COMPARISON PRINCIPLE FOR UNBOUNDED VISCOSITY SOLUTIONS OF DEGENERATE ELLIPTIC PDES WITH GRADIENT SUPERLINEAR TERMS.

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ABSTRACT. We are concerned with fully nonlinear possibly degenerate elliptic partial differential equations (PDEs) with superlinear terms with respect to Du. We prove several comparison principles among viscosity solutions which may be unbounded under some polynomial-type growth conditions. Our main result applies to PDEs with convex superlinear terms but we also obtain some results in nonconvex cases. Applications to monotone systems of PDEs are given.

1. INTRODUCTION

We are concerned with the comparison principle for viscosity solutions of fully nonlinear elliptic partial differential equations:

$$\lambda u + F(x, Du, D^2u) + H(x, Du) = f(x) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $\lambda > 0$, $F : \mathbb{R}^N \times \mathbb{R}^N \times S^N \to \mathbb{R}$, $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \to \mathbb{R}$ are given functions. Here S^N denotes the set of $N \times N$ symmetric matrices equipped with the standard order.

We will suppose that F satisfies the standard hypothesis called structure condition. In particular, F is degenerate elliptic, that is

$$F(x, \xi, X) \leq F(x, \xi, Y) \quad \text{when } X \geq Y, \quad x, p \in \mathbb{R}^N, \ X, Y \in S^N. \tag{1.2}$$

On the contrary, we will suppose that the mapping $\xi \to H(x, \xi)$ has superlinear growth. A typical example is

$$H(x, \xi) = \langle A(x)\xi, \xi\rangle^{q/2}, \tag{1.3}$$

where $q > 1$, and $A : \mathbb{R}^N \to S^N$.

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When we consider unbounded solutions of PDEs with superlinear growth terms in Du, we may not expect solutions to be unique in general. In fact, for $N = 1$, the equation

$$\lambda u - u'' + |u'|^2 = 0 \quad \text{in } \mathbb{R} \tag{1.4}$$

admits at least two solutions; $u_1 \equiv 0$ and $u_2(x) = -\frac{\lambda}{4}x^2 - \frac{1}{2}$.

In [3], Alvarez introduced bounded-from-below solutions to avoid u_2 in this case. He showed the uniqueness of strong bounded-from-below solutions of

$$u - \Delta u + |Du|^q = f(x) \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

We will mention this result after introducing some notations in Section 2.

We also refer to [4] and [14] for comparison results, which yield the uniqueness among bounded-from-below viscosity solutions of Hamilton-Jacobi equations.

On the other hand, the uniqueness of unbounded viscosity solutions has been studied under certain growth condition on solutions. In this direction, H. Ishii [13] first established the comparison principle for unbounded viscosity solutions of Hamilton-Jacobi equations. For nonlinear elliptic PDEs, Aizawa-Tomita [1, 2], Crandall-Newcomb-Tomita [10] and K. Ishii-Tomita [16] obtained comparison results for unbounded viscosity solutions satisfying certain growth condition. However, unfortunately, we cannot apply these results to PDEs having variable coefficients to superlinear terms in Du. For instance, it seems difficult to treat typical H as (1.3) unless A is constant.

To avoid this technical difficulty, we will adapt a "linearization" technique, which Da Lio and the second author [12] used to show the uniqueness of unbounded viscosity solutions of parabolic Bellman equations with quadratic nonlinearity.

More recently, we are informed that Barles and Porretta [7] proved that (1.5) with $q = 2$ admits at most one bounded-from-below solution if f is bounded from below. In the case of (1.4), u_1 is the only bounded-from-below solution. However, their proof seems to be specific to (1.4) since if we perturb this equation with a transport term as in

$$\lambda u - u'' + |u'|^2 + txu' = 0 \quad \text{in } \mathbb{R}, \tag{1.6}$$

then there is at least two solutions $u_1 \equiv 0$ and $u_2(x) = -\frac{\lambda+2t}{4}x^2 - \frac{\lambda+2t}{2\lambda}$. Thus, for $t < -\frac{\lambda}{2}$, u_1 and u_2 are bounded-from-below solutions of (1.6).

In this paper, we study the comparison principle for viscosity solutions of (1.1) under certain growth condition on f and solutions. We obtained two types of results depending on whether $H(x, \xi)$ is convex in ξ or not. The convex case is typically (1.3) with positive $A(x) \in S^N$. Then we consider two nonconvex cases. The first one is when $H(x,\xi)$ is convex in ξ in some subset $\Omega_0 \subset \Omega$ and is concave in its complement. The second one is when $H(x,\xi)$ is defined as a minimum of convex Hamiltonians, that is,

$$H(x,\xi) = \min\{H_k(x,\xi) \mid k = 1,\dots,m\},$$

where $\xi \to H_k(x,\xi)$ is convex for $x \in \Omega$. We will discuss a generalization of the above H, which appears in differential games (See [18] for applications). Some applications to monotone systems of PDEs are also given.

