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## A SIMPLE SEMI-INTRUSIVE METHOD FOR UNCERTAINTY QUANTIFICATION OF SHOCKED FLOWS, COMPARISON WITH A NON-INTRUSIVE POLYNOMIAL CHAOS METHOD

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**Abstract.** *The purpose of this paper is to provide a description and a comparison of a method already described in [?] with more standard UQ tools.*

### 1 INTRODUCTION

This paper is devoted to the comparison of two methods of uncertainty quantification (UQ) in the context of compressible inviscid flows. More specifically, we aim at comparing a well documented non-intrusive Polynomial Chaos (PC) method with a newly developed approach [?] offering the potential advantage of being less dependent on the probability distribution functions (pdf) of the random variables of the problem. This new approach is said semi-intrusive (SI) because it requires only a limited amount of modification in a deterministic flow solver to quantify the uncertainty on the flow state when the flow solver includes uncertain variables. The SI and PC methods will be first compared by computing the compressible flow of perfect gas in a nozzle, using the quasi-1D Euler model, with uncertainty on the heat coefficient ratio  $\gamma$ . This test-problem has been retained because it encompasses the essential features of compressible flows with a reduced computational cost allowing an extensive comparison of the UQ strategies, including convergence studies on the polynomial order for the PC approach and on the refinement level of the stochastic space discretization for the newly proposed approach. To allow further comparison with existing PC techniques described in [?], we also compute shock-tube problems governed by the 1D Euler equations where the uncertainty is injected in the initial conditions, namely on the location of the interface between the light and heavy fluid initially at rest. The paper is organized as follows : Section 2 describes in some details the method recently proposed in [?]; Section 3 reviews the non-intrusive Polynomial Chaos approach also employed in this study as a reference method; Section 4 describes the test-cases

retained for assessing the performances of both UQ approaches and analyzes the results with emphasis on the choice of the pdf for the uncertain variables. Conclusions are drawn and work perspectives given in a concluding section.

## 2 A FLEXIBLE SEMI NON INTRUSIVE METHOD

### 2.1 A general strategy

Let us start from a PDE of the type

$$\mathcal{L}(u, \omega) = 0 \tag{1}$$

defined in a domain  $K$  of  $\mathbb{R}^d$ , subjected to boundary conditions. Since the discussion of this section is formal, we put the different boundary conditions of the problem in the symbol  $\mathcal{L}$ . The operator  $\mathcal{L}$  may depend on some random variables  $X$  depending themselves on a random parameter  $\omega$ , *i.e.* an element of a set  $\Omega$  equipped with a probability measure  $d\mu$ . We assume, using abuse of notations, that the measure is such that  $\Omega$  can be identified to a subset of  $\mathbb{R}^d$ , still denoted by  $\Omega$ , for some  $d$ , and the expectancy of any  $d\mu$  measurable function  $\Psi(X)$  is

$$\mathcal{E}(\Psi(X)) = \int_{\Omega} \Psi(x) d\mu(x)$$

where  $d\mu$  may or may not have a density. In (1), the uncertainty is weakly coupled with the PDE, *i.e.* the random variable  $X$  does not depend on any space variables. However, the measure  $d\mu$  may depend on some space variable. The examples we have in mind are such that for a given realisation  $\omega_0 \in \Omega$ ,  $\mathcal{L}(u, X(\omega_0)) = 0$  is a “standard” PDE, such as the Laplace equation, Burgers equation, the Navier Stokes equations, etc. To make things even clearer let us consider the quasi-1D Euler equations modelling the compressible flows in nozzle geometries :

$$\left\{ \begin{array}{l} \frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x} = S(x) \quad t > 0, x \in [0, L] \\ \text{initial and boundary conditions} \end{array} \right. \tag{2}$$

where  $U = (\rho A, \rho u A, EA)^T$  with  $\rho$  the density,  $u$  the velocity,  $E = \rho \varepsilon + \frac{\rho}{2} u^2$  the total energy with  $\varepsilon$  the specific internal energy,  $A(x)$  is the known nozzle section area distribution. The physical flux reads  $f(U) = (\rho u A, \rho u^2 A + p A, H u A)^T$  with  $p$  the pressure and  $H = E + p$  the total enthalpy. The source term is given by  $S(x) = (0, p dA/dx, 0)$ . System (2) is closed by the equation of state  $p = p(\rho, \varepsilon)$ ; for the perfect gas flows considered in this study,  $p = (\gamma - 1)\rho \varepsilon$  with  $\gamma$  the ratio of specific heats. The test-cases considered in this work will solve (2) either with  $dA/dx \neq 0$  and  $\gamma$  considered as a random variable (stochastic nozzle flow) or with  $A = cst$ ,  $\gamma$  fixed to 1.4 but uncertain initial conditions (stochastic shock-tube problem taken from [?]).

