\mathbf{1}|^\alpha - 1) + 1\right) P(X > u)\ ,$$

and

$$\mu(A) = \frac{\lambda^{(0)}|\mathbf{1}|^\alpha \mu_0(A) + \sum_{j=1}^d \lambda^{(j)}\mu_j(A)}{\lambda^{(0)}\left(|\mathbf{1}|^\alpha - 1\right) + \overline{\lambda}}$$

where

$$\mu_0(A) = \lim_{u\to\infty} \frac{P(X\mathbf{1} \in uA)}{P(|X\mathbf{1}| > u)}\ ,$$

and, for $1 \le j \le d$,

$$\mu_j(A) = \lim_{u\to\infty} \frac{P(X\mathbf{e}_j \in uA)}{P(|X\mathbf{e}_j| > u)}\ .$$

We get the result using 2.6, Case 1 and Case 3.
⋄

Corollary 3.6 *When no transfer is allowed ($\beta = 0$), we get, for large u and $T > 0$,*

$$\psi_{d,0}(u,T) \sim \left\{\lambda^{(0)}\left[\min_{1\le j\le d}\{a^{(j)}\}\right]^{-\alpha} + \sum_{j=1}^d \lambda^{(j)}\left[a^{(j)}\right]^{-\alpha}\right\} T\overline{F}(u)\ .$$

This result corresponds to ψ_{or} (2.6).

Corollary 3.7 *When transfer is allowed without restriction ($\beta = 1$), we get, for large u and $T > 0$,*

$$\psi_{d,1}(u,T) \sim \left\{\lambda^{(0)}(d^\alpha - 1) + 1\right\} T\overline{F}(u)\ .$$

This result corresponds to ψ_{sum} (2.4).

4 Optimal allocation problems

Throughout this Section, we assume $\beta = 0$. So we write $\psi_{d,\beta=0} = \psi_d$. This case corresponds to the probability that at least one of the line business becomes negative before T without money transfer.

In this section, we suppose that the company owns a global initial reserve u to allocate to the d lines of business in order to minimize its finite-time ruin probability. Explicitly, we have the following optimal problem :

$$\begin{cases} \min_{\mathbf{a}\in(0,1)^d} \psi_d(u,T)\ , \\ \text{under the constraint } a^{(1)} + \ldots + a^{(d)} = 1\ . \end{cases} \tag{4.1}$$

Here, we are going to minimize the asymptotics of $\psi_d(u,T)$ we denote by $\tilde{\psi}_d(u,T)$. Thus the problem (4.1) becomes :

$$\begin{cases} \min_{\mathbf{a}\in(0,1)^d} \tilde{\psi}_d(u,T) \ , \\ \text{under the constraint } a^{(1)} + \ldots + a^{(d)} = 1 \ . \end{cases} \tag{4.2}$$

We investigate the following cases.

- **Case 1 :** the company is composed of d lines of business and they are mutually independent; explicitly, the model is the Subsection 3.2 one with $\lambda^{(0)} = 0$.
- **Case 2 :** the company is composed of two lines of business and their dependence structure is described by the Poisson shock model of Subsection 3.2.
- **Case 3 :** the company is composed of three lines of business, one is independent from the others and the two others are dependent via the Poisson shock model of Subsection 3.2.

4.1 Case 1

In this Subsection, we start with the model of Subsection 3.2 wherein $\lambda^{(0)}$ is assumed to be equal to zero. That is to say that $\mathbf{X}$ is composed with d mutually independent random variables, so the business lines are mutually independent too.

Proposition 4.1 *Under the above assumptions, we have for $T > 0$ and large u,*

$$\psi_d(u,T) \sim \left\{ \sum_{j=1}^{d} \lambda^{(j)} \left[a^{(j)}\right]^{-\alpha} \right\} T\overline{F}(u) \ .$$

Proof. Take $\lambda^{(0)} = 0$ in Corollary 3.6.
⋄

The following proposition gives the optimal allocation of our optimization problem (4.2).