Let us mention that we restrict ourselves to comparison principles since it is the main ingredient to obtain existence and uniqueness in the theory of viscosity solutions.

This paper is organized as follows: In Section 2, we give our hypothesis on F and H. Section 3 is devoted to the case when H is strictly convex in ξ. We then discuss on the case when H may be nonconvex in Section 4. In section 5, we extend our results to monotone systems.

2. Preliminaries

First of all, we recall the definition of viscosity solutions of general PDEs:

$$G(x, u, Du, D^2u) = 0 \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

where $G : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S^N \to \mathbb{R}$ is continuous.

Definition 2.1. *We call $u : \mathbb{R}^N \to \mathbb{R}$ a viscosity subsolution (resp., supersolution) of (2.1) if for $\phi \in C^2(\mathbb{R}^N)$,*

$$G(\hat{x}, u^*(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \le 0$$

$$(\textit{resp., } G(\hat{x}, u_*(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \ge 0)$$

provided $u^* - \phi$ (resp., $u_* - \phi$) attains its local maximum (resp., minimum) at $\hat{x} \in \mathbb{R}^N$.

We also call u a viscosity solution of (2.1) if it is both a viscosity sub- and supersolution of (2.1).

Here u^* and u_* denote upper and lower semicontinuous envelopes of u, respectively. We refer to [9, 6, 5, 17] for their definitions, and the basic theory of viscosity solutions.

In order to explain our hypotheses below, we give a typical example:

$$u - \mathrm{Tr}(\sigma(x)\sigma^T(x)D^2u) + \langle b(x), Du\rangle + \langle A(x)Du, Du\rangle^{\frac{q}{2}} = f(x) \tag{2.2}$$

in $\mathbb{R}^N$, where $\sigma, A : \mathbb{R}^N \to S^N$, and $b : \mathbb{R}^N \to \mathbb{R}^N$ are given functions. In this example, $G(x,\xi,X) = F(x,\xi,X) + H(x,\xi) - g(x)$ with $F(x,\xi,X) = -\mathrm{Tr}(\sigma(x)\sigma^T(x)X) + \langle b(x), \xi\rangle$, and $H(x,\xi) = \langle A(x)\xi,\xi\rangle^{q/2}$.

We denote by $\mathcal{M}$ the set of modulus of continuity; $m \in \mathcal{M}$ if $m(s) \to 0$ as $s \to 0^+$ and $m(s+t) \le m(s) + m(t)$ for all $s, t > 0$.

We present a list of hypothesis on F: The first one is a modification of the structure condition, under which we may consider (2.2) when σ and b are locally Lipschitz continuous.

$$(F1)\quad \begin{cases} \text{For } R>0, \text{ there exists } m_R \in \mathcal{M} \text{ such that} \\ F\left(x, \varepsilon^{-1}(x-y), X\right) - F\left(y, \varepsilon^{-1}(x-y), Y\right) \\ \le m_R\left(|x-y| + \varepsilon^{-1}|x-y|^2\right) \\ \text{provided } \varepsilon > 0,\ x, y \in B_R \text{ and } (X,Y) \in S^N \times S^N \text{ satisfies} \\ -\dfrac{3}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \le \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \le \dfrac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \end{cases}$$

Here $B_r = \{x \in \mathbb{R}^N \mid |x| < r\}$ and $B_r(x) = x + B_r$ for $r > 0$ and $x \in \mathbb{R}^N$. Notice that (F1) implies the degenerate ellipticity (1.2).

We next suppose homogeneity of F in $(\xi, X) \in \mathbb{R}^N \times S^N$:

$$(F2)\qquad F(x,\theta\xi,\theta X) = \theta F(x,\xi,X) \quad \text{for } \theta \ge 0,\ x,\xi \in \mathbb{R}^N,\ X \in S^N.$$

To state further hypotheses, we introduce two subsets of functions having superlinear growth of order r;

