Periodic points and homoclinic classes

Flavio Abdenur, Christian Bonatti, Sylvain Crovisier, Lorenzo J. Díaz, Lan Wen

Periodic points and homoclinic classes

F. Abdenur, Ch. Bonatti, S. Crovisier, L. J. Díaz and L. Wen *

November 22, 2010

Abstract

We prove that there is a residual subset $\mathcal{I}$ of $\mathrm{Diff}^1(\mathrm{M})$ such that any homoclinic class of a diffeomorphism $f \in \mathcal{I}$ having saddles of indices α and β contains a dense subset of saddles of index τ for every $\tau \in [\alpha, \beta] \cap \mathbb{N}$. We also derive some consequences from this result about the Lyapunov exponents of periodic points and the sort of bifurcations inside homoclinic classes of generic diffeomorphisms.

Keywords: chain recurrence class, dominated splitting, heterodimensional cycle, homoclinic class, homoclinic tangency, index of a saddle.
MSC 2000: 37C05, 37C20, 37C25, 37C29, 37D30.

1 Introduction

Hyperbolic periodic points play a key role in the study of the dynamics of diffeomorphisms. Naively speaking, one tries to structure the dynamics using these points as a spine. This strategy is supported by the Closing Lemma of Pugh ([Pu]), which claims that, for generic C^1-diffeomorphisms (i.e., diffeomorphisms in a *residual* subset of $\mathrm{Diff}^1(\mathrm{M})$, that is, a set containing a countable intersection of open and dense subsets of $\mathrm{Diff}^1(\mathrm{M})$), hyperbolic periodic points form a dense subset of the limit set of f. Using hyperbolic periodic points one attempts to split the dynamics of C^1-generic diffeomorphisms into elementary pieces (whose archetypal models are the basic sets of the Smale's theory [Sm]). Ideally, these elementary pieces should be pairwise disjoint, indecomposable (i.e., each piece is in some sense a dynamical unity) and not contained in bigger pieces (this corresponds to the notion of maximality).

Recently, in the C^1-generic context, substantial progress has been made in the direction of finding good candidates for the role of these elementary pieces. The results in [BC] state that, for generic diffeomorphisms of $\mathrm{Diff}^1(\mathrm{M})$, these elementary pieces are the *chain recurrence classes.* Moreover, every chain recurrence class containing a periodic point p is the *homoclinic class* of p. See also some preliminary results in [Ab, Ar, GW, We_1]. Furthermore, due to the closing lemma

*This paper was partially supported by CAPES/Prodoc, CNPq (Bolsa de Pesquisa and Edital Universal) and Faperj (Cientistas do Nosso Estado), PRONEX (Brazil), the Agreement Brazil-France in Mathematics, the program *Special Funds for Major State Basic Research Projects* of the Ministry of Science and Technology of China, and the Institute of Mathematics of Peking University, China. The authors thank the warm hospitality of the IMPA (Rio de Janeiro, Brazil), Institut Henri Poincaré (Trimester Time at Work) (Paris, France), the Institut de Mathématiques de Bourgogne (Dijon, France), and the School of Mathematics of Peking University (Beijing, China) during their visits while preparing this paper.

mentioned above and the results in [CMP], the homoclinic classes constitute a partition of a dense part of the limit set of generic diffeomorphisms. Furthermore, [Cr] states that chain recurrence classes of generic diffeomorphisms are Hausdorff limits of homoclinic classes. These results evidence the importance of the homoclinic classes in dynamics. For a discussion of the notion of elementary piece of dynamics and a survey on recent progress in the study of C^1-generic dynamics we refer to [BDV, Chapter 10].

The previous results motivate our interest in obtaining as complete as possible description of homoclinic classes, especially in the non-hyperbolic context. Homoclinic classes were introduced by Newhouse in [Ne_1] as a generalization of the basic sets in the Smale Decomposition Theorem (see [Sm]). The homoclinic class of a (hyperbolic) saddle p of a diffeomorphism f, denoted by $H(p, f)$, is the closure of the transverse intersections of the invariant manifolds (stable and unstable ones) of the orbit of p. A homoclinic class can be also (equivalently) defined as the closure of the set of (hyperbolic) saddles q *homoclinically related* to p (the stable manifold of the orbit of q transversely meets the unstable one of the orbit of p and vice-versa). This implies that the saddles having the same *index* (dimension of the stable bundle) as p form a dense subset of the whole class $H(p, f)$.

Every homoclinic class $H(p, f)$ is f-invariant and transitive (in fact, the points of $H(p, f)$ whose orbits are dense in the whole class form a residual subset of it): the homoclinic class is the ω-limit set of some $z \in H(p, f)$ (see for instance, [Ne_4]). But a homoclinic class may fail to be uniformly hyperbolic (for instance, it may contain in a robust way hyperbolic saddles having different indices as p, see the constructions in [$\mathrm{Dí}_1$, $\mathrm{Dí}_2$, DR] and the examples of non-hyperbolic robustly transitive diffeomorphisms in [BD_1, BV]) and locally maximal (for instance, a class can be contained with some persistence in the closure of an infinite set of sinks or sources, see [BD_2]).

Bearing in mind that homoclinic classes may fail to be hyperbolic and may indeed contain hyperbolic periodic points having different indices in a robust way, it is natural to wonder whether these indices form an interval in $\mathbb{N}$. We give a positive answer to this question for homoclinic classes of C^1-generic diffeomorphisms. Our main result is the following:

Theorem 1. *There is a residual subset $\mathcal{I}$ of* $\mathrm{Diff}^1(\mathrm{M})$ *of diffeomorphisms f such that, for every $f \in \mathcal{I}$, any homoclinic class $H(p, f)$ containing hyperbolic saddles of indices α and β contains a dense subset of saddles of index τ for all $\tau \in [\alpha, \beta] \cap \mathbb{N}$.*

This paper can be viewed as a continuation of [BDPR], where an analogous result was proved for *robustly transitive sets*[1]: among the diffeomorphisms $f \in \mathrm{Diff}^1(\mathrm{M})$ having a robustly transitive set Λ_f, the property of the indices of the saddles of Λ_f forming an interval in $\mathbb{N}$ holds open and densely. Our proof involves a lot of the C^1-generic machinery developed recently. Moreover, as in [BDPR], a key ingredient in our constructions is the notion of *heterodimensional cycle* (i.e., there are saddles p and q of different indices such that the stable manifold of p intersects the unstable one of q and vice-versa). We analyze the creation of new saddles (of intermediate indices) via heterodimensional cycles associated to saddles of the homoclinic class (see Sections 3.2 and 3.3).

Let us briefly explain this point and summarize the main ingredients of our proof. Suppose that a diffeomorphism f has a saddle p_f of index α whose homoclinic class contains a saddle q_f of index β. In our generic context, we can assume that this property holds *locally residually* (i.e., in a

[1]Recall that an f-invariant compact set Λ_f is *transitive* if there is some $x \in \Lambda_f$ whose forward orbit is dense in the whole Λ_f. The set Λ_f is a *robustly transitive set* of the diffeomorphism f if there are an open neighborhood U of Λ_f in the ambient manifold M and a neighborhood $\mathcal{U}_f$ of f in $\mathrm{Diff}^1(\mathrm{M})$ such that, for every $g \in \mathcal{U}_f$, the set $\Lambda_g(U) = \bigcap_{k\in\mathbb{Z}} g^k(\overline{U})$ is transitive, contained in U, and $\Lambda_f(U) = \Lambda_f$.

residual set of an open neighborhood of f in $\mathrm{Diff}^1(\mathrm{M})$) and that, in fact, the homoclinic classes of p_f and q_f locally residually coincide (this follows from [CMP], see Section 2.2). Using the Connecting Lemma of Hayashi (see Lemma 2.7), this allows us to create heterodimensional cycles associated to the continuations of p_f and q_f (see Section 2.4). We then show that the unfolding of these heterodimensional cycles leads to diffeomorphisms g having saddles r_g of any index τ in between α and β (Section 3.2). In principle, this new saddle r_g might not belong to the homoclinic class $H(p_g, g) = H(q_g, g)$. Let us point out that in the context of robustly transitive sets in [BDPR] this problem does not appear: the new saddle r_g automatically belongs to the robustly transitive set. So an extra difficulty of this paper is to see that the saddle r_g, obtained after a perturbation, can be taken inside the homoclinic class.

We see that, after an appropriate perturbation, the new saddle r_g can be taken in $H(p_g, g)$. Generically, this implies that $H(r_g, g) = H(p_g, g)$. This is done in two steps. We first see that the new saddle can be obtained in such a way $W^u(p_g, g)$ transversely meets $W^s(r_g, g)$ and that $W^u(r_g, g)$ transversely meets $W^s(q_g, g)$ (this is done in Section 3.3). We see that this implies that the three saddles p_g, r_g and q_g are in the same chain recurrence class (see Section 2.1 for the precise definition). Finally, a genericity argument we borrow from [BC] assures that, for generic diffeomorphisms, any chain recurrence class containing a periodic point is the homoclinic class of that point. This guarantees that generically $H(p_g, g) = H(q_g, g) = H(r_g, g)$, which implies Theorem 1 (see Section 4 for details).

The proof of Theorem 1 allows us to obtain the following property about Lyapunov exponents of the saddles inside the homoclinic classes of generic diffeomorphisms. Let us define the *Lyapunov exponent vector* of a hyperbolic point p of period $\pi(p)$ by

$$v = \left(\frac{\log(|\mu_1|)}{\pi(p)}, \dots, \frac{\log(|\mu_n|)}{\pi(p)} \right),$$

where $\mu_1, \dots, \mu_n$ are the eigenvalues of $Df^{\pi(p)}(p)$ ordered by their moduli.

Corollary 2. *Let $\mathcal{I}$ be the residual subset of* $\mathrm{Diff}^1(\mathrm{M})$ *in Theorem 1. For every $f \in \mathcal{I}$ and any homoclinic class $H(p, f)$ of f, the closure of the set of Lyapunov vectors of the saddles of $H(p, f)$ is convex.*

Using Markov partitions, it is easy to see that this result holds for any hyperbolic homoclinic class.

We now derive further consequences from our main result and state some related problems. We first study the sort of bifurcations associated to a homoclinic class in terms of its dominated splitting. A *dominated splitting* of an f-invariant set K is a Df-invariant splitting

$$T_K M = E_1 \oplus \cdots \oplus E_k$$

over K such that the dimensions of the fibers $E_i(x)$ of E_i do not depend on the base point $x \in K$, and the expansion rate of Df on E_i is uniformly smaller than the expansion rate on E_{i+1}, that is, there are $C > 0$ and $\lambda > 1$ such that for any integer $n > 0$, any $x \in K$, one has:

$$\frac{\|Df^n(u)\|}{\|Df^n(v)\|} < C\lambda^{-n}, \qquad \text{for every unitary vectors } u \in E_i(x),\ v \in E_j(x) \text{ with } i < j.$$

In this case one writes $E_1 \prec \cdots \prec E_k$.