Proposition 4.2 *Under the assumptions of the Subsection 4.1, the solution of (4.2) is, for all $1 \le i \le d$,*

$$a^{(i)^*} = \left\{ \frac{\lambda^{(i)\frac{1}{\alpha+1}}}{\sum_{j=1}^{d} \lambda^{(j)\frac{1}{\alpha+1}}} \right\} .$$

Proof. Let $g : \mathbf{a} \in (0,1)^d \mapsto g(\mathbf{a}) = \sum_{i=1}^{d} \lambda^{(i)} \left[a^{(i)}\right]^{-\alpha}$. g is a continuous, differentiable and strictly convex function on $(0,1)^d$. Using the method of Lagrange multipliers, we find one $\mathbf{a}^*$ which minimizes g on $\{(a^{(1)},\ldots,a^{(d)}) \in (0,1)^d \ , \ a^{(1)} + \ldots + a^{(d)} = 1\}$. Since g is strictly convex, on the non empty open convex set $\Omega = \{a^{(1)} + \ldots + a^{(d)} = 1\}$ this minimum is unique.
⋄

In Figure 2, for $d = 2$, we represent, $a^{(1)}$ and $a^{(2)}$ as a function of $\lambda^{(1)}/\overline{\lambda}$ ($\lambda^{(1)}/\overline{\lambda}$ varies from 0 to 1). Both cases $\alpha = 2$ and $\alpha = 5$ are plotted. As expected, we allow a larger part of the initial reserve to the riskier line of business. We can also note that when α is increasing, then initial reserves become more similar.