A continuous function $h : \mathbb{R}^N \to \mathbb{R}$ belongs to $\mathcal{SSG}_r^\pm$ if and only if

$$\liminf_{|x|\to\infty} \frac{\pm h(x)}{|x|^r} \ge 0.$$

Notice that $h \in \mathcal{SSG}_r^+$ (resp., $\mathcal{SSG}_r^-$) if, for any $\varepsilon > 0$, there exists $C_\varepsilon = C_\varepsilon(h) > 0$ such that

$$h(x) \ge -\varepsilon|x|^r - C_\varepsilon \quad (\text{resp., } h(x) \le \varepsilon|x|^r + C_\varepsilon) \quad \text{in } \mathbb{R}^N.$$

We define $\mathcal{SSG}_r = \mathcal{SSG}_r^+ \cap \mathcal{SSG}_r^-$. Notice that $h \in \mathcal{SSG}_r$ if and only if

$$\lim_{|x|\to\infty} \frac{|h(x)|}{|x|^r} = 0.$$

A continuous function $h : \mathbb{R}^N \to \mathbb{R}$ belongs to $\mathcal{SG}_r^\pm$ if and only if

$$\liminf_{|x|\to\infty} \frac{\pm h(x)}{|x|^r} > -\infty.$$

Notice that $h \in \mathcal{SG}_r^+$ (resp., $\mathcal{SG}_r^-$) if, for any $\varepsilon > 0$, there exist positive constants $\varepsilon = \varepsilon(h), C = C(h)$ such that

$$h(x) \ge -\varepsilon|x|^r - C \quad (\text{resp., } h(x) \le \varepsilon|x|^r + C) \quad \text{in } \mathbb{R}^N.$$

We define $\mathcal{SG}_r = \mathcal{SG}_r^+ \cap \mathcal{SG}_r^-$. Notice that, if a continuous function h belongs to $\mathcal{SG}_r$, then there exists $M > 0$ such that, for all $x \in \mathbb{R}^N$,

$$|h(x)| \leq M(1 + |x|^r).$$

The next assumptions indicate that the coefficients to the second and first derivatives are in $\mathcal{SSG}_2$ and $\mathcal{SSG}_1$, respectively.

$$(F3) \quad \begin{cases} \text{There exists } \sigma_0 : \mathbb{R}^N \to S^N \text{ such that } |\sigma_0| \in \mathcal{SSG}_1 \text{ and} \\ F(x,\xi,X) - F(x,\xi,Y) \geq -\mathrm{Tr}(\sigma_0(x)\sigma_0(x)^T(X-Y)) \\ \text{for } x,\xi \in \mathbb{R}^N, X, Y \in S^N. \end{cases}$$

$$(F4) \quad \begin{cases} \text{There exists } b_0 : \mathbb{R}^N \to \mathbb{R} \text{ such that } |b_0| \in \mathcal{SSG}_1 \text{ and} \\ |F(x,\xi,X) - F(x,\eta,X)| \leq b_0(x)|\xi - \eta| \\ \text{for } x,\xi,\eta \in \mathbb{R}^N, X \in S^N. \end{cases}$$

We shall write $\mathcal{P}(x,X) = -\mathrm{Tr}(\sigma_0(x)\sigma_0^T(x)X)$.

We next give a list of hypothesis on $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$:

$$(H1) \qquad \xi \in \mathbb{R}^N \to H(x,\xi) \text{ is convex for } x \in \mathbb{R}^N,$$

which will be violated in Section 4 when we treat PDEs (2.2) with matrices $A(\cdot)$ which are not positive definite everywhere. Under $(H1)$, we need to suppose strict positivity and boundedness of H with respect to $x \in \mathbb{R}^N$. For a fixed $q > 1$,

$$(H2) \quad \begin{cases} \text{There exist } \delta \in C(\mathbb{R}^N) \text{ and } C_0 > 0 \text{ such that } \delta(x) > 0, \\ \text{and } \delta(x)|\xi|^q \leq H(x,\xi) \leq C_0|\xi|^q \text{ for } x,\xi \in \mathbb{R}^N. \end{cases}$$

$$(H3) \qquad H(x,\theta\xi) = \theta^q H(x,\xi) \quad \text{for } x,\xi \in \mathbb{R}^N,\ \theta \geq 0.$$

We also suppose continuity of H in $x \in \mathbb{R}^N$.

$$(H4) \quad \begin{cases} \text{For } R > 0, \text{ there exists } \omega_R \in \mathcal{M} \text{ such that} \\ |H(x,\xi) - H(y,\xi)| \leq \omega_R(|x-y|)|\xi|^q \\ \text{for } x, y \in B_R \text{ and } \xi \in \mathbb{R}^N. \end{cases}$$

In the sequel, we denote by q' the conjugate of $q > 1$;

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Now, we shall come back to the result in [3] for (1.5). Roughly speaking, the comparison result in [3] is as follows: if we suppose that

$$f - g \in \mathcal{SSG}_{q'} \quad \text{for a nonnegative convex function } g : \mathbb{R}^N \to \mathbb{R},$$

then the uniqueness holds among strong solutions in $W^{2,N}_{loc}(\mathbb{R}^N) \cap \mathcal{SSG}^+_{q'}$. Thus, if one restricts f to be nonnegative and convex, then one does not need to suppose any growth condition on f to obtain the comparison principle. In this paper, we generalize the uniqueness result by assuming only that $f \in \mathcal{SSG}^+_{q'}$, i.e., f may have any growth from above and need not to be "close" to a convex function.