Let us recall two results relating dominated splitting and homoclinic tangencies. Consider a diffeomorphism f and an f-invariant compact set K contained in the closure of a set of periodic saddles having the same index τ. [We$_2$] proved that if the natural splitting over this set of periodic orbits given by the stable and unstable bundles cannot be extended to a dominated splitting over the whole K, then there is g arbitrarily C^1-close to f having a homoclinic tangency (in an arbitrarily small neighborhood of K).

The second result deals with robustly transitive sets. First, every robustly transitive set $\Lambda_f(U)$ has a finest dominated splitting $T_{\Lambda_f(U)}M = E_1 \oplus \cdots \oplus E_m$, (see [BDP]): that is, a dominated splitting whose bundles cannot be split in a dominated way. Let now $\Lambda_f(U)$ be a robustly transitive set defined for any f in an open set $\mathcal{U}$ of $\mathrm{Diff}^1(\mathrm{M})$. Then, open and densely in $\mathcal{U}$, the maximum β and the minimum α of the indices of the saddles in $\Lambda_f(U)$ are locally constant functions. Now, if for some k the dimension of $E = E_1 \oplus \cdots \oplus E_k$ is less than or equal to α then this bundle E is uniformly contracting. Similarly, if the dimension of $F = E_{m-k} \oplus \cdots \oplus E_m$ is less than or equal to $(\dim(M) - \beta)$ then the bundle F is uniformly expanding. Finally, suppose that $j \in [\alpha, \beta] \cap \mathbb{N}$ is such that there is $k \in \{1, \ldots, m-1\}$ with

$$\dim(E_1 \oplus \cdots \oplus E_k) < j < \dim(E_1 \oplus \cdots \oplus E_{k+1}).$$

Then there exists g arbitrarily close to f with a homoclinic tangency associated to a saddle of index j of $\Lambda_g(U)$. This result is stated in [BDPR, Theorem F].

These two results motivate the following conjecture[2]:

Conjecture 1. *For every C^1-generic diffeomorphism f and every homoclinic class $H(p_f, f)$ of f there is the following dichotomy:*

- *either there is g arbitrarily C^1-close to f having a homoclinic tangency associated to the continuation of some saddle of $H(p_f, f)$;*
- *or there is a dominated splitting*

$$T_{H(p_f,f)}M = E^s \oplus E_1^c \oplus \cdots \oplus E_k^c \oplus E^u,$$

  where E^s is uniformly contracting, E^u is uniformly expanding, and every E_i^c is not hyperbolic and has dimension one.

Let us observe that the arguments in [BDPR] in fact prove this conjecture for isolated homoclinic classes. For non-isolated homoclinic classes one of the main difficulties comes from the fact that the homoclinic class is necessarily accumulated by periodic saddles which do not belong to the class. Moreover, the indices of some of these saddles may not belong to the interval of indices of the saddles of the class. Thus one of the difficulties for proving this conjecture is that the homoclinic tangency that one obtains may correspond to a saddle outside of the homoclinic class. This difficulty illustrates the importance of the periodic saddles in a neighborhood of the homoclinic class.

As a direct consequence of Theorem 1 and the result in [We$_2$] above we obtain a partial answer to this conjecture:

[2] This conjecture was proposed by the second author during the Second Latin American Conference of Mathematicians (Cancún, Mexico, 2004).

Corollary 3. *Let $\mathcal{I}$ be the residual subset of* $\mathrm{Diff}^1(\mathrm{M})$ *in Theorem 1 and f a diffeomorphism in $\mathcal{I}$ having a homoclinic class $H(p_f, f)$ which contains hyperbolic saddles of indices α and β. Then at least one of the two following possibilities holds:*

1. *For any neighborhood U of $H(p_f, f)$ and any C^1-neighborhood $\mathcal{U}$ of f there is a diffeomorphism $g \in \mathcal{U}$ with a homoclinic tangency associated to a periodic orbit contained in U.*
2. *There is a dominated splitting*

$$T_{H(p_f,f)}M = E \oplus F_1 \oplus \cdots \oplus F_{\beta-\alpha} \oplus G,$$

with $\dim(E) = \alpha$ *and* $\dim(F_i) = 1$ *for all i.*

A result recently announced by Gourmelon implies that the homoclinic tangency in the first item of the corollary is associated to a saddle of the homoclinic class $H(p_f, f)$ having an index in between α and β; this implies in particular that the two cases of the conjecture cannot occur simultaneously.

Let us state some questions related to Theorem 1 and to the conjecture, in the broader setting of *chain recurrence classes*, which we now briefly recall.

A point x is *chain recurrent* if for any $\varepsilon > 0$, there are periodic ε-pseudo-orbits containing x. The set of chain recurrent points is denoted by $\mathcal{R}(f)$. This set admits a natural partition into invariant compact sets called the *chain recurrence classes*: two points $x, y \in \mathcal{R}(f)$ are in the same class if, for any $\varepsilon > 0$, there are ε-pseudo-orbits going from x to y and from y to x. The Conley theory, [Co, Ro], associates a filtration to this partition. [BC] proved that, for C^1-generic diffeomorphisms, the chain recurrent set $\mathcal{R}(f)$ is the closure of the hyperbolic periodic orbits. Furthermore, for C^1-generic f, there are two types of chain recurrence classes: those containing periodic points (which are the homoclinic classes), and those without periodic orbits (called therefore *aperiodic classes*).

Moreover, [Cr] proved that any chain recurrence class of a generic diffeomorphisms is the Hausdorff limit of a sequence of periodic orbits. This result, however, does not yield any information on the indices of these periodic orbits. One may ask many questions regarding chain recurrence classes, indices of saddles, Hausdorff limits of periodic orbits, and the closure of periodic points of a prescribed index. Let us propose two such questions. First, we do not know whether chain recurrence classes Σ of generic diffeomorphisms are *index complete*, that is, whether if whenever Σ is the Hausdorff limit of periodic points of indices α and β, $\alpha < \beta$, then it is also the Hausdorff limit of periodic points of index τ for any $\tau \in [\alpha, \beta] \cap \mathbb{N}$.

Question 1. *Are all the chain recurrence classes of C^1-generic diffeomorphism index complete?*

The next question relates the local dynamics (the dynamics around an invariant compact set of the class) and the global dynamics in the whole class:

Question 2. *Let f be a C^1-generic diffeomorphism and Σ any chain recurrence class of f such that every neighborhood of the whole set Σ contains periodic orbits of index τ.*

Is Σ the Hausdorff limit of periodic orbits of index τ? Or a weaker question: is Σ contained in the closure of the saddles of index τ?

We observe that these two questions have positive answers in the case of isolated chain recurrence classes (which are also the isolated homoclinic classes). On the other hand, they are open for non-isolated homoclinic classes.

Let us explain briefly how the previous questions are related to the conjecture and why positive answers to both of them would imply a weaker version of the conjecture: in this weaker version, the first case involves the creation of some tangency associated to a saddle *near* – but not necessarily inside – the homoclinic class $H(p_f, f)$.

Let us assume that we are not in the first case: that is, that there are no homoclinic tangencies associated to saddles whose orbit is contained in a neighborhood of the class $H(p_g, g)$, for g close to f. We consider the set of indices of periodic points arbitrarily close to the class. By Question 2, the class is the Hausdorff limit of periodic orbits of these indices. By Question 1 this set of indices is an interval $\{\alpha, \alpha+1, \dots, \alpha+k\}$). By [We$_2$] this implies a dominated splitting of the form

$$T_{H(p_f,f)}M = E \oplus E_1^c \oplus \cdots \oplus E_k^c \oplus F,$$

where E has dimension α and every E_i^c is not hyperbolic and has dimension one. It now follows, essentially from Mañé's Ergodic Closing Lemma, [Ma], (see also [BDPR, Theorem B and Section 4] and [We$_3$]), that E is uniformly contracting (the argument for F is similar); the crucial assumption here is that it is forbidden to decrease the index of a periodic point of index α by perturbation. This means that we are in the second case of the conjecture, as claimed.

We finish this introduction discussing briefly how homoclinic classes may be accumulated by periodic orbits.

Non-isolated homoclinic classes of C^1-generic diffeomorphisms are accumulated by infinitely many disjoint homoclinic classes and are called *wild homoclinic classes*. There are no known examples of C^1-wild homoclinic classes for surface diffeomorphisms (there are, however, examples of C^2-wild classes associated to homoclinic tangencies, see [Ne$_2$, Ne$_3$]). Moreover, the only known examples of C^1-wild homoclinic classes occur in dimension equal to or higher than three. In such examples the classes are either (a) contained in the closure of an infinite set of sinks or sources, or else (b) obtained by considering the product of a three dimensional diffeomorphism f exhibiting a wild class accumulated by (say) sinks by a strong expansion. In case (b) one obtains a diffeomorphism F having a normally hyperbolic wild homoclinic class accumulated by saddles (in this case the dimension of the ambient manifold is at least 4), see [BD$_2$, BDP, CM]. But the existence of wild homoclinic classes accumulated only by infinitely many *true* saddles (i.e., not obtained via a product) remains an open problem. As the nature of this problem is three dimensional (and to avoid tricky solutions considering multiplications by a strong expansion/contraction), we formulate the following question:

Question 3. *Let M be a closed manifold of dimension 3. Do there exist locally generic diffeomorphisms $f \in \mathrm{Diff}^1(M)$ having wild homoclinic classes which are (a) not contained in a normally hyperbolic locally invariant submanifold and (b) disjoint from the closure of the set of sinks and sources?*

Finally, there is a special type of non-isolated homoclinic classes, which we call *pelliculaire classes*: those classes which are accumulated by *true* periodic saddles whose indices do not belong to the index interval corresponding to the class given by Theorem 1. We do not know whether these classes exist locally generically. The archetypal model of a pellicular class in dimension 3 is the following: suppose that the index interval of the class $H(p_f, f)$ is $\{1\}$ but that the class is accumulated simultaneously by saddles of index 2. In fact, this type of homoclinic classes needs to be considered when trying to solving the questions we posed above.