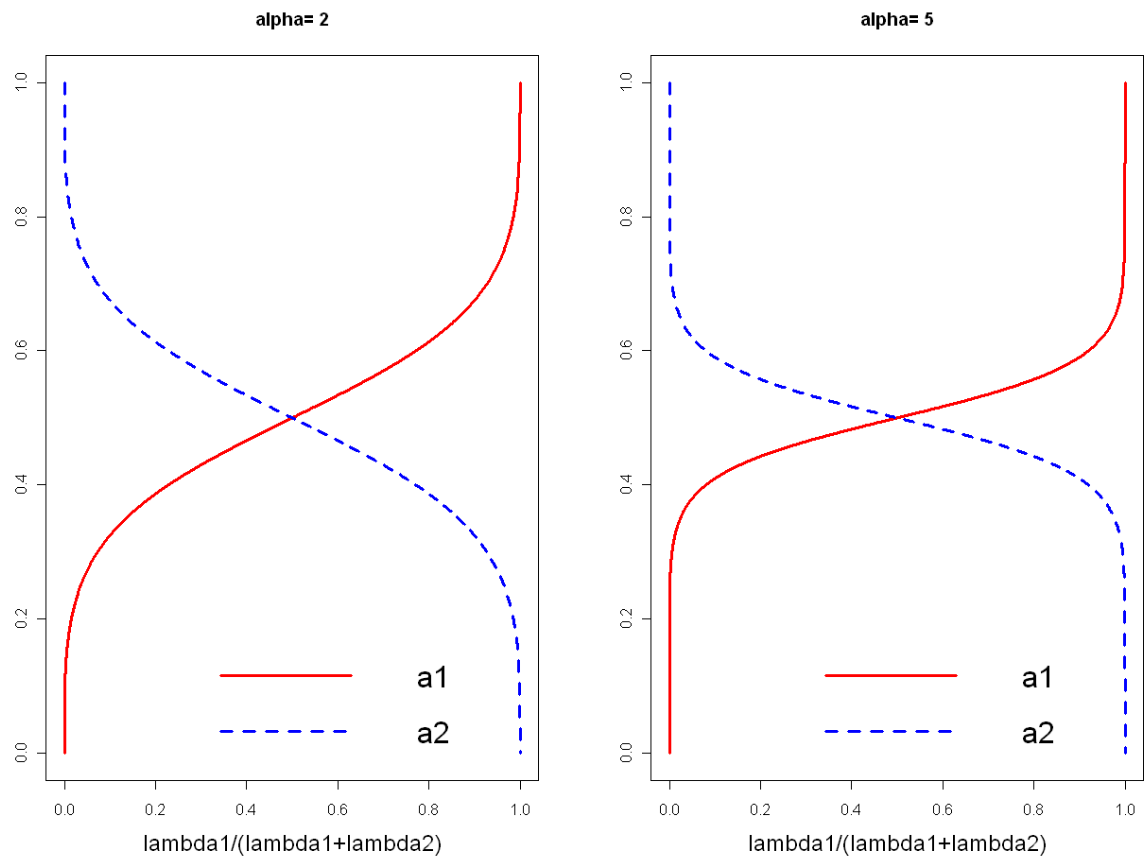

Figure 2: Optimal solution of the Case 1 for $d = 2$

4.2 Case 2

In this Subsection, we investigate the two dimensional case wherein the dependence structure is described by the model of Subsection 3.2. Let $a \in (0,1)$ such that $a^{(1)} = a$ and $a^{(2)} = 1 - a$.

Proposition 4.3 *Under the above assumptions, we have for $T > 0$ and large u,*

$$\psi_2(u,T) \sim \left\{\lambda^{(0)}\left[\min(a;1-a)\right]^{-\alpha} + \lambda^{(1)}a^{-\alpha} + \lambda^{(2)}(1-a)^{-\alpha}\right\} T\overline{F}(u) .$$

Proof. Take $d = 2$ in Corollary 3.6.
◇

Proposition 4.4 *Under the assumptions of the Subsection 4.2, the solution of (4.2) is*

$$a* = \begin{cases} \frac{1}{2} & \text{if } \lambda^{(0)} > |\lambda^{(1)} - \lambda^{(2)}| , \\ \frac{\lambda^{(1)\frac{1}{\alpha+1}}}{\lambda^{(1)\frac{1}{\alpha+1}} + \left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}}} & \text{if } \lambda^{(0)} \le \lambda^{(1)} - \lambda^{(2)} , \\ \frac{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(2)\frac{1}{\alpha+1}}} & \text{if } \lambda^{(0)} \le \lambda^{(2)} - \lambda^{(1)} . \end{cases}$$

Proof. Let, for $0 < a \le 1/2$