3. Comparison principle

We denote by $USC(\mathbb{R}^N)$ (resp. $LSC(\mathbb{R}^N)$) the set of upper (resp., lower) semicontinuous functions in $\mathbb{R}^N$. We first establish the comparison principle when given data are of $\mathcal{SG}_q$.

Theorem 3.1. *Fix any $\lambda > 0$. Assume that (F1−4) and (H1−4) hold. Let $u \in USC(\mathbb{R}^N) \cap \mathcal{SSG}^-_{q'}$ and $v \in LSC(\mathbb{R}^N) \cap \mathcal{SSG}^+_{q'}$ be, respectively, a viscosity subsolution and a viscosity supersolution of (1.1). If $f \in \mathcal{SSG}^+_{q'}$, then for any $\lambda > 0$, we have $u \le v$ in $\mathbb{R}^N$.*

Proof. *Step 1: Linearization procedure.*

For $\mu \in (0,1)$, it is easy to verify that $u_\mu := \mu u$ is a viscosity subsolution of

$$\lambda u_\mu + F(x, Du_\mu, D^2 u_\mu) + \mu^{1-q} H(x, Du_\mu) = \mu f(x) \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

We shall show that $w = w_\mu := u_\mu - v$ is a viscosity subsolution of an extremal PDE

$$\lambda w + \mathcal{P}(x, D^2 w) - b_0(x)|Dw| - \beta_\mu |Dw|^q \le (\mu - 1) f(x) \quad \text{in } \mathbb{R}^N, \tag{3.2}$$

where $\beta_\mu := (\frac{1-\mu}{2})^{1-q} C_0 > 0$.

For $\phi \in C^2(\mathbb{R}^N)$, we suppose that $w - \phi$ attains a local maximum at $\hat{x} \in \mathbb{R}^N$. We may suppose that $(w-\phi)(\hat{x}) = 0 > (w-\phi)(x)$ for $x \in B_r(\hat{x}) \setminus \{\hat{x}\}$ with a small $r \in (0,1)$.

Let $(x_\varepsilon, y_\varepsilon) \in B := \overline{B}_r(\hat{x}) \times \overline{B}_r(\hat{x})$ be a maximum point of $u_\mu(x) - v(y) - (2\varepsilon)^{-1}|x-y|^2 - \phi(y)$ over B. Since we may suppose $\lim_{\varepsilon\to 0}(x_\varepsilon, y_\varepsilon) = (\hat{x}, \hat{x})$, and moreover $\lim_{\varepsilon \to 0}(u_\mu(x_\varepsilon), v(y_\varepsilon)) = (u_\mu(\hat{x}), v(\hat{x}))$, it follows that $(x_\varepsilon, y_\varepsilon) \in \mathrm{int}(B)$ for small ε. Hence, in view of Ishii's lemma (e.g. Theorem 3.2 in [9]), setting $p_\varepsilon = \varepsilon^{-1}(x_\varepsilon - y_\varepsilon)$, we find $X_\varepsilon, Y_\varepsilon \in S^N$ such that $(p_\varepsilon, X_\varepsilon) \in \overline{J}^{2,+} u_\mu(x_\varepsilon)$, $(p_\varepsilon - D\phi(y_\varepsilon), Y_\varepsilon - D^2\phi(y_\varepsilon)) \in \overline{J}^{2,-} v(y_\varepsilon)$, and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \le \begin{pmatrix} X_\varepsilon & O \\ O & -Y_\varepsilon \end{pmatrix} \le \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Thus, from the definition, we have

$$\lambda u_\mu(x_\varepsilon) + F(x_\varepsilon, p_\varepsilon, X_\varepsilon) + \mu^{1-q} H(x_\varepsilon, p_\varepsilon) \le \mu f(x_\varepsilon)$$

and

$$\lambda v(y_\varepsilon) + F(y_\varepsilon, p_\varepsilon - D\phi(y_\varepsilon), Y_\varepsilon - D^2\phi(y_\varepsilon)) + H(y_\varepsilon, p_\varepsilon - D\phi(y_\varepsilon)) \ge f(y_\varepsilon).$$