We close this introduction by explaining the organization of this paper and giving an outline of the proof of Theorem 1. This paper is organized as follows. In Section 2.1 we summarize the basic properties of homoclinic and chain recurrence classes of generic diffeomorphisms we use throughout the paper. In Section 2.2, we state a generic dichotomy result: generically homoclinic class either persistently coincide or are persistently disjoint. In Section 2.4, we explain how saddles of a homoclinic class having different indices can be related via a heterodimensional cycle. In Section 2.3, we state a key technical result (Proposition 2.3): *for homoclinic classes of generic diffeomorphisms, the saddles having positive real multipliers of multiplicity one form a dense subset of the class.* The importance of this proposition is the following: the saddles of intermediate indices are obtained through heterodimensional cycles associated to saddles in the homoclinic class. But the creation of such saddles is only well understood when the saddles in the cycle have real multipliers with multiplicity one. This is the reason we need this preparatory step. In fact, in Section 3.1, we show how heterodimensional cycles can be perturbed to create new cycles along which the dynamics is essentially affine. This allows us to analyze the dynamics in a heterodimensional cycle in a rather simple way. Using these affine heterodimensional cycles we obtain the saddles having intermediate indices in Section 3.2. In Sections 3.3 and 4, we see that these saddles of intermediate indices can be taken inside the original homoclinic class. This ends the proof of Theorem 1.

2 Homoclinic and chain recurrence classes of C^1-generic diffeomorphisms

In this section we collect some properties of homoclinic and chain recurrence classes of C^1-generic diffeomorphisms.

2.1 Summary of generic properties of C^1-diffeomorphisms

The results in [Ku, Sm_1, Pu, CMP] give a residual subset $\mathcal{G}$ of $\text{Diff}^1(\text{M})$ of diffeomorphisms f verifying the following properties:

(G1) f is Kupka-Smale (hyperbolicity of the periodic points and general position of the invariant manifolds);

(G2) the periodic points of f are dense in the non-wandering set of f;

(G3) for any pair of saddles p and q of f, either $H(p,f) = H(q,f)$ or $H(p,f) \cap H(q,f) = \emptyset$; and

(G4) for every saddle p of f, the homoclinic class $H(p_g, g)$ depends continuously on $g \in \mathcal{G}$, where p_g is the continuation of the saddle p of f for g close to f.

To state the generic conditions (G5) and (G6) we need the notion of *chain recurrence class.* A point y is *f-chain attainable* from the point x if for every $\varepsilon > 0$ there is an *ε-pseudo-orbit* going from x to y (i.e., there is a finite sequence $(x_i)_{i=0}^m$, $m \geq 1$, such that $x_0 = x$, $x_m = y$, and $\text{dist}(f(x_i), x_{i+1}) < \varepsilon$ for all $i = 0, \ldots, (m-1)$). The points x and y are *f-bi-chain attainable* if

x is chain attainable from y and vice-versa. An f-invariant set Λ is *chain recurrent* if every pair of points of Λ are bi-chain attainable. The bi-chain attainability relation defines an equivalence relation on the *chain recurrent set* $R(f)$ of f (i.e., the set of points x which are chain attainable from themselves). The *chain recurrence classes* are the equivalence classes of $R(f)$ for the bi-chain attainability relation. These sets are the maximal recurrent sets.

We also have the following two generic properties:

(G5) The chain recurrence classes of f form a partition of the chain recurrent set of f (i.e., they are pairwise disjoint and cover $R(f)$)[3].

(G6) Every chain recurrence class Λ containing a (hyperbolic) periodic point p satisfies $\Lambda = H(p, f)$, see [BC, Remarque 1.10].

2.2 Coincidence of homoclinic classes

It is well-known that any hyperbolic periodic point p_f of f has a unique *continuation*: there are a neighborhood U of the orbit of p_f in the ambient manifold M and a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(\mathrm{M})$ such that every g in $\mathcal{U}$ has a unique periodic point p_g close to p_f whose orbit is contained in U. The point p_g is called the *continuation of* p_f.

The next lemma is perhaps well-known in C^1-dynamics. We include its proof for completeness.

Lemma 2.1. *There is a residual subset $\mathcal{G}_0 \subset \mathcal{G}$ of* $\mathrm{Diff}^1(\mathrm{M})$ *such that, for every diffeomorphism $f \in \mathcal{G}_0$ and every pair of saddles p_f and q_f of f, there is a neighborhood $\mathcal{U}_f$ of f in $\mathcal{G}_0$ such that either $H(p_g, g) = H(q_g, g)$ for all $g \in \mathcal{U}_f$, or $H(p_g, g) \cap H(q_g, g) = \emptyset$ for all $g \in \mathcal{U}_f$.*

Proof: We first fix some integer $N \geq 1$. Given $f \in \mathcal{G}$ let $\mathrm{Per}_N(f)$ be the (finite) set of periodic points of f of period less than N. This set varies continuously in a C^1-neighborhood $\mathcal{U}_N(f)$ of f: let $\mathrm{Per}_N(f) = \{p_f^1, \ldots, p_f^{k(N)}\}$, then for every $g \in \mathcal{U}_N(f)$ one has $\mathrm{Per}_N(g) = \{p_g^1, \ldots, p_g^{k(N)}\}$ (each p_g^i is the continuation of p_f^i).

We now fix $f \in \mathcal{G}$ and its neighborhood $\mathcal{U}_N(f)$. For each $i, j \in \{1, \ldots, k(N)\}$ let

- $\mathcal{V}_{i,j} = \{g \in \mathcal{U}_N(f) \cap \mathcal{G} \colon H(p_g^i, g) \cap H(p_g^j, g) = \emptyset\}$;
- $\mathcal{B}_{i,j} = (\mathcal{U}_N(f) \setminus \overline{\mathcal{V}_{i,j}}) \cap \mathcal{G}$.

Claim 2.2. *The set $\mathcal{V}_{i,j}$ is open in $\mathcal{G}$.*

We postpone the proof of the claim and finish the proof of the lemma. First, note that for every $g \in \mathcal{B}_{i,j}$ the homoclinic classes $H(p_g^i, g)$ and $H(p_g^j, g)$ have non-empty intersection so that, by (G3), the two homoclinic classes coincide.

By Claim 2.2, the set $\mathcal{V}_{i,j} \cup \mathcal{B}_{i,j}$ is open and dense in $\mathcal{U}_N(f)$. Let $\mathcal{O}_N(f)$ be the finite intersection $\mathcal{O}_N(f) = \cap_{i,j}(\mathcal{V}_{i,j} \cup \mathcal{B}_{i,j})$. By construction, this set is open and dense in $\mathcal{U}_N(f)$.

Observe that for every diffeomorphism $g \in \mathcal{O}_N(f)$ we have the conclusions of the lemma for the saddles of period less than N of g. Let now $\mathcal{O}_N$ be the union of all the sets $\mathcal{O}_N(f)$, $f \in \mathcal{G}$. The set $\mathcal{O}_N$ is open and dense in $\mathcal{G}$. Finally, the set $\mathcal{G}_0$ is the intersection of the open and dense sets $\mathcal{O}_N$. By construction, this set is residual in $\mathrm{Diff}^1(\mathrm{M})$ and every $g \in \mathcal{G}_0$ satisfies the conclusions in the lemma.

[3] In fact, by the Conley Theory, see [Co, Ro], this property holds for all C^1 diffeomorphisms.

Proof of the claim: Let $g \in \mathcal{V}_{i,j}$. By (G6) the disjoint homoclinic classes $H(p_g^i, g)$ and $H(p_g^j, g)$ are also disjoint chain recurrence classes. By the Conley Theory, these sets are separated by a filtration (see [Co, Ro]): there is an open set U of M with $g(\overline{U}) \subset U$, such that one of the classes, say $H(p_g^i, g)$, is contained in U, and the other one is contained in $M \setminus \overline{U}$. Clearly, for every h close to g it holds $h(\overline{U}) \subset U$. Therefore U contains the stable manifold of the orbit of p_h^i. Similarly, $M \setminus \overline{U}$ contains the unstable manifold of the orbit of p_h^j. Thus, by definition of homoclinic class, $H(p_h^i, h) \subset U$ and $H(p_h^j, h) \subset (M \setminus \overline{U})$. Hence $H(p_h^i, h) \cap H(p_h^j, h) = \emptyset$. This ends the proof of the claim. □

The proof of the lemma is now complete. □

2.3 Saddles of homoclinic classes

Given a homoclinic class $H(p, f)$, denote by $\mathrm{Per}_h(H(p, f))$ the set of hyperbolic saddles q homoclinically related to p, and by $\mathrm{Per}_{\mathbb{R}}(H(p, f))$ the subset of $\mathrm{Per}_h(H(p, f))$ of points q such that all the eigenvalues of the derivative $Df^{\pi(q)}(q)$ are real, positive, and have multiplicity one; here $\pi(q)$ denotes the period of q.

Proposition 2.3. *There is a residual subset $\mathcal{G}_1$ of $\mathrm{Diff}^1(\mathrm{M})$ consisting of diffeomorphisms f such that $\mathrm{Per}_{\mathbb{R}}(H(p_f, f))$ is dense in $H(p_f, f)$ for every nontrivial homoclinic class $H(p_f, f)$ of f.*

Proof of the proposition: This proposition is just a translation of the results in [BDP] for *periodic linear systems (cocycles) with transitions* to the context of homoclinic classes. Recall that a periodic linear system is a 4-uple $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$, where f is a diffeomorphism, Σ is an infinite set of periodic points of f, $\mathcal{E}$ an Euclidean vector bundle defined over Σ, and $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ is such that $A(x)\colon \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a linear isomorphism for each x ($\mathcal{E}_x$ is the fiber of $\mathcal{E}$ at x). We refer to [BDP, Section 1] for the precise definition (we do not need it here). Naively speaking, such systems are cocycles where the extra structure of the transitions allows us to compose linear maps at different points. In a rough terms, this guarantees the existence of new periodic orbits visiting certain prescribed different periodic orbits of the system (i.e., there is a shadowing-like property in the bundle). Let us now explain how Proposition 2.3 follows:

Lemma 2.4. ([BDP, Lemma 1.9]) *Let $H(p_f, f)$ be a nontrivial homoclinic class. Then the derivative Df of f induces a periodic linear system with transitions over $\mathrm{Per}_h(H(p_f, f))$.*

We say that a periodic linear system with transitions $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$ is *diagonalizable* at the point $x \in \Sigma$ if the linear map

$$M_A(x)\colon \mathcal{E}_x \to \mathcal{E}_x, \quad M_A(x) = A(f^{\pi(x)-1}(x)) \circ \cdots \circ A(f^2(x)) \circ A(x),$$

only has positive real eigenvalues of multiplicity one.