$$g_1(a) = (\lambda^{(0)} + \lambda^{(1)})a^{-\alpha} + \lambda^{(2)}[1-a]^{-\alpha} ,$$

and for $1/2 \ge a < 1$

$$g_2(a) = (\lambda^{(0)} + \lambda^{(2)})\,[1-a]^{-\alpha} + \lambda^{(1)}a^{-\alpha} .$$

g_1 (resp. g_2) is differentiable and strictly convex on $(0,1/2)$ (resp. $(1/2,1)$). Moreover $g_1(1/2) = g_2(1/2)$ for all $a_1 \in (0,1/2)$ and $a_2 \in (1/2,1)$; $g'(a_1) < g'(a_2)$. Let

$$g(a) = \begin{cases} g_1(a) & 0 < x \le 1/2 \\ g_2(a) & 1/2 < a < 1 \end{cases} .$$

Thus, g is continuous on $(0,1)$ and g' is strictly increasing on $(0,1/2) \cup (1/2,1)$. So, g is strictly convex on $(0,1)$. As a consequence, on the non empty open convex set $(0,1)$, there exists a unique a^* which minimizes g. Since $g(0) = g(1) = +\infty$, we have

$$a^* = \begin{cases} \arg\min\limits_{(0,1/2)} g_1 & \text{if } g_1'(1/2) > 0 , \\ \arg\min\limits_{(1/2,1)} g_2 & \text{if } g_2'(1/2) < 0 , \\ 1/2 & \text{if } g_1'(1/2) < 0 \text{ and } g_2'(1/2) > 0 , \end{cases}$$

that is to say

$$a^* = \begin{cases} a_1^* \ / \ g_1'(a_1^*) = 0 & \text{if } g_1'(1/2) > 0 , \\ a_2^* \ / \ g_2'(a_2^*) = 0 & \text{if } g_2'(1/2) < 0 , \\ 1/2 & \text{if } g_1'(1/2) < 0 \text{ and } g_2'(1/2) > 0 . \end{cases}$$