Since $(F3)$ and $(F4)$ imply

$$\begin{aligned}&\mathcal{P}(y_\varepsilon, D^2\phi(y_\varepsilon)) - b_0(y_\varepsilon)|D\phi(y_\varepsilon)|\\ \le\ & F(y_\varepsilon, p_\varepsilon, Y_\varepsilon) - F(y_\varepsilon, p_\varepsilon - D\phi(y_\varepsilon), Y_\varepsilon - D^2\phi(y_\varepsilon)),\end{aligned}$$

by $(F1)$, we have

$$\begin{aligned}&\lambda(u_\mu(x_\varepsilon) - v(y_\varepsilon)) + \mathcal{P}(y_\varepsilon, D^2\phi(y_\varepsilon)) - b_0(y_\varepsilon)|D\phi(y_\varepsilon)|\\ \le\ & H(y_\varepsilon, p_\varepsilon - D\phi(y_\varepsilon)) - \mu^{1-q}H(x_\varepsilon, p_\varepsilon) + \mu f(x_\varepsilon) - f(y_\varepsilon)\\ & + m_R(|x_\varepsilon - y_\varepsilon| + \varepsilon^{-1}|x_\varepsilon - y_\varepsilon|^2),\end{aligned}$$

where $R = r + |\hat{x}|$.

We shall estimate the first two terms in the right hand side of the above. By $(H1)$, we have

$$H(y_\varepsilon, p_\varepsilon - D\phi(y_\varepsilon)) \le (\frac{1+\mu}{2})^{1-q}H(y_\varepsilon, p_\varepsilon) + (\frac{1-\mu}{2})^{1-q}H(y_\varepsilon, -D\phi(y_\varepsilon)).$$

Thus, due to $(H2)$ and $(H4)$, we find $\omega_R \in \mathcal{M}$ such that

$$\begin{aligned}&H(y_\varepsilon, p_\varepsilon - D\phi(y_\varepsilon)) - \mu^{1-q}H(x_\varepsilon, p_\varepsilon)\\ \le\ & -(\mu^{1-q} - (\tfrac{1+\mu}{2})^{1-q})\delta(y_\varepsilon)|p_\varepsilon|^q + \mu^{1-q}\omega_R(|x_\varepsilon - y_\varepsilon|)|p_\varepsilon|^q\\ & + (\tfrac{1-\mu}{2})^{1-q}H(y_\varepsilon, -D\phi(y_\varepsilon)).\end{aligned} \tag{3.3}$$

Since the positivity of $\delta(\hat{x})$ implies $\mu^{1-q}\omega_R(|x_\varepsilon - y_\varepsilon|) \le (\mu^{1-q} - (\frac{1-\mu}{2})^{1-q})\delta(y_\varepsilon)$ for small $\varepsilon > 0$, we have

$$\begin{aligned}&\lambda(u_\mu(x_\varepsilon) - v(y_\varepsilon)) + \mathcal{P}(y_\varepsilon, D^2\phi(y_\varepsilon)) - b_0(y_\varepsilon)|D\phi(y_\varepsilon)| - \beta_\mu|D\phi(y_\varepsilon)|^q\\ &\qquad \le \mu f(x_\varepsilon) - f(y_\varepsilon) + m_R(|x_\varepsilon - y_\varepsilon| + \varepsilon^{-1}|x_\varepsilon - y_\varepsilon|^2),\end{aligned}$$

where $\beta_\mu = (\frac{1-\mu}{2})^{1-q}C_0$. Therefore, sending $\varepsilon \to 0$ and using that $(2\varepsilon)^{-1}|x_\varepsilon - y_\varepsilon|^2 \to 0$, we have

$$\lambda w(\hat{x}) + \mathcal{P}(\hat{x}, D^2\phi(\hat{x})) - b_0(\hat{x})|D\phi(\hat{x})| - \beta_\mu|D\phi(\hat{x})|^q \le (\mu - 1)f(\hat{x}),$$

which proves that w is a viscosity subsolution of (3.2).

Step 2: Construction of smooth strict supersolutions of (3.2).

Let $\Phi(x) = (1-\mu)\{C_1 + \alpha\langle x\rangle^{q'}\}$, where $\langle x\rangle = (1+|x|^2)^{1/2}$, and $C_1, \alpha > 0$ will be chosen later.