Lemma 2.5. ([BDP, Lemma 4.16]) *For every periodic linear system with transitions $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$ and every $\varepsilon > 0$ there is a dense subset Σ' of Σ and an ε-perturbation A' of A defined on Σ' which is diagonalizable, that is, $M_{A'}(x)$ has positive real eigenvalues of multiplicity one for every $x \in \Sigma'$.*

The next result allows us to perform dynamically the perturbations of a cocycle:

Lemma 2.6. (Franks, [Fr]). *Consider a diffeomorphism f and an f-invariant finite set Σ. Let A be an ε-perturbation of the derivative of f in Σ (i.e., the linear maps $Df(x)$ and $A(x)$ are ε-close for all $x \in \Sigma$). Then, for every neighborhood U of Σ, there is g ε-C^1-close to f such that*

- $f(x) = g(x)$ *for every* $x \in \Sigma$ *and every* $x \notin U$,
- $Dg(x) = A(x)$ *for all* $x \in \Sigma$.

We are now ready to prove the proposition. By Lemma 2.4, the derivative of f induces a periodic linear system with transitions over $\Sigma = \mathrm{Per}_h(H(p_f, f))$. Applying Lemma 2.5 to such a system, we get that fixing any $\varepsilon > 0$ there is a dense subset Σ' of $\mathrm{Per}_h(H(p_f, f))$ such that for every $r_f \in \Sigma'$ there is an ε-perturbation A of Df throughout the orbit of r_f such that

$$M_A(r_f) = A(f^{\pi(r_f)-1}(r_f)) \circ A(f^{\pi(r_f)-2}(r)) \circ \cdots \circ A(f(r_f)) \circ A(r_f)$$

has positive real eigenvalues of multiplicity one. Applying Lemma 2.6 to the orbit of r_f and the perturbation A of Df, we get g close to f such that $r_f = r_g$ is a periodic point of g (of period $\pi(r_f)$) and $Dg^{\pi(r_f)}(r_g) = M_A(r_f)$. Thus, r_g is a periodic point of g having real positive eigenvalues of multiplicity one.

Note that, from the proof of [BDP, Lemma 4.16], the orbit of r_f can be taken arbitrarily close in the Hausdorff metric to the orbit of a transverse homoclinic point x of p_f. More precisely, let $\Lambda = \{\mathcal{O}(p_f) \cup \mathcal{O}(x)\}$, then the orbit of r_f can be taken arbitrarily close to Λ in the Hausdorff metric. Since Λ is a hyperbolic set of f, there is $\delta > 0$ such that for any r_f whose orbit $\mathcal{O}(r_f, f)$ is Hausdorff δ-close to Λ and any g that is C^1-ϵ close to f, r_g and p_g are homoclinically related. Thus $r_g \in H(p_g, g)$.

The proof of the proposition concludes as follows. For each n, take a finite covering $\mathbb{B}_n$ of $H(p_f, f)$ by open balls B_i of radius $1/n$ (each ball intersecting $H(p_f, f)$). Arguing as above, in each ball B_i we will obtain, after a small C^1-perturbation, $g \in \mathcal{G}$ with a saddle $r_g \in \mathrm{Per}_g(H(p_g, g))$ in B_i having only positive real eigenvalues of multiplicity one.

Considering perturbations at each ball B_i (there are finitely many) and recalling condition (G4) (in $\mathcal{G}$ the homoclinic classes depend continuously), we get $g \in \mathcal{G}$ close to f such that $\mathrm{Per}_{\mathbb{R}}(H(p_g, g))$ is $2/n$-dense in $H(p_g, g)$ (by continuity, $H(P_g, g)$ is contained in the union of the balls B_i). This also implies that $\mathrm{Per}_{\mathbb{R}}(H(p_h, h))$ is $2/n$-dense in $H(p_h, h)$ for all h close to g in $\mathcal{G}$. Thus there are a neighborhood $\mathcal{U}_f$ of f in $\mathcal{G}$ and an open and dense subset $\mathcal{D}_n$ of $\mathcal{U}_f$ of diffeomorphisms g such that $\mathrm{Per}_{\mathbb{R}}(H(p_g, g))$ is $2/n$-dense in $H(p_g, g)$. The proof of the proposition now follows using a genericity argument identical to the one in Lemma 2.1. The proof of the proposition is now complete. □

2.4 Creation of intersections between invariant manifolds

Let $\mathcal{G}_2 = \mathcal{G}_0 \cap \mathcal{G}_1$, where $\mathcal{G}_0$ and $\mathcal{G}_1$ are the residual subsets of $\mathrm{Diff}^1(\mathrm{M})$ in Lemma 2.1 and Proposition 2.3. Consider f in $\mathcal{G}_2$ having a homoclinic class $H(p_f, f)$ and a saddle q_f of different index from that of p_f with $q_f \in H(p_f, f)$. In particular, $H(p_f, f)$ is nontrivial. Without loss of generality we can assume that the indices of p_f and q_f are α and β, with $\alpha < \beta$. By Lemma 2.1, there is a neighborhood $\mathcal{U}_f$ of f in $\mathcal{G}_2$ such that $H(p_g, g) = H(q_g, g)$ for all $g \in \mathcal{U}_f$.

By hypotheses, recall Proposition 2.3, there is a saddle p_g^1 (resp. q_g^1) homoclinically related to p_g (resp. q_g) such that $Dg^{\pi(p_g^1)}(p_g^1)$ (resp. $Dg^{\pi(q_g^1)}(q_g^1)$) has positive real eigenvalues of multiplicity one. Observe that the indices of p_g^1 and q_g^1 are α and β.

We need the following lemma about the creation of cycles:

Lemma 2.7. (Hayashi's Connecting Lemma, [Ha]) *Let a_f and b_f be a pair of saddles of a diffeomorphism f such that there are sequences of points y_n and of natural numbers k_n such that*

- $y_n \to y \in W^u_{loc}(a_f, f)$, $y \neq a_f$, *and*
- $f^{k_n}(y_n) \to z \in W^s_{loc}(b_f, f)$, $z \neq b_f$.

Then there is a diffeomorphism g arbitrarily C^1-close to f such that $W^u(a_g, g)$ and $W^s(b_g, g)$ have an intersection arbitrarily close to y.

Lemma 2.8. *Consider a homoclinic class $H(a_f, f)$, any saddle $b_f \in H(a_f, f)$, and any transverse homoclinic point y of a_f. Then there is g arbitrarily C^1-close to f such that $W^u(a_g, g)$ and $W^s(b_g, g)$ have an intersection arbitrarily close to y.*

Proof: Recall that $H(a_f, f)$ is the ω-limit set of some $w \in H(a_f, f)$. Thus the forward orbit of w passes arbitrarily close to a_f, b_f and y and it accumulates to some $z \in W^s_{loc}(b_f)$. Hence there are sequences $f^{m_n}(w)$ converging to some $z \in W^s_{loc}(b_f)$ and $f^{r_n}(w) \to y$, where $m_n > r_n$ and $m_n \to \infty$. Taking $y_n = f^{r_n}(w)$ and $k_n = m_n - r_n$ we obtain the hypotheses of Lemma 2.7, which implies the result. □

By Lemma 2.8 (taking $a_f = p_f$, $b_f = q_f$ and any homoclinic point y of a_f), there is g arbitrarily C^1-close to f such that $W^s(q_g, g)$ and $W^u(p_g, g)$ have some intersection close to y. Observe that

$$\dim(W^s(q_g, g)) + \dim(W^u(p_g, g)) = \beta + (n - \alpha) > \beta + (n - \beta) = n.$$

Thus we can assume (after a perturbation) that the previous intersection between $W^u(p_g, g)$ and $W^s(q_g, g)$ is transverse. Since the property of having a transverse intersection is open, using a Baire argument similar to the one in the proof of Lemma 2.1, one immediately obtains:

Lemma 2.9. *There is a residual subset $\mathcal{G}_3$ of $\mathrm{Diff}^1(\mathrm{M})$ of diffeomorphisms f such that for every pair of saddles p_f and q_f of f of indices α and β, $\alpha < \beta$, with $H(p_f, f) = H(q_f, f)$ it holds that $W^u(p_f, f)$ and $W^s(q_f, f)$ have some transverse intersection.*

3 Heterodimensional cycles and creation of periodic orbits

Let us recall that, by convention, the *index* of a hyperbolic periodic point x is the dimension of its stable manifold. Given a pair of hyperbolic points p and q we write $p <_{\mathrm{us}} q$ if the unstable manifold $W^u(\mathcal{O}_p)$ of the orbit $\mathcal{O}_p$ of p intersects transversally the stable manifold $W^s(\mathcal{O}_q)$ of the orbit $\mathcal{O}_q$ of q: there exists a point $x \in W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ such that $T_xM = T_xW^u(\mathcal{O}_p) + T_yW^s(\mathcal{O}_q)$.

Remark 3.1. *The property $<_{\mathrm{us}}$ is open in $\mathrm{Diff}^1(\mathrm{M})$: let p_f and q_f be hyperbolic periodic points of a diffeomorphism f with $p_f <_{\mathrm{us}} q_f$, then there is a neighborhood $\mathcal{U}_f$ of f in $\mathrm{Diff}^1(\mathrm{M})$ such that $p_g <_{\mathrm{us}} q_g$ for every $g \in \mathcal{U}_f$ (here p_g and q_g are the continuations of p_f and q_f).*

Recall that a diffeomorphism f has a *heterodimensional cycle* associated to the saddles p and q if p and q have different indices and both intersections $W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$ and $W^s(\mathcal{O}_q) \cap W^u(\mathcal{O}_p)$ are non-empty.

We say that a periodic point p of period $\pi(p)$ of a diffeomorphism f has *real eigenvalues* if every eigenvalue of the linear isomorphism $Df^{\pi(p)}(p)\colon T_pM \to T_pM$ is real.

Theorem 3.2. *Let f be a diffeomorphism having a heterodimensional cycle associated to periodic saddles p_f and q_f, of indices α and β with $\alpha < \beta - 1$, with real eigenvalues. Then, for any C^1-neighborhood $\mathcal{U}$ of f and for any integer τ with $\alpha \le \tau \le \beta$, there exists $g \in \mathcal{U}$ having a periodic point r_g of index τ such that $p_g <_{\rm us} r_g <_{\rm us} q_g$ (p_g and q_g are the continuations of p_f and q_f).*

In fact, we will see that under some additional hypotheses (in our case, $H(p_f, f) = H(q_f, f)$), the saddle r_g can be taken such that $r_g \in H(p_g, q) = H(q_g, g)$, see Propositions 3.10 and 4.1.

The proof of Theorem 3.2 is the aim of this whole section.

