A simplified model for elastic thin shells.

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Abstract. We introduce a simplified model for the minimization of the elastic energy in thin shells. This model is not obtained by an asymptotic analysis. The nonlinear simplified model admits always minimizers by contrast with the original one. We show the relevance of our approach by proving that the rescaled minimum of the simplified model and the rescaled infimum of the full model have the same limit as the thickness tends to 0. The simplified energy can be expressed as a functional acting over fields defined on the mid-surface of the shell and where the thickness remains as a parameter.

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1. Introduction

This paper is devoted to introduce and justify a simplified model for nonlinear elastic shells. Let ω be a bounded Lipschitz domain of $\mathbb{R}^2$ and ϕ be a smooth function from $\overline{\omega}$ into $\mathbb{R}^3$ (see the detailed assumptions on ϕ in Section 2) and set $S=\phi(\omega)$. We denote by $\mathbf{n}$ an unit vector field normal to S and by Φ the map $(s_1,s_2,s_3)\longrightarrow \phi(s_1,s_2)+s_3\mathbf{n}(s_1,s_2)$. The elastic shell is defined by $\mathcal{Q}_\delta=\Phi(\omega\times]-\delta,\delta[)$ and we consider that it is clamped on a part of its lateral boundary $\Gamma_{0,\delta}=\Phi(\gamma_0\times]-\delta,\delta[)$, where $\gamma_0\subset\partial\omega$. The energy density is denoted W and we assume that $\mathcal{Q}_\delta$ is submitted to applied body forces $f_{\kappa,\delta}$ whose order with respect to δ depends upon a parameter κ (see the order of $f_{\kappa,\delta}$ below). The total energy is given by $J_{\kappa,\delta}(v)=\displaystyle\int_{\mathcal{Q}_\delta}W(E(v))-\int_{\mathcal{Q}_\delta}f_{\kappa,\delta}\cdot(v-I_d)$ if $\det(\nabla v)>0$ and where $E(v)=1/2\big((\nabla v)^T\nabla v-\mathbf{I}_3\big)$ is the Green-St Venant's tensor and I_d is the identity map. We set

$$m_{\kappa,\delta}=\inf_{v\in\mathbf{U}_\delta}J_{\kappa,\delta}(v),$$

where $\mathbf{U}_\delta$ is the set of admissible deformations (which are equal to the identity map on $\Gamma_{0,\delta}$). The Korn's type inequalities established in [6] (see also [12]) allow us to prove that if the order of $f_{\kappa,\delta}$ is equal to $\delta^{2\kappa-2}$ for $1\le\kappa\le 2$ (or δ^κ for $\kappa\ge 2$), then the order of $m_{\kappa,\delta}$ is $\delta^{2\kappa-1}$.

Even for a classical St-Venant-Kirchhoff's material, proving the existence of a minimizer for $J_{\kappa,\delta}$ is still an open problem. The aim of this paper is to replace the above minimization problem by a minimization problem for a simplified functional $J^s_{\kappa,\delta}$ defined on a new set $\mathbb{D}_{\delta,\gamma_0}$ and which admits a minimum

$$m^s_{\kappa,\delta}=\min_{\mathbf{v}\in\mathbb{D}_{\delta,\gamma_0}}J^s_{\kappa,\delta}(\mathbf{v})$$

of the same order as $m_{\kappa,\delta}$. This approximation is justified if one shows that

$$\lim_{\delta\to 0}\frac{m_{\kappa,\delta}-m^s_{\kappa,\delta}}{\delta^{2\kappa-1}}=0.$$

In the present paper we show this result in the case $\kappa = 2$ (other critical cases will be investigated in forthcoming papers).

The expression of $J^s_{\kappa,\delta}$ and the choice of $\mathbb{D}_{\delta,\gamma_0}$ rely on the decomposition technique introduced in [6]. Let us recall that a deformation v of the shell $\mathcal{Q}_\delta$, whose "geometrical energy" -$||dist(\nabla v, SO(3))||_{L^2(\mathcal{Q}_\delta)}$- is at most of order $\delta^{3/2}$, is decomposed as (see [6] or Theorem 3.1 below)

$$v(x) = \mathcal{V}(s_1, s_2) + s_3\mathbf{R}(s_1, s_2)\mathbf{n}(s_1, s_2) + \overline{v}(s_1, s_2, s_3), \quad x = \Phi(s), \quad \text{for a.e. } s = (s_1, s_2, s_3) \in \omega\times] - \delta, \delta[.$$

The field $\mathcal{V}$ stands for the mid-surface deformation, the matrix field $\mathbf{R}$ takes its values in $SO(3)$ and represents the rotations of the fibers and $\overline{v}$ is the warping of theses fibers. It is also shown in [6] that the fields $\mathcal{V}$, $\mathbf{R}$ and $\overline{v}$ satisfy the natural boundary conditions on γ_0 and on $\gamma_0\times] - \delta, \delta[$ and that they are estimated in terms of $||dist(\nabla v, SO(3))||_{L^2(\mathcal{Q}_\delta)}$ and δ. With the help of these estimates, we justify the simplification of the Green-St Venant's strain tensor $E(v)$ in order to give a simplified matrix $\widehat{E}(\mathbf{v})$ which depends on the triplet $\mathbf{v} = (\mathcal{V}, \mathbf{R}, \overline{v})$ associated to a deformation v. This matrix depends linearly upon $\dfrac{\partial \overline{v}}{\partial s_3}$ and on the first partial derivatives of $\mathcal{V}$ and $\mathbf{R}$ (see Section 5) but which is nonlinear with respect to $(\mathcal{V}, \mathbf{R}, \overline{v})$.

Then we define the set $\mathbb{D}_{\delta,\gamma_0}$ of admissible triplets $\mathbf{v} = (\mathcal{V}, \mathbf{R}, \overline{v})$ and we derive the simplified total energy $J^s_{\kappa,\delta}(\mathbf{v})$ as follows. Firstly we replace $\int_{\mathcal{Q}_\delta} W(E(v))$ by $\int_{\mathcal{Q}_\delta} Q(E(\mathbf{v}))$ where Q is a quadratic form which is assumed to approximate W near the origin. Secondly we add two penalization terms in order to approach the usual limit kinematic condition $\dfrac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R}\mathbf{t}_\alpha$ and to insure the coerciveness of $J^s_{\kappa,\delta}$. Finally in the term involving the forces we neglect the contribution of the warping $\overline{v}$. As announced above we prove that $J^s_{\kappa,\delta}$ admit minimizers on $\mathbb{D}_{\delta,\gamma_0}$. We justify the approximation process described above in the case $\kappa = 2$.

In some sense, the introduction of $J^s_{\kappa,\delta}$ can be seen as a nonlinear version of the approach which leads to the simplified Timoshenko's model for rods, the Reisner-Mindlin's model for plates and the Koiter's model for shells in linear elasticity.

As general references on the theory of nonlinear elasticity, we refer to [8] and [24] and to the extensive bibliographies of these works. A general theory for the existence of minimizers of nonlinear elastic energies can be found in [1]. For the justification of plate or shell models in nonlinear elasticity we refer to [9], [10], [11], [13], [15], [18], [23], [25], [26]. The derivation of limit energies for thin domains using Γ-convergence arguments are developed in [14], [15], [22], [23]. The decomposition of the deformations in thin structures is introduced in [17], [18] and a few applications to the junctions of multi-structures and homogenization are given in [2], [3], [4]. The justification of simplified models for rods and plates in linear elasticity, based on a decomposition technique of the displacement, is presented in [19], [20]. In this linear case, error estimates between the solution of the initial model and the one of the simplified model are also established. In some sense, these works give a mathematical justification of Timoshenko's model for rods and Reisner-Mindlin's model for plates.

The paper is organized as follows. Section 2 is devoted to describe the geometry of the shell and to give a few notations. In Section 3 we recall the results of [6]: decomposition of a deformation of a thin shell, estimates on the terms of this decomposition and two nonlinear Korn's type inequalities. Section 4 is concerned with a standard rescaling. We present the simplification of the Green-St Venant's strain tensor of a deformation in Section 5. We also introduce the set $\mathbb{D}_{\delta,\gamma_0}$ of admissible triplets $\mathbf{v} = (\mathcal{V}, \mathbf{R}, \overline{V})$ and we prove Korn's type inequalities for the elements of $\mathbb{D}_{\delta,\gamma_0}$ (see Corollary 5.3). In Section 6 we consider nonlinear elastic shells and we use the results of [6] to scale the applied forces in order to obtain a priori estimates on $m_{\kappa,\delta}$. Section 7 is devoted to introduce the simplified energy $J^s_{\kappa,\delta}$ and to prove the existence of minimizers.

In Sections 8 and 9, we restrict the analysis to $\kappa=2$. We prove that

$$\lim_{\delta\to 0}\frac{m_{2,\delta}}{\delta^3}=\lim_{\delta\to 0}\frac{m^s_{2,\delta}}{\delta^3}=m^s_2$$

where m^s_2 is the minimum of a functional defined over a set of triplets. In Section 10, we give an alternative formulation of the minimization problem for $J^s_{\kappa,\delta}$ through elimination of the variable $\overline{v}$. Then we obtain that $m^s_{\kappa,\delta}$ is the minimum of a functional which depends only upon $(\mathcal{V},\mathbf{R})$. At last an appendix contains an approximation result for the elements of $\mathbb{D}_{\delta,\gamma_0}$ and an algebraic elimination process for quadratic forms. The results of this paper were announced in [7].

2. The geometry and notations.

Let us introduce a few notations and definitions concerning the geometry of the shell.

Let ω be a bounded domain in $\mathbb{R}^2$ with lipschitzian boundary and let ϕ be an injective mapping from $\overline{\omega}$ into $\mathbb{R}^3$ of class $\mathcal{C}^2$. We denote S the surface $\phi(\overline{\omega})$. We assume that the two vectors $\dfrac{\partial\phi}{\partial s_1}(s_1,s_2)$ and $\dfrac{\partial\phi}{\partial s_2}(s_1,s_2)$ are linearly independent at each point $(s_1,s_2)\in\overline{\omega}$.

We set

$$\mathbf{t}_1=\frac{\partial\phi}{\partial s_1},\qquad \mathbf{t}_2=\frac{\partial\phi}{\partial s_2},\qquad \mathbf{n}=\frac{\mathbf{t}_1\wedge\mathbf{t}_2}{\|\mathbf{t}_1\wedge\mathbf{t}_2\|_2}. \tag{2.1}$$

The vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are tangential vectors to the surface S and the vector $\mathbf{n}$ is a unit normal vector to this surface. We set

$$\Omega_\delta=\omega\times]-\delta,\delta[.$$

Now we consider the mapping $\Phi:\overline{\omega}\times\mathbb{R}\longrightarrow\mathbb{R}^3$ defined by

$$\Phi\ :\ (s_1,s_2,s_3)\longmapsto x=\phi(s_1,s_2)+s_3\mathbf{n}(s_1,s_2). \tag{2.2}$$

There exists $\delta_0\in(0,1]$ depending only on S, such that the restriction of Φ to the compact set $\overline{\Omega}_{\delta_0}=\overline{\omega}\times[-\delta_0,\delta_0]$ is a $\mathcal{C}^1-$ diffeomorphism of that set onto its range (see e.g. [21]). Hence, there exist two constants $c_0>0$ and $c_1\geq c_0$, which depend only on ϕ, such that

$$\forall s\in\Omega_{\delta_0},\quad c_0\leq|||\nabla_s\Phi(s))|||\leq c_1,\quad \text{and for } x=\Phi(s)\qquad c_0\leq|||\nabla_x\Phi^{-1}(x))|||\leq c_1. \tag{2.3}$$

Definition 2.1. *For $\delta\in(0,\delta_0]$, the shell $\mathcal{Q}_\delta$ is defined as follows:*

$$\mathcal{Q}_\delta=\Phi(\Omega_\delta).$$

The mid-surface of the shell is S. The fibers of the shell are the segments $\Phi\big(\{(s_1,s_2)\}\times]-\delta,\delta[\big)$, $(s_1,s_2)\in\omega$. The lateral boundary of the shell is $\Gamma_\delta=\Phi(\partial\omega\times]-\delta,\delta[)$. In the following sections the shell will be fixed on a part of its lateral boundary. Let γ_0 be an open subset of $\partial\omega$ which made of a finite number of connected components (whose closure are disjoint). We assume that the shell is clamped on

$$\Gamma_{0,\delta}=\Phi(\gamma_0\times]-\delta,\delta[).$$

The admissible deformations v of the shell must then satisfy

$$v = I_d \qquad \text{on} \quad \Gamma_{0,\delta} \tag{2.4}$$

where I_d is the identity map of $\mathbb{R}^3$.

Notation. From now on we denote by c and C two positive generic constants which do not depend on δ. We respectively note by x and s the generic points of $\mathcal{Q}_\delta$ and of Ω_δ. A field v defined on $\mathcal{Q}_\delta$ can be also considered as a field defined on Ω_δ that, as a convention, we will also denote by v. As far as the gradients of field v, say in $(W^{1,1}(\mathcal{Q}_\delta))^3$, are concerned we have $\nabla_x v$ and $\nabla_s v = \nabla_x v.\nabla\Phi$ for a.e. $x = \Phi(s)$ and (2.3) shows that

$$c|||\nabla_x v(x)||| \le |||\nabla_s v(s)||| \le C|||\nabla_x v(x)|||.$$

3. Korn's type inequalities for shells. Decomposition of a deformation.

We first recall the Korn's type inequalities for shells established in Section 4 of [6]. Let v be an admissible deformation belonging to $\big(H^1(\mathcal{Q}_\delta)\big)^3$ and satisfying the boundary condition (2.4). Setting $\mathcal{V}(s_1,s_2) = \dfrac{1}{2\delta}\displaystyle\int_{-\delta}^{\delta} v(s_1,s_2,t)dt$ a.e. $(s_1,s_2)\in\omega$, we have

$$\begin{cases} ||v-I_d||_{(L^2(\mathcal{Q}_\delta))^3} + ||\nabla_x v - \mathbf{I}_3||_{(L^2(\mathcal{Q}_\delta))^9} \le C\big(\delta^{1/2} + ||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\big),\\ ||(v-I_d) - (\mathcal{V}-\phi)||_{(L^2(\mathcal{Q}_\delta))^3} \le C\delta\big(\delta^{1/2} + ||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\big), \end{cases} \tag{3.1}$$

and

$$\begin{cases} ||v-I_d||_{(L^2(\mathcal{Q}_\delta))^3} + ||\nabla_x v - \mathbf{I}_3||_{(L^2(\mathcal{Q}_\delta))^9} \le \dfrac{C}{\delta}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)},\\ ||(v-I_d) - (\mathcal{V}-\phi)||_{(L^2(\mathcal{Q}_\delta))^3} \le C||\mathrm{dist}(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}. \end{cases} \tag{3.2}$$

Inequalities (3.1) are better than those (3.2) if the order of the geometric energy $||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}$ is greater than $\delta^{3/2}$.

Now the theorem of decomposition of the deformations established in [6] (see Theorem 3.4 of Section 3) is given below.

Theorem 3.1. *There exists a constant $C(S)$ which depends only on the mid-surface of the shell such that for all deformation v belonging to $\big(H^1(\mathcal{Q}_\delta)\big)^3$ and satisfying*

$$||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \le C(S)\delta^{3/2}, \tag{3.3}$$

then, there exist $\mathcal{V}\in(H^1(\omega))^3$, $\mathbf{R}\in\big(H^1(\omega)\big)^{3\times3}$ satisfying $\mathbf{R}(s_1,s_2)\in SO(3)$ for a.e. $(s_1,s_2)\in\omega$ and $\overline{v}$ belonging to $\big(H^1(\mathcal{Q}_\delta)\big)^3$ such that for a.e. $s\in\Omega_\delta$

$$v(s) = \mathcal{V}(s_1,s_2) + s_3\mathbf{R}(s_1,s_2)\mathbf{n}(s_1,s_2) + \overline{v}(s), \tag{3.4}$$

where we can choose $\mathcal{V}(s_1,s_2) = \dfrac{1}{2\delta}\displaystyle\int_{-\delta}^{\delta} v(s_1,s_2,t)dt$ a.e. $(s_1,s_2)\in\omega$, and such that the following estimates hold:

$$\begin{cases} ||\overline{v}||_{(L^2(\Omega_\delta))^3} \le C\delta||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ ||\nabla_s\overline{v}||_{(L^2(\Omega_\delta))^9} \le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ \Big\|\dfrac{\partial\mathbf{R}}{\partial s_\alpha}\Big\|_{(L^2(\omega))^9} \le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ \Big\|\dfrac{\partial\mathcal{V}}{\partial s_\alpha} - \mathbf{R}\mathbf{t}_\alpha\Big\|_{(L^2(\omega))^3} \le \dfrac{C}{\delta^{1/2}}||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ \big\|\nabla_x v - \mathbf{R}\big\|_{(L^2(\Omega_\delta))^9} \le C||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}. \end{cases} \tag{3.5}$$

□

Due to (3.4) and to the definition of $\mathcal{V}$, the field $\overline{v}$ satisfies $\displaystyle\int_{-\delta}^{\delta}\overline{v}(s_1,s_2,t)dt=0$ a.e. $(s_1,s_2)\in\omega$.