Since $\arg\min g = \arg\min \tilde{\psi}_{2,\beta}$, we get the result.
◇

In Figures 3, 4 and 5, for $d = 2$, we represent, $a^{(1)}$ and $a^{(2)}$ as a function of $\lambda^{(1)}/\overline{\lambda}$ (we fix $\lambda^{(0)}$ and $\lambda^{(1)}/\overline{\lambda}$ varies from 0 to $1 - \lambda^{(0)}$). in the three figures, both cases $\alpha = 2$ and $\alpha = 5$ are plotted. Figures 3, 4, and 5 represent respectively cases $\lambda^{(0)} = 0.1$, $\lambda^{(0)} = 0.3$ and $\lambda^{(0)} = 0.5$.

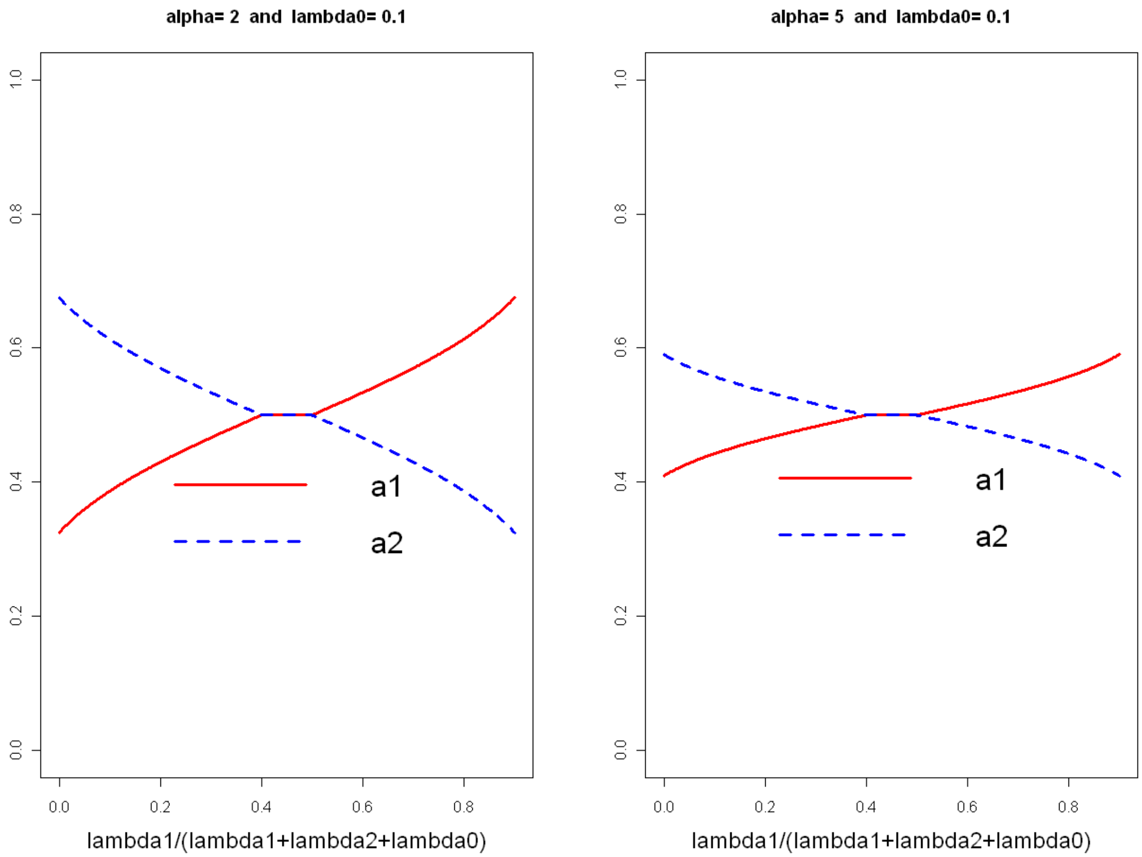

Figure 3: Optimal solution of the Case 2 for $d = 2$ and $\lambda^{(0)} = 0.1$

Remark 4.5 *There are three different forms of the optimal allocation in Proposition 4.4.*

- *When $\lambda^{(0)}$ is large, or when the two lines of business are very similar, we allocate half of the reserve to each line. Actually, both high positive dependence and close parameters conduce to a similar behavior of the two processes. We can observe this behavior on the figures. In Figures 3, 4, we observe a plateau when $\lambda^{(1)}$ is closed to $\lambda^{(2)}$ and this plateau becomes wider when $\lambda^{(0)}$, so the dependence, increases. When the dependence is too high, as in Figure 5, we only observe a plateau.*
- *When $\lambda^{(1)}$ is large, compared to $\lambda^{(0)}$ and $\lambda^{(2)}$, the optimal solution is the same as in the case where the two lines of business are independent and where $\lambda^{(2)}$ is switched with $\lambda^{(2)} + \lambda^{(0)}$, and, as expected, we allocate more to the first line of business. In Figures 3, 4, it corresponds to the part after the plateau. We have also the symmetric case, when $\lambda^{(2)}$ is large compared to $\lambda^{(0)}$ and $\lambda^{(1)}$, which is corresponds to the part before the plateau in Figures 3, 4.*

We can also note that an increase of α conduces to a decrease of the difference between the two reserves.

4.3 Case 3

In this Subsection, we assume that the insurance company has three lines of business, two dependent through the common shock model of Subsection 3.2, and one independent from the