3.1 Affine heterodimensional cycles

Definition 3.3 (Affine heterodimensional cycle)**.** *Let f be a diffeomorphism having a heterodimensional cycle associated to periodic points p, q of indices α and β, $\alpha < \beta$, and to heteroclinic points $x \in W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ and $y \in W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$. We say that the* heterodimensional cycle is affine *if all the following properties are satisfied:*

(A1) *The eigenvalues associated to the saddles p and q are all real and different in modulus and have multiplicity one: denote the eigenvalues of $Df^{\pi(p)}(p)$ and $Df^{\pi(q)}(q)$ by $\lambda_1, \dots, \lambda_n$ and $\sigma_1, \dots, \sigma_n$, respectively, so that we have $0 < |\lambda_1| < \cdots < |\lambda_\alpha| < 1 < |\lambda_{\alpha+1}| < \cdots < |\lambda_n|$ and $0 < |\sigma_1| < \cdots < |\sigma_\beta| < 1 < |\sigma_{\beta+1}| < \cdots < |\sigma_n|$.*

(A2) *There are local charts $\varphi_p \colon U_p \to \mathbb{R}^n$ and $\varphi_q \colon U_q \to \mathbb{R}^n$ centered at the points p and q such that the open sets $U_p, f(U_p), \dots, f^{\pi(p)-1}(U_p), U_q, f(U_q), \dots, f^{\pi(q)-1}(U_q)$ are pairwise disjoint. Moreover, these charts linearize the dynamics locally: in these local coordinates, the maps*

$$f^{\pi(p)} \colon U_p \cap f^{-\pi(p)}(U_p) \to U_p \quad \text{and} \quad f^{\pi(q)} \colon U_q \cap f^{-\pi(q)}(U_q) \to U_q$$

are the diagonal linear maps whose k^{th} diagonal entries are λ_k and σ_k, respectively. Furthermore, $\varphi_p\left(U_p \cap f^{-\pi(p)}(U_p)\right)$ and $\varphi_q\left(U_q \cap f^{-\pi(q)}(U_q)\right)$ contain the cube $[-2,2]^n$ of $\mathbb{R}^n$.

(A3) *The heteroclinic point $x \in W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ has two iterates $x^- = f^{n_x^-}(x) \in U_p$ and $x^+ = f^{n_x^+}(x) \in U_q$, where $n_x^- < n_x^+$, whose local coordinates are*

$$x^- = (\underbrace{0, \dots, 0}_{\alpha}, 1, \underbrace{0 \dots, 0}_{n-\alpha-1}), \quad \text{and} \quad x^+ = (\underbrace{0, \dots, 0}_{\beta-1}, 1, \underbrace{0 \dots, 0}_{n-\beta}).$$

(A4) *There is a neighborhood $U_x^- \subset U_p$ of x^- such that the neighborhood $U_x^+ = f^{n_x}(U_x^-)$ of x^+ is contained in U_q, where $n_x = n_x^+ - n_x^-$. Furthermore, in the correspondig local coordinates, U_x^- and U_x^+ are contained in the cube $[-2,2]^n$, the map $T_x = f^{n_x} \colon U_x^- \to U_x^+$ is affine and its linear part $\mathcal{T}_x$ is diagonal. We denote the diagonal entries of $\mathcal{T}_x$ by $(t_{x,1}, \dots, t_{x,n})$. Note that by definition of T_x one has*

$$T_x(\underbrace{0, \dots, 0}_{\alpha}, 1, \underbrace{0 \dots, 0}_{n-\alpha-1}) = (\underbrace{0, \dots, 0}_{\beta-1}, 1, \underbrace{0 \dots, 0}_{n-\beta}).$$

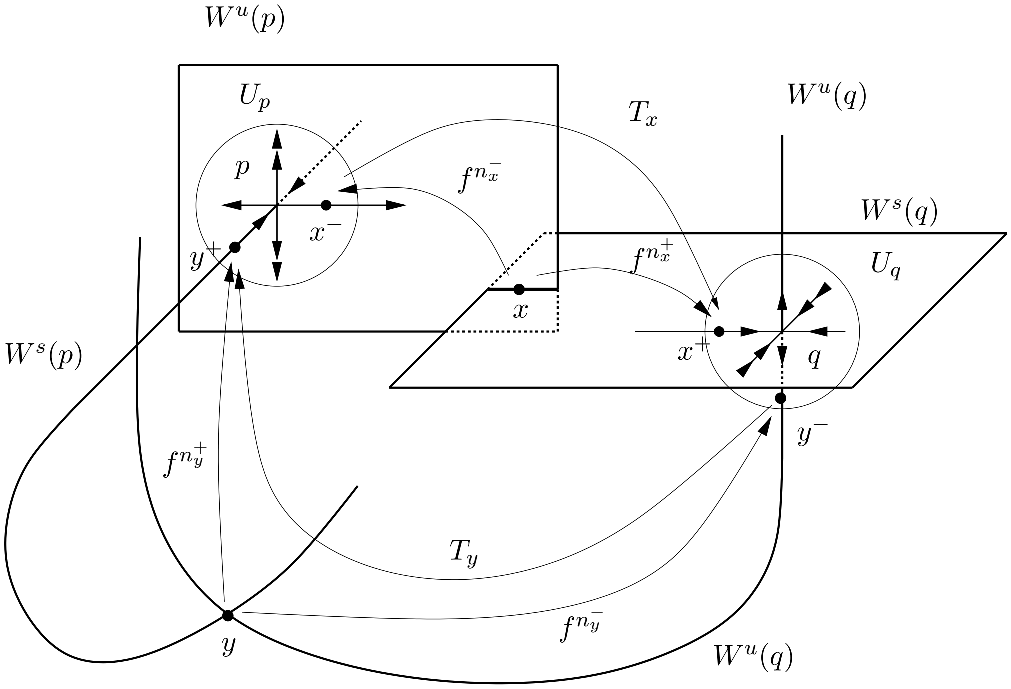

Figure 1: An affine heterodimensional cycle

(A5) *The heteroclinic point* $y \in W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$ *has two iterates* $y^- = f^{n_y^-}(y) \in U_q$ *and* $y^+ = f^{n_y^+}(y) \in U_p$, *where* $n_y^- < n_y^+$, *whose local coordinates are*

$$y^- = (\underbrace{0,\dots,0}_{\beta},1,\underbrace{0\dots,0}_{n-\beta-1}) \quad \text{and} \quad y^+ = (\underbrace{0,\dots,0}_{\alpha-1},1,\underbrace{0\dots,0}_{n-\alpha}).$$

(A6) *There is a neighborhood* $U_y^- \subset U_q$ *of* y^- *such that the neighborhood* $U_y^+ = f^{n_y}(U_y^-)$ *of* y^+ *is contained in* U_p, *where* $n_y = n_y^+ - n_y^-$. *Moreover, in the corresponding local coordinates,* U_y^- *and* U_y^+ *are contained in the cube* $[-2,2]^n$, *the map* $T_y = f^{n_y}\colon U_y^- \to U_y^+$ *is affine and its linear part* $\mathcal{T}_y$ *is diagonal. We denote the diagonal entries of* $\mathcal{T}_y$ *by* $(t_{y,1},\dots,t_{y,n})$. *Observe that by definition of* T_y,

$$T_y(\underbrace{0,\dots,0}_{\beta},1,\underbrace{0\dots,0}_{n-\beta-1}) = (\underbrace{0,\dots,0}_{\alpha-1},1,\underbrace{0\dots,0}_{n-\alpha}).$$

Lemma 3.4. *Let* f *be a diffeomorphism having a heterodimensional cycle associated to the saddles* p *and* q, *having real eigenvalues and indices* α *and* β, $\alpha < \beta$, *and to the heteroclinic points* $x \in W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ *and* $y \in W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$. *Then, for any* C^1*-neighborhood* $\mathcal{U}$ *of* f, *there is* $g \in \mathcal{U}$ *having an affine heterodimensional cycle associated to the saddles* p *and* q *and to the heteroclinic points* x *and* y.

Proof: The proof is essentially the same as that of [BDPR, Lemma 3.2], so we just sketch it. We first note that, after an arbitrarily small C^1-perturbation, we can assume that f is linearizable in some small neighborhoods of the saddles p and q, while keeping the heteroclinic points x and y. Moreover, we can also suppose that all the eigenvalues of $Df^{\pi(p)}(p)$ and of $Df^{\pi(q)}(q)$ have different moduli. In this way we obtain local coordinates verifying (A1) and (A2) above.

Next, by a local perturbation in the neighborhood of the heteroclinic points x and y, one makes the intersection $W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ at x transverse and the intersection $W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$ at y quasi-transverse (i.e., $T_y W^s(\mathcal{O}_p) \cap T_y W^u(\mathcal{O}_q) = \bar{0}$). Furthermore, one may prevent these intersections from belonging to the strong stable or strong unstable manifolds, so that the heteroclinic intersections belong to the weak stable and the weak unstable manifolds. Let us explain this point more precisely. In the linearizing local coordinates we define local weak unstable and stable manifolds by

$$W^{cu}_{loc}(\mathcal{O}_q) = \{(\underbrace{0,\dots,0}_{\beta}, s, \underbrace{0,\dots,0}_{n-\beta-1}), s \in [-2,2]\}, \quad W^{cs}_{loc}(\mathcal{O}_p) = \{(\underbrace{0,\dots,0}_{\alpha-1}, s, \underbrace{0,\dots,0}_{n-\alpha}), s \in [-2,2]\}.$$

Considering iterations by f we obtain a global invariant weak unstable manifold $W^{cu}(\mathcal{O}_q)$. Similarly, iterating by f^{-1} we obtain a weak stable manifold $W^{cs}(\mathcal{O}_p)$.

Analogously, we first define local strong unstable and stable manifolds by

$$W^{uu}_{loc}(\mathcal{O}_q) = \{(\underbrace{0,\dots,0}_{\beta+1})\} \times [-2,2]^{n-\beta-1}, \quad W^{ss}_{loc}(\mathcal{O}_p) = [-2,2]^{\alpha-1} \times \{(\underbrace{0,\dots,0}_{n-\alpha+1})\}.$$

We next extend these manifolds by iterations of $f^{\pm 1}$ to global strong unstable and strong stable manifolds $W^{uu}(\mathcal{O}_q)$ and $W^{ss}(\mathcal{O}_p)$.

After a new perturbation, we can assume that $y \notin W^{ss}(\mathcal{O}_p)$, therefore (using domination) its forward iterates *accumulate* to $W^{cs}(\mathcal{O}_p)$. Therefore, after a new arbitrarily small perturbation, we can assume that $y \in W^{cs}(\mathcal{O}_p)$. Using similar arguments, but now considering backward iterations, we can assume that (after an arbitrarily small perturbation) $y \in W^{cu}(\mathcal{O}_q)$. Hence $y \in W^{cs}(\mathcal{O}_p) \cap W^{cu}(\mathcal{O}_q)$. This implies (A5) above. To obtain (A3) one proceeds analogously.