If the deformation v as in Theorem 3.1 satisfies the boundary condition (2.4) then indeed

$$\mathcal{V}=\phi \qquad \text{on} \quad \gamma_0. \tag{3.6}$$

Moreover due to Lemma 4.1 of [6], we can choose $\mathbf{R}$ and $\overline{v}$ in Theorem 3.1 above such that

$$\mathbf{R}=\mathbf{I}_3 \qquad \text{on } \gamma_0, \qquad \overline{v}=0 \qquad \text{on } \Gamma_{0,\delta}. \tag{3.7}$$

From estimates (3.5) we also derive the following ones:

$$\begin{cases} ||\mathbf{R}-\mathbf{I}_3||_{(L^2(\omega))^9}\le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ ||\mathcal{V}-\phi||_{(L^2(\omega))^3}\le \dfrac{C}{\delta^{3/2}}||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)}.\end{cases} \tag{3.8}$$

4. Rescaling Ω_δ

As usual, we rescale Ω_δ using the operator

$$(\Pi_\delta w)(s_1,s_2,S_3)=w(s_1,s_2,\delta S_3) \text{ for any } (s_1,s_2,S_3)\in\Omega$$

defined for e.g. $w\in L^2(\Omega_\delta)$ for which $(\Pi_\delta w)\in L^2(\Omega)$. Let v be a deformation decomposed as (3.4), by transforming by Π_δ we obtain

$$\Pi_\delta(v)(s_1,s_2,S_3)=\mathcal{V}(s_1,s_2)+\delta S_3\mathbf{R}(s_1,s_2)\mathbf{n}(s_1,s_2)+\Pi_\delta(\overline{v})(s_1,s_2,S_3), \qquad \text{for a.e. } (s_1,s_2,S_3)\in\Omega.$$

The estimates (3.5) of $\overline{v}$ transposed over Ω are (notice that $\Pi_\delta\Big(\dfrac{\partial\overline{v}}{\partial s_3}\Big)=\dfrac{1}{\delta}\dfrac{\partial\Pi_\delta(\overline{v})}{\partial S_3}$)

$$\begin{cases} ||\Pi_\delta(\overline{v})||_{(L^2(\Omega))^3}\le C\delta^{1/2}||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ \Big\|\dfrac{\partial\Pi_\delta(\overline{v})}{\partial s_1}\Big\|_{(L^2(\Omega))^3}\le \dfrac{C}{\delta^{1/2}}||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ \Big\|\dfrac{\partial\Pi_\delta(\overline{v})}{\partial s_2}\Big\|_{(L^2(\Omega))^3}\le \dfrac{C}{\delta^{1/2}}||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)}\\ \Big\|\dfrac{\partial\Pi_\delta(\overline{v})}{\partial S_3}\Big\|_{(L^2(\Omega))^3}\le C\delta^{1/2}||dist(\nabla_x v,SO(3))||_{L^2(\mathcal{Q}_\delta)},\end{cases} \tag{4.1}$$

5. Simplification in the Green-St Venant's strain tensor.

In this section we introduce a simplification of the Green-St Venant's strain tensor $E(v)=1/2\big((\nabla v)^T\nabla v-\mathbf{I}_3\big)$. Let v be a deformation of the shell belonging to $(H^1(\mathcal{Q}_\delta))^3$ and satisfying the condition (3.3). We decompose v as (3.4). We have the identity

$$(\nabla_x v)^T\nabla_x v-\mathbf{I}_3=(\nabla_x v-\mathbf{R})^T\mathbf{R}+\mathbf{R}^T(\nabla_x v-\mathbf{R})+(\nabla_x v-\mathbf{R})^T(\nabla_x v-\mathbf{R}).$$

In order to compare the orders (of the norms) of the different terms in the above equality, we work in the fix domain Ω using the operator Π_δ. Thanks to estimates (3.5) we get

$$
(5.1)\qquad \begin{cases} ||\Pi_\delta\big((\nabla_x v-\mathbf{R})^T\mathbf{R}+\mathbf{R}^T(\nabla_x v-\mathbf{R})\big)||_{(L^1(\Omega))^{3\times 3}} \le C\dfrac{||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}}{\delta^{1/2}},\\ ||\Pi_\delta\big((\nabla_x v-\mathbf{R})^T(\nabla_x v-\mathbf{R})\big)||_{(L^1(\Omega))^{3\times 3}} \le C\Big[\dfrac{||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}}{\delta^{1/2}}\Big]^2. \end{cases}
$$

In view of (3.1), these estimates show that the term $\Pi_\delta\big((\nabla_x v-\mathbf{R})^T(\nabla_x v-\mathbf{R})\big)$ can be neglected in $E(v)$.

Now we have

$$
\frac{\partial v}{\partial s_1}=\nabla_x v\,\big(\mathbf{t}_1+s_3\frac{\partial \mathbf{n}}{\partial s_1}\big),\qquad \frac{\partial v}{\partial s_2}=\nabla_x v\,\big(\mathbf{t}_2+s_3\frac{\partial \mathbf{n}}{\partial s_2}\big),\qquad \frac{\partial v}{\partial s_3}=\nabla_x v\,\mathbf{n}.
$$

Then

$$
\Pi_\delta(\nabla_x v-\mathbf{R})\,\big(\mathbf{t}_\alpha+\delta S_3\frac{\partial \mathbf{n}}{\partial s_\alpha}\big)=\Big(\frac{\partial \mathcal{V}}{\partial s_\alpha}-\mathbf{R}\mathbf{t}_\alpha\Big)+\delta S_3\frac{\partial \mathbf{R}}{\partial s_\alpha}\mathbf{n}+\frac{\partial \Pi_\delta\overline{v}}{\partial s_\alpha},\qquad \Pi_\delta(\nabla_x v-\mathbf{R})\,\mathbf{n}=\frac{1}{\delta}\frac{\partial \Pi_\delta\overline{v}}{\partial S_3}.
$$

First, we can neglect the term $\delta S_3\dfrac{\partial \mathbf{n}}{\partial s_\alpha}$ which is of order δ in the quantity $\mathbf{t}_\alpha+\delta S_3\dfrac{\partial \mathbf{n}}{\partial s_\alpha}$. Secondly, as a consequence of these equalities and the following estimates (obtained from (3.5) and (4.1)):

$$
(5.2)\qquad \begin{cases} \Big\|\dfrac{\partial \mathcal{V}}{\partial s_\alpha}-\mathbf{R}\mathbf{t}_\alpha\Big\|_{(L^2(\Omega))^3}\le C\dfrac{||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}}{\delta^{1/2}},\\ \Big\|\delta S_3\dfrac{\partial \mathbf{R}}{\partial s_\alpha}\mathbf{n}\Big\|_{(L^2(\Omega))^3}\le C\dfrac{||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}}{\delta^{1/2}},\\ \Big\|\dfrac{1}{\delta}\dfrac{\partial \Pi_\delta\overline{v}}{\partial S_3}\Big\|_{(L^2(\Omega))^3}\le C\dfrac{||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}}{\delta^{1/2}},\\ \Big\|\dfrac{\partial \Pi_\delta\overline{v}}{\partial s_\alpha}\|_{(H^{-1}(\Omega))^3}\le C\delta\Big[\dfrac{||dist(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}}{\delta^{1/2}}\Big] \end{cases}
$$

we deduce that in the quantity $\Pi_\delta\big((\nabla_x v-\mathbf{R})^T\mathbf{R}+\mathbf{R}^T(\nabla_x v-\mathbf{R})\big)$ we can neglect the terms $\dfrac{\partial \Pi_\delta\overline{v}}{\partial s_\alpha}$.

Now, if in the Green-St Venant's strain tensor of v we carry out the simplifications mentioned above, we are brought to replace

$$
\begin{aligned} &\frac{1}{2}\Pi_\delta\big((\nabla_x v)^T\nabla_x v-\mathbf{I}_3\big)\quad\text{by}\quad(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\Pi_\delta\big(E^s(v)\big)(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\\ \text{or}\quad &\frac{1}{2}\big((\nabla_x v)^T\nabla_x v-\mathbf{I}_3\big)\quad\text{by}\quad(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}E^s(v)(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1} \end{aligned}
$$

where the symmetric matrix $E^s(v)\in(L^2(\Omega_\delta))^{3\times 3}$ is equal to

$$
(5.3)\qquad \begin{aligned} &E^s(v)=\begin{pmatrix} s_3\Gamma_{11}(\mathbf{R})+\mathcal{Z}_{11} & s_3\Gamma_{12}(\mathbf{R})+\mathcal{Z}_{12} & \dfrac{1}{2}\mathbf{R}^T\dfrac{\partial\overline{v}}{\partial s_3}\cdot\mathbf{t}_1+\dfrac{1}{2}\mathcal{Z}_{31}\\ * & s_3\Gamma_{22}(\mathbf{R})+\mathcal{Z}_{22} & \dfrac{1}{2}\mathbf{R}^T\dfrac{\partial\overline{v}}{\partial s_3}\cdot\mathbf{t}_2+\dfrac{1}{2}\mathcal{Z}_{32}\\ * & * & \mathbf{R}^T\dfrac{\partial\overline{v}}{\partial s_3}\cdot\mathbf{n} \end{pmatrix}\\ &\Gamma_{\alpha\beta}(\mathbf{R})=\frac{1}{2}\Big[\frac{\partial \mathbf{R}}{\partial s_\alpha}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_\beta+\frac{\partial \mathbf{R}}{\partial s_\beta}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_\alpha\Big],\\ &\mathcal{Z}_{\alpha\beta}=\frac{1}{2}\Big[\Big(\frac{\partial \mathcal{V}}{\partial s_\alpha}-\mathbf{R}\mathbf{t}_\alpha\Big)\cdot\mathbf{R}\mathbf{t}_\beta+\Big(\frac{\partial \mathcal{V}}{\partial s_\beta}-\mathbf{R}\mathbf{t}_\beta\Big)\cdot\mathbf{R}\mathbf{t}_\alpha\Big],\quad \mathcal{Z}_{3\alpha}=\frac{\partial \mathcal{V}}{\partial s_\alpha}\cdot\mathbf{R}\mathbf{n}, \end{aligned}
$$

where $(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})$ denotes the 3×3 matrix with first column $\mathbf{t}_1$, second column $\mathbf{t}_2$ and third column $\mathbf{n}$ and where $(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}=\big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\big)^T$. Let us notice that $E^s(v)$ belongs to $\big(L^2(\Omega_\delta)\big)^{3\times 3}$ for any deformation

$$v(s)=\mathcal{V}(s_1,s_2)+s_3\mathbf{R}(s_1,s_2)\mathbf{n}(s_1,s_2)+\overline{v}(s),\qquad \text{for a.e. } s\in\Omega_\delta$$

where $\mathcal{V}\in(H^1(\omega))^3$, $\mathbf{R}\in H^1(\omega;SO(3))$ and $\overline{v}\in L^2(\omega;H^1(-\delta,\delta))^3$.

Remark 5.1. From the last estimate in (3.5) we deduce that

$$||\Pi_\delta\big(\nabla_x v-\mathbf{R}\big)||_{(L^2(\Omega))^{3\times 3}}\le C\delta$$

and then we get that the set

$$\big\{s\in\Omega\ |\ |||\Pi_\delta\big(\nabla_x v-\mathbf{R}\big)(s)|||\ge 1\big\}$$

has a measure less than $C\delta^2$. It follows that the measure of the set

$$\big\{s\in\Omega\ |\ \det\big(\Pi_\delta\big[\nabla_x v\big](s)\big)\le 0\big\}$$

tends to 0 as δ goes to 0. □

Now, we introduce the following closed subset $\mathbb{D}_\delta$ of $(H^1(\omega))^3\times(H^1(\omega))^{3\times 3}\times(L^2(\omega;H^1(-\delta,\delta)))^3$

$$\begin{aligned}\mathbb{D}_\delta=\Big\{&\mathbf{v}=(\mathcal{V},\mathbf{R},\overline{v})\in(H^1(\omega))^3\times(H^1(\omega))^{3\times 3}\times(L^2(\omega;H^1(-\delta,\delta)))^3\ |\\ &\mathbf{R}(s_1,s_2)\in SO(3),\qquad \int_{-\delta}^{\delta}\overline{v}(s_1,s_2,s_3)ds_3=0,\\ &\int_{-\delta}^{\delta}s_3\overline{v}(s_1,s_2,s_3)\cdot\mathbf{t}_\alpha(s_1,s_2)ds_3=0,\ \text{ for a.e. } (s_1,s_2)\in\omega,\ \ \alpha=1,2.\Big\}\end{aligned}$$

The last condition on $\overline{v}$ in $\mathbb{D}_\delta$ is not satisfied in general (if $\overline{v}$ is the warping introduced in Theorem 3.1), loosely speaking this new condition will allow to decouple the estimates of $\overline{v}$ and $\mathcal{Z}_{i\alpha}$ (see the proof of Proposition 5.2).

For any $\mathbf{v}\in\mathbb{D}_\delta$, we consider v defined by

$$v(s)=\mathcal{V}(s_1,s_2)+s_3\mathbf{R}(s_1,s_2)\mathbf{n}(s_1,s_2)+\overline{v}(s),\qquad \text{for a.e. } s\in\Omega_\delta. \tag{5.4}$$

The deformation v belongs to $(L^2(\omega;H^1(-\delta,\delta)))^3$ so that, in general, the Green-St Venant's tensor of v is not defined. Nevertheless, the tensor field $E^s(v)$ belongs to $(L^2(\Omega_\delta))^{3\times 3}$ and we set

$$\widehat{E}(\mathbf{v})=E^s(v),\qquad\qquad \widehat{E}(\mathbf{v})\in(L^2(\Omega_\delta)^{3\times 3}. \tag{5.5}$$

Let us point out that if a triplet $\mathbf{v}$ satisfies the limit kinematic condition $\dfrac{\partial\mathcal{V}}{\partial s_\alpha}=\mathbf{R}\mathbf{t}_\alpha$, then it is easy to obtain

$$\frac{1}{\delta}||\overline{v}||_{(L^2(\Omega_\delta)^3}+\Big\|\frac{\partial\overline{v}}{\partial s_3}\Big\|_{(L^2(\Omega_\delta)^3}\le||\widehat{E}(\mathbf{v})||_{(L^2(\Omega_\delta))^{3\times 3}},\qquad \Big\|\frac{\partial\mathbf{R}}{\partial s_\alpha}\Big\|_{(L^2(\omega))^{3\times 3}}\le\frac{C}{\delta^{3/2}}||\widehat{E}(\mathbf{v})||_{(L^2(\Omega_\delta))^{3\times 3}}$$

which permits with some boundary conditions to control the product norm of $\mathbf{v}$ in term of $||\widehat{E}(\mathbf{v})||_{(L^2(\Omega_\delta))^{3\times 3}}$ and δ. In order to define an energy which have this property for any $\mathbf{v}\in\mathbb{D}_\delta$, we are led to add two penalization terms, which vanish as $\delta \to 0$, to $||\widehat{E}(\mathbf{v})||^2_{(L^2(\Omega_\delta))^{3\times 3}}$. This is why for every deformation $\mathbf{v} \in \mathbb{D}_\delta$ we set

$$\mathcal{E}_\delta(\mathbf{v}) = ||\widehat{E}(\mathbf{v})||^2_{(L^2(\Omega_\delta))^{3\times 3}} + \delta^3 \Big\| \frac{\partial \mathbf{R}}{\partial s_1}\mathbf{t}_2 - \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{t}_1 \Big\|^2_{(L^2(\omega))^3} + \delta \Big\| \frac{\partial \mathcal{V}}{\partial s_1}\cdot \mathbf{R}\mathbf{t}_2 - \frac{\partial \mathcal{V}}{\partial s_2}\cdot \mathbf{R}\mathbf{t}_1 \Big\|^2_{L^2(\omega)}. \tag{5.6}$$

Proposition 5.2. *There exists a positive constant C which does not depend on δ such that for all $\mathbf{v} \in \mathbb{D}_\delta$*

$$\begin{aligned} \frac{1}{\delta}||\overline{v}||_{(L^2(\Omega_\delta))^3} + \Big\| \frac{\partial \overline{v}}{\partial s_3} \Big\|_{(L^2(\Omega_\delta))^3} &\le ||\widehat{E}(\mathbf{v})||_{(L^2(\Omega_\delta))^{3\times 3}} \\ \Big\| \frac{\partial \mathbf{R}}{\partial s_\alpha} \Big\|^2_{(L^2(\omega))^{3\times 3}} + \frac{1}{\delta^2}\Big\| \frac{\partial \mathcal{V}}{\partial s_\alpha} - \mathbf{R}\mathbf{t}_\alpha \Big\|^2_{(L^2(\omega))^3} &\le \frac{C}{\delta^3}\mathcal{E}_\delta(\mathbf{v}). \end{aligned}$$

Proof. First of all there exists a positive constant C independent of δ such that

$$\begin{cases} \delta \Big\| \frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot \mathbf{R}\mathbf{t}_1 \Big\|_{L^2(\omega)} + \delta \Big\| \frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot \mathbf{R}\mathbf{t}_2 + \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot \mathbf{R}\mathbf{t}_1 \Big\|_{L^2(\omega)} + \delta \Big\| \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot \mathbf{R}\mathbf{t}_2 \Big\|_{L^2(\omega)} \\ \quad + ||\mathcal{Z}_{11}||_{L^2(\omega)} + ||\mathcal{Z}_{12}||_{L^2(\omega)} + ||\mathcal{Z}_{22}||_{L^2(\omega)} \le \frac{C}{\delta^{1/2}}||\widehat{E}(\mathbf{v})||_{(L^2(\Omega_\delta))^{3\times 3}}, \\ \Big\| \mathbf{R}^T\frac{\partial \overline{v}}{\partial s_3}\cdot \mathbf{t}_\alpha + \mathcal{Z}_{3\alpha} \Big\|_{L^2(\Omega_\delta)} + \Big\| \mathbf{R}^T\frac{\partial \overline{v}}{\partial s_3}\cdot \mathbf{n} \Big\|_{L^2(\Omega_\delta)} \le C||\widehat{E}(\mathbf{v})||_{(L^2(\Omega_\delta))^{3\times 3}}. \end{cases} \tag{5.7}$$