We consider a spatial discretisation for (2) with node points  $x_i = i\Delta x$  where  $i$  belongs to some subset of  $\mathbb{Z}$ , a time step  $\Delta t > 0$  and set  $t_n = n\Delta t$ ,  $n \in \mathbb{N}$ . The control volumes are as usual the intervals  $\mathcal{C}_i = [x_{i-1/2}, x_{i+1/2}]$  with  $x_{i+1/2} = \frac{x_i + x_{i+1}}{2}$ . We start from a finite volume scheme, and for the simplicity of exposure, we only consider a first order in time and space scheme. The generalisation to more accurate scheme is obvious. Thus we define the *deterministic scheme* as

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} \left( F(U_{i+1}^n, U_i^n) - F(U_i^n, U_{i-1}^n) \right) \quad (3)$$

with  $U_i^0$  being an approximation of  $\int_{\mathcal{C}_i} U_0(x) dx / \Delta x$  and  $F$  a consistent approximation of the continuous flux  $f$ . In all what follows,  $F$  is the Roe flux.

When  $U_0$  or the flux  $f$  depends on a random variable  $X$ , we propose the following modifications. First the set  $\Omega \subset \mathbb{R}^d$  is subdivided into non overlapping subset  $\Omega_j$ ,  $j = 1, \dots, n_p$  and the variables are represented by their conditional expectancies in the  $\Omega_j$  subsets. More precisely, our set of variables is

$$U_{i,j}^n \approx \frac{\mathcal{E}(\int_{x_{i-1/2}}^{x_{i+1/2}} U(x, t_n, X(\omega)) | \omega \in \Omega_j)}{\Delta x P(\Omega_j)}.$$

which evolves as

$$U_{i,j}^{n+1} = U_{i,j}^n - \frac{\Delta x}{\Delta t} \left( \mathcal{E}(F(U_{i+1}^n, U_i^n) | \Omega_j) - \mathcal{E}(F(U_i^n, U_{i-1}^n) | \Omega_j) \right).$$

The scheme is fully defined provided the “flux”  $\mathcal{E}(F(U_{l+1}^n, U_l^n) | \Omega_k)$  can be evaluated for any  $l$  and  $k$ . This will be the topic of a further section. The choice of the time step also has to be discussed.

## 2.2 Case of a Nozzle flow

In the case of a nozzle flow, the problem is steady, so that an implicit scheme is used to achieve faster convergence to a steady-state. The *deterministic* scheme is a linearized implicit one : we look for the solution on the cell  $\mathcal{C}_i$  as the limit of a sequence  $U_i^n$  when  $n \rightarrow +\infty$ . In the example we consider, the iteration for cell  $\mathcal{C}_i$  is defined by

$$\frac{\Delta x}{\Delta t} \delta U_i^n - A_{i-1/2}^- \delta U_{i-1}^n - A_i \delta U_i - A_{i+1/2}^+ \delta U_{i+1}^n = F(U_{i+1}^n, U_i^n) - F(U_i^n, U_{i-1}^n) - S(U_i)^n. \quad (4)$$

where  $\delta U_i^n = U_i^{n+1} - U_i^n$ ,  $F$  is the Roe flux,  $A_{i+1/2}$  is the Roe matrix between states  $U_i$  and  $U_{i+1}$  at  $t_n$ , and  $A^+$  (resp.  $A^-$ ) is the positive (resp. negative) part of  $A$ . For the sake of simplicity, the boundary conditions are omitted in the description. In the test case we have considered (shocked nozzle), the inlet and outlet boundary conditions are subsonic and approximated via a characteristic approach. System (4) is solved by Gauss

Seidel iteration with a CFL number (hence time-step  $\Delta t$ ) increased after each iteration. To summarize, the state vector  $U^{n+1} = (U_1^{n+1}, \dots, U_i^{n+1}, \dots, U_{imax}^{n+1})^T$  is obtained by

$$U^{n+1} = \mathcal{G}(U^n)$$

where the operator  $\mathcal{G}$  gathers the implicit phase, the Gauss-Seidel iteration, the flux formulation and the boundary conditions.