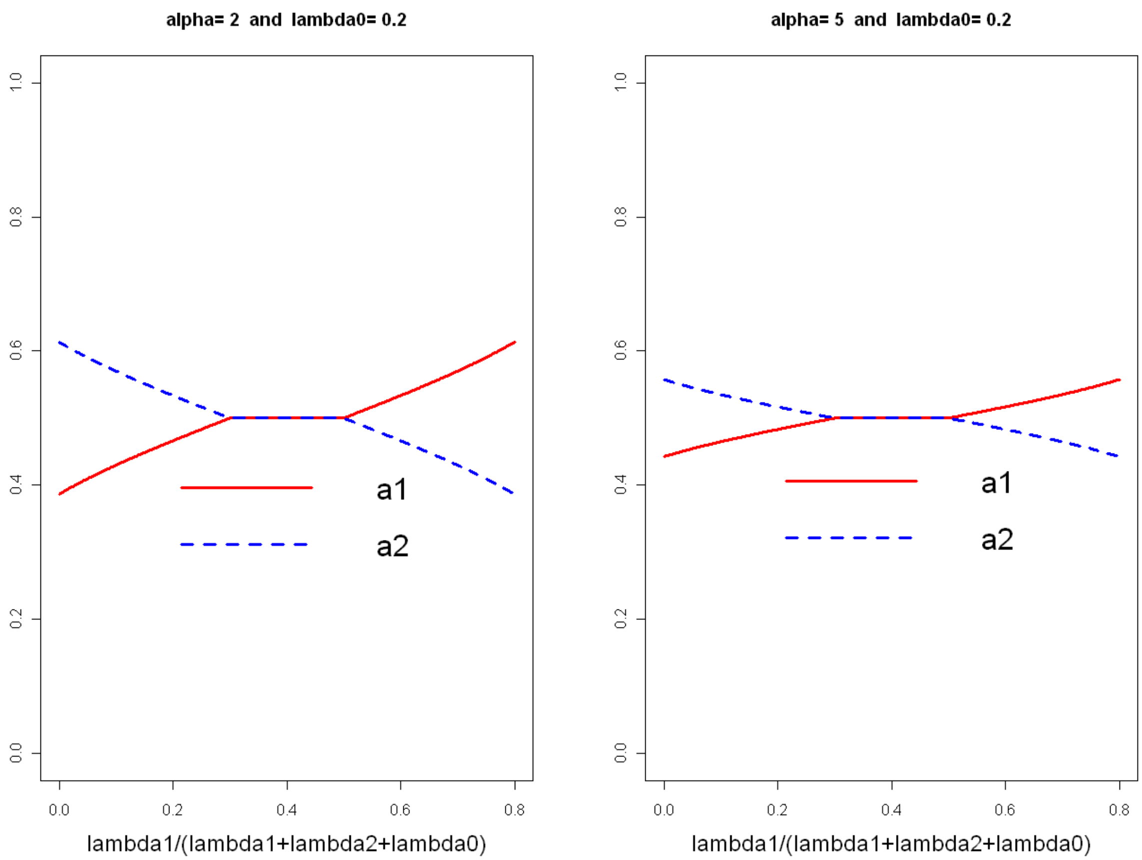

Figure 4: Optimal solution of the Case 2 for $d = 2$ and $\lambda^{(0)} = 0.3$

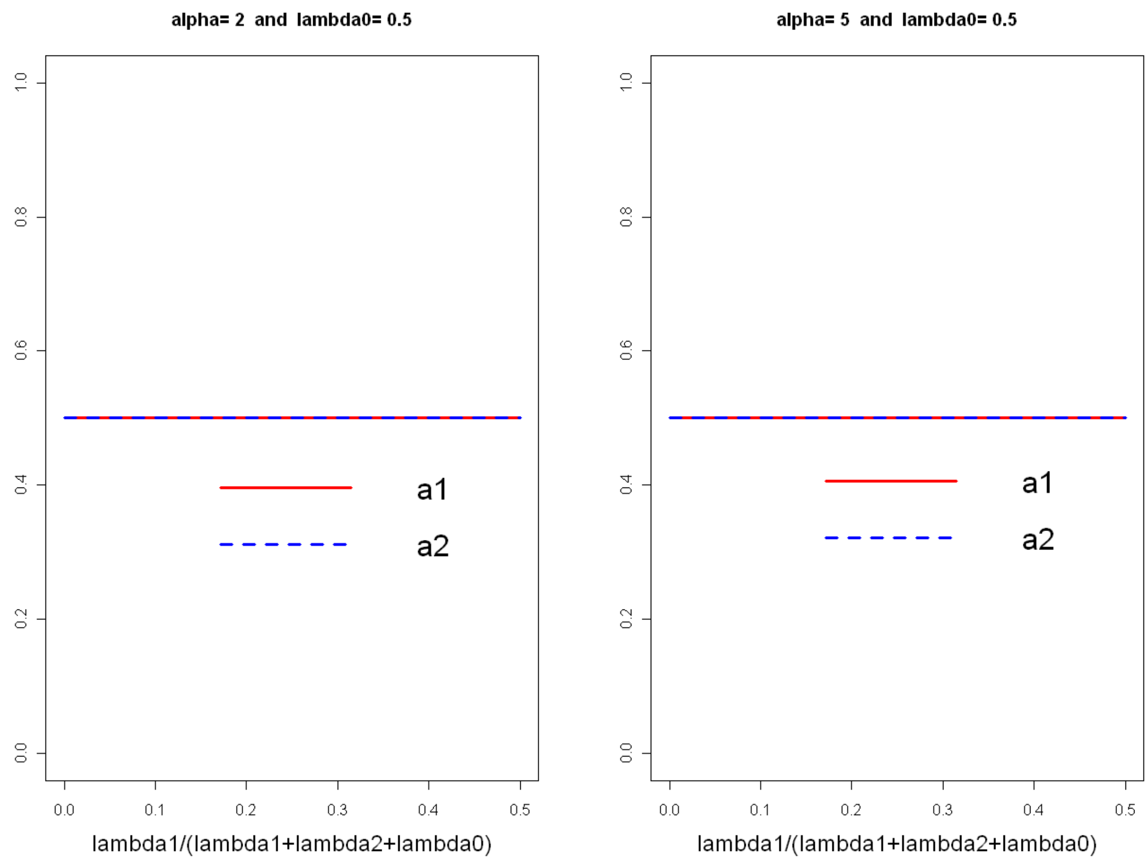

Figure 5: Optimal solution of the Case 2 for $d = 2$ and $\lambda^{(0)} = 0.5$

two others. Explicitly, we have (with a simple adaptation of the Subsection 3.2 model) :

$$\mathbf{S}_t = \sum_{k=1}^{N(t)} \mathbf{X}_k ,$$

where $N(t)$ is a Poisson process with intensity $\overline{\lambda} = \lambda^{(0)} + \lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)}$, and $\mathbf{X}_k = X_k^0 \mathbf{1}_2 \delta_0(\xi_k) + \sum_{i=1}^{3} X_k^i \mathbf{e}_i \delta_i(\xi_k)$ with,

- $\lambda_i > 0 \ , \ 0 \le i \le 3 \ ,$
- for $0 \le i \le 3$, $(X_k^i)_{k\ge1}$ is an i.i.d sequence with common distribution $X \in \mathcal{R}_{-\alpha}$,
- and with all $(X_k^i)_{k\ge1}$, $0 \le i \le 3$ independent and independent from $N(t)$,
- $(\xi_k)_{k\ge1}$ an i.i.d. sequence of random variables independent of all others random variables and with $P(\xi_k = i) = \lambda^{(i)}/\overline{\lambda}$ for $k \ge 1$ and $0 \le i \le 3$.

Denote by $\psi_{\tilde{3}}$ the ruin probability associated with the above model.