We use the definition of $\mathbb{D}_\delta$ to estimate the field $\mathbf{R}^T\overline{v}\cdot \mathbf{t}_\alpha$. Introducing the function $\mathbf{R}^T\overline{v}\cdot \mathbf{t}_\alpha + s_3\mathcal{Z}_{3\alpha}$, using Poincaré-Wirtinger's inequality and the first condition on $\mathbf{R}^T\overline{v}$ in $\mathbb{D}_\delta$ give

$$\Big\| \mathbf{R}^T\overline{v}\cdot \mathbf{t}_\alpha + s_3\mathcal{Z}_{3\alpha} \Big\|_{L^2(\Omega_\delta)} + \Big\| \mathbf{R}^T\overline{v}\cdot \mathbf{n} \Big\|_{L^2(\Omega_\delta)} \le C\delta||\widehat{E}(\mathbf{v})||_{(L^2(\Omega_\delta))^{3\times 3}}. \tag{5.8}$$

Now we use the second condition on $\overline{v}\cdot \mathbf{t}_\alpha$ (in the definition of $\mathbb{D}_\delta$) in the above estimates and again (5.7) to get the estimates on $\mathbf{R}^T\dfrac{\partial \overline{v}}{\partial s_3}$ and $\mathcal{Z}_{3\alpha}$

$$\sum_{\alpha=1}^{2}\Big\{ \Big\| \mathbf{R}^T\frac{\partial \overline{v}}{\partial s_3}\cdot \mathbf{t}_\alpha \Big\|_{(L^2(\Omega_\delta))^3} + \delta^{1/2}||\mathcal{Z}_{3\alpha}||_{L^2(\omega)} \Big\} + \Big\| \mathbf{R}^T\frac{\partial \overline{v}}{\partial s_3}\cdot \mathbf{n} \Big\|_{(L^2(\Omega_\delta))^3} \le C||\widehat{E}(\mathbf{v})||_{(L^2(\Omega_\delta))^{3\times 3}}.$$

Finally (5.8) gives the L^2 estimate on $\overline{v}$. Let us notice that due to the last condition on $\overline{v}$ in $\mathbb{D}_\delta$, we obtain the same estimates that in the case where $\mathbf{v}$ satisfies the limit kinematic condition $\dfrac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R}\mathbf{t}_\alpha$.
There exist two antisymmetric matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ in $(L^2(\omega))^{3\times 3}$ such that

$$\frac{\partial \mathbf{R}}{\partial s_1} = \mathbf{R}\mathbf{A}_1 \qquad \frac{\partial \mathbf{R}}{\partial s_2} = \mathbf{R}\mathbf{A}_2.$$

From (5.7) we get

$$\big\| \mathbf{A}_1\mathbf{n}\cdot \mathbf{t}_1 \big\|_{L^2(\omega)} + \big\| \mathbf{A}_1\mathbf{n}\cdot \mathbf{t}_2 + \mathbf{A}_2\mathbf{n}\cdot \mathbf{t}_1 \big\|_{L^2(\omega)} + \big\| \mathbf{A}_2\mathbf{n}\cdot \mathbf{t}_2 \big\|_{L^2(\omega)} \le \frac{C}{\delta^{3/2}}||\widehat{E}(\mathbf{v})||_{(L^2(\Omega_\delta))^{3\times 3}}.$$

Besides there exists a positive constant such

$$\begin{aligned} &||\mathbf{A}_1||_{(L^2(\omega))^{3\times 3}} + ||\mathbf{A}_2||_{(L^2(\omega))^{3\times 3}} \\ \le &C\Big\{ \big\| \mathbf{A}_1\mathbf{n}\cdot \mathbf{t}_1 \big\|_{L^2(\omega)} + \big\| \mathbf{A}_1\mathbf{n}\cdot \mathbf{t}_2 + \mathbf{A}_2\mathbf{n}\cdot \mathbf{t}_1 \big\|_{L^2(\omega)} + \big\| \mathbf{A}_2\mathbf{n}\cdot \mathbf{t}_2 \big\|_{L^2(\omega)} + ||\mathbf{A}_1\mathbf{t}_2 - \mathbf{A}_2\mathbf{t}_1||_{(L^2(\omega))^3} \Big\}. \end{aligned}$$

Hence we get

$$\Big\|\frac{\partial \mathbf{R}}{\partial s_1}\Big\|_{(L^2(\omega))^{3\times 3}}+\Big\|\frac{\partial \mathbf{R}}{\partial s_2}\Big\|_{(L^2(\omega))^{3\times 3}}\le C\Big\{\frac{1}{\delta^{3/2}}||\widehat{E}(\mathbf{v})||_{(L^2(\Omega_\delta))^{3\times 3}}+\Big\|\frac{\partial \mathbf{R}}{\partial s_1}\mathbf{t}_2-\frac{\partial \mathbf{R}}{\partial s_2}\mathbf{t}_1\Big\|_{(L^2(\omega))^3}\Big\}.$$

Due to the estimates concerning the $\mathcal{Z}_{i\alpha}$ and the definition of $\mathcal{E}_\delta(\mathbf{v})$ we finally obtain

$$\Big\|\frac{\partial \mathcal{V}}{\partial s_\alpha}-\mathbf{R}\mathbf{t}_\alpha\Big\|^2_{(L^2(\omega))^3}\le \frac{C}{\delta}\mathcal{E}_\delta(\mathbf{v}).$$

□

We define now the set of the admissible triplets

$$\mathbb{D}_{\delta,\gamma_0}=\Big\{\mathbf{v}=(\mathcal{V},\mathbf{R},\overline{v})\in\mathbb{D}_\delta \mid \mathcal{V}=\phi,\;\; \mathbf{R}=\mathbf{I}_3 \quad \text{on } \gamma_0\Big\}.$$

Notice that the triplet $\mathbf{I}_d=(\phi,\mathbf{I}_3,0)$ belongs to $\mathbb{D}_{\delta,\gamma_0}$ and it is associated to the deformation $v=I_d$.

In some sense, the following corollary gives two Korn's type inequalities on the set $\mathbb{D}_{\delta,\gamma_0}$ with respect to the quantity $\mathcal{E}_\delta(\mathbf{v})$, the more accurate of which depending on the order of $\mathcal{E}_\delta(\mathbf{v})$.

Corollary 5.3. *There exists a positive constant C which does not depend on δ such that for all $\mathbf{v}\in\mathbb{D}_{\delta,\gamma_0}$*

$$\begin{aligned}&||\mathcal{V}-\phi||^2_{(H^1(\omega))^3}+||\mathbf{R}-\mathbf{I}_3||^2_{(H^1(\omega))^{3\times 3}}\le \frac{C}{\delta^3}\mathcal{E}_\delta(\mathbf{v}),\\ &||\mathcal{V}-\phi||^2_{(H^1(\omega))^3}\le C\Big(1+\frac{1}{\delta}\mathcal{E}_\delta(\mathbf{v})\Big).\end{aligned}$$

Proof. Recall that $\mathbf{R}=\mathbf{I}_3$ and $\mathcal{V}=\phi$ on γ_0, then from Proposition 5.1 we obtain

$$||\mathbf{R}-\mathbf{I}_3||^2_{(H^1(\omega))^{3\times 3}}\le \frac{C}{\delta^3}\mathcal{E}_\delta(\mathbf{v}).$$

Using the above estimate and again Proposition 5.1 we obtain the first estimate on $\mathcal{V}-\phi$ (recall that $\mathbf{t}_\alpha=\dfrac{\partial\phi}{\partial s_\alpha}$). To obtain the second estimate on $\mathcal{V}-\phi$, notice that $||\mathbf{R}-\mathbf{I}_3||^2_{(L^2(\omega))^{3\times 3}}\le C$. □

6. Elastic shells

In this section we consider a shell made of an elastic material. Its thickness 2δ is fixed and belongs to $]0,2\delta_0]$. The local energy $W\;:\;\mathbf{S}_3\longrightarrow\mathbb{R}^+$ is a continuous function of symmetric matrices which satisfies the following assumptions which are similar to those adopted in [14], [15] and [16] (the reader is also referred to [8] for general introduction to elasticity)

(6.1) $\quad \exists c>0 \quad \text{such that} \quad \forall E\in\mathbf{S}_3 \quad W(E)\ge c|||E|||^2,$

(6.2) $\quad \forall\varepsilon>0,\quad \exists\theta>0,\quad \text{such that}\quad \forall E\in\mathbf{S}_3\quad |||E|||\le\theta \implies |W(E)-Q(E)|\le\varepsilon|||E|||^2,$

where Q is a positive quadratic form defined on the set of 3×3 symmetric matrices. Remark that Q satisfies (6.1) with the same constant c.

Still following [8], for any 3×3 matrix F, we set

$$\widehat{W}(F)=\begin{cases} W\Big(\dfrac{1}{2}(F^TF-\mathbf{I}_3)\Big) & \text{if}\quad \det(F)>0\\ +\infty & \text{if}\quad \det(F)\le 0.\end{cases} \tag{6.3}$$

Remark that due to (6.1), (6.3) and to the inequality $|||F^TF - \mathbf{I}_3||| \geq dist(F, SO(3))$ if $\det(F) > 0$, we have for any 3×3 matrix F

$$\widehat{W}(F) \geq \frac{c}{4} dist(F, SO(3))^2. \tag{6.4}$$

Remark 6.1. As a classical example of a local elastic energy satisfying the above assumptions, we mention the following St Venant-Kirchhoff's law (see [8]) for which

$$\widehat{W}(F) = \begin{cases} \frac{\lambda}{8}\big(tr(F^TF - \mathbf{I}_3)\big)^2 + \frac{\mu}{4} tr\big((F^TF - \mathbf{I}_3)^2\big) & \text{if} \quad \det(F) > 0 \\ +\infty \qquad \text{if} \qquad \det(F) \leq 0. \end{cases}$$

In order to take into account the boundary condition on the admissible deformations we introduce the space

$$\mathbf{U}_\delta = \Big\{ v \in (H^1(\mathcal{Q}_\delta))^3 \mid v = I_d \quad \text{on} \quad \Gamma_{0,\delta} \Big\}. \tag{6.5}$$

Let $\kappa \geq 1$. Now we assume that the shell is submitted to applied body forces $f_{\kappa,\delta} \in (L^2(\Omega_\delta))^3$ and we define the total energy $J_{\kappa,\delta}(v)$* over $\mathbf{U}_\delta$ by

$$J_{\kappa,\delta}(v) = \int_{\mathcal{Q}_\delta} \widehat{W}(\nabla_x v)(x)dx - \int_{\mathcal{Q}_\delta} f_{\kappa,\delta}(x) \cdot (v(x) - I_d(x))dx. \tag{6.6}$$

To introduce the scaling on $f_{\kappa,\delta}$, let us consider f and g in $(L^2(\omega))^3$ and assume that the force $f_{\kappa,\delta}$ is given by

$$f_{\kappa,\delta}(x) = \delta^{\kappa'} f(s_1, s_2) + \delta^{\kappa' - 2} s_3 g(s_1, s_2) \qquad \text{for a.e. } x = \Phi(s) \in \mathcal{Q}_\delta. \tag{6.7}$$

where

$$\kappa' = \begin{cases} 2\kappa - 2 & \text{if} \quad 1 \leq \kappa \leq 2, \\ \kappa & \text{if} \quad \kappa \geq 2. \end{cases} \tag{6.8}$$

Notice that $J_{\kappa,\delta}(I_d) = 0$. So, in order to minimize $J_{\kappa,\delta}$ we only need to consider deformations v of $\mathbf{U}_\delta$ such that $J_{\kappa,\delta}(v) \leq 0$.

Now from (6.1), (6.3), (6.4), the two Korn's type inequalities (3.1)-(3.2), the assumption (6.7) of the body forces and the definition (6.8) of κ', we obtain the following bound for $||\text{dist}(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)}$

$$||\text{dist}(\nabla_x v, SO(3))||_{L^2(\mathcal{Q}_\delta)} \leq C\delta^{\kappa - 1/2} \quad \text{and} \quad \int_{\mathcal{Q}_\delta} f_{\kappa,\delta} \cdot (v - I_d) \leq C\delta^{2\kappa - 1} \tag{6.9}$$

which in turn imply that

$$c\delta^{2\kappa-1} \leq J_{\kappa,\delta}(v) \leq 0. \tag{6.10}$$

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* For later convenience, we have added the term $\int_{\mathcal{Q}_\delta} f_{\kappa,\delta}(x) \cdot I_d(x)dx$ to the usual standard energy, indeed this does not affect the minimizing problem for $J_{\kappa,\delta}$.

Again from (6.3)-(6.4) and the estimates (6.9) we deduce

$$\frac{c}{4}||(\nabla_x v)^T\nabla_x v-\mathbf{I}_3||^2_{(L^2(\mathcal{Q}_\delta))^{3\times 3}}\le J_{\kappa,\delta}(v)+\int_{\mathcal{Q}_\delta} f_{\kappa,\delta}\cdot(v-I_d)\le C\delta^{2\kappa-1}.$$

Hence, the following estimate of the Green-St Venant's tensor:

$$\Big|\Big|\frac{1}{2}\big\{(\nabla_x v)^T\nabla_x v-\mathbf{I}_3\big\}\Big|\Big|_{(L^2(\mathcal{Q}_\delta))^{3\times 3}}\le C\delta^{\kappa-1/2}.$$

We deduce from the above inequality that $v\in(W^{1,4}(\mathcal{Q}_\delta))^3$ with

$$||\nabla_x v||_{(L^4(\mathcal{Q}_\delta))^{3\times 3}}\le C\delta^{\frac{1}{4}}.$$

We set

$$m_{\kappa,\delta}=\inf_{v\in\mathbf{U}_\delta} J_{\kappa,\delta}(v).$$

As a consequence of (6.10) we have

$$c\le\frac{m_{\kappa,\delta}}{\delta^{2\kappa-1}}\le 0.$$

In general, a minimizer of $J_{\kappa,\delta}$ does not exist on $\mathbf{U}_\delta$. In what follows, we replace the elastic functional $v\longmapsto J_{\kappa,\delta}(v)$ on $\mathbf{U}_\delta$ by a simplified functional defined on $\mathbb{D}_\delta$ which admits a minimum.

From now on we assume $\kappa>1$.

7. The simplified elastic model for shells

The aim of this section is to define a functional $J^s_{\kappa,\delta}$ on the set $\mathbb{D}_{\delta,\gamma_0}$, which will appear as a simplification of the total energy $J_{\kappa,\delta}$ defined on the set $\mathbf{U}_\delta$. In order to perform this task, we use the results of Section 5 and we proceed in three steps. Let us first consider an admissible deformation v satisfying (3.3), decomposed as in (3.4) and such that $J_{\kappa,\delta}(v)\le 0$. It is convenient to express the energy $J_{\kappa,\delta}(v)$ over the domain Ω_δ

$$\begin{aligned}J_{\kappa,\delta}(v)=&\int_{\Omega_\delta} W\Big(\frac{1}{2}\big((\nabla_x v)^T\nabla_x v-\mathbf{I}_3\big)\Big)\det\Big(\mathbf{t}_1+s_3\frac{\partial\mathbf{n}}{\partial s_1}\,|\,\mathbf{t}_2+s_3\frac{\partial\mathbf{n}}{\partial s_2}\,|\,\mathbf{n}\Big)ds_1ds_2ds_3\\&-\int_{\Omega_\delta}\big(\delta^{\kappa'}f+\delta^{\kappa'-2}s_3g\big)\cdot(v-I_d)\det\Big(\mathbf{t}_1+s_3\frac{\partial\mathbf{n}}{\partial s_1}\,|\,\mathbf{t}_2+s_3\frac{\partial\mathbf{n}}{\partial s_2}\,|\,\mathbf{n}\Big)ds_1ds_2ds_3.\end{aligned}\tag{7.1}$$

The triplet associated to v by the decomposition (3.4) is denoted $\mathbf{v}=\big(\mathcal{V},\mathbf{R},\overline{v}\big)$. The following estimate has been proved in Section 6

$$\Big|\Big|\frac{1}{2}\big\{(\nabla_x v)^T\nabla_x v-\mathbf{I}_3\big\}\Big|\Big|_{(L^2(\Omega_\delta))^{3\times 3}}\le C\delta^{\kappa-1/2}.$$

Then, for all $\theta>0$, the set $A^\theta_\delta=\{s\in\Omega;\,|||\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)(s)|||\ge\theta\}$ has a measure satisfying

$$\text{meas}(A^\theta_\delta)\le C\frac{\delta^{2\kappa-2}}{\theta^2}.$$

Now, according to assumptions (6.2) and $\kappa>1$ and the above estimate, in the first term of the total energy $J_{\kappa,\delta}(v)$ we replace the quantity $W\Big(\frac{1}{2}\big((\nabla_x v)^T\nabla_x v-\mathbf{I}_3\big)\Big)$ by $Q\Big(\frac{1}{2}\big((\nabla_x v)^T\nabla_x v-\mathbf{I}_3\big)\Big)$. Following the analysis of Section 5, we then replace $Q\Big(\frac{1}{2}\big((\nabla_x v)^T\nabla_x v-\mathbf{I}_3\big)\Big)$ by $Q\big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\widehat{E}(\mathbf{v})(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\big)$ where $\widehat{E}(\mathbf{v})$ is defined by (5.3) and (5.5). At last, we replace $\det\big(\mathbf{t}_1 + s_3\dfrac{\partial \mathbf{n}}{\partial s_1}|\mathbf{t}_2 + s_3\dfrac{\partial \mathbf{n}}{\partial s_2}|\mathbf{n}\big)$ by $\det\big(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}\big)$. Setting for all 3×3 symmetric matrix F

$$W^s(F) = Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\,F\,(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big) \tag{7.2}$$

all the above considerations lead us to replace the first term in the right hand side of (7.1) by

$$\int_{\Omega_\delta} W^s\big(\widehat{E}(\mathbf{v})\big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2ds_3. \tag{7.3}$$