Note that

$$D\langle x\rangle^{q'} = q'\langle x\rangle^{q'-2}x, \text{ and } D^2\langle x\rangle^{q'} = q'\langle x\rangle^{q'-4}\left(\langle x\rangle^2 I + (q'-2)x\otimes x\right).$$

Since $\sigma_0, b_0 \in \mathcal{SSG}_1$ and $f \in \mathcal{SSG}^+_{q'}$, for any $\varepsilon, \varepsilon' > 0$, we can find $C_\varepsilon = C_\varepsilon(\sigma_0, b_0) > 0$ and $C_{\varepsilon'} = C_{\varepsilon'}(f) > 0$ (independent of $\alpha > 0$) such that

$$\begin{aligned}&\mathcal{P}(x, D^2\Phi) - b_0(x)|D\Phi| + (1-\mu)f(x)\\ \ge\ & (1-\mu)\{-\alpha(\varepsilon\langle x\rangle^{q'} + C_\varepsilon\langle x\rangle^{q'-1}) - \varepsilon'\langle x\rangle^{q'} - C_{\varepsilon'}\},\end{aligned}$$

and

$$-\beta_\mu|D\Phi|^q \ge -(1-\mu)\alpha^q C_0'\langle x\rangle^{q(q'-1)} = -(1-\mu)\alpha^q C_0'\langle x\rangle^{q'},$$

where $C_0' = 2^{q-1}(q')^q$. Hence, we have

$$
\begin{array}{rl}
 & \lambda\Phi + \mathcal{P}(x, D^2\Phi) - b_0(x)|D\Phi| - \beta_\mu |D\Phi|^q + (1-\mu)f(x) \\
(3.4) \quad \geq & (1-\mu)\{\lambda C_1 + \alpha(\lambda - \varepsilon - C_\varepsilon \langle x\rangle^{-1} - \alpha^{q-1}C_0')\langle x\rangle^{q'} \\
 & \quad -\varepsilon'\langle x\rangle^{q'} - C_{\varepsilon'}\}.
\end{array}
$$

Fix $\varepsilon, \alpha \in (0,1)$ such that $\varepsilon \leq \lambda/4$ and $\alpha^{q-1}C_0' \leq \lambda/4$. We then choose $\varepsilon' \leq \lambda\alpha/4$ to estimate the right hand side of the above from below by

$$(1-\mu)\{\lambda C_1 - C_{\varepsilon'} + \alpha(\frac{\lambda}{4} - C_\varepsilon\langle x\rangle^{-1})\langle x\rangle^{q'}\}.$$

Hence, taking $C_1 = \lambda^{-1}[C_{\varepsilon'} + \max\{C_\varepsilon \langle x\rangle^{q'-2} \mid \langle x\rangle \leq 4C_\varepsilon/\lambda\}] + 1$, we see that Φ satisfies

$$\lambda\Phi + \mathcal{P}(x, D^2\Phi) - b_0(x)|D\Phi| - \beta_\mu|D\Phi|^q > (\mu - 1)f(x) \quad \text{in } \mathbb{R}^N. \tag{3.5}$$

Step 3: Conclusion.

Since $w \in \mathcal{SSG}^-_{q'}$, $w - \Phi$ takes its maximum at $\hat{x} \in \mathbb{R}^N$. Thus, we have

$$\lambda w(\hat{x}) + \mathcal{P}(\hat{x}, D^2\Phi(\hat{x})) - b_0(\hat{x})|D\Phi(\hat{x})| - \beta_\mu|D\Phi(\hat{x})|^q \leq (\mu-1)f(\hat{x}).$$

If $(w - \Phi)(\hat{x}) \geq 0$, then we get a contradiction to (3.5). Hence, we have

$$w(x) \leq (1-\mu)(C_1 + \alpha\langle x\rangle^{q'}) \quad \text{for } x \in \mathbb{R}^N,$$

which concludes the assertion in the limit $\mu \nearrow 1$. □

Note that, if we suppose $\sigma_0 \in \mathcal{SG}_1$ or $b_0 \in \mathcal{SG}_1$ in (F3-4), then the comparison principle for (1.1) fails among solutions in $\mathcal{SG}_{q'}$ in general. In fact, we recall the example (1.6) stated in the Introduction. In this example, $b_0 \in \mathcal{SG}_1$ but does not belong to $\mathcal{SSG}_1$ unless $t = 0$, and the comparison obviously fails since one does not have uniqueness.

Also, if we consider

$$u - (1+x^2)u'' + |u'|^2 = 0 \quad \text{in } \mathbb{R}, \tag{3.6}$$

then it is easy to check that $v_1 \equiv 0$ and $v_2(x) = \frac{1}{2} + \frac{1}{4}x^2$ are solutions of (3.6) in $\mathcal{SG}_2$ but $v_2 \notin \mathcal{SSG}_2$. This nonuniqueness comes from $\sigma_0 \in \mathcal{SG}_1$.