If some parameter is random, the iteration writes

$$U^{n+1} = \mathcal{G}(U^n, X(\omega))$$

and we can apply the same procedure as before, *i.e.* evolve the conditional expectancies

$$U_{i,j}^n \approx \frac{\mathcal{E}(\int_{x_{i-1/2}}^{x_{i+1/2}} U(x, t_n, X(\omega)) | \omega \in \Omega_j)}{\Delta x P(\Omega_j)}.$$

via the “scheme”

$$U_{i,j}^{n+1} = \mathcal{E}(\mathcal{G}(U^n, X(\omega)),$$

We arrive exactly at the same problem, namely, given a set of constant values  $\{U_{j=0}, \dots, U_{j=n_p}\}$  that are interpreted as the conditional expectancies of a function  $U$  in  $\Omega_j$ ,

$$U_j = \frac{\mathcal{E}(U | \Omega_j)}{P(\Omega_j)}$$

and a measurable function  $f$ , find an accurate approximation of  $E(f(U) | \Omega_j)$  for any  $j$ . A solution to this problem is described in the next section.

### 2.3 Some computational remarks

Let  $d\mu$  be a probability measure and  $X$  be a random variable defined on the probability space  $(\Omega, d\mu)$ . Assume we have a decomposition of  $\Omega$  by non overlapping subsets  $\Omega_i$ ,  $i = 1, N$  of strictly positive measure:

$$\Omega = \cup_{i=1}^N \Omega_i.$$

We are given the conditional expectancies  $E(X | \Omega_i)$ . Can we estimate for a given  $f$ ,  $E(f(X))$ ? Assuming  $X = (X_1, \dots, X_n)$ , the idea is the following : for each  $\Omega_i$ , we wish to evaluate a polynomial  $P_i \in \mathbb{R}^n[x_1, \dots, x_n]$  of degree  $n$  such that

$$E(X | \Omega_j) = \frac{\int_{\mathbb{R}^n} 1_{\Omega_j}(x_1, \dots, x_n) P(x_1, \dots, x_n) d\tilde{\mu}}{\mu(\Omega_j)} \quad \text{for } j \in \mathcal{S}_i \quad (5)$$

where  $d\tilde{\mu}$  is the image of  $d\mu$  and  $\mathcal{S}_i$  is a stencil associated to  $\Omega_i$ .<sup>1</sup>

<sup>1</sup>For example,  $d\mu$  is the sum of a Gaussian and a Dirac at  $x_0$ ,

$$\int_{\mathbb{R}^n} P(x) d\tilde{\mu} = \alpha \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} P(x) e^{-\frac{(x-m)^2}{2\sigma}} dx + (1 - \alpha) P(x_0)$$

This problem is reminiscent of what is done in finite volume schemes to compute a polynomial reconstruction in order to increase the accuracy of the flux evaluation thanks to the MUSCL extrapolation. Among the many references that have dealt with this problem, with the Lebesgue measure  $dx_1 \dots dx_n$ , one may quote [?] and for general meshes [?, ?]. A systematic method for computing the solution of problem (5) is given in [?].

Assume that the stencil  $\mathcal{S}_i$  is defined, the technical condition that ensures a unique solution to problem (5) is that the Vandermonde-like determinant (given here for one random variable for the sake of simplicity)

$$\Delta_i = \det \left( E(x^l | \Omega_j) \right)_{0 \leq l \leq n, j \in \mathcal{S}_i}. \quad (6)$$

is non zero. In the case of several random variable, the exponent  $l$  in (6) is replaced by a multi-index.

Once the solution of (5) is known, we can estimate

$$E(f(X)) \approx \sum_{j=1}^N \int_{\mathbb{R}^n} 1_{\Omega_j}(x_1, \dots, x_n) f\left(P(x_1, \dots, x_n)\right) d\tilde{\mu}.$$

The following approximation results hold : if  $f \in C^p(\mathbb{R}^n)$  with  $p \geq n$  then

$$\left| E(f(X)) - \sum_{j=1}^N \int_{\mathbb{R}^n} 1_{\Omega_j}(x_1, \dots, x_n) f\left(P(x_1, \dots, x_n)\right) d\tilde{\mu} \right| \leq C(\mathcal{S}) \max_j [\mu(\Omega_j)^{\frac{p+1}{p}}]$$

for a set of regular stencil; the proof is a straightforward generalization of the approximation results contained in [?].