Observe now the term involving the forces in (7.1). We have

$$\begin{aligned}
&\Big|\int_{\Omega_\delta}\big(\delta^{\kappa'} f + \delta^{\kappa'-2}s_3 g\big)\cdot\big(v - I_d\big)\det\big(\mathbf{t}_1 + s_3\frac{\partial \mathbf{n}}{\partial s_1}|\mathbf{t}_2 + s_3\frac{\partial \mathbf{n}}{\partial s_2}|\mathbf{n}\big)ds_1ds_2ds_3\\
&\quad - 2\delta^{\kappa'+1}\int_\omega\Big[f\cdot(\mathcal{V}-\phi) + \frac{1}{3}g\cdot(\mathbf{R}-\mathbf{I}_3)\mathbf{n}\Big]\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2\\
&\quad - \frac{2}{3}\delta^{\kappa'+1}\int_\omega g\cdot(\mathcal{V}-\phi)\Big[\det\Big(\frac{\partial \mathbf{n}}{\partial s_1}|\mathbf{t}_2\mid\mathbf{n}\Big) + \det\Big(\mathbf{t}_1|\frac{\partial \mathbf{n}}{\partial s_2}\mid\mathbf{n}\Big)\Big]ds_1ds_2\Big|\\
&\le C\delta^{\kappa'+2}\big(||f||_{(L^2(\omega))^3} + ||g||_{(L^2(\omega))^3}\big)\big(||\mathcal{V}-\phi||_{(L^2(\omega))^3} + ||\mathbf{R}-\mathbf{I}_3||_{(L^2(\omega))^{3\times 3}} + \frac{1}{\delta^{5/2}}||\overline{v}||_{(L^2(\Omega_\delta))^3}\big).
\end{aligned}$$

Then, in view of the first estimate in (3.5) we replace the term involving the forces by

$$\mathcal{L}_{\kappa,\delta}(\mathcal{V},\mathbf{R}) = \delta^{\kappa'+1}\mathcal{L}(\mathcal{V},\mathbf{R}) \tag{7.4}$$

where

$$\begin{aligned}
\mathcal{L}(\mathcal{V},\mathbf{R}) = 2\int_\omega&\Big[f\cdot(\mathcal{V}-\phi) + \frac{1}{3}g\cdot(\mathbf{R}-\mathbf{I}_3)\mathbf{n}\Big]\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2\\
&+\frac{2}{3}\int_\omega g\cdot(\mathcal{V}-\phi)\Big[\det\Big(\frac{\partial \mathbf{n}}{\partial s_1}|\mathbf{t}_2\mid\mathbf{n}\Big) + \det\Big(\mathbf{t}_1|\frac{\partial \mathbf{n}}{\partial s_2}\mid\mathbf{n}\Big)\Big]ds_1ds_2.
\end{aligned}$$

At the end of this first step, we obtain a simplified energy for a deformation $v\in\mathbf{U}_\delta$ which satisfies (3.3) and $J_{\kappa,\delta}(v)\le 0$

$$J^{simpl}_{\kappa,\delta}(v) = \int_{\Omega_\delta} W^s\big(\widehat{E}(\mathbf{v})\big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2ds_3 - \delta^{\kappa'+1}\mathcal{L}(\mathcal{V},\mathbf{R}).$$

Indeed the energy $J^{simpl}_{\kappa,\delta}(v)$ can be seen as a functional of $\mathbf{v}$ defined over $\mathbb{D}_{\delta,\gamma_0}$ since we have already notice that $\widehat{E}(\mathbf{v})$ belongs to $(L^2(\Omega_\delta))^{3\times 3}$. As a consequence, in a second step we are in a position to extend the above energy to the whole set $\mathbb{D}_{\delta,\gamma_0}$ and to put

$$\forall\mathbf{v}\in\mathbb{D}_{\delta,\gamma_0},\qquad J^{\mathbf{simpl}}_{\kappa,\delta}(\mathbf{v}) = \int_{\Omega_\delta} W^s\big(\widehat{E}(\mathbf{v})\big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2ds_3 - \delta^{\kappa'+1}\mathcal{L}(\mathcal{V},\mathbf{R}).$$

As observed in Section 5, the functional $J^{\mathbf{simpl}}_{\kappa,\delta}$ is not coercive on $\mathbb{D}_{\delta,\gamma_0}$. In a third step, in view of Proposition 5.2 and in order to obtain the coerciveness of the simplified energy, the two terms $\delta^3\Big\|\dfrac{\partial\mathbf{R}}{\partial s_1}\mathbf{t}_2 - \dfrac{\partial\mathbf{R}}{\partial s_2}\mathbf{t}_1\Big\|^2_{(L^2(\omega))^3}$, $\delta\Big\|\dfrac{\partial\mathcal{V}}{\partial s_1}\cdot\mathbf{R}\mathbf{t}_2 - \dfrac{\partial\mathcal{V}}{\partial s_2}\cdot\mathbf{R}\mathbf{t}_1\Big\|^2_{L^2(\omega)}$ are added to $J^{\mathbf{simpl}}_{\kappa,\delta}$.

Using all the above considerations, we are able to define the simplified elastic energy on $\mathbb{D}_{\delta,\gamma_0}$ by setting for any $\mathbf{v}$ in $\mathbb{D}_{\delta,\gamma_0}$

$$(7.5)\qquad \begin{cases} J^s_{\kappa,\delta}(\mathbf{v}) = \displaystyle\int_{\Omega_\delta} W^s\big(\widehat{E}(\mathbf{v})\big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2ds_3 + \delta^3\Big\|\frac{\partial \mathbf{R}}{\partial s_1}\mathbf{t}_2 - \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{t}_1\Big\|^2_{(L^2(\omega))^3} \\ \qquad + \delta\Big\|\dfrac{\partial \mathcal{V}}{\partial s_1}\cdot\mathbf{R}\mathbf{t}_2 - \dfrac{\partial \mathcal{V}}{\partial s_2}\cdot\mathbf{R}\mathbf{t}_1\Big\|^2_{L^2(\omega)} - \delta^{\kappa'+1}\mathcal{L}(\mathcal{V},\mathbf{R}). \end{cases}$$

The end of this section is dedicated to show that the functional $J^s_{\kappa,\delta}$ admits a minimizer on $\mathbb{D}_{\delta,\gamma_0}$. Let $\mathbf{v}$ be in $\mathbb{D}_{\delta,\gamma_0}$ we have

$$(7.6)\qquad \Big|\mathcal{L}(\mathcal{V},\mathbf{R})\Big| \le C\big(||f||_{(L^2(\omega))^3} + ||g||_{(L^2(\omega))^3}\big)\big(||\mathcal{V}-\phi||_{(L^2(\omega))^3} + ||\mathbf{R}-\mathbf{I}_3||_{(L^2(\omega))^{3\times 3}}\big).$$

The quadratic form Q being positive, the definition (5.6) of $\mathcal{E}_\delta(\mathbf{v})$ and (7.5)-(7.6) give

$$C\mathcal{E}_\delta(\mathbf{v}) - C\delta^{\kappa'+1}\big(||f||_{(L^2(\omega))^3} + ||g||_{(L^2(\omega))^3}\big)\big(||\mathcal{V}-\phi||_{(L^2(\omega))^3} + ||\mathbf{R}-\mathbf{I}_3||_{(L^2(\omega))^{3\times 3}}\big) \le J^s_{\kappa,\delta}(\mathbf{v}).$$

Now thanks to Corollary 5.3 and (6.8), we get, if $J^s_{\kappa,\delta}(\mathbf{v}) \le 0\big(= J^s_{\kappa,\delta}(\mathbf{I}_d)\big)$

$$(7.7)\qquad \mathcal{E}_\delta(\mathbf{v}) \le C\delta^{2\kappa-1}(||f||_{(L^2(\omega))^3} + ||g||_{(L^2(\Omega))^3})^2.$$

Hence, there exists a constant c which does not depend on δ such that for any $\mathbf{v}\in\mathbb{D}_{\delta,\gamma_0}$ satisfying $J^s_{\kappa,\delta}(\mathbf{v}) \le 0$, we have

$$c\delta^{2\kappa-1} \le J^s_{\kappa,\delta}(\mathbf{v}).$$

We set

$$(7.8)\qquad m^s_{\kappa,\delta} = \inf_{\mathbf{v}\in\mathbb{D}_{\delta,\gamma_0}} J^s_{\kappa,\delta}(\mathbf{v}).$$

As a consequence of the above inequality, we have

$$c \le \frac{m^s_{\kappa,\delta}}{\delta^{2\kappa-1}} \le 0.$$

In the following theorem we prove that for κ and δ fixed the minimization problem (7.8) has at least a solution.

Theorem 7.1. *There exists $\mathbf{v}_\delta \in \mathbb{D}_{\delta,\gamma_0}$ such that*

$$(7.9)\qquad m^s_{\kappa,\delta} = J^s_{\kappa,\delta}(\mathbf{v}_\delta) = \min_{v\in\mathbb{D}_{\delta,\gamma_0}} J^s_{\kappa,\delta}(\mathbf{v}).$$

Proof. Since $J^s_{\kappa,\delta}(\mathbf{I}_d) = 0$, we can consider a minimizing sequence $\mathbf{v}_n$ in $\mathbb{D}_{\delta,\gamma_0}$ such that $J^s_{\kappa,\delta}(\mathbf{v}_n) \le 0$ and

$$m^s_{\kappa,\delta} = \lim_{n\to+\infty} J^s_{\kappa,\delta}(\mathbf{v}_n).$$

From (7.7) we get

$$\mathcal{E}_\delta(\mathbf{v}_n) \le C\delta^{2\kappa-1}(||f_{(L^2(\omega))^3} + ||g||_{(L^2(\Omega))^3})^4.$$

Thanks to Corollary 5.3 and Proposition 5.2, the above estimate show that there exists a subsequence still denoted n such that (recall that $||\mathbf{R}_n||_{(L^\infty(\omega))^{3\times 3}} = \sqrt{3}$)

$$\begin{aligned} &\mathcal{V}_n \rightharpoonup \mathcal{V}_\delta \quad \text{weakly in } (H^1(\omega))^3\\ &\mathbf{R}_n \rightharpoonup \mathbf{R}_\delta \quad \text{weakly in } (H^1(\omega))^{3\times 3}\\ &\mathbf{R}_n \longrightarrow \mathbf{R}_\delta \quad \text{strongly in } (L^2(\omega))^{3\times 3} \text{ and a.e. in } \omega\\ &\overline{v}_n \rightharpoonup \overline{v}_\delta \quad \text{weakly in } (L^2(\omega; H^1(-\delta,\delta)))^3.\end{aligned}$$

Then setting $\mathbf{v}_\delta = (\mathcal{V}_\delta, \mathbf{R}_\delta, \overline{v}_\delta) \in \mathbb{D}_{\delta,\gamma_0}$, we get

$$\begin{aligned} &\widehat{E}(\mathbf{v}_n) \rightharpoonup \widehat{E}(\mathbf{v}_\delta) \quad \text{weakly in } (L^2(\Omega_\delta))^{3\times 3},\\ &\frac{\partial \mathcal{V}_n}{\partial s_1}\cdot \mathbf{R}_n\mathbf{t}_2 - \frac{\partial \mathcal{V}_n}{\partial s_2}\cdot \mathbf{R}_n\mathbf{t}_1 \rightharpoonup \frac{\partial \mathcal{V}_\delta}{\partial s_1}\cdot \mathbf{R}_\delta\mathbf{t}_2 - \frac{\partial \mathcal{V}_\delta}{\partial s_2}\cdot \mathbf{R}_\delta\mathbf{t}_1 \quad \text{weakly in } (L^2(\omega))^3.\end{aligned}$$

Now, passing to the limit inf in $J^s_{\kappa,\delta}(\mathbf{v}_n)$, we obtain

$$m^s_{\kappa,\delta} \le J^s_{\kappa,\delta}(\mathbf{v}_\delta) \le \varliminf_{n\to+\infty} J^s_{\kappa,\delta}(\mathbf{v}_n) = \lim_{n\to+\infty} J^s_{\kappa,\delta}(\mathbf{v}_n) = m^s_{\kappa,\delta}.$$

□

8. Asymptotic behavior of the simplified model. Case $\kappa = 2$.

In this section we study the asymptotic behavior of the sequence $(\mathbf{v}_\delta)$ of minimizer given in Theorem 7.1 and we characterize the limit of the minima $\dfrac{m^s_{2,\delta}}{\delta^3}$ as a minimum of a new functional. AS usual, to perform this task, we work on the fixed domain Ω and we use the operator Π_δ defined in Section 4. We denote $\mathbb{D}$ the following closed subset of $\mathbb{D}_{1,\gamma_0}$ (i.e. $\mathbb{D}_{\delta,\gamma_0}$ for $\delta = 1$ or $\mathbb{D}_{1,\gamma_0} = \Pi_\delta\big(\mathbb{D}_{\delta,\gamma_0}\big)$):

$$\mathbb{D} = \Big\{\mathbf{v} = (\mathcal{V}, \mathbf{R}, \overline{V}) \in \mathbb{D}_{1,\gamma_0} \;\mid\; \frac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R}\mathbf{t}_\alpha\Big\}.$$

Notice that $\mathcal{V} \in (H^2(\omega))^3$. Then we define the following functional over $\mathbb{D}$

$$\mathcal{J}_2(\mathbf{v}) = \int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{v})(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}) - \mathcal{L}(\mathcal{V}, \mathbf{R}). \tag{8.1}$$

where

$$\mathbf{E}(\mathbf{v}) = \begin{pmatrix} S_3\dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1 & S_3\dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 & \dfrac{1}{2}\mathbf{R}^T\dfrac{\partial \overline{V}}{\partial S_3}\cdot\mathbf{t}_1\\ * & S_3\dfrac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 & \dfrac{1}{2}\mathbf{R}^T\dfrac{\partial \overline{V}}{\partial S_3}\cdot\mathbf{t}_2\\ * & * & \mathbf{R}^T\dfrac{\partial \overline{V}}{\partial S_3}\cdot\mathbf{n}\end{pmatrix} \tag{8.2}$$

As in Theorem 7.1 we easily prove that there exists $\mathbf{v}_2 = (\mathcal{V}_2, \mathbf{R}_2, \overline{V}_2) \in \mathbb{D}$ such that

$$m^s_2 = \mathcal{J}_2(\mathbf{v}_2) = \min_{\mathbf{v}\in\mathbb{D}} \mathcal{J}_2(\mathbf{v}). \tag{8.3}$$

Theorem 8.1. *We have*

$$m^s_2 = \lim_{\delta\to 0} \frac{m^s_{2,\delta}}{\delta^3}.$$

Moreover, let $\mathbf{v}_\delta = (\mathcal{V}_\delta, \mathbf{R}_\delta, \overline{v}_\delta) \in \mathbb{D}_{\delta,\gamma_0}$ *be a minimizer of the functional* $J^s_{2,\delta}(\cdot)$*, there exists a subsequence still denoted* δ *such that*

$$
(8.4)\qquad \begin{cases} \mathcal{V}_\delta \longrightarrow \mathcal{V}_0 \quad \text{strongly in } (H^1(\omega))^3, \\ \mathbf{R}_\delta \longrightarrow \mathbf{R}_0 \quad \text{strongly in } (H^1(\omega))^{3\times 3}, \\ \dfrac{1}{\delta}\mathcal{Z}_{i\beta,\delta} \longrightarrow 0 \quad \text{strongly in } L^2(\omega), \\ \dfrac{1}{\delta^2}\Pi_\delta(\overline{v}_\delta) \longrightarrow \overline{V}_0 \quad \text{strongly in } (L^2(\omega; H^1(-1,1)))^3. \end{cases}
$$

The triplet $\mathbf{v}_0 = (\mathcal{V}_0, \mathbf{R}_0, \overline{V}_0)$ *belongs to* $\mathbb{D}$ *and we have*

$$
m^s_2 = \mathcal{J}_2(\mathbf{v}_0).
$$

Proof. For all $\mathbf{v} = (\mathcal{V}, \mathbf{R}, \overline{V}) \in \mathbb{D}$, we have $(\mathcal{V}, \mathbf{R}, \overline{v}_\delta) \in \mathbb{D}_{\delta,\gamma_0}$ where

$$
\overline{v}_\delta(s_1, s_2, s_3) = \delta^2 \overline{V}\big(s_1, s_2, \frac{s_3}{\delta}\big) \qquad \text{for a.e. } (s_1, s_2, s_3) \in \Omega_\delta.
$$

Using the fact that $\mathbf{v} \in \mathbb{D}$, which implies that $\dfrac{\partial \mathbf{R}}{\partial s_1}\mathbf{n} \cdot \mathbf{R}\mathbf{t}_2 = \dfrac{\partial \mathbf{R}}{\partial s_2}\mathbf{n} \cdot \mathbf{R}\mathbf{t}_1$, we have

$$
(8.5)\qquad \frac{J^s_{2,\delta}\big(\mathcal{V}, \mathbf{R}, \overline{v}_\delta\big)}{\delta^3} = \int_\Omega Q\Big((\mathbf{t}_1 \,|\, \mathbf{t}_2 \,|\, \mathbf{n})^{-T}\mathbf{E}(\mathbf{v})(\mathbf{t}_1 \,|\, \mathbf{t}_2 \,|\, \mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}) - \mathcal{L}(\mathcal{V}, \mathbf{R}) = \mathcal{J}_2(\mathbf{v}).
$$