In [16], they may suppose that given functions belong to $\mathcal{SG}_1$ for the comparison principle. However, they need to suppose that λ is large enough. We can extend their results following the above arguments.

$$
(F3') \quad \left\{ \begin{array}{c}
\text{There exists } \sigma_0 : \mathbb{R}^N \to S^N \text{ such that } |\sigma_0| \in \mathcal{SG}_1 \text{ and} \\
F(x,\xi,X) - F(x,\xi,Y) \geq -\mathrm{Tr}(\sigma_0(x)\sigma_0(x)^T(X-Y)) \\
\text{for } x,\xi \in \mathbb{R}^N, X, Y \in S^N,
\end{array} \right.
$$

$$(F4') \qquad \begin{cases} \text{There exists } b_0 : \mathbb{R}^N \to \mathbb{R} \text{ such that } |b_0| \in \mathcal{SG}_1 \text{ and} \\ |F(x,\xi,X) - F(x,\eta,X)| \le b_0(x)|\xi - \eta| \\ \text{for } x,\xi,\eta \in \mathbb{R}^N, X \in S^N. \end{cases}$$

Theorem 3.2. *Assume that* $(F1,2)$, $(F3',4')$ *and* $(H1-4)$ *hold. For* $f \in \mathcal{SG}_{q'}^+$, *there exists* $\lambda_0 > 0$ *such that for* $\lambda \ge \lambda_0$, *if* $u \in USC(\mathbb{R}^N) \cap \mathcal{SSG}_{q'}^-$ *and* $v \in LSC(\mathbb{R}^N) \cap \mathcal{SSG}_{q'}^+$ *are, respectively, a viscosity subsolution and a viscosity supersolution of* (1.1), *then* $u \le v$ *in* $\mathbb{R}^N$.

Proof. We do not need any change in Step 1 of proof of Theorem 3.1.

In view of $(F3')$ and $(F4')$, we can get (3.4) for some $\varepsilon, \varepsilon', C_\varepsilon, C_{\varepsilon'} > 0$ which are not necessary small. Therefore, we can choose $\lambda_0 > 0$ such that for $\lambda \ge \lambda_0$, we can show Φ is a strict supersolution of (3.2). The rest of proof can be done by the same argument. □

In the above Theorem, we need to assume that $u, -v \in \mathcal{SSG}_{q'}^-$ to be sure that $w - \Phi$ achieves a maximum in $\mathbb{R}^N$ (recall that $(1-\mu)$ in front of Φ is arbitrarily small). If we are concerned with PDEs (1.1) without superlinear terms, that is

$$\lambda u + F(x, Du, D^2u) = f(x) \quad \text{in } \mathbb{R}^N, \tag{3.7}$$

then we can obtain slightly stronger results.

Proposition 3.3. *Assume that* $(F1-4)$ *holds. Let* $u \in USC(\mathbb{R}^N) \cap \mathcal{SG}_{q'}^-$ *and* $v \in LSC(\mathbb{R}^N) \cap \mathcal{SG}_{q'}^+$ *be, respectively, a viscosity subsolution and a viscosity supersolution of* (3.7). *If* $f \in \mathcal{SSG}_{q'}^+$, *then* $u \le v$ *in* $\mathbb{R}^N$.

Proof. Following the argument in the proof of Theorem 3.1, we verify that $w := u - v$ is a viscosity subsolution of

$$\lambda w + \mathcal{P}(x, D^2 w) - b_0(x)|Dw| = 0 \quad \text{in } \mathbb{R}^N. \tag{3.8}$$

Now, setting $\Phi(x) = \alpha\langle x\rangle^{q'} + C_1$ for $\alpha, C_1 \ge 1$, we see that Φ satisfies

$$\lambda\Phi(x) + \mathcal{P}(x,\Phi(x)) - b_0(x)|D\Phi(x)| \ge (\lambda C_1 - C_\varepsilon) + \alpha\langle x\rangle^{q'}(\lambda - \varepsilon\langle x\rangle^{-2} - \varepsilon\alpha^{-1}), \tag{3.9}$$

where $\varepsilon > 0$ is small enough so that the second term of the right hand side is positive. We then choose $C_1 \ge C_\varepsilon/\lambda$ to show that Φ is a strict supersolution of (3.8). Since we may take α large enough so that $w - \Phi$ attains its maximum at a point in $\mathbb{R}^N$, we conclude the proof. □

Finally, we treat the case when given functions are in $\mathcal{SG}_1$.