In all the following practical illustrations, we will use only one or two sources of uncertainty even though the method can be used for any number of uncertain parameters - a large number of uncertain parameters leads however to other known problems such as the so-called curse of dimensionality. The space  $\Omega$  is subdivided into non overlapping measurable subsets. In the case of one source of uncertainty, the subsets can be identified, via the measure  $d\mu$ , to  $N$  intervals of  $\mathbb{R}$  which are denoted by  $[\omega_j, \omega_{j+1}]$ . The case of multiple sources can be considered by tensorisation of the probabilistic mesh. This formalism enables to consider correlated random variables.

Let us describe in details what is done for a single source of uncertainties. In the cell  $[\omega_j, \omega_{j+1}]$ , the polynomial  $P_{i+1/2}$  is fully described by a stencil  $\mathcal{S}_{i+1/2} = \{i + 1/2, i_1 + 1/2, \dots\}$  such that in the cell  $[\omega_j, \omega_{j+1}]$  with  $j + 1/2 \in \mathcal{S}_{i+1/2}$  we have

$$E(P_{i+1/2} | [\omega_j, \omega_{j+1}]) = E(u | [\omega_j, \omega_{j+1}]).$$

It is easy to see that there is a unique solution to that problem provided that the elements of  $\{[\omega_j, \omega_{j+1}]\}_{j+1/2 \in \mathcal{S}_{i+1/2}}$  do not overlap, which is the case. In the numerical examples, we consider three reconstruction mechanisms :

- a first order reconstruction: we simply take  $\mathcal{S}_{i+1/2} = \{i+1/2\}$  and the reconstruction is piecewise constant,
- a centered reconstruction: the stencil is  $\mathcal{S}_{i+1/2} = \{i-1/2, i+1/2, i+3/2\}$  and the reconstruction is piecewise quadratic. At the boundary of  $\Omega$ , we use the reduced stencils  $\mathcal{S}_{1/2} = \{1/2, 3/2\}$  for the first cell  $[\omega_0, \omega_1]$  and  $\mathcal{S}_{N-1/2} = \{N-1/2, N-3/2\}$  for the last cell  $[\omega_{N-1}, \omega_N]$ , *i.e.* we use a linear reconstruction at the boundaries.
- An ENO reconstruction : for the cell  $[\omega_i, \omega_{i+1}]$ , we first evaluate two polynomials of degree 1. The first one,  $p_i^-$ , is constructed using the cells  $\{[\omega_{i-1}, \omega_i], [\omega_i, \omega_{i+1}]\}$  and the second one,  $p_i^+$ , on  $\{[\omega_i, \omega_{i+1}], [\omega_{i+1}, \omega_{i+2}]\}$ . We can write (with  $\omega_{i+1/2} = \frac{\omega_i + \omega_{i+1}}{2}$ )

$$p_i^+(\xi) = a_i^+(\xi - \omega_{i+1/2}) + b_i^+ \text{ and } p_i^-(\xi) = a_i^-(\xi - \omega_{i+1/2}) + b_i^-.$$

We choose the least oscillatory one, *i.e.* the one which realises the oscillation  $\min(|a_i^+|, |a_i^-|)$ . In that case, we take a first order reconstruction on the boundary of  $\Omega$ .

Other choices are possible such as WENO-like interpolants. Again, the case of multiple source of uncertainties can be handled by tensorisation.

### 3 POLYNOMIAL CHAOS METHOD

Polynomial Chaos (PC) expansions are derived from the original theory of Wiener on spectral representation of stochastic processes using Gaussian random variables. PC expansions have been used for UQ by Ghanem and Spanos [?] and extended by Xiu and Karniadakis [?] to non-Gaussian processes. Any well-behaved process  $y$  (e.g. a second-order process) can be expanded in a convergent (in the mean square sense, see Cameron and Martin [?]) series of the form :