Then, taking the minimum in the right hand side w.r.t. $\mathbf{v} \in \mathbb{D}$, we immediately deduce that $\dfrac{m^s_{2,\delta}}{\delta^3} \le m^s_2$. We recall that $\mathbf{v}_\delta = \big(\mathcal{V}_\delta, \mathbf{R}_\delta, \overline{v}_\delta\big) \in D_{\delta,\gamma_0}$ is a minimizer of J^s_δ

$$
c \le \frac{m^s_{2,\delta}}{\delta^3} = \frac{J^s_{2,\delta}(\mathbf{v}_\delta)}{\delta^3} = \min_{\mathbf{v} \in \mathbb{D}_{\delta,\gamma_0}} \frac{J^s_{2,\delta}(\mathbf{v})}{\delta^3}
$$

and moreover with (7.7)

$$
\mathcal{E}_\delta(\mathbf{v}_\delta) \le C\delta^3(||f_{(L^2(\omega))^3} + ||g||_{(L^2(\Omega))^3})^2.
$$

Thanks to the estimates in Proposition 5.2, Corollary 5.3 and the above estimate we can extract a subsequence still denoted δ such that

$$
(8.6)\qquad \begin{cases} \mathcal{V}_\delta \longrightarrow \mathcal{V}_0 \quad \text{strongly in } (H^1(\omega))^3, \\ \mathbf{R}_\delta \rightharpoonup \mathbf{R}_0 \quad \text{weakly in } (H^1(\omega))^{3\times 3} \quad \text{and a.e. in } \omega, \\ \dfrac{1}{\delta^2}\Pi_\delta(\overline{v}_\delta) \rightharpoonup \overline{V}_0 \quad \text{weakly in } (L^2(\omega; H^1(-1,1)))^3, \\ \dfrac{1}{\delta}\mathcal{Z}_{i\alpha,\delta} \rightharpoonup \mathcal{Z}_{i\alpha,0} \quad \text{weakly in } L^2(\omega), \\ \Big(\dfrac{\partial \mathbf{R}_\delta}{\partial s_1}\mathbf{t}_2 - \dfrac{\partial \mathbf{R}_\delta}{\partial s_2}\mathbf{t}_1\Big) \rightharpoonup Y \quad \text{weakly in } (L^2(\omega))^3, \\ \dfrac{1}{\delta}\Big(\dfrac{\partial \mathcal{V}_\delta}{\partial s_1} \cdot \mathbf{R}_\delta\mathbf{t}_2 - \dfrac{\partial \mathcal{V}_\delta}{\partial s_2} \cdot \mathbf{R}_\delta\mathbf{t}_1\Big) \rightharpoonup X \quad \text{weakly in } L^2(\omega). \end{cases}
$$

Then from the fifth convergence we obtain $\dfrac{\partial \mathcal{V}_0}{\partial s_\alpha} = \mathbf{R}_0\mathbf{t}_\alpha$. So we have $\mathcal{V}_0 \in (H^2(\omega))^3$ and $\mathbf{v}_0 = (\mathcal{V}_0, \mathbf{R}_0, \overline{V}_0)$ belongs to $\mathbb{D}$. From the above convergences, and upon extracting another subsequence, we also get

$$
\frac{1}{\delta}\Pi_\delta\big(\widehat{E}(\mathbf{v}_\delta)\big) \rightharpoonup \mathbf{E}_0 \quad \text{weakly in } (L^2(\Omega))^{3\times 3}
$$

where

$$\mathbf{E}_0 = \begin{pmatrix} S_3 \frac{\partial \mathbf{R}_0}{\partial s_1}\mathbf{n}\cdot \mathbf{R}_0\mathbf{t}_1 + \mathcal{Z}_{11,0} & S_3 \frac{\partial \mathbf{R}_0}{\partial s_1}\mathbf{n}\cdot \mathbf{R}_0\mathbf{t}_2 + \mathcal{Z}_{12,0} & \frac{1}{2}\mathbf{R}_0^T\frac{\partial \overline{W}_0}{\partial S_3}\cdot \mathbf{t}_1 \\ * & S_3 \frac{\partial \mathbf{R}_0}{\partial s_2}\mathbf{n}\cdot \mathbf{R}_0\mathbf{t}_2 + \mathcal{Z}_{22,0} & \frac{1}{2}\mathbf{R}_0^T\frac{\partial \overline{W}_0}{\partial S_3}\cdot \mathbf{t}_2 \\ * & * & \mathbf{R}_0^T\frac{\partial \overline{W}_0}{\partial S_3}\cdot \mathbf{n} \end{pmatrix}$$

with

$$\overline{W}_0 = \overline{V}_0 + S_3\mathcal{Z}_{31,0}\mathbf{R}_0\mathbf{t}_1^{'} + S_3\mathcal{Z}_{32,0}\mathbf{R}_0\mathbf{t}_2^{'}.$$

Due to the expression of $J^s_{2,\delta}$ we have

$$\begin{aligned}\frac{J^s_{2,\delta}(\mathbf{v}_\delta)}{\delta^3} = &\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\,\frac{1}{\delta}\Pi_\delta(\widehat{E}(\mathbf{v}_\delta))\,(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2dS_3 + \Big\|\frac{\partial \mathbf{R}_\delta}{\partial s_1}\mathbf{t}_2 - \frac{\partial \mathbf{R}_\delta}{\partial s_2}\mathbf{t}_1\Big\|^2_{(L^2(\omega))^3}\\ &+\frac{1}{\delta^2}\Big\|\frac{\partial \mathcal{V}_\delta}{\partial s_1}\cdot\mathbf{R}_\delta\mathbf{t}_2 - \frac{\partial \mathcal{V}_\delta}{\partial s_2}\cdot\mathbf{R}_\delta\mathbf{t}_1\Big\|^2_{L^2(\omega)} - \mathcal{L}(\mathcal{V}_\delta,\mathbf{R}_\delta).\end{aligned}$$

With the convergences (8.6), since Q is quadratic and thanks to the expression of $\mathcal{L}$, we are in a position to pass to the limit-inf in the above equality which gives

$$\begin{aligned}\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}_0(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2dS_3 + ||X||^2_{L^2(\omega)} + ||Y||^2_{(L^2(\omega))^3} - \mathcal{L}(\mathcal{V}_0,\mathbf{R}_0)\\ \le \varliminf_{\delta\to 0}\frac{J^s_{2,\delta}(\mathbf{v}_\delta)}{\delta^3} = \varliminf_{\delta\to 0}\frac{m^s_{2,\delta}}{\delta^3}.\end{aligned}$$

Hence we get

$$\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}_0(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2dS_3 - \mathcal{L}(\mathcal{V}_0,\mathbf{R}_0) \le \varliminf_{\delta\to 0}\frac{m^s_{2,\delta}}{\delta^3}.$$

First, notice that if $\mathbf{v} = (\mathcal{V},\mathbf{R},\overline{V}) \in \mathbb{D}$ then $\overline{V}$ satisfies

$$\int_{-1}^1 \frac{\partial \overline{V}}{\partial S_3}(s_1,s_2,S_3)\cdot\mathbf{t}_\alpha(s_1,s_2)(S_3^2-1)dS_3 = 0 \qquad \text{for a.e. } (s_1,s_2)\in\omega.$$

Now we apply Lemma A with $\mathbf{a} = \Big(\frac{\partial \mathbf{R}_0}{\partial s_1}\mathbf{n}\cdot\mathbf{R}_0\mathbf{t}_1, \frac{\partial \mathbf{R}_0}{\partial s_1}\mathbf{n}\cdot\mathbf{R}_0\mathbf{t}_2, \frac{\partial \mathbf{R}_0}{\partial s_2}\mathbf{n}\cdot\mathbf{R}_0\mathbf{t}_2\Big)$, $\mathbf{b} = (\mathcal{Z}_{11,0},\mathcal{Z}_{12,0},\mathcal{Z}_{22,0})$, $\mathbf{c} = \Big(\frac{1}{2}\mathbf{R}_0^T\frac{\partial \overline{W}_0}{\partial S_3}\cdot\mathbf{t}_1, \frac{1}{2}\mathbf{R}_0^T\frac{\partial \overline{W}_0}{\partial S_3}\cdot\mathbf{t}_2, \mathbf{R}_0^T\frac{\partial \overline{W}_0}{\partial S_3}\cdot\mathbf{n}\Big)$ and with the quadratic form defined by

$$\mathcal{Q}_m(\mathbf{a},\mathbf{b},\mathbf{c}) = \int_{-1}^1 Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}_0(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})dS_3 \qquad \text{for a.e. } (s_1,s_2)\in\omega.$$

We obtain

$$\min_{\mathbf{v}\in\mathbb{D}}\mathcal{J}_2(\mathbf{v}) \le \int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}_0(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2dS_3 - \mathcal{L}(\mathcal{V}_0,\mathbf{R}_0). \tag{8.7}$$

Hence $m^s_2 \le \varliminf_{\delta\to 0}\frac{m^s_{2,\delta}}{\delta^3}$. Recall that we have $\frac{m^s_{2,\delta}}{\delta^3} \le m^s_2$, so we get

$$\lim_{\delta\to 0}\frac{m^s_{2,\delta}}{\delta^3} = m^s_2.$$

Finally, from convergences (8.6) we obtain $\mathcal{Z}_{i\alpha,0} = 0$, $X = Y = 0$ and moreover we have the strong convergences in (8.4). □

9. Justification of the simplified model. Case $\kappa = 2$

In this section, the introduction of the simplified energy is justified in the sense that we prove that both the minima of the elastic energy and of the simplified energy have the same limit as δ tends to 0.

Theorem 9.1. *We have*

$$\lim_{\delta\to 0}\frac{m_{2,\delta}}{\delta^3} = \lim_{\delta\to 0}\frac{m^s_{2,\delta}}{\delta^3} = m^s_2.$$

Proof.

Step 1. In this step we prove that $m^s_2 \leq \underline{\lim}_{\delta\to 0}\dfrac{m_{2,\delta}}{\delta^3}$. Let $\big(v_\delta\big)_{0<\delta\leq\delta_0}$ be a minimizing sequence of deformations belonging to $\mathbf{U}_\delta$ and such that

$$\underline{\lim}_{\delta\to 0}\frac{m_{2,\delta}}{\delta^3} = \lim_{\delta\to 0}\frac{J_{2,\delta}(v_\delta)}{\delta^3}. \tag{9.1}$$

From the estimates of Section 6 we get

$$\begin{cases} ||\mathrm{dist}(\nabla_x v_\delta, SO(3))||_{L^2(\mathcal{Q}_\delta)} \leq C\delta^{3/2}, \\ \Big|\Big|\dfrac{1}{2}\big\{\nabla_x v_\delta^T\nabla_x v_\delta - \mathbf{I}_3\big\}\Big|\Big|_{(L^2(\mathcal{Q}_\delta))^{3\times 3}} \leq C\delta^{3/2}, \\ ||\nabla_x v_\delta||_{(L^4(\mathcal{Q}_\delta))^{3\times 3}} \leq C\delta^{1/4}. \end{cases} \tag{9.2}$$

We still denote by $\mathcal{V}_\delta(s_1,s_2) = \dfrac{1}{2\delta}\displaystyle\int_{-\delta}^{\delta} v_\delta(s_1,s_2,s_3)ds_3$ the mean of v_δ over the fibers of the shell. Upon extracting a subsequence (still indexed by δ), the results of [6] show that there exist $\mathcal{V}\in(H^2(\omega))^3$, $\mathbf{R}\in(H^1(\omega))^{3\times 3}$ with $\mathbf{R}(s_1,s_2)\in SO(3)$ for a.e. $(s_1,s_2)\in\omega$, $\mathcal{Z}_{\alpha\beta}\in L^2(\omega)$ and $\overline{V}\in\big(L^2(\omega;H^1(-1,1)\big)^3$ satisfying

$$\int_{-1}^{1}\overline{V}(s_1,s_2,S_3)dS_3 = 0 \qquad \text{for a.e.}\quad (s_1,s_2)\in\omega, \qquad \frac{\partial\mathcal{V}}{\partial s_\alpha} = \mathbf{R}\mathbf{t}_\alpha \tag{9.3}$$

together with the boundaries conditions $\mathcal{V}=\phi$, $\mathbf{R}=\mathbf{I}_3$ on γ_0, and with the following convergences

$$\begin{cases} \Pi_\delta(v_\delta) \longrightarrow \mathcal{V} \quad \text{strongly in} \quad \big(H^1(\Omega)\big)^3, \\ \Pi_\delta(\nabla_x v_\delta) \longrightarrow \mathbf{R} \quad \text{strongly in} \quad \big(L^2(\Omega)\big)^{3\times 3}, \\ \dfrac{\Pi_\delta(v_\delta - \mathcal{V}_\delta)}{\delta} \longrightarrow S_3(\mathbf{R}-\mathbf{I}_3)\mathbf{n} \quad \text{strongly in} \quad \big(L^2(\Omega)\big)^3, \\ \dfrac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big) \rightharpoonup (\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}\,(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1} \qquad \text{weakly in} \quad (L^2(\Omega))^9, \end{cases} \tag{9.4}$$

where

$$\mathbf{E} = \begin{pmatrix} S_3\dfrac{\partial\mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1 + \mathcal{Z}_{11} & S_3\dfrac{\partial\mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 + \mathcal{Z}_{12} & \dfrac{1}{2}\mathbf{R}^T\dfrac{\partial\overline{V}}{\partial S_3}\cdot\mathbf{t}_1 \\ * & S_3\dfrac{\partial\mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 + \mathcal{Z}_{22} & \dfrac{1}{2}\mathbf{R}^T\dfrac{\partial\overline{V}}{\partial S_3}\cdot\mathbf{t}_2 \\ * & * & \mathbf{R}^T\dfrac{\partial\overline{V}}{\partial S_3}\cdot\mathbf{n} \end{pmatrix}$$

Now, recall that

$$\frac{J_{2,\delta}(v_\delta)}{\delta^3} = \int_\Omega\frac{1}{\delta^2}W\Big(\frac{1}{2}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big)\Big)\Pi_\delta\big(\det(\nabla\Phi)\big) - \frac{1}{\delta^3}\int_{\mathcal{Q}_\delta} f_{\kappa,\delta}\cdot(v_\delta - I_d). \tag{9.5}$$

In order to pass to the lim-inf in (9.5) we first notice that $\det(\nabla\Phi)=\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})+s_3\det\Big(\dfrac{\partial\mathbf{n}}{\partial s_1}|\mathbf{t}_2\mid\mathbf{n}\Big)+s_3\det\Big(\mathbf{t}_1|\dfrac{\partial\mathbf{n}}{\partial s_2}\mid\mathbf{n}\Big)+s_3^2\det\Big(\dfrac{\partial\mathbf{n}}{\partial s_1}|\dfrac{\partial\mathbf{n}}{\partial s_2}\mid\mathbf{n}\Big)$ so that indeed $\Pi_\delta\big(\det(\nabla\Phi)\big)$ strongly converges to $\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})$ in $L^\infty(\Omega)$ as δ tends to 0.

We now consider the first term of the right hand side. Let $\varepsilon>0$ be fixed. Due to (6.2), there exists $\theta>0$ such that

$$\forall E\in\mathbf{S}_3,\ \ |||E|||\leq\theta,\ \ W(E)\geq Q(E)-\varepsilon|||E|||^2. \tag{9.6}$$

We now use a similar argument given in [5]. Let us denote by χ^θ_δ the characteristic function of the set $A^\theta_\delta=\{s\in\Omega;|||\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)(s)|||\geq\theta\}$. Due to (9.2), we have

$$\text{meas}(A^\theta_\delta)\leq C\frac{\delta^2}{\theta^2}. \tag{9.7}$$

Using the positive character of W, (9.2) and (9.6) give

$$\begin{aligned}\int_\Omega\frac{1}{\delta^2}\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)|\Pi_\delta\big(\det(\nabla\Phi)\big)|&\geq\int_\Omega\frac{1}{\delta^2}W\Big(\frac{1}{2}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)\Big)(1-\chi^\theta_\delta)\Pi_\delta\big(\det(\nabla\Phi)\big)\\&\geq\int_\Omega Q\Big(\frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)(1-\chi^\theta_\delta)\Big)\Pi_\delta\big(\det(\nabla\Phi)\big)-C\varepsilon\end{aligned}$$

In view of (9.7), the function χ^θ_δ converges a.e. to 0 as δ tends to 0 while the weak limit of $\dfrac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)(1-\chi^\theta_\delta)$ is given by (9.4). As a consequence and also using the convergence of $\Pi_\delta\big(\det(\nabla\Phi)\big)$ obtained above, we have

$$\underline{\lim}_{\delta\to0}\int_\Omega\frac{1}{\delta^2}\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta\big(\det(\nabla\Phi)\big)\geq\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}\,(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})-C\varepsilon.$$

As ε is arbitrary, this gives

$$\underline{\lim}_{\delta\to0}\int_\Omega\frac{1}{\delta^2}\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta\big(\det(\nabla\Phi)\big)\geq\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}\,(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}). \tag{9.8}$$

Using the convergences (9.4), it follows that

$$\lim_{\delta\to0}\Big(\frac{1}{\delta^3}\int_{\mathcal{Q}_\delta}f_{2,\delta}\cdot(v_\delta-I_d)\Big)=\mathcal{L}(\mathcal{V},\mathbf{R})$$

where $\mathcal{L}(\cdot,\cdot)$ is defined by (8.5). From (9.5), (9.8) and the above limit, we conclude that

$$\underline{\lim}_{\delta\to0}\frac{m_{2,\delta}}{\delta^3}=\underline{\lim}_{\delta\to0}\frac{J_{2,\delta}(v_\delta)}{\delta^3}\geq\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}\,(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})-\mathcal{L}(\mathcal{V},\mathbf{R}). \tag{9.9}$$

Proceeding as in the proof of (8.7) in Section 8, we get

$$\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}\,(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})-\mathcal{L}(\mathcal{V},\mathbf{R})\geq\min_{\mathbf{v}\in\mathbb{D}}\mathcal{J}_2(\mathbf{v})=m_2^s.$$

Finally we have proved that $m_2^s\leq\underline{\lim}_{\delta\to0}\dfrac{m_{2,\delta}}{\delta^3}$.