A more subtle point consists in perturbing f in such a way that the differential of f along the heteroclinic orbits preserves the ordered eigenspaces. This was done in [BDPR, Lemma 3.2], so we will just sketch this point. One first shows (using domination) that (after an arbitrarily small perturbation) it is possible to preserve the one-dimensional bundles corresponding to the strong stable eigenvalues λ_1 and σ_1 (say $E_1(p)$ and $E_1(q)$) and the $(n-1)$-dimensional bundles corresponding to the remainder eigenvalues $\lambda_2, \dots, \lambda_n$ and $\sigma_2, \dots, \sigma_n$ (say $E_1^{n-1}(p)$ and $E_1^{n-1}(q)$). Now, a new perturbation, keeping invariant the previous bundles $E_1(p)$, $E_1(q)$, $E_1^{n-1}(p)$ and $E_1^{n-1}(q)$ but now focusing on $E_1^{n-1}(p)$ and $E_1^{n-1}(q)$, allows us to preserve the one-dimensional bundles corresponding to λ_2 and σ_2 (say $E_2(p)$ and $E_2(q)$) and the $(n-2)$-dimensional bundles corresponding to the remaining eigenvalues $\lambda_3, \dots, \lambda_n$ and $\sigma_3, \dots, \sigma_n$ (say $E_2^{n-2}(p)$ and $E_2^{n-2}(q)$). This step is identical to the first one and also involves domination. The proof now follows inductively.

In this way we obtain a diffeomorphism having a cycle verifying (A1)–(A6) above, ending the proof of the lemma. □

3.2 Creation of saddles of intermediate indices

Consider a diffeomorphism f with an affine heterodimensional cycle associated to the saddles p and q, of indices α and β, $\alpha < \beta$, and the heteroclinic points $x \in W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ and $y \in W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$. We use the notations introduced in Section 3.1.

Consider a sequence of points $(r_{\ell,m})_{\ell,m\in\mathbb{N}}$ having local coordinates in U_p as follows:

$$r_{\ell,m} = (\underbrace{\overbrace{0,\dots,0}^{\alpha-1},1,\lambda_{\alpha+1}^{-\ell},0,\dots,0}_{\beta},\lambda_{\beta+1}^{-\ell}\,t_{x,\beta+1}^{-1}\,\sigma_{\beta+1}^{-m},\underbrace{0,\dots,0}_{n-\beta-1}).$$

We next define the points

$$s_{\ell,m} = f^{\pi_{\ell,m}}(r_{\ell,m}), \quad \text{where} \quad \pi_{\ell,m} = \ell\,\pi(p) + n_x + m\,\pi(q) + n_y.$$

We perturb f in a small neighborhood D_y^+ of $y^+ = f^{n_y^+}(y)$ which is relatively compact in U_y^+. We can assume that D_y^+ is disjoint from the cube $f^{\pi(p)}([-1-\varepsilon,1+\varepsilon]^n) \subset U_p$, where $\varepsilon > 0$ is a small constant: this is possible because $f^{\pi(p)}([-1-\varepsilon,1+\varepsilon]^n)$ is a small neighborhood of the cube whose expression in the coordinates of U_p is $[-|\lambda_1|,|\lambda_1|]\times\dots\times[-|\lambda_n|,|\lambda_n|]$ and because $|\lambda_\alpha|<1$ and D_y^+ is a small neighborhood of the point $y^+ = (\underbrace{0,\dots,0}_{\alpha-1},1,\underbrace{0\dots,0}_{n-\alpha})$. Moreover, we can also choose D_y^+ disjoint from the orbits $(f^k(x^-))$, $k\in\{0,\dots,n_x-1\}$, and $f^k(y^-)$, $k\in\{0,\dots,n_y-1\}$ and from the open sets $f(U_p),\dots,f^{\pi(p)-1}(U_p)$, $U_q, f(U_q),\dots,f^{\pi(q)-1}(U_q)$.

Proposition 3.5. *For every ℓ and m large enough, the points $r_{\ell,m}$ and $s_{\ell,m}$ are well-defined, belong to U_p, and the sequences $(r_{\ell,m})$ and $(s_{\ell,m})$ converge to the point $y^+ = f^{n_y^+}(y)$ as $\ell,m\to\infty$. Moreover, the intermediate iterates $f^k(r_{\ell,m})$, $k\in\{1,\dots,\pi_{\ell,m}-1\}$, do not intersect the set D_y^+.*

Proof: As $|\lambda_{\alpha+1}|,|\lambda_{\beta+1}|$ and $|\sigma_{\beta+1}|$ are greater than 1, the sequence $(r_{\ell,m})$ converges to the point $y^+ = (\underbrace{0,\dots,0}_{\alpha-1},1,0\dots,0)$. In particular, the point $r_{\ell,m}$ is well-defined for ℓ,m large, and belongs to U_p. Now, for ℓ,m large, $f^{\ell\,\pi(p)}(r_{\ell,m})$ is the point in U_p whose coordinates are

$$f^{\ell\,\pi(p)}(r_{\ell,m}) = (\underbrace{\overbrace{0,\dots,0}^{\alpha-1},\lambda_\alpha^\ell,1,0,\dots,0}_{\beta},t_{x,\beta+1}^{-1}\,\sigma_{\beta+1}^{-m},\underbrace{0,\dots,0}_{n-\beta-1}).$$

The intermediate points $f^k(r_{\ell,m})$, $k\in\{1,\dots,\ell\,\pi(p)\}$, do not belong to U_p when k is different from 0 modulo $\pi(p)$ (just note that the sets $U_p,\dots,f^{\pi(p)-1}(U_p)$ are pairwise disjoint) and belong to $f^{\pi(p)}([-1-\varepsilon,1+\varepsilon]^n)$ when k is equal to 0 modulo $\pi(p)$. In all these cases, these iterates do not meet D_y^+.

Thus, for ℓ and m large enough, the point $f^{\ell\,\pi(p)}(r_{\ell,m})$ is very close to $x^- = (\underbrace{0,\dots,0}_{\alpha},1,0,\dots,0)$. Hence $T_x\left(f^{\ell\,\pi(p)}(r_{\ell,m})\right)$ is well-defined and close to $x^+ = (\underbrace{0,\dots,0}_{\beta-1},1,0,\dots,0)\in U_q$. Moreover, by the definition of T_x in (A4), in the local coordinates we have

$$T_x\left(f^{\ell\,\pi(p)}(r_{\ell,m})\right) = (\underbrace{\overbrace{0,\dots,0}^{\alpha-1},t_{x,\alpha}\,\lambda_\alpha^\ell,,0,\dots,0}_{\beta-1},1,\sigma_{\beta+1}^{-m},\underbrace{0,\dots,0}_{n-\beta-1}).$$

Moreover, the points $f^k\left(f^{\ell\,\pi(p)}(r_{\ell,m})\right)$, $k \in \{0,\dots,n_x\}$, are very close to the points $f^k(x^-)$ and are disjoint from D_y^+.

Repeating these arguments, one shows that, for ℓ, m large enough, $f^k\left(T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})\right)$, $k \in \{0,\dots,m\,\pi(q)\}$, is disjoint from D_y^+ and $f^{m\,\pi(q)}\left(T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})\right) \in U_q$ and has coordinates

$$f^{m\,\pi(q)}\left(T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})\right) = (\underbrace{\overbrace{0,\dots,0}^{\alpha-1}, \sigma_\alpha^m\, t_{x,\alpha}\, \lambda_\alpha^\ell, 0, \dots, 0}_{\beta-1}, \sigma_\beta^m, 1, \underbrace{0,\dots,0}_{n-\beta-1}).$$

As $|\sigma_\beta|, |\sigma_\alpha|, |\lambda_\alpha| < 1$, this point is arbitrarily close to $y^- = (\underbrace{0,\dots,0}_{\beta}, 1, 0, \dots, 0)$ if m is large enough. So, the points

$$s_{\ell,m} = T_y\left(f^{m\,\pi(q)} \circ T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})\right)$$

are well-defined and close to $y^+ \in U_p$. Finally, they belong to D_y^+ and their local coordinates are:

$$s_{\ell,m} = (\underbrace{\overbrace{0,\dots,0}^{\alpha-1}, 1 + t_{y,\alpha}\, \sigma_\alpha^m\, t_{x,\alpha}\, \lambda_\alpha^\ell, 0, \dots, 0}_{\beta-1}, t_{y,\beta}\, \sigma_\beta^m, \underbrace{0,\dots,0}_{n-\beta}).$$

Moreover, the intermediate points $f^k\left(f^{m\,\pi(q)} \circ T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})\right)$, $k \in \{0,\dots,n_y-1\}$, are close to the points $f^k(y^-)$ and do not meet D_y^+.

As $|\lambda_\alpha|, |\sigma_\alpha|, \sigma_\beta| < 1$, one easily checks that, when $m \to \infty$, the sequence $(s_{\ell,m})$ converges to $y^+ = (\underbrace{0,\dots,0}_{\alpha-1}, 1, 0, \dots, 0)$. This completes the proof of the proposition. □

For ℓ, m large enough, we use the coordinates of the chart φ_p to define the vector $\theta_{\ell,m} \in \mathbb{R}^n$ by $r_{\ell,m} = s_{\ell,m} + \theta_{\ell,m}$. We denote by $\Theta_{\ell,m}$ the local diffeomorphism defined on D_y^+ whose expression in the local coordinates of U_p is the translation $z \mapsto z + \theta_{\ell,m}$.

We fix a neighborhood V_y of y^+, relatively compact in the interior of D_y^+. The following lemma yields a sequence $(g_{\ell,m})$ of local C^1-perturbations of f, $g_{\ell,m} \to f$ as $\ell, m \to \infty$, each one closing the orbit of the corresponding point $r_{\ell,m}$.