$$y(x, t, \xi) = \sum_{\alpha} y_{\alpha}(x, t) \Psi_{\alpha}(\xi) \tag{7}$$

where  $\xi$  is a set of  $nx$  independent random variables  $\xi = (\xi_1, \xi_2, \dots, \xi_n)$  and  $\alpha$  a multi-index  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  with each component  $\alpha_i = 0, 1, \dots$ . The multivariate polynomial function  $\Psi_{\alpha}$  is defined by a product of orthogonal polynomials  $\Phi_i^{\alpha_i}(\xi_i)$  in relation to the probability density of the random variable  $\xi_i$ , namely  $\Psi_{\alpha}(\xi) = \prod_{i=1}^{nx} \Phi_i^{\alpha_i}(\xi_i)$ . A one-to-one correspondence exists between the choice of stochastic variable  $\xi_i$  and the polynomials  $\Phi_i^{\alpha_i}(\xi_i)$  of degree  $\alpha_i$ . For instance if  $\xi_i$  is a normal/uniform variable, the corresponding  $\Phi_i^{\alpha_i}(\xi_i)$  are Hermite/Legendre polynomials of degree  $\alpha_i$ ; the degree of  $\Psi_{\alpha}$  is  $|\alpha|_1 = \sum_{i=1}^{nx} \alpha_i$ . The multivariate polynomial functions  $\Psi_{\alpha}$  are orthogonal with respect to the probability distribution function of the vector  $\xi$  of standard independent random variables  $\xi_i, i = 1, 2, \dots, nx$ . Coefficients  $y_{\alpha}(x, t)$  are the PC coefficients or stochastic modes of the random process  $y$ . Defining the scalar product by the expectation operator yields :

$$y_{\alpha}(x, t) = \langle y(x, t), \Psi_{\alpha} \rangle \|\Psi_{\alpha}\|^{-2} \tag{8}$$

For practical use, the PC expansions have to be truncated in term of degree polynomial  $no$  :

$$y(x, t, \xi) = \sum_{|\alpha|_1 \leq no} y_\alpha(x, t) \Psi_\alpha(\xi) \quad (9)$$

The number of multivariate polynomials  $\Psi_\alpha$ , that is the dimension of the expansion basis, is related to the stochastic dimension  $nx$  and the degree  $no$  of polynomials and is given by the formula  $(nx + no)!/(nx!no!)$ . Several approaches can be used to estimate PC coefficients. The first approach is based on a Galerkin projection of the model equations; it leads to a set of coupled equations and requires an adaptation of the deterministic code in order to be applied. Alternative non-intrusive approaches are based on Monte Carlo simulations or quadrature formulae to evaluate PC coefficients (see for instance [?]). When the number  $d$  of variables is large, quadrature formulae based on tensor product of a 1D formula require too many numerical evaluations and Sparse Grids integration based on Smolyak's construction [?] are preferred. For these non intrusive approaches, PC coefficients are therefore evaluated from a set of points and weights  $(\xi_i, \omega_i)$  by formulae of the form :

$$y_\alpha(x, t) = \|\Psi_\alpha\|^{-2} \sum_{i=1}^n y(x, t, \xi_i) \Psi_\alpha(\xi_i) \omega_i \quad (10)$$

From the PC expansion (9), it is easy to derive the mean  $E$  and variance  $V$  of the random process. To simplify notations, an orthonormal basis is chosen,  $\|\Psi_\alpha\| = 1 \forall \alpha$ ; the following equalities hold in that case :

$$E(y(x, t)) = y_0(x, t) \quad (11)$$

$$V(y(x, t)) = \sum_{\alpha} y_\alpha^2(x, t) \quad (12)$$

## 4 APPLICATION

Two test-cases are considered in this section to assess the performance of the semi-intrusive method described in Section 2 :

- the steady shocked flow in a convergent-divergent nozzle, of fixed (deterministic) geometry :

$$A(x) = \begin{cases} 1 + 6(x - \frac{1}{2})^2 & \text{for } 0 < x \leq \frac{1}{2} \\ 1 + 2(x - \frac{1}{2})^2 & \text{for } \frac{1}{2} < x \leq 1 \end{cases}$$

with fixed (deterministic) boundary conditions (subsonic inlet flow with a stagnation pressure  $p_0 = 2 \text{ bar}$  and a stagnation temperature  $T_0 = 300 \text{ K}$ , subsonic outlet flow with  $p_e = 1.6529 \text{ bar}$  yielding an exact shock position at  $x = 0.75$  in the nozzle divergent) but an uncertain heat coefficient ratio  $\gamma$ . Various choices of pdf (uniform, Gaussian and discontinuous) for  $\gamma$  will be used to describe this stochastic variable.