Step 2. In this step we prove that $m_2^s \geq \overline{\lim_{\delta\to 0}} \dfrac{m_{2,\delta}}{\delta^3}$.

Let us now consider a minimizer $\mathbf{v}_0 = \big(\mathcal{V}_0, \mathbf{R}_0, \overline{V}_0\big) \in \mathbb{D}$ of $\mathcal{J}_2$ and the sequence $\Big(\big(\mathcal{V}_\delta, \mathbf{R}_\delta, \overline{V}_\delta\big)\Big)_{\delta>0}$ of approximation of $\mathbf{v}_0$ given by Lemma C constructed in the Appendix. The deformation v_δ is now defined by

$$v_\delta(s) = \mathcal{V}_\delta(s_1,s_2) + s_3\mathbf{R}_\delta(s_1,s_2)\mathbf{n}(s_1,s_2) + \delta^2\overline{V}_\delta\Big(s_1,s_2,\frac{s_3}{\delta}\Big), \quad \text{for } s\in\Omega_\delta. \tag{9.10}$$

Step 2.1. Estimate on $||\Pi_\delta\big(\nabla_x v_\delta - \mathbf{R}_\delta\big)||_{(L^\infty(\Omega))^{3\times 3}}$ and $||dist\big(\nabla_x v_\delta, SO(3)\big)||_{L^\infty(\omega)}$.

From (9.10) and trough simple calculations, we first have

$$\begin{cases} (\nabla_x v_\delta - \mathbf{R}_\delta)\mathbf{t}_\alpha = \dfrac{\partial \mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_\delta\mathbf{t}_\alpha + s_3\dfrac{\partial \mathbf{R}_\delta}{\partial s_\alpha}\mathbf{n} + \delta^2\dfrac{\partial \overline{V}_\delta}{\partial s_\alpha} - (\nabla_x v_\delta - \mathbf{R}_\delta)s_3\dfrac{\partial \mathbf{n}}{\partial s_\alpha} \\ (\nabla_x v_\delta - \mathbf{R}_\delta)\mathbf{n} = \delta\dfrac{\partial \overline{V}_\delta}{\partial S_3}, \end{cases} \tag{9.11}$$

then

$$\begin{aligned} &\Pi_\delta(\nabla_x v_\delta - \mathbf{R}_\delta)\cdot\Pi_\delta(\nabla_s\Phi) \\ =&\Big(\frac{\partial \mathcal{V}_\delta}{\partial s_1} - \mathbf{R}_\delta\mathbf{t}_1 + S_3\delta\frac{\partial \mathbf{R}_\delta}{\partial s_\alpha}\mathbf{n} + \delta^2\frac{\partial \overline{V}_\delta}{\partial s_\alpha} \mid \frac{\partial \mathcal{V}_\delta}{\partial s_2} - \mathbf{R}_\delta\mathbf{t}_2 + S_3\delta\frac{\partial \mathbf{R}_\delta}{\partial s_\alpha}\mathbf{n} + \delta^2\frac{\partial \overline{V}_\delta}{\partial s_\alpha} \mid \delta\frac{\partial \overline{V}_\delta}{\partial S_3}\Big). \end{aligned} \tag{9.12}$$

Thanks to (2.3) and the estimates of Lemma C in Appendix we obtain

$$||\Pi_\delta\big(\nabla_x v_\delta - \mathbf{R}_\delta\big)||_{(L^\infty(\Omega))^{3\times 3}} \leq \frac{1}{4} \tag{9.13}$$

and we deduce that there exists a positive constant C_0 such that

$$||\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big)||_{(L^\infty(\Omega))^{3\times 3}} \leq C_0. \tag{9.14}$$

Again using the estimates in Lemma C we get

$$||dist\big(\nabla_x v_\delta, SO(3)\big)||_{L^\infty(\omega)} \leq \frac{1}{2}$$

and then we obtain

$$\text{for a.e.} \quad s\in\Omega_\delta \qquad \det\big(\nabla_x v_\delta(s)\big) > 0. \tag{9.15}$$

Step 2.2. Strong limit of $\dfrac{1}{2\delta}\Pi_\delta\Big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\Big)$.

Thanks to the estimates and convergences of Lemma C and (9.12) we have

$$||\Pi_\delta\big(\nabla_x v_\delta - \mathbf{R}_\delta\big)||_{(L^2(\Omega))^{3\times 3}} \leq C\delta. \tag{9.16}$$

We write the identity $(\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3 = (\nabla_x v_\delta - \mathbf{R}_\delta)^T\mathbf{R}_\delta + \mathbf{R}_\delta^T(\nabla_x v_\delta - \mathbf{R}_\delta) + (\nabla_x v_\delta - \mathbf{R}_\delta)^T(\nabla_x v_\delta - \mathbf{R}_\delta) + (\mathbf{R}_\delta - \mathbf{R})^T\mathbf{R}_\delta + \mathbf{R}^T(\mathbf{R}_\delta - \mathbf{R})$. So, from (9.13) and (9.16) we get

$$||\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big)||_{(L^2(\Omega))^{3\times 3}} \leq C\delta. \tag{9.17}$$

In view of (9.11), the strong convergences of Lemma C and (9.16) we deduce that

$$(9.18)\qquad \begin{cases} \dfrac{1}{\delta}\Pi_\delta\big((\nabla_x v_\delta-\mathbf{R})\mathbf{t}_\alpha\big)\longrightarrow S_3\dfrac{\partial\mathbf{R}}{\partial s_\alpha}\mathbf{n}\quad\text{strongly in}\quad (L^2(\Omega))^3\\ \dfrac{1}{\delta}\Pi_\delta\big((\nabla_x v_\delta-\mathbf{R})\mathbf{n}\big)\longrightarrow \dfrac{\partial\overline{V}}{\partial S_3}\cdot\mathbf{n}\quad\text{strongly in}\quad (L^2(\Omega))^3\end{cases}$$

Now thanks (9.13) and the strong convergences (9.18) we obtain

$$\frac{1}{\sqrt{\delta}}\Pi_\delta(\nabla_x v_\delta-\mathbf{R})\longrightarrow 0\quad\text{strongly in}\quad (L^4(\Omega))^3$$

and then using again Lemma C, (9.18) and the above decomposition of $(\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3$, we get

$$(9.19)\qquad \frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)\longrightarrow(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})^{-T}\mathbf{E}(\mathbf{v}_0)(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})^{-1}\quad\text{strongly in}\quad (L^2(\Omega))^{3\times 3},$$

where $\mathbf{E}(\mathbf{v}_0)$ is given by (8.2).

Step 2.3. Let ε be a fixed positive constant and let θ given by (7.2). We denote χ^θ_δ the characteristic function of the set $A^\theta_\delta=\{s\in\Omega;\,|||\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)(s)|||\ge\theta\}$. Due to (9.17), we have

$$(9.20)\qquad \text{meas}(A^\theta_\delta)\le C\frac{\delta^2}{\theta^2}$$

and from (9.15) we have $\det\big(\nabla_x v_\delta(s)\big)>0$ for a. e. $s\in\Omega_\delta$. Due to (6.2), (6.4) and (9.19) we deduce that

$$\begin{aligned}\overline{\lim_{\delta\to 0}}\int_\Omega\frac{1}{\delta^2}(1-\chi^\theta_\delta)\,\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta\big(\det(\nabla\Phi)\big)\le&\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{v}_0)(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})\\ &+\varepsilon\int_\Omega|||(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{v}_0)(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}|||^2\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})\end{aligned}$$

where $\mathbf{E}(\mathbf{v}_0)$ is given by (8.2). Notice that there exists a positive constant C_1 such that for all $E\in\mathbf{S}_3$ satisfying $\theta\le|||E|||\le C_0$ we have

$$W(E)\le C_1|||E|||.$$

Thanks to (6.3), (6.4), (9.17), the strong convergence (9.19) and the weak convergence $\dfrac{1}{\delta}\chi^\theta_\delta\rightharpoonup 0$ in $L^2(\Omega)$ we obtain

$$\lim_{\delta\to 0}\int_\Omega\frac{1}{\delta^2}\chi^\theta_\delta\,\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta\big(\det(\nabla\Phi)\big)\le C_1\lim_{\delta\to 0}\int_\Omega\frac{1}{\delta}\chi^\theta_\delta\,|||\frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta-\mathbf{I}_3\big)|||\,\Pi_\delta\big(\det(\nabla\Phi)\big)=0$$

Hence for any $\varepsilon>0$ we get

$$\begin{aligned}\overline{\lim_{\delta\to 0}}\int_\Omega\frac{1}{\delta^2}\,\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta\big(\det(\nabla\Phi)\big)\le&\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{v}_0)(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})\\ &+\varepsilon\int_\Omega|||(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{v}_0)(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}|||^2\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})\end{aligned}$$

Finally

$$(9.21)\qquad \overline{\lim_{\delta\to 0}}\int_\Omega\frac{1}{\delta^2}\,\widehat{W}\big(\Pi_\delta(\nabla_x v_\delta)\big)\Pi_\delta\big(\det(\nabla\Phi)\big)\le\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{v}_0)(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}).$$

As far as the contribution of the applied forces is concerned, we use the convergences of Lemma C to obtain

$$\lim_{\delta\to 0}\Big(\frac{1}{\delta^3}\int_{\mathcal{Q}_\delta} f_{2,\delta}\cdot(v_\delta - I_d)\Big) = \mathcal{L}(\mathcal{V},\mathbf{R}). \tag{9.22}$$

From (9.21) and (9.22), we conclude that

$$\overline{\lim_{\delta\to 0}}\,\frac{J_{2,\delta}(v_\delta)}{\delta^3} \le \int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{v}_0)(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}) - \mathcal{L}(\mathcal{V},\mathbf{R}) = \mathcal{J}_2(\mathbf{v}_0) = m_2^s.$$

Then we get $\overline{\lim_{\delta\to 0}}\,\dfrac{m_{2,\delta}}{\delta^3} \le m_2^s$. □

10. Alternative formulations of the minima $m^s_{\kappa,\delta}$ and m_2^s.

In the following theorem we characterize the minimum of the functional $J^s_{\kappa,\delta}(\cdot)$ over $\mathbb{D}_{\delta,\gamma_0}$, respectively $\mathcal{J}_2$ over $\mathbb{D}$, as the minima of two functionals which depend on the mid-surface deformation $\mathcal{V}$ and on the matrix $\mathbf{R}$ which gives the rotation of the fibers.

The first theorem of this section shows that the variable $\overline{v}$ can be eliminated in the minimization problem (7.9).

We set

$$\mathbb{E} = \Big\{(\mathcal{V},\mathbf{R})\in (H^1(\omega))^3\times(H^1(\omega))^{3\times 3}\ \mid\ \mathcal{V}=\phi,\quad \mathbf{R}=\mathbf{I}_3 \ \text{ on } \gamma_0,$$
$$\mathbf{R}(s_1,s_2)\in SO(3)\quad \text{for a.e. } (s_1,s_2)\in\omega\Big\}.$$

We recall (see (5.3)) that for all $(\mathcal{V},\mathbf{R})\in\mathbb{E}$ we have set

$$\mathcal{Z}_{\alpha\beta} = \frac{1}{2}\Big[\Big(\frac{\partial\mathcal{V}}{\partial s_\alpha} - \mathbf{R}\mathbf{t}_\alpha\Big)\cdot\mathbf{R}\mathbf{t}_\beta + \Big(\frac{\partial\mathcal{V}}{\partial s_\beta} - \mathbf{R}\mathbf{t}_\beta\Big)\cdot\mathbf{R}\mathbf{t}_\alpha\Big],\qquad \mathcal{Z}_{3\alpha} = \frac{\partial\mathcal{V}}{\partial s_\alpha}\cdot\mathbf{R}\mathbf{n}$$
$$\Gamma_{\alpha\beta}(\mathbf{R}) = \frac{1}{2}\Big\{\frac{\partial\mathbf{R}}{\partial s_\alpha}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_\beta + \frac{\partial\mathbf{R}}{\partial s_\beta}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_\alpha\Big\}.$$

Theorem 10.1. *Let $\mathbf{v}_\delta = (\mathcal{V}_\delta,\mathbf{R}_\delta,\overline{V}_\delta)\in\mathbb{D}_{\delta,\gamma_0}$ such that $m^s_{\kappa,\delta} = J^s_{\kappa,\delta}(\mathbf{v}_\delta) = \min_{v\in\mathbb{D}_{\delta,\gamma_0}} J^s_{\kappa,\delta}(\mathbf{v})$. We have*

$$m^s_{\kappa,\delta} = \mathcal{F}^s_{\kappa,\delta}(\mathcal{V}_\delta,\mathbf{R}_\delta) = \min_{(\mathcal{V},\mathbf{R})\in\mathbb{E}}\mathcal{F}^s_{\kappa,\delta}(\mathcal{V},\mathbf{R}) \tag{10.1}$$

where

$$\begin{cases}\mathcal{F}^s_{\kappa,\delta}(\mathcal{V},\mathbf{R}) = \delta^3\displaystyle\int_\omega a_{\alpha\beta\alpha'\beta'}\Gamma_{\alpha\beta}(\mathbf{R})\Gamma_{\alpha'\beta'}(\mathbf{R}) + \delta\int_\omega b_{i\alpha i'\alpha'}\mathcal{Z}_{i\alpha}\mathcal{Z}_{i'\alpha'}\\ \qquad + \delta^3\Big\|\dfrac{\partial\mathbf{R}}{\partial s_1}\mathbf{t}_2 - \dfrac{\partial\mathbf{R}}{\partial s_2}\mathbf{t}_1\Big\|^2_{(L^2(\omega))^3} + \delta\Big\|\dfrac{\partial\mathcal{V}}{\partial s_1}\cdot\mathbf{R}\mathbf{t}_2 - \dfrac{\partial\mathcal{V}}{\partial s_2}\cdot\mathbf{R}\mathbf{t}_1\Big\|^2_{L^2(\omega)} - \delta^{\kappa'+1}\mathcal{L}(\mathcal{V},\mathbf{R}).\end{cases} \tag{10.2}$$

The $a_{\alpha\beta\alpha'\beta'}$ and $b_{i\alpha i'\alpha'}$ are constants which depend only of the quadratic form Q and the vectors $(\mathbf{t}_1,\mathbf{t}_2,\mathbf{n})$.