Proposition 4.6 *Under the above assumptions, we have, for $T > 0$ and large u,*

$$\psi_{\tilde{3}}(u,T) \sim \left\{\lambda^{(0)} \left[\min(a^{(1)}; a^{(2)})\right]^{-\alpha} + \lambda^{(1)} a^{(1)-\alpha} + \lambda^{(2)} a^{(2)-\alpha} + \lambda^{(3)} a^{(3)-\alpha}\right\} T\overline{F}(u) .$$

Proof. Let $A = \mathbf{a} - F_\beta$.
We have, for large u

$$P(|\mathbf{X}| > u) \sim \frac{\lambda^{(0)}}{\overline{\lambda}} P(|X^0 \mathbf{1}_2| > u) + \sum_{i=1}^{3} \frac{\lambda^{(i)}}{\overline{\lambda}} P(|X^i \mathbf{e}_i| > u) \sim \left(\frac{\lambda^{(0)}}{\overline{\lambda}}(|\mathbf{1}_2|^\alpha - 1) + 1\right) P(X > u) .$$

Moreover $\mathbf{X} \in \mathcal{MR}_{-\alpha,\mu}$ with

$$\mu(A) = \frac{\lambda^{(0)} |\mathbf{1}_2|^\alpha \mu_{(1,2)}(A) + \sum_{i=1}^{3} \lambda^{(i)} \mu_i(A)}{\lambda^{(0)}(|\mathbf{1}_2|^\alpha - 1) + \overline{\lambda}} ,$$

where

$$\mu_{(1,2)}(A) = \lim_{u\to\infty} \frac{P(X\mathbf{1}_2 \in uA)}{P(|X\mathbf{1}_2| > u)} ,$$

and, for $1 \le j \le d$,

$$\mu_j(A) = \lim_{u\to\infty} \frac{P(X\mathbf{e}_j \in uA)}{P(|X\mathbf{e}_j| > u)} .$$