- the Sod shock tube ( $x \in [0, 1]$ ) with fixed heavy and light fluid properties (deterministic initial states), a fixed heat coefficient ratio  $\gamma = 1.4$  in the perfect gas equation of state but an uncertain position  $x_d$  of the diaphragm initially separating the heavy and light fluids :

$$\begin{pmatrix} \rho^0(x) \\ \rho^0(x)u^0(x) \\ E^0(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2.5 \end{pmatrix} \text{ if } x \leq x_d, \quad \begin{pmatrix} \rho^0(x) \\ \rho^0(x)u^0(x) \\ E^0(x) \end{pmatrix} = \begin{pmatrix} 0.125 \\ 0 \\ 0.25 \end{pmatrix} \text{ otherwise}$$

The initial position of the diaphragm is such that  $x_d = 0.5 + 0.05\xi$  where the stochastic parameter  $\xi \in [-1, 1]$  is described by various pdf (uniform, Gaussian, discontinuous).

#### 4.1 Nozzle flow with uncertain $\gamma$

The two stochastic methods previously described, SI (semi-intrusive) and PC (Polynomial Chaos) are used to compute statistic solutions of a supersonic nozzle. The outlet pressure is chosen in order to have a compression shock in the divergent part of the nozzle, exactly located at  $x = 0.75$ . The mean  $\gamma$  is 1.4, and a maximal variation of 5% is considered. Different pdf are used for  $\gamma$ , *i.e.* uniform and Gaussian in order to compare SI and PC, and a discontinuous pdf to demonstrate the flexibility offered by the SI method. After a study on the grid convergence, the 1D physical space is divided in 201 points (with the normalized geometric domain that varies from 0 to 1). A preliminary convergence study with respect to the stochastic estimation has been realized, by using an increasing refinement of the probabilistic space discretization in the case of the SI method, and an increasing polynomial order in the case of PC method. The probabilistic space discretization varies from 5 to 160 points (5, 10, 20, 40, 80, 160), while the polynomial order varies from 2 to 5. Next, the stochastic solutions are compared computing the mean and the variance of the Mach number and pressure distributions along the nozzle using various choices of pdf for  $\gamma$ . Finally, a comparison in terms of computational cost is performed. In Figure ??, the mean solutions of Mach number and the pressure along the 1D nozzle are reported, where the mean stochastic solutions are computed with the SI method using 160 points in the probabilistic space and the PC method using a 5th order polynomial, with  $\gamma$  described by a uniform pdf ( $\gamma$  varying between 1.33 and 1.47). As can be observed in figure ??, the mean flow is characterized by an isentropic region of increasing speed or Mach number between  $x = 0$  and the mean shock location in the divergent (the flow becoming supersonic at the nozzle throat located in  $x = 0.5$ ), followed by a subsonic flow behind the shock with decreasing speed. The mean solutions computed by the two UQ methods are coincident. The same trend is obtained if a Gaussian pdf is considered for  $\gamma$  (the figure is not reported for brevity). Next the standard deviation of the Mach number is computed along the nozzle by using different refinement levels for the probabilistic space in the case of the SI method (Figure ??) and different polynomial order in the case of the PC method (Figure ??), always keeping a uniform pdf for  $\gamma$ . In Figure ??, it is

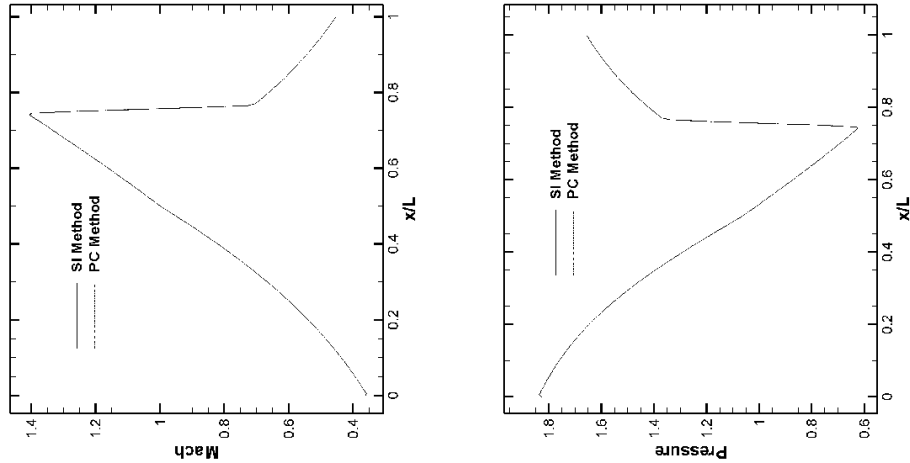


Figure 1: Nozzle flow with uncertain  $\gamma$  (uniform pdf). Computed mean distribution for the Mach number (left) and the static pressure (right) using the semi-intrusive method with 160 points in the probabilistic space and the PC method with a 5th order polynomial.