Proof. We have

$$m^s_{\kappa,\delta} = \min_{\mathbf{v}\in\mathbb{D}_{1,\gamma_0}}\Big[\delta\int_\Omega Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\Pi_\delta(\widehat{E}(\mathbf{v}))(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})ds_1ds_2dS_3$$
$$+\delta^3\Big\|\frac{\partial\mathbf{R}}{\partial s_1}\mathbf{t}_2 - \frac{\partial\mathbf{R}}{\partial s_2}\mathbf{t}_1\Big\|^2_{(L^2(\omega))^3} + \delta\Big\|\frac{\partial\mathcal{V}}{\partial s_1}\cdot\mathbf{R}\mathbf{t}_2 - \frac{\partial\mathcal{V}}{\partial s_2}\cdot\mathbf{R}\mathbf{t}_1\Big\|^2_{L^2(\omega)} - \delta^{\kappa'+1}\mathcal{L}(\mathcal{V},\mathbf{R})\Big].$$

In order to eliminate $\overline{v}$, we first fix $\big(\mathcal{V},\mathbf{R}\big)\in\mathbb{E}$. We set

$$\int_{-1}^{1} Q\Big((\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\Pi_\delta(\widehat{E}(\mathbf{v}))(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-1}\Big)\det(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n})dS_3=\mathcal{Q}_m(\mathbf{a},\mathbf{b},\mathbf{c}+\mathbf{d})$$

where

$$\mathbf{a}=\delta\begin{pmatrix}\Gamma_{11}(\mathbf{R})\\ \Gamma_{12}(\mathbf{R})\\ \Gamma_{22}(\mathbf{R})\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}\mathcal{Z}_{11}\\ \mathcal{Z}_{12}\\ \mathcal{Z}_{22}\end{pmatrix},\quad \mathbf{d}=\begin{pmatrix}\mathcal{Z}_{31}\\ \mathcal{Z}_{32}\\ 0\end{pmatrix},\qquad \mathbf{c}=\frac{1}{\delta}\begin{pmatrix}\dfrac{1}{2}\mathbf{R}^T\dfrac{\partial\Pi_\delta(\overline{v})}{\partial S_3}\cdot\mathbf{t}_1\\ \dfrac{1}{2}\mathbf{R}^T\dfrac{\partial\Pi_\delta(\overline{v})}{\partial S_3}\cdot\mathbf{t}_2\\ \mathbf{R}^T\dfrac{\partial\Pi_\delta(\overline{v})}{\partial S_3}\cdot\mathbf{n}\end{pmatrix}$$

and we apply Lemma B in Appendix to obtain the theorem. □

The next theorem is similar to Theorem 10.1 for the limit energy and the minimization problem (8.3). We set

$$\mathbb{E}_{lim}=\Big\{\big(\mathcal{V},\mathbf{R}\big)\in\mathbb{E}\ \mid\ \frac{\partial\mathcal{V}}{\partial s_\alpha}=\mathbf{R}\mathbf{t}_\alpha,\ \ \alpha=1,2\Big\}.$$

Theorem 10.2. *Let* $\mathbf{v}_0=(\mathcal{V}_0,\mathbf{R}_0,\overline{V}_0)\in\mathbb{D}$ *such that* $m_2^s=\mathcal{J}_2(\mathbf{v}_0)=\min_{v\in\mathbb{D}}\mathcal{J}_2(\mathbf{v})$. *We have*

$$m_2^s=\mathcal{F}_2\big(\mathcal{V}_0,\mathbf{R}_0\big)=\min_{(\mathcal{V},\mathbf{R})\,\in\,\mathbb{E}_{lim}}\mathcal{F}_2\big(\mathcal{V},\mathbf{R}\big)\tag{10.3}$$

where

$$\mathcal{F}_2\big(\mathcal{V},\mathbf{R}\big)=\int_\omega a_{\alpha\beta\alpha'\beta'}\Big(\frac{\partial\mathbf{R}}{\partial s_\alpha}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_\beta\Big)\Big(\frac{\partial\mathbf{R}}{\partial s'_\alpha}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_{\beta'}\Big)-\mathcal{L}(\mathcal{V},\mathbf{R})\tag{10.4}$$

The $a_{\alpha\beta\alpha'\beta'}$ *are the same constants as the one in Theorem 10.1.*

Proof. We proceed as in Theorem 10.1. In order to eliminate $\overline{V}$, we fix $\big(\mathcal{V},\mathbf{R}\big)\in\mathbb{E}_{lim}$ and we minimize the functional $\mathcal{J}_2\big(\mathcal{V},\mathbf{R},\cdot\big)$ over the space $\mathbf{V}$. Thanks to Lemma B in Appendix we obtain the minimum with respect to $\overline{V}$ and then the new characterization of the minimum m_2^s. □

Of course, for all $(\mathcal{V},\mathbf{R})\in\mathbb{E}_{lim}$, we get

$$\mathcal{F}_{2,\delta}^s\big(\mathcal{V},\mathbf{R}\big)=\delta^3\mathcal{F}_2\big(\mathcal{V},\mathbf{R}\big).$$

Let us give the explicit expression of the limit energies $\mathcal{F}_{\kappa,\delta}^s$ and $\mathcal{F}_2$ in the case where S is a developable surface such that the parametrization ϕ is locally isometric

$$\forall(s_1,s_2)\in\overline{\omega}\qquad ||\mathbf{t}_\alpha(s_1,s_2)||_2=1\qquad \mathbf{t}_1(s_1,s_2)\cdot\mathbf{t}_2(s_1,s_2)=0.$$

We consider a St Venant-Kirchhoff's law for which we have

$$\widehat{W}(F)=\begin{cases}\dfrac{\lambda}{8}\big(tr(F^TF-\mathbf{I}_3)\big)^2+\dfrac{\mu}{4}tr\big((F^TF-\mathbf{I}_3)^2\big) & \text{if}\quad \det(F)>0\\ +\infty\qquad \text{if}\qquad \det(F)\le 0,\end{cases}$$

so that $Q=W=W^s$.

Expression of $\mathcal{F}^s_{\kappa,\delta}$. For any $\mathbf{v} = \big(\mathcal{V}, \mathbf{R}, \overline{v}\big) \in \mathbb{D}_{\delta,\gamma_0}$, the expression (7.5) gives

$$
(10.5) \qquad \begin{cases} J^s_{\kappa,\delta}(\mathbf{v}) = \delta \displaystyle\int_\Omega \Big[\frac{\lambda}{2}\big(tr(\widehat{E}(\mathbf{v}))\big)^2 + \mu\, tr\big((\widehat{E}(\mathbf{v}))^2\big)\Big] + \delta^3 \Big\| \frac{\partial \mathbf{R}}{\partial s_1}\mathbf{t}_2 - \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{t}_1 \Big\|^2_{(L^2(\omega))^3} \\ \qquad + \delta \Big\| \dfrac{\partial \mathcal{V}}{\partial s_1}\cdot \mathbf{R}\mathbf{t}_2 - \dfrac{\partial \mathcal{V}}{\partial s_2}\cdot \mathbf{R}\mathbf{t}_1 \Big\|^2_{L^2(\omega)} - \delta^{\kappa'+1}\mathcal{L}(\mathcal{V}, \mathbf{R}). \end{cases}
$$

where $\widehat{E}(\mathbf{v})$ is defined by (5.3). It follows that the elimination of $\overline{V}$ in Theorem 10.1 gives the partial derivatives of $\overline{V}$ with respect to S_3

$$
(10.6) \qquad \begin{pmatrix} \dfrac{\partial \overline{v}}{\partial s_3}(.,.,s_3)\cdot \mathbf{R}\mathbf{t}_1 \\ \dfrac{\partial \overline{v}}{\partial s_3}(.,.,s_3)\cdot \mathbf{R}\mathbf{t}_2 \\ \dfrac{\partial \overline{v}}{\partial s_3}(.,.,s_3)\cdot \mathbf{R}\mathbf{n} \end{pmatrix} = \begin{pmatrix} -\dfrac{\mathcal{Z}_{31}}{\delta^2}\Big(\delta^2 + \dfrac{5}{4}(s_3^2 - \delta^2)\Big) \\ -\dfrac{\mathcal{Z}_{32}}{\delta^2}\Big(\delta^2 + \dfrac{5}{4}(s_3^2 - \delta^2)\Big) \\ -\dfrac{\nu}{1-\nu}\Big(s_3\big[\Gamma_{11}(\mathbf{R}) + \Gamma_{22}(\mathbf{R})\big] + \big[\mathcal{Z}_{11} + \mathcal{Z}_{22}\big]\Big) \end{pmatrix}
$$

and then

$$
\begin{aligned}
\mathcal{F}^s_{\kappa,\delta}\big(\mathcal{V}, \mathbf{R}\big) = & \frac{E\delta^3}{3(1-\nu^2)}\int_\omega \Big[(1-\nu)\sum_{\alpha,\beta=1}^{2}\big(\Gamma_{\alpha\beta}(\mathbf{R})\big)^2 + \nu\big(\Gamma_{11}(\mathbf{R}) + \Gamma_{22}(\mathbf{R})\big)^2\Big] \\
& + \frac{E\delta}{(1-\nu^2)}\int_\omega \Big[(1-\nu)\sum_{\alpha,\beta=1}^{2}\big(\mathcal{Z}_{\alpha\beta}\big)^2 + \nu\big(\mathcal{Z}_{11} + \mathcal{Z}_{22}\big)^2\Big] + \frac{5E\delta}{12(1+\nu)}\int_\omega \big(\mathcal{Z}_{31}^2 + \mathcal{Z}_{32}^2\big) \\
& + \delta^3 \Big\| \frac{\partial \mathbf{R}}{\partial s_1}\mathbf{t}_2 - \frac{\partial \mathbf{R}}{\partial s_2}\mathbf{t}_1 \Big\|^2_{(L^2(\omega))^3} + \delta \Big\| \frac{\partial \mathcal{V}}{\partial s_1}\cdot \mathbf{R}\mathbf{t}_2 - \frac{\partial \mathcal{V}}{\partial s_2}\cdot \mathbf{R}\mathbf{t}_1 \Big\|^2_{L^2(\omega)} - \delta^{\kappa'+1}\mathcal{L}(\mathcal{V}, \mathbf{R}).
\end{aligned}
$$

Expression of $\mathcal{F}_2$. For any $\mathbf{v} = \big(\mathcal{V}, \mathbf{R}, \overline{V}\big) \in \mathbb{D}$, the expression (8.1) gives

$$
\mathcal{J}_2\big(\mathbf{v}\big) = \int_\Omega \Big[\frac{\lambda}{2}\big(tr(\mathbf{E}(\mathbf{v}))\big)^2 + \mu\, tr\big((\mathbf{E}(\mathbf{v}))^2\big)\Big] - \mathcal{L}(\mathcal{V}, \mathbf{R})
$$

where $\mathbf{E}(\mathbf{v})$ is defined by (8.2). It follows that the elimination of $\overline{V}$ in Theorem 10.2 is identical to that of standard linear elasticity (see [18]) hence we have

$$
(10.7) \qquad \overline{V}(.,.,S_3) = -\frac{\nu}{2(1-\nu)}\Big(S_3^2 - \frac{1}{3}\Big)\Big[\Gamma_{11}(\mathbf{R}) + \Gamma_{22}(\mathbf{R})\Big]\mathbf{R}\mathbf{n}
$$

and then

$$
\mathcal{F}_2\big(\mathcal{V}, \mathbf{R}\big) = \frac{E}{3(1-\nu^2)}\int_\omega \Big[(1-\nu)\sum_{\alpha,\beta=1}^{2}\big(\Gamma_{\alpha\beta}(\mathbf{R})\big)^2 + \nu\big(\Gamma_{11}(\mathbf{R}) + \Gamma_{22}(\mathbf{R})\big)^2\Big] - \mathcal{L}(\mathcal{V}, \mathbf{R}).
$$

Remark 10.1. In the case of a St-Venant-Kirchhoff material a classical energy argument show that if $\big(v_\delta\big)_{0<\delta\le\delta_0}$ is a sequence such that

$$
m_2^s = \lim_{\delta\to 0}\frac{J_{2,\delta}(v_\delta)}{\delta^3},
$$

then there exists a subsequence and $(\mathcal{V}_0, \mathbf{R}_0) \in \mathbb{E}$, which is a solution of Problem (10.3), such that the sequence of the Green-St Venant's deformation tensors satisfies

$$
\frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T\nabla_x v_\delta - \mathbf{I}_3\big) \longrightarrow (\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T}\mathbf{E}(\mathbf{v}_0)(\mathbf{t}_1\,|\,\mathbf{t}_2\,|\,\mathbf{n})^{-T} \qquad \text{strongly in} \quad (L^2(\Omega))^{3\times 3},
$$

where $\mathbf{E}(\mathbf{v}_0)$ is defined in (8.2) with $\overline{V}_0$ given by (10.7) (replacing $\mathbf{R}$ by $\mathbf{R}_0$).

Remark 10.2. It is well known that the constraint $\dfrac{\partial \mathcal{V}}{\partial s_1} = \mathbf{R}\mathbf{t}_1$ and $\dfrac{\partial \mathcal{V}}{\partial s_2} = \mathbf{R}\mathbf{t}_2$ together the boundary conditions are strong limitations on the possible deformation for the limit 2d shell. Actually for a plate or as soon as S is a developable surface, the configuration after deformation must also be a developable surface. In the general case, it is an open problem to know if the set $\mathbb{E}_{lim}$ contains other deformations than identity mapping or very special isometries (as for example symetries).

Appendix.

Lemma A. *Let $\mathcal{Q}_m$ be the positive definite quadratic form defined on the space $\mathbb{R}^3 \times \mathbb{R}^3 \times \big(L^2(-1,1)\big)^3$ by*

$$\forall (\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \big(L^2(-1,1)\big)^3, \qquad \mathcal{Q}_m(\mathbf{a},\mathbf{b},\mathbf{c}) = \int_{-1}^{1} \mathbf{A}(S_3) \begin{pmatrix} S_3\mathbf{a}_1 + \mathbf{b}_1 \\ S_3\mathbf{a}_2 + \mathbf{b}_2 \\ S_3\mathbf{a}_3 + \mathbf{b}_3 \\ \mathbf{c}_1(S_3) \\ \mathbf{c}_2(S_3) \\ \mathbf{c}_3(S_3) \end{pmatrix} \cdot \begin{pmatrix} S_3\mathbf{a}_1 + \mathbf{b}_1 \\ S_3\mathbf{a}_2 + \mathbf{b}_2 \\ S_3\mathbf{a}_3 + \mathbf{b}_3 \\ \mathbf{c}_1(S_3) \\ \mathbf{c}_2(S_3) \\ \mathbf{c}_3(S_3) \end{pmatrix} dS_3$$

where $\mathbf{A}(S_3)$ is a symmetric positive definite 6×6 matrix satisfying

$$(A.1) \qquad \mathbf{A}(S_3) = \mathbf{A}(-S_3) \qquad \text{for a.e. } S_3 \in]-1,1[$$

and moreover there exists a positive constant c such that

$$(A.2) \qquad \forall \xi \in \mathbb{R}^6, \qquad \mathbf{A}(S_3)\xi \cdot \xi \geq c|\xi|^2 \qquad \text{for a.e. } S_3 \in]-1,1[.$$

For all $\mathbf{a} \in \mathbb{R}^3$, we have

$$\min_{(\mathbf{b},\mathbf{c}) \in \mathbb{R}^3 \times (L^2(-1,1))^3} \mathcal{Q}_m(\mathbf{a},\mathbf{b},\mathbf{c}) = \min_{\mathbf{c} \in \mathbf{L}_2} \mathcal{Q}_m(\mathbf{a},0,\mathbf{c})$$

where

$$\mathbf{L}_2 = \Big\{ \mathbf{c} \in (L^2(-1,1))^3 \mid \int_{-1}^{1} \mathbf{c}_\alpha(S_3)(S_3^2 - 1)dS_3 = 0, \ \ \alpha \in \{1,2\} \Big\}.$$

Proof. We write

$$\mathbf{A}(S_3) = \begin{pmatrix} \mathbf{A}_1(S_3) & \vdots & \mathbf{A}_2(S_3) \\ \cdots & & \cdots \\ \mathbf{A}_2^T(S_3) & \vdots & \mathbf{A}_3(S_3) \end{pmatrix}$$

where for a.e. $S_3 \in]-1,1[$, $\mathbf{A}_1(S_3)$ and $\mathbf{A}_3(S_3)$ are symmetric positive definite 3×3 matrices. The both minimum are obtained with

$$\mathbf{c}_0(S_3) = -\mathbf{A}_3^{-1}(S_3)\mathbf{A}_2^T(S_3)S_3\mathbf{a}, \qquad \mathbf{b}_0 = 0.$$

We have

$$(A.3) \qquad \mathcal{Q}_m(\mathbf{a},0,\mathbf{c}_0) = \Big(\int_{-1}^{1} S_3^2 \big(\mathbf{A}_1(S_3) - \mathbf{A}_2(S_3)\mathbf{A}_3^{-1}(S_3)\mathbf{A}_2^T(S_3) \big) dS_3 \Big) \mathbf{a} \cdot \mathbf{a}.$$

□

In the following lemma we use the same notation as in Lemma A.

Lemma B. *Let* $\mathbf{a}$, $\mathbf{b}$ *be two fixed vectors in* $\mathbb{R}^3$ *and let* $\mathbf{d}$ *be a fixed vector in* $\mathbb{R}^2 \times \{0\}$. *We have*

$$(B.1) \qquad \min_{\mathbf{c}\in\mathbf{L}_2} \mathcal{Q}_m(\mathbf{a},\mathbf{b},\mathbf{c}+\mathbf{d}) = \Big(\int_{-1}^{1} S_3^2\big[\mathbf{A}_1(S_3) - \mathbf{A}_2(S_3)\mathbf{A}_3^{-1}(S_3)\mathbf{A}_2^T(S_3)\big]\Big)\mathbf{a}\cdot\mathbf{a} + Q_m^{'}(\mathbf{b},\mathbf{d})$$

where $Q_m^{'}$ *is a positive definite quadratic form which depends only on the matrix* $\mathbf{A}$.