We get the result using 2.6, Case 2 and Case 3.
$\diamond$

Proposition 4.7 *Under the assumptions of Subsection 4.3, the solution of (4.2) is as follows.*

- *If $\lambda^{(0)} > |\lambda^{(1)} - \lambda^{(2)}|$, then*

$$\begin{cases} a^{(1)^*} = a^{(2)^*} = \frac{1}{2} \dfrac{\left(2^\alpha\left(\lambda^{(0)}+\lambda^{(1)}+\lambda^{(2)}\right)\right)^{\frac{1}{\alpha+1}}}{\left(2^\alpha\left(\lambda^{(0)}+\lambda^{(1)}+\lambda^{(2)}\right)\right)^{\frac{1}{\alpha+1}} + \lambda^{(3)\,\frac{1}{\alpha+1}}} , \\ a^{(3)^*} = \dfrac{\lambda^{(3)\,\frac{1}{\alpha+1}}}{\left(2^\alpha\left(\lambda^{(0)}+\lambda^{(1)}\right)\right)^{\frac{1}{\alpha+1}} + \lambda^{(3)\,\frac{1}{\alpha+1}}} . \end{cases}$$

- *If* $\lambda^{(0)} \leq \lambda^{(1)} - \lambda^{(2)}$, *then*

$$\begin{cases} a^{(1)^*} = \frac{\lambda^{(1)\frac{1}{\alpha+1}}}{\lambda^{(1)\frac{1}{\alpha+1}} + \left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(3)\frac{1}{\alpha+1}}}, \\ a^{(2)^*} = \frac{\left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}}}{\lambda^{(1)\frac{1}{\alpha+1}} + \left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(3)\frac{1}{\alpha+1}}}, \\ a^{(3)^*} = \frac{\lambda^{(3)\frac{1}{\alpha+1}}}{\lambda^{(1)\frac{1}{\alpha+1}} + \left(\lambda^{(0)}+\lambda^{(2)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(3)\frac{1}{\alpha+1}}}. \end{cases}$$

- *If* $\lambda^{(0)} \leq \lambda^{(2)} - \lambda^{(1)}$, *then*

$$\begin{cases} a^{(1)^*} = \frac{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(2)\frac{1}{\alpha+1}} + \lambda^{(3)\frac{1}{\alpha+1}}}, \\ a^{(2)^*} = \frac{\lambda^{(2)\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(2)\frac{1}{\alpha+1}} + \lambda^{(3)\frac{1}{\alpha+1}}}, \\ a^{(3)^*} = \frac{\lambda^{(3)\frac{1}{\alpha+1}}}{\left(\lambda^{(0)}+\lambda^{(1)}\right)^{\frac{1}{\alpha+1}} + \lambda^{(2)\frac{1}{\alpha+1}} + \lambda^{(3)\frac{1}{\alpha+1}}}. \end{cases}$$

Proof. Fix $a^{(3)} \in (0,1)$. Let, for $a^{(1)} \in (0, 1-a_3)$,

$$g_1(a^{(1)}) = \lambda^{(0)} \min(a^{(1)}, 1-a^{(3)}-a^{(1)})^{-\alpha} + \lambda^{(1)} a^{(1)^{-\alpha}} + \lambda^{(2)}(1-a^{(3)}-a^{(2)})^{-\alpha} + \lambda^{(3)} a^{(3)^{-\alpha}}.$$

Using the same way as in the proof of Proposition 4.4, we get $a^{(1)^*} = g_3(a^3) = \arg\min_{(0,1)} g_1$. Then $a^{(3)^*} = \arg\min_{(0,1)} g_3$ and we get the result.

⋄

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