shown that a 10-point discretization of the probabilistic space is sufficient to obtain a converged solution. A close-up on the shock region indicates very small differences between a 10-point and a 40-point discretization but a perfect match between this latter and the finest 160-point discretization. When analyzing the PC method in Figure ??, the solutions seem very similar for different polynomial orders; however, a closer observation of the shock region leads to conclude the maximal standard deviation is not properly estimated as long as the highest-order (here 5th order) polynomial is used. In Figure ?? and Figure ??, the standard deviation of the Mach is reported along the nozzle for an uniform pdf and a Gaussian pdf respectively. For what concerns the uniform pdf, the standard deviation distributions computed by means of SI and PC are nearly coincident, except in the shock region, where there is a difference of 5% in the maximal standard deviation. If a Gaussian pdf is considered (Figure ??), there is a stronger variation near the shock and the difference in the maximal standard deviation increases up to 12%. The stochastic estimation remains globally very similar for the newly proposed SI approach and the well-established PC method, which allows to validate the SI method results for the case of a uniform and Gaussian pdf on  $\gamma$ . It is interesting to show the kind of innovative contribution the SI method can bring with respect to the PC method. To this end, in figure ?? the standard deviation of Mach is reported along the nozzle when a

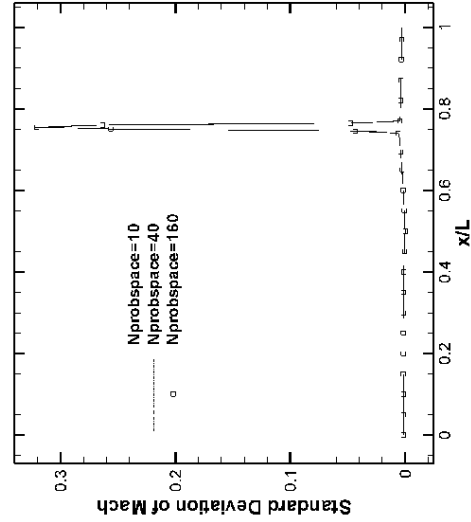


Figure 2: Nozzle flow with uncertain  $\gamma$  (uniform pdf). Convergence study for the standard deviation on the Mach number distribution computed using the semi-intrusive method.

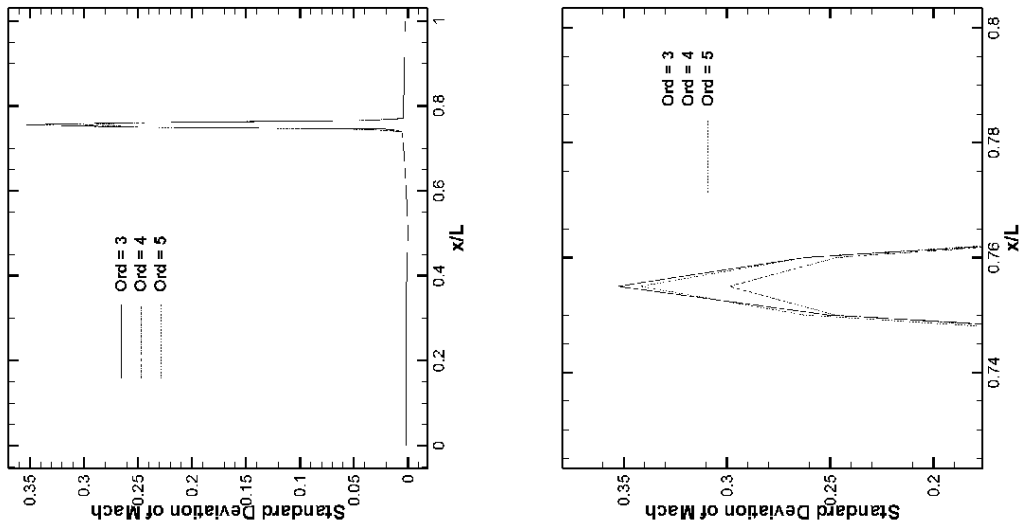


Figure 3: Nozzle flow with uncertain  $\gamma$  (uniform pdf). Convergence study for the standard deviation on the Mach number distribution computed using the NIP method.