Lemma 3.6. *For any C^1-neighborhood $\mathcal{U}$ of f and for every ℓ, m large enough, there is a diffeomorphism $h_{\ell,m}$ coinciding with $\Theta_{\ell,m}$ on V_y and with the identity map outside of D_y^+, such that the diffeomorphism $g_{\ell,m} = h_{\ell,m} \circ f$ belongs to $\mathcal{U}$.*

Proof: This comes from the fact that the vectors $\theta_{\ell,m}$ go to $\bar{0}$ as ℓ and m go to ∞. □

One now obtains the announced periodic points for $g_{\ell,m}$:

Proposition 3.7. *For every ℓ, m large enough, the $r_{\ell,m}$ is a periodic point of $g_{\ell,m}$ whose period is $\pi_{\ell,m}$. Furthermore, the derivative $Dg^{\pi_{\ell,m}}(r_{\ell,m})$ is*

$$Dg^{\pi_{\ell,m}}(r_{\ell,m}) = \mathcal{T}_y \circ \left(Df^{\pi(q)}(q)\right)^m \circ \mathcal{T}_x \circ \left(Df^{\pi(p)}(p)\right)^\ell.$$

In other words, $Dg^{\pi_{\ell,m}}(r_{\ell,m})$ is the diagonal linear map whose k^{th} diagonal entry is

$$t_{y,k}\, \sigma_k^m\, t_{x,k}\, \lambda_k^\ell.$$

Proof: For ℓ and m large enough, the points $r_{\ell,m}$ and $s_{\ell,m}$ belong to V_y so that $\Theta_{\ell,m}(s_{\ell,m}) = r_{\ell,m}$. As $r_{\ell,m}$ and $s_{\ell,m} = f^{\pi_{\ell,m}}(r_{\ell,m})$ are the only points of the segment of orbit $r_{\ell,m}, f(r_{\ell,m}), \dots, f^{\pi_{\ell,m}}(r_{\ell,m})$ in the support D_y^+ of the perturbation $h_{\ell,m}$, the point $r_{\ell,m}$ is $\pi_{\ell,m}$-periodic for $g_{\ell,m}$.

In the local coordinates, the derivative $Df^{\pi_{\ell,m}}(r_{\ell,m})$ is $\mathcal{T}_y \circ \left(Df^{\pi(q)}(q)\right)^m \circ \mathcal{T}_x \circ \left(Df^{\pi(p)}(p)\right)^\ell$. In the coordinates of $V_y \subset U_p$, the map $h_{\ell,m}$ is a translation, hence, one obtains the same expression for the derivative of $Dg^{\pi_{\ell,m}}(r_{\ell,m})$. □

By choosing carefully the integers ℓ and m, one obtains any index τ between α and β:

Corollary 3.8. *For any integer $\tau \in \{\alpha, \dots, \beta\}$, there exists a sequence (ℓ_k, m_k), with $\lim_{k\to+\infty} \ell_k = \lim_{k\to+\infty} m_k = +\infty$ such that, for every k, the point r_{ℓ_k,m_k} is a hyperbolic saddle of g_{ℓ_k,m_k} having index τ.*

Proof: Let us assume that τ belongs to $\{\alpha+1, \dots, \beta\}$, the case $\tau = \alpha$ follows similarly. As $|\sigma_\tau| < 1 < |\lambda_\tau|$, one can choose ℓ and m arbitrarily large such that the modulus of the τ^{th} eigenvalue of $r_{\ell,m}$, which is $|t_{y,\tau}\, \sigma_\tau^m\, t_{x,\tau}\, \lambda_\tau^\ell|$, belongs to $[|\sigma_\tau^2|, |\sigma_\tau|]$, in particular is less than one. On the other hand, the ratio between the moduli of the $(\tau+1)^{\text{th}}$ and $(\tau)^{\text{th}}$ eigenvalues of $r_{\ell,m}$ is

$$\frac{|t_{y,\tau+1}\, t_{x,\tau+1}|}{|t_{y,\tau}\, t_{y,\tau}|} \left(\frac{|\lambda_{\tau+1}|}{|\lambda_\tau|}\right)^m \left(\frac{|\sigma_{\tau+1}|}{|\sigma_\tau|}\right)^\ell.$$

Since $|\lambda_{\tau+1}| > |\lambda_\tau|$ and $|\sigma_{\tau+1}| > |\sigma_\tau|$, if ℓ and m are big enough, the ratio between the $(\tau+1)^{\text{th}}$ and the $(\tau)^{\text{th}}$ eigenvalues above is strictly bigger than $|\sigma_\tau^{-3}|$. This implies that the modulus of the $(\tau+1)^{\text{th}}$ eigenvalue of $r_{\ell,m}$ is greater that $|\sigma_\tau^{-1}| > 1$. This shows that the index of $r_{\ell,m}$ is exactly τ, ending the proof of the lemma. □

Remark 3.9. *The arguments in the proof of Corollary 3.8 imply the following: for every $\varepsilon > 0$ and every $\rho \in (0,1)$, there is a saddle $r = r_{\ell,m}$ of $g = g_{\ell,m}$ (for some appropriate large ℓ and m) whose Lyapunov exponents $(\log|\mu_1|/\pi(r)), \dots, (\log|\mu_n|/\pi(r))$ (here the μ_i are the eigenvalues of $Dg^{\pi(r)}(r)$) verify*

$$\left|\frac{\log|\mu_i|}{\pi(r)} - \left(\rho\,\frac{\log|\lambda_i|}{\pi(p)} + (1-\rho)\,\frac{\log|\sigma_i|}{\pi(q)}\right)\right| < \varepsilon.$$

This remark and Theorem 1 immediately imply Corollary 2.

3.3 Control of the heteroclinic intersections

In this section we will complete the proof of Theorem 3.2.

Note that the diffeomorphisms $g_{\ell,m}$ coincide with f in the neighborhood of the orbits of p and q, thus the orbits of p and q by f and $g_{\ell,m}$ coincide. Thus, according to Corollary 3.8, to prove Theorem 3.2 it is enough to see the following:

Proposition 3.10. *For ℓ and m large enough, the diffeomorphism $g_{\ell,m}$ satisfies*

$$p <_{\text{us}} r_{\ell,m} <_{\text{us}} q.$$

Proof: Given ℓ and m big enough, denote by $\Delta^s_{\ell,m}$ the α-dimensional disk contained in U_p defined in the corresponding local coordinates by

$$\Delta^s_{\ell,m} = [-1-\varepsilon, 1+\varepsilon]^\alpha \times (\lambda^{-\ell}_{\alpha+1}, \underbrace{0,\dots,0}_{\beta-\alpha-1}, \lambda^{-\ell}_{\beta+1}\, t^{-1}_{x,\beta+1}\, \sigma^{-m}_{\beta+1}, \underbrace{0,\dots,0}_{n-\beta-1}).$$

Similarly, consider the $(n-\beta)$-dimensional cube $\Delta^u_{\ell,m}$ defined in the local coordinates of U_q by

$$\Delta^u_{\ell,m} = (\underbrace{0,\dots,0}_{\alpha-1}, t_{x,\alpha}\, \lambda^\ell_\alpha\, \sigma^m_\alpha, \underbrace{0,\dots,0}_{\beta-\alpha-1}, \sigma^m_\beta) \times [-1-\varepsilon, 1+\varepsilon]^{n-\beta}.$$

Note that the disks $\Delta^s_{\ell,m}$ and $\Delta^u_{\ell,m}$ contain the points $r_{\ell,m}$ and $f^{m\,\pi(q)} \circ T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m}) = T_y^{-1}(r_{\ell,m})$, respectively.

Lemma 3.11. *For every ℓ, m large enough,*

- $\Delta^s_{\ell,m} \subset W^s(r_{\ell,m}, g_{\ell,m})$;
- $\Delta^u_{\ell,m} \subset W^u(T_y^{-1}(r_{\ell,m}))$.

Before proving this lemma let us complete the proof of the proposition assuming it. Observe that when ℓ and m go to $+\infty$, the disks $\Delta^s_{\ell,m}$ converge (in the C^1 topology) to the disk

$$\Delta^s_p = [-1-\varepsilon, 1+\varepsilon]^\alpha \times \{0\}^{n-\alpha},$$

which is a local stable manifold of p. This implies that $\Delta^s_{\ell,m}$ intersects transversally the unstable manifold of p when ℓ and m are large enough. Lemma 3.11 implies that the unstable manifold of p transversally intersects the stable one of $r_{\ell,m}$, thus $p <_{\text{us}} r_{\ell,m}$. In the same way, one shows that the disk $\Delta^u_{\ell,m}$, containing $f^{m\,\pi(q)} \circ T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m}) = T_y^{-1}(r_{\ell,m})$, intersects transversally the local stable manifold of q. Therefore $r_{\ell,m} <_{\text{us}} q$. Thus to complete the proof of the proposition it remains to prove Lemma 3.11.

Proof of the lemma: We just prove the first item of the lemma, the second one follows similarly by considering backward iterates of f.

The images of $\Delta^s_{\ell,m}$ by f^k, $k \in \{1, \dots, \ell\,\pi(p)\}$, are disjoint from D^+_y: when k is different from 0 modulo $\pi(p)$, the image is disjoint from U_p and, when k is equal to 0 modulo $\pi(p)$, the image is a disk contained in the cube $f^{\pi(p)}([-1-\varepsilon, 1+\varepsilon]^n) \subset U_p$. In particular, this shows that $g^{\ell\,\pi(p)}_{\ell,m}$ coincides with $f^{\ell\,\pi(p)}$ on $\Delta^s_{\ell,m}$ and that $g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m})$ is an α-dimensional disk contained in U^-_x of size *less* than $|(\lambda_\alpha)^\ell(1+\varepsilon)|$ containing $g^{\ell\,\pi(p)}_{\ell,m}(r_{\ell,m}) = f^{\ell\,\pi(p)}(r_{\ell,m})$. As the diameter of this disk tends to 0 when ℓ goes to $+\infty$, one has that $g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m}) \subset U^-_x$, so that T_x is defined on $g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m})$ and $T_x \circ g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m}) = g^{\ell\,\pi(p)+n_x}_{\ell,m}(\Delta^s_{\ell,m})$ is an arbitrarily small disk containing the point $T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})$ in U^+_x.

In the same way, one verifies that $g^{m\,\pi(q)}_{\ell,m} \circ T_x \circ g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m})$ is a very small disk that contains $f^{m\,\pi(q)} \circ T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})$ in U^-_y. So (for ℓ, m large enough) $g^{m\,\pi(q)}_{\ell,m} \circ T_x \circ g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m})$ is contained in the small neighborhood $T_y^{-1}(V_y)$ of y^-. As a consequence, $g^{n_y}_{\ell,m}$ coincides with $\Theta_{\ell,m} \circ T_y$ on $g^{m\,\pi(q)}_{\ell,m} \circ T_x \circ g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m})$.

The previous arguments show that $g_{\ell,m}^{\pi_{\ell,m}}(\Delta_{\ell,m}^s)$ is an α-dimensional disk containing $r_{\ell,m} = g_{\ell,m}^{\pi_{\ell,m}}(r_{\ell,m})$ and whose diameter tends to 0 as ℓ, m tend to $+\infty$. So for ℓ, m large enough, the map $g_{\ell,m}^{\pi_{\ell,m}}$ maps $\Delta_{\ell,m}^s$ into itself and is a linear contraction on this disk, having $r_{\ell,m}$ as a fixed point. This concludes the proof of the lemma. □

The proof of Proposition 3.10 is now complete. □

4 Heterodimensional cycles and periodic points. Proof of Theorem 1

In this section we finish the proof of Theorem 1, which follows immediately from the proposition below: just note that if r_f is a saddle of index τ then the saddles of index τ in $H(r_f, f)$ constitute a dense subset of $H(r_f, f)$.