Proof. Through solving a simple variational problem, we find that the minimum of the functional $\mathbf{c} \longmapsto Q_m(\mathbf{a},\mathbf{b},\mathbf{c}+\mathbf{d})$ over the space $\mathbf{L}_2$ is obtained with

$$\mathbf{c}(S_3) = -\mathbf{d} - \mathbf{A}_3^{-1}(S_3)\mathbf{A}_2^T(S_3)(S_3\mathbf{a}+\mathbf{b}) + (S_3^2-1)\mathbf{A}_3^{-1}(S_3)\mathbf{e}$$

where $\mathbf{e} \in \mathbb{R}^2 \times \{0\}$

$$\mathbf{e} = e_1\mathbf{e}_1 + e_2\mathbf{e}_2, \qquad \mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

is the solution of the system

$$\Big[\frac{4}{3}\mathbf{d} - \Big(\int_{-1}^{1}(S_3^2-1)\mathbf{A}_3^{-1}(S_3)\mathbf{A}_2^T(S_3)dS_3\Big)\mathbf{b} + \Big(\int_{-1}^{1}(S_3^2-1)^2\mathbf{A}_3^{-1}(S_3)dS_3\Big)\mathbf{e}\Big]\cdot\mathbf{e}_\alpha = 0, \quad \alpha = 1,2.$$

Notice that the matrix $\displaystyle\int_{-1}^{1}(S_3^2-1)^2\mathbf{A}_3^{-1}(S_3)dS_3$ is a 3×3 symmetric positive definite matrix. Replacing $\mathbf{c}$ and $\mathbf{e}$ by their values we obtain $(B.1)$. $\square$

Lemma C. *Let* $(\mathcal{V},\mathbf{R},\overline{V})$ *be in* $\mathbf{D}_2$, *there exists a sequence* $\Big((\mathcal{V}_\delta,\mathbf{R}_\delta,\overline{V}_\delta)\Big)_{\delta>0}$ *of* $(W^{2,\infty}(\omega))^3\times(W^{1,\infty}(\omega))^{3\times3}\times(W^{1,\infty}(\Omega))^3$ *such that*

$$(C.1) \qquad \mathcal{V}_\delta = \phi, \quad \mathbf{R}_\delta = \mathbf{I}_3 \quad on \ \gamma_0, \qquad \overline{V}_\delta = 0, \quad on \ \gamma_0\times]-1,1[,$$

with

$$(C.2) \qquad \begin{cases} \mathcal{V}_\delta \longrightarrow \mathcal{V} \quad strongly\ in \quad (H^2(\omega))^3 \\ \mathbf{R}_\delta \longrightarrow \mathbf{R} \quad strongly\ in \quad (H^1(\omega))^{3\times3} \\ \dfrac{1}{\delta}(\mathbf{R}_\delta - \mathbf{R}) \longrightarrow 0 \quad strongly\ in \quad (L^2(\omega))^{3\times3} \\ \dfrac{1}{\delta}\Big(\dfrac{\partial\mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_\delta\mathbf{t}_\alpha\Big) \longrightarrow 0 \quad strongly\ in \quad (L^2(\omega))^3 \\ \overline{V}_\delta \longrightarrow \overline{V} \quad strongly\ in \quad (L^2(\omega;H^1((-1,1)))^3, \\ \delta\dfrac{\partial\overline{V}_\delta}{\partial s_\alpha} \longrightarrow 0 \quad strongly\ in \quad (L^2(\Omega))^3, \end{cases}$$

and moreover

$$(C.3) \qquad \begin{cases} ||dist(\mathbf{R}_\delta, SO(3))||_{L^\infty(\omega)} \le \dfrac{1}{8}, \qquad \Big\|\dfrac{\partial\mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_\delta\mathbf{t}_\alpha\Big\|_{(L^\infty(\omega))^3} \le \dfrac{1}{8}, \\ ||\mathbf{R}_\delta||^2_{(W^{1,\infty}(\omega))^{3\times3}} + ||\overline{V}_\delta||^2_{((W^{1,\infty}(\Omega))^3} \le \dfrac{1}{(4c_1^{'}\delta)^2}. \end{cases}$$

The constant $c_1^{'}$ *is given by (2.3).*

Proof. For $h>0$ small enough, consider a $\mathcal{C}_0^\infty(\mathbb{R}^2)$-function ψ_h such that $0\le\psi_h\le 1$

$$\begin{cases}\psi_h(s_1,s_2)=1 \text{ if } dist\big((s_1,s_2),\gamma_0\big)\le h\\ \psi_h(s_1,s_2)=0 \text{ if } dist\big((s_1,s_2),\gamma_0\big)\ge 2h.\end{cases}$$

Indeed we can assume that

$$(C.4)\qquad ||\psi_h||_{W^{1,\infty}(\mathbb{R}^2)}\le\frac{C}{h},\qquad ||\psi_h||_{W^{2,\infty}(\mathbb{R}^2)}\le\frac{C}{h^2}.$$

Since ω is bounded with a Lipschitz boundary, we first extend the fields $\mathcal{V}$ and $\mathbf{R_n}=\mathbf{Rn}$ into two fields of $(H^2(\mathbb{R}^2))^3$ and $(H^1(\mathbb{R}^2))^3$ (and we use the same notations for these extentions). We define the 3×3 matrix field $\mathbf{R}^{'}\in(H^1(\mathbb{R}^2))^{3\times 3}$ by the formula

$$(C.5)\qquad \mathbf{R}^{'}=\Big(\frac{\partial\mathcal{V}}{\partial s_1}|\frac{\partial\mathcal{V}}{\partial s_2}|\mathbf{R_n}\Big)\Big(\mathbf{t}_1|\mathbf{t}_2|\mathbf{n}\Big)^{-1}.$$

By construction we have $\dfrac{\partial\mathcal{V}}{\partial s_\alpha}=\mathbf{R}^{'}\mathbf{t}_\alpha$ in $\mathbb{R}^2$ and $\mathbf{R}^{'}=\mathbf{R}$ in ω. At least, we introduce below the approximations $\mathcal{V}_h$ and $\mathbf{R}_h$ of $\mathcal{V}$ and $\mathbf{R}$ as restrictions to $\overline{\omega}$ of the following fields defined into $\mathbb{R}^2$:

$$(C.6)\qquad \begin{cases}\mathcal{V}_h^{'}(s_1,s_2)=\dfrac{1}{9\pi h^2}\displaystyle\int_{B(0,3h)}\mathcal{V}(s_1+t_1,s_2+t_2)dt_1dt_2,\\ \mathbf{R}_h^{'}(s_1,s_2)=\dfrac{1}{9\pi h^2}\displaystyle\int_{B(0,3h)}\mathbf{R}^{'}(s_1+t_1,s_2+t_2)dt_1dt_2,\end{cases}\qquad \text{a.e. } (s_1,s_2)\in\mathbb{R}^2$$

and

$$(C.7)\qquad \mathcal{V}_h=\phi\psi_h+\mathcal{V}_h^{'}(1-\psi_h),\qquad \mathbf{R}_h=\mathbf{I}_3\psi_h+\mathbf{R}_h^{'}(1-\psi_h),\quad\text{in } \omega.$$

Notice that we have

$$(C.8)\qquad \begin{aligned}&\mathcal{V}_h^{'}\in(W^{2,\infty}(\mathbb{R}^2))^3,\qquad \mathbf{R}_h^{'}\in(W^{1,\infty}(\mathbb{R}^2))^{3\times 3},\\ &\mathcal{V}_h\in(W^{2,\infty}(\omega))^3,\qquad \mathbf{R}_h\in(W^{1,\infty}(\omega))^{3\times 3},\qquad \mathcal{V}_h=\phi,\quad \mathbf{R}_h=\mathbf{I}_3 \text{ on } \gamma_0.\end{aligned}$$

Due to the definition (C.5) of $\mathbf{R}^{'}$ and in view of (C.6) we have

$$(C.9)\qquad \begin{cases}\mathcal{V}_h^{'}\longrightarrow\mathcal{V}\quad\text{strongly in } (H^2(\mathbb{R}^2))^3,\\ \mathbf{R}_h^{'}\longrightarrow\mathbf{R}^{'}\quad\text{strongly in } (H^1(\mathbb{R}^2))^{3\times 3}\end{cases}$$

and thus using estimates (C.4)

$$(C.10)\qquad \begin{cases}\mathcal{V}_h\longrightarrow\mathcal{V}\quad\text{strongly in } (H^2(\omega))^3,\\ \mathbf{R}_h\longrightarrow\mathbf{R}\quad\text{strongly in } (H^1(\omega))^{3\times 3}\end{cases}$$

Moreover using again (C.6) and the fact that $\mathbf{R}^{'}-\mathbf{R}_h$ strongly converges to 0 in $(H^1(\mathbb{R}^2))^{3\times 3}$ we deduce that

$$\frac{1}{h}\big(\mathbf{R}_h^{'}-\mathbf{R}^{'}\big)\longrightarrow 0\quad\text{strongly in } (L^2(\mathbb{R}^2))^{3\times 3}$$

and then together with (C.4), (C.5), (C.7) and (C.10) we get

$$\frac{1}{h}\big(\mathbf{R}_h-\mathbf{R}\big)\longrightarrow 0 \quad \text{strongly in } (L^2(\omega))^{3\times 3},$$
$$\frac{1}{h}\big(\frac{\partial \mathcal{V}_h}{\partial s_\alpha}-\mathbf{R}_h\mathbf{t}_\alpha\big)\longrightarrow 0 \quad \text{strongly in } (L^2(\omega))^{3}.$$

We now turn to the estimate of the distance between $\mathbf{R}_h(s_1,s_2)$ and $SO(3)$ for a.e. $(s_1,s_2)\in\omega$. We apply the Poincaré-Wirtinger's inequality to the function $(u_1,u_2)\longmapsto \mathbf{R}^{'}(u_1,u_2)$ in the ball $B((s_1,s_2),3h)$. We obtain

$$\int_{B((s_1,s_2),3h)}|||\mathbf{R}^{'}(u_1,u_2)-\mathbf{R}^{'}_h(s_1,s_2)|||^2du_1du_2\le Ch^2||\nabla\mathbf{R}^{'}||^2_{(L^2(B((s_1,s_2),3h)))^3}$$

where C is the Poincaré-Wirtinger's constant for a ball. Since the open set ω is boundy with a Lipschitz boundary, there exists a positive constant $c(\omega)$, which depends only on ω, such that

$$|\big(B((s_1,s_2),3h)\setminus B((s_1,s_2),2h)\big)\cap\omega|\ge c(\omega)h^2.$$

Setting $m_h(s_1,s_2)$ the essential infimum of the function $(u_1,u_2)\mapsto |||\mathbf{R}(u_1,u_2)-\mathbf{R}^{'}_h(s_1,s_2)|||$ into the set $\big(B((s_1,s_2),3h)\setminus B((s_1,s_2),2h)\big)\cap\omega$, we then obtain

$$c(\omega)h^2m_h(s_1,s_2)^2\le Ch^2||\nabla\mathbf{R}^{'}||^2_{(L^2(B((s_1,s_2),3h)))^3}$$

Hence, thanks to the strong convergence of $\mathbf{R}^{'}_h$ given by (C.9), the above inequality shows that there exists $h^{'}_0$ which does not depend on $(s_1,s_2)\in\overline{\omega}$ such that for any $h\le h^{'}_0$

$$dist(\mathbf{R}^{'}_h(s_1,s_2),SO(3))\le 1/8 \qquad \text{for any } (s_1,s_2)\in\overline{\omega}.$$

Now,

• in the case $dist\big((s_1,s_2),\gamma_0\big)>2h$, $(s_1,s_2)\in\omega$, by definition of $\mathbf{R}_h$ and thanks to the above inequality we have $dist(\mathbf{R}_h(s_1,s_2),SO(3))\le 1/8$,

• in the case $dist\big((s_1,s_2),\gamma_0\big)<h$, $(s_1,s_2)\in\omega$, by definition of $\mathbf{R}_h$ we have $\mathbf{R}_h(s_1,s_2)=\mathbf{I}_3$ and then $dist(\mathbf{R}_h(s_1,s_2),SO(3))=0$,

• in the case $h\le dist\big((s_1,s_2),\gamma_0\big)\le 2h$, $(s_1,s_2)\in\omega$, due to the fact that $\mathbf{R}^{'}=\mathbf{I}_3$ onto γ_0, firstly we have

$$||\mathbf{R}^{'}-\mathbf{I}_3||^2_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}\le Ch^2||\nabla\mathbf{R}^{'}||^2_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}$$

where $\omega_{kh,\gamma_0}=\{(s_1,s_2)\in\mathbb{R}^2\mid dist\big((s_1,s_2),\gamma_0\big)\le kh\}$, $k\in\mathbb{N}^*$. Hence

$$||\mathbf{R}^{'}_h-\mathbf{I}_3||^2_{(L^2(\omega_{3h,\gamma_0}))^{3\times 3}}\le Ch^2||\nabla\mathbf{R}^{'}||^2_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}.$$

The constants depend only on $\partial\omega$.

Secondly, we set M_h the maximum of the function $(u_1,u_2)\mapsto|||\mathbf{I}_3-\mathbf{R}^{'}_h(u_1,u_2)|||$ into the closed set $\big\{(u_1,u_2)\in\omega\mid h\le dist\big((u_1,u_2),\gamma_0\big)\le 2h\big\}$, and let (s_1,s_2) be in this closed subset of ω such that

$$M_h=|||\mathbf{I}_3-\mathbf{R}^{'}_h(s_1,s_2)|||.$$

Applying the Poincaré-Wirtinger's inequality in the ball $B\big((s_1,s_2),4h\big)$ we deduce that

$$\forall (s_1^{'},s_2^{'})\in B\big((s_1,s_2),h\big),\qquad |||\mathbf{R}_h^{'}(s_1^{'},s_2^{'})-\mathbf{R}_h^{'}(s_1,s_2)|||\le C||\nabla\mathbf{R}^{'}||_{(L^2(B((s_1,s_2),4h)))^3}.$$

The constant depends only on the Poincaré-Wirtinger's constant for a ball.
If M_h is larger than $C||\nabla\mathbf{R}^{'}||_{(L^2(B((s_1,s_2),4h)))^3}$ we have

$$\begin{aligned}&\pi h^2\big(M_h-C||\nabla\mathbf{R}^{'}||_{(L^2(B((s_1,s_2),4h)))^3}\big)^2\le||\mathbf{R}_h^{'}-\mathbf{I}_3|||^2_{(L^2(B((s_1,s_2),h)))^3}\\ \le&||\mathbf{R}_h^{'}-\mathbf{I}_3||^2_{(L^2(\omega_{3h,\gamma_0}))^{3\times 3}}\le Ch^2||\nabla\mathbf{R}^{'}||^2_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}\end{aligned}$$

then, in all the cases we obtain

$$M_h\le C||\nabla\mathbf{R}^{'}||_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}.$$

The constant does not depend on h and $\mathbf{R}^{'}$. The above inequalities show that there exists $h_0^{''}$ such that for any $h\le h_0^{''}$

$$|||\mathbf{R}_h^{'}(s_1,s_2)-\mathbf{I}_3|||\le C||\nabla\mathbf{R}^{'}||_{(L^2(\omega_{6h,\gamma_0}))^{3\times 3}}\le 1/8\qquad\text{for any } (s_1,s_2)\in\omega\quad\text{such that}\quad h\le dist\big((s_1,s_2),\gamma_0\big)\le 2h.$$

By definition of $\mathbf{R}_h$, that gives $|||\mathbf{R}_h(s_1,s_2)-\mathbf{I}_3|||\le 1/8$.

Finally, for any $h\le\max(h_0^{'},h_0^{''})$ and for any $(s_1,s_2)\in\overline{\omega}$ we have

$$dist(\mathbf{R}_h(s_1,s_2),SO(3))\le 1/8.$$

Using (C.5) and (C.6) we obtain (recall that $\big\|\cdot\big\|_2$ is the euclidian norm in $\mathbb{R}^3$)

$$\forall (s_1,s_2)\in\omega,\qquad \Big\|\frac{\partial\mathcal{V}_h^{'}}{\partial s_\alpha}(s_1,s_2)-\mathbf{R}_h^{'}(s_1,s_2)\mathbf{t}_\alpha(s_1,s_2)\Big\|_2\le Ch||\phi||_{(W^{2,\infty}(\omega))^3}+C\big(||\mathcal{V}||_{(H^2(\omega_{3h}))^3}+||\mathbf{R}^{'}||_{(H^1(\omega_{3h}))^{3\times 3}}\big)$$

where $\omega_{3h}=\{(s_1,s_2)\in\mathbb{R}^2\mid dist\big((s_1,s_2),\partial\omega\big)\le 3h\}$.
We have

$$\frac{\partial\mathcal{V}_h}{\partial s_\alpha}-\mathbf{R}_h\mathbf{t}_\alpha=(1-\psi_h)\Big(\frac{\partial\mathcal{V}_h^{'}}{\partial s_\alpha}-\mathbf{R}_h^{'}\mathbf{t}_\alpha\Big)+\frac{\partial\psi_h}{\partial s_\alpha}\big(\phi-\mathcal{V}_h^{'}\big).$$

Thanks to the above inequality, (C.4) and again the estimate of $|||\mathbf{R}_h^{'}-\mathbf{I}_3|||$ in the edge strip $h\le dist\big((s_1,s_2),\gamma_0\big)\le 2h$ we obtain for all $(s_1,s_2)\in\omega$

$$\begin{aligned}&\Big\|\frac{\partial\mathcal{V}_h}{\partial s_\alpha}(s_1,s_2)-\mathbf{R}_h(s_1,s_2)\mathbf{t}_\alpha(s_1,s_2)\Big\|_2\\ \le&C\big(h||\phi||_{(W^{2,\infty}(\omega))^3}+||\mathcal{V}||_{(H^2(\omega_{3h}))^3}+||\mathbf{R}^{'}||_{(H^1(\omega_{3h}))^{3\times 3}}+||\phi-\mathcal{V}||_{(H^2(\omega_{5h,\gamma_0}))^{3\times 3}}\big).\end{aligned}$$

The same argument as above imply that there exists $h_0\le\max(h_0^{'},h_0^{''})$ such that for any $0<h\le h_0$ and for any $(s_1,s_2)\in\omega$ we have

$$(C.11)\qquad \Big\|\frac{\partial\mathcal{V}_h}{\partial s_\alpha}(s_1,s_2)-\mathbf{R}_h(s_1,s_2)\mathbf{t}_\alpha(s_1,s_2)\Big\|_2\le\frac{1}{8}.$$

From (C.4), (C.5), (C.6) and (C.7) there exists a positive constant C which does not depend on h such that

$$(C.12)\qquad ||\mathbf{R}_h||_{(W^{1,\infty}(\omega))^{3\times 3}}\le\frac{C}{h}\big\{||\mathcal{V}||_{(H^2(\omega))^3}+||\mathbf{R}||_{(H^1(\omega))^{3\times 3}}\big\}.$$

Now we can choose h in term of δ. We set

$$h = \theta\delta, \qquad \delta \in (0, \delta_0]$$

and we fixed θ in order to have $h \leq h_0$ and to obtain the right hand side in (C.12) less than $\dfrac{1}{4\sqrt{2}c_1^{'}\delta}$ ($c_1^{'}$ is given by (2.3)). It is well-known that there exists a sequence $\big(\overline{V}_\delta\big)_{\delta\in(0,\delta_0]}$ such that $(\mathcal{V}_\delta, \mathbf{R}_\delta, \overline{V}_\delta) \in \mathbb{D}_{\delta,\gamma_0}$ and satisfying the convergences in (C.1) and the estimate in (C.3). □

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