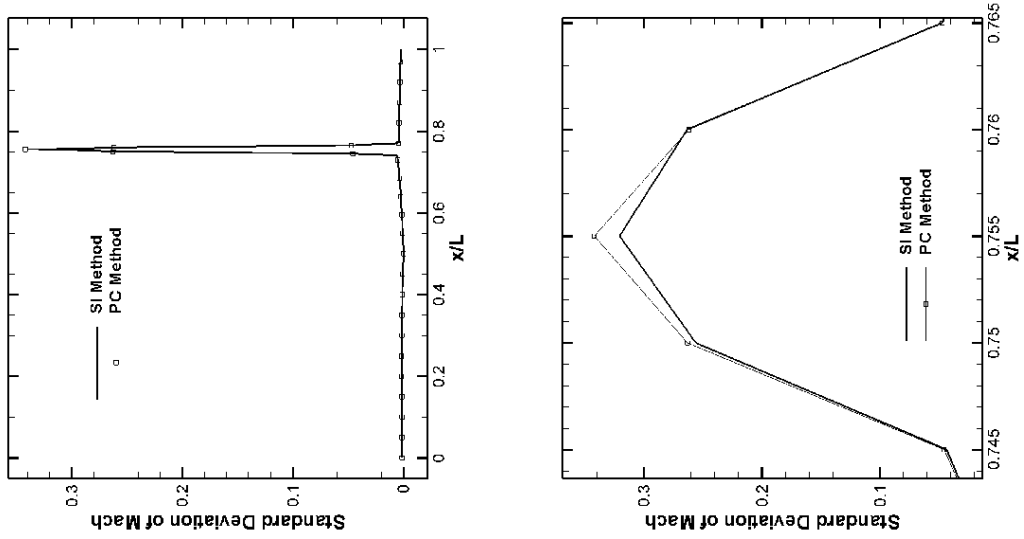


Figure 4: Nozzle flow with uncertain  $\gamma$  (uniform pdf). Standard deviation for the Mach number distribution. Left : global view; right : close-up on the shock region.

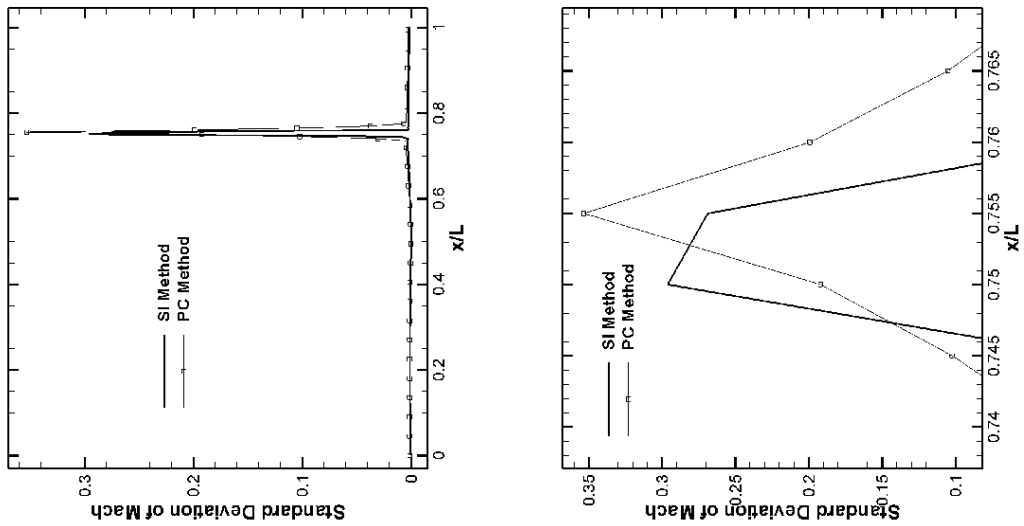


Figure 5: Nozzle flow with uncertain  $\gamma$  (Gaussian pdf). Standard deviation for the Mach number distribution. Left : global view; right : close-up on the shock region.