Proposition 4.1. *There is a residual subset $\mathcal{G}_4$ of $\mathrm{Diff}^1(\mathrm{M})$ with the following property: for every $f \in \mathcal{G}_4$ and every pair of saddles p_f and q_f of f having indices α and β, $\alpha < \beta$, such that $H(p_f, f) = H(q_f, f)$, given any $\tau \in (\alpha, \beta) \cap \mathbb{N}$ there is a hyperbolic periodic saddle r_f of index τ such that*

$$H(p_f, f) = H(q_f, f) = H(r_f, f).$$

The proposition immediately follows from a standard Baire argument (analogous to the one in the proof of Lemma 2.1) and the lemma below.

Lemma 4.2. *Let $\mathcal{U}$ be an open subset of $\mathcal{G}_3$ (the residual subset of $\mathrm{Diff}^1(M)$ in Lemma 2.9) such that for every $f \in \mathcal{U}$ there are saddles p_f and q_f of indices α and β, $\alpha < \beta$, depending continuously on f such that $H(p_f, f) = H(q_f, f)$. Then for every $\tau \in (\alpha, \beta) \cap \mathbb{N}$ there is an open and dense subset $\mathcal{V}_\tau$ of $\mathcal{U}$ such that every $g \in \mathcal{V}_\tau$ has a saddle r_g of index τ with $H(p_g, g) = H(q_g, g) = H(r_g, g)$.*

Proof: Let $f \in \mathcal{U}$. By Proposition 2.3, we can assume (after replacing the initial saddles by saddles homoclinically related to them) that p_f and q_f both have real positive eigenvalues of multiplicity one (i.e., $p_f, q_f \in \mathrm{Per}_{\mathbb{R}}(H(p_f, f))$). As $f \in \mathcal{G}_3$, one has that $W^u(p_f, f)$ and $W^s(q_f, f)$ have some non-empty transverse intersection. This property holds for every g close to f. Using Lemma 2.8, we get h close to f having a heterodimensional cycle associated to p_h and q_h.

Fix now $\tau \in (\alpha, \beta) \cap \mathbb{N}$. Using Theorem 3.2 we obtain some g close to h, thus close to f, having a saddle r_g of index τ with $p_g <_{\mathrm{us}} r_g <_{\mathrm{us}} q_g$. By Remark 3.1 this relation persists under C^1-perturbations. This gives an open and dense subset $\mathcal{V}_\tau$ of $\mathcal{U}$ such that every $g \in \mathcal{V}_\tau$ has a saddle r_g of index τ with $p_g <_{\mathrm{us}} r_g <_{\mathrm{us}} q_g$.

Claim 4.3. *Let $g \in \mathcal{V}_\tau$. For every $\varepsilon > 0$ there is a closed ε-pseudo-orbit containing p_g, r_g and q_g.*

This claim implies that the saddles p_g, q_g and r_g are in the same chain recurrence class Λ of g. By (G6),

$$\Lambda = H(p_g, g), \quad \Lambda = H(r_g, g), \quad \Lambda = H(q_g, g).$$

Thus the three homoclinic classes coincide. To end the proof of the lemma it remains only to prove the claim.

Proof of the claim: Fix any $\varepsilon > 0$. Since $H(p_g, g) = H(q_g, g)$ is a transitive set, there is a (finite) segment $S_{q,p}$ of ε-pseudo-orbit going from q_g to p_g. On the other hand, since $W^u(p_g, g)$ intersects $W^s(r_g, g)$ there is an orbit going from p_g to r_g. Thus there is is a (finite) segment $S_{p,r}$ of ε-pseudo-orbit going from p_g to r_g. Similarly, as $W^u(r_g, g)$ intersects $W^s(q_g, g)$ there is a (finite) segment $S_{r,q}$ of ε-pseudo-orbit going from r_g to q_g. The announced pseudo-orbit is obtained concatenating the segments of ε-pseudo-orbits $S_{q,p}$, $S_{p,r}$ and $S_{r,q}$. □

The proof of Lemma 4.2 is now complete. □

References

[Ab] F. Abdenur, *Generic robustness of spectral decompositions,* Ann. Scient. Éc. Norm. Sup., **36**, 212-224, (2003).

[Ar] M.-C. Arnaud, *Creation de connexions en topologie C^1,* Ergod. Th. and Dynam. Syst., **21**, 339-381, (2001).

[BC] Ch. Bonatti and S. Crovisier, *Récurrence et généricité,* Inventiones Math., **158**, 33-104, (2004).

[BD_1] Ch. Bonatti and L.J. Díaz, *Persistence of transitive diffeomorphims,* Ann. Math., **143**, 367-396, (1995).

[BD_2] Ch. Bonatti and L.J. Díaz, *Connexions hétéroclines et généricité d'une infinité de puits ou de sources,* Ann. Scient. Éc. Norm. Sup., **32**, 135-150, (1999).

[BD_3] Ch. Bonatti and L.J. Díaz, *On maximal transitive sets of generic diffeomorphis,* Publ. Math. Inst. Hautes Études Sci., **96**, 171-197, (2002).

[BDP] Ch. Bonatti, L.J. Díaz, and E.R. Pujals, *A $\mathcal{C}^1$-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicicity or infinitely many sinks or sources,* Ann. of Math., **158**, 355-418, (2003).

[BDPR] Ch. Bonatti, L.J. Díaz, E.R. Pujals, and J. Rocha, *Robustly transitive sets and heterodimensional cycles,* Astérisque, **286**, 187-222, (2003).

[BDV] Ch. Bonatti, L.J. Díaz, and M. Viana, *Dynamics beyond uniform hyperbolicity,* Encyclopaedia of Mathematical Sciences (Mathematical Physics), **102**, Springer Verlag, (2004).

[BV] Ch. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting,* Israel J. Math., **115**, 157-193, (2000).

[CM] C. Carballo and C. Morales, *Homoclinic classes and finitude of attractors for vectors fields on n manifolds,* Bull. London Math. Soc., **35**, 85-91, (2003).

[CMP] C. Carballo, C. Morales, and M.J. Pacífico, *Homoclinic classes for $\mathcal{C}^1$-generic vector fields,* Ergodic Th. and Dynan. Syst., **23**, 403-415, (2003).

[Co] C. Conley, *Isolated invariant sets and the Morse index,* CBMS Regional Conference Series in Mathematics, **38**, AMS, Providence, R.I., (1978).

[Cr] S. Crovisier, *Periodic orbits and chain transitive sets of C^1-diffeomorphisms,* preprint num. 368, Institut de Mathématiques de Bourgogne (France) (2004).

[Dí$_1$] L.J. Díaz, *Robust nonhyperbolic dynamics and heterodimensional cycles,* Ergodic. Th. and Dynam. Sys., **15**, 291-315, (1995).

[Dí$_2$] L.J. Díaz, *Persistence of cycles and non-hyperbolic dynamics at the unfolding of heterodimensional cycles,* Nonlinearity, **8**, 693-715, (1995).

[DR] L.J. Díaz and J. Rocha, *Partially hyperbolic and transitive dynamics generated by heteroclinic cycles,* Ergodic Th. and Dynam. Syst., **25**, 25-76, (2001).

[DPU] L.J. Díaz, E.R. Pujals, and R. Ures, *Partial hyperbolicity and robust transitivity,* Acta Mathematica, **183**, 1-43, (1999).

[Fr] J. Franks, *Necessary conditions for stability of diffeomorphisms,* Trans. A.M.S., **158**, 301-308, (1971).

[GW] S. Gan and L. Wen, *Heteroclinic cycles and homoclinic closures for generic diffeomorphisms,* J. Dynam. Differential Equations, **15**, 451–471, (2003).

[Ha] S. Hayashi, *Connecting invariant manifolds and the solution of the C^1-stability and Ω-stability conjectures for flows,* Ann. of Math., **145**, 81-137, (1997).

[Ku] I. Kupka, *Contributions à la théorie des champs génériques,* Contributions to Differential Equations, **2**, 457-484, (1963).

[Ma] R. Mañé, *An ergodic closing lemma,* Ann. of Math., **116**, 503-540, (1982).

[Ne$_1$] S. Newhouse, *Hyperbolic Limit Sets,* Trans. Amer. Math. Soc., **167**, 125-150, (1972).

[Ne$_2$] S. Newhouse, *Diffeomorphisms with infinitely many sinks,* Topology, **13**, 9-18, (1974).

[Ne$_3$] S. Newhouse, *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms,* Publ. Math. I. H.E.S, **50**, 101-151, (1979).

[Ne$_4$] S. Newhouse, *Lectures on dynamical systems, CIME Lectures, Bressanone, Italy, June 1978,* Progress in Mathematics, **81**, Birkhauser, 1-114, (1980).

[Ro] C. Robinson, *Dynamical systems. Stability, symbolic dynamics, and chaos,* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, (1995).

[Pu] C. Pugh, *The closing lemma,* Amer. Jour. of Math., **89**, 956-1009, (1967).

[Sm$_1$] S. Smale, *Stable manifolds for differential equations and diffeomorphisms,* Ann. Scuola Norm. Sup. Pisa, **17**, 97-116, (1963).

[Sm] S. Smale, *Differentiable dynamical systems,* Bull. A.M.S., **73**, 747-817, (1967).

[We$_1$] L. Wen, *On the preperiodic sets,* Disc. and Cont. Dyn. Syst., **1**, 237-241, (2000).

[We$_2$] L. Wen, *Homoclinic tangencies and dominated splittings,* Nonlinearity, **15**, 1445-1469, (2002).

[We$_3$] L. Wen, *Generic diffeomorphisms away from homoclinic tangencies and heterodimensional cycles,* Bull. Braz. Math. Soc. **35**(3), 419–452, (2004).

Flavio Abdenur (flavio@impa.br)
IMPA
Estrada Dona Castorina 110
22460-320 Rio de Janeiro RJ
Brazil

Christian Bonatti (bonatti@u-bourgogne.fr)
Institut de Mathématiques de Bourgogne
B.P. 47 870
21078 Dijon Cedex
France

Sylvain Crovisier (crovisie@math.univ-paris13.fr)
CNRS - LAGA, UMR 7539, Université Paris 13
Av J.-B. Clément
93430 Villetaneuse
France.

Lorenzo J. Díaz (lodiaz@mat.puc-rio.br)
Depto. Matemática, PUC-Rio
Marquês de S. Vicente 225
22453-900 Rio de Janeiro RJ
Brazil

Lan Wen (lwen@math.pku.edu.cn)
School of Mathematics
Peking University
Beijing 100871
People's Republic of China