Proposition 3.4. *Assume that* $(F1,2)$ *and* $(F3',4')$ *hold. For* $f \in \mathcal{SG}_{q'}^+$, *there exists* $\lambda_0 > 0$ *such that if* $u \in USC(\mathbb{R}^N) \cap \mathcal{SG}_{q'}^-$ *and* $v \in LSC(\mathbb{R}^N) \cap \mathcal{SG}_{q'}^+$ *are, respectively, a viscosity subsolution and a viscosity supersolution of* (3.7), *then* $u \le v$ *in* $\mathbb{R}^N$.

Proof. As above, we can show (3.9) but $\varepsilon > 0$ may not be small. However, again, for large $\lambda > 0$, we can show that Φ is a strict supersolution of (3.8) when α, C_1 are any large numbers. Thus, we can conclude the proof even for $w \in \mathcal{SG}^-_{q'}$. $\square$

4. Non-convex H

In this section, we deal with some case when $(H1)$ is not satisfied.

We denote by $\Gamma \subset \mathbb{R}^N$ the zero-level set of $H(\cdot, \xi)$ for all $\xi \in \mathbb{R}^N$;

$$\Gamma = \{x \in \mathbb{R}^N \mid H(x, \xi) = 0 \text{ for any } \xi \in \mathbb{R}^N\}.$$

Our assumptions are as follows. For σ_0 in $(F3)$ and b_0 in $(F4)$,

$$(A1) \qquad \Gamma \subset \{x \in \mathbb{R}^N \mid \sigma_0(x) = 0,\ b_0(x) = 0\}.$$

Assumption $(A1)$ is a kind of degeneracy condition on the coefficients of F.

$$(A2) \qquad \left\{ \begin{array}{c} \text{There exist open sets } \Omega^\pm \subset \mathbb{R}^N, \delta^\pm \in C(\mathbb{R}^N) \text{ and} \\ C_0^\pm > 0 \text{ such that } \mathbb{R}^N = \Gamma \cup \Omega^+ \cup \Omega^-, \delta^\pm(x) > 0, \\ \delta^\pm(x)|\xi|^q \le \pm H(x,\xi) \le C_0^\pm |\xi|^q \text{ for } x \in \Omega^\pm, \xi \in \mathbb{R}^N, \\ \text{and } \xi \to \pm H(x,\xi) \text{ are convex for } x \in \Omega^\pm. \end{array} \right.$$

It means that we can divide $\mathbb{R}^N \setminus \Gamma$ into two open subsets: Ω^+ where $H(x, \cdot)$ is convex and Ω^- where $H(x, \cdot)$ is concave.

When $A(x) = a(x)I$ in (2.2) for some $a : \mathbb{R}^N \to \mathbb{R}$, $\Omega^\pm = \{x \in \mathbb{R}^N \mid \pm a(x) > 0\}$, and $\Gamma = \{x \in \mathbb{R}^N \mid a(x) = 0\}$.

We also suppose that σ_0 in $(F3)$ and b_0 in $(F4)$ satisfy that

$$(A3) \qquad \sigma_0, b_0 \in W^{1,\infty}_{loc}(\mathbb{R}^N).$$

Finally, we need some degeneracy condition for H on Γ.

$$(A4) \qquad \left\{ \begin{array}{c} \text{For each } x_0 \in \Gamma, \text{ there exist } r, C_1 > 0 \text{ such that} \\ |H(x,\xi)| \le C_1 |x - x_0|^q |\xi|^q \text{ for } x \in B_r(x_0). \end{array} \right.$$

Theorem 4.1. *Assume that $(F1-4)$, $(H3,4)$ and $(A1-4)$ hold. Let $u \in USC(\mathbb{R}^N) \cap \mathcal{SSG}^-_{q'}$ and $v \in LSC(\mathbb{R}^N) \cap \mathcal{SSG}^+_{q'}$ be, respectively, a viscosity subsolution and a viscosity supersolution of* (1.1)*. If $f \in \mathcal{SSG}^+_{q'}$, then $u \le v$ in $\mathbb{R}^N$.*

Proof. We first notice that the comparison principle holds if $\xi \to H(x, \xi)$ is concave instead of $(H1)$. In fact, we may take $w_\mu = u - \mu v$ for $\mu \in (0,1)$, and then we can follow the argument in the proof of Theorem 3.2.

Step 1: $u \le f/\lambda \le v$ on Γ.

We only prove the first inequality since the second one can be shown similarly. For $x_0 \in \Gamma$, let $x_\varepsilon \in \overline{B}_1(x_0)$ be the maximum point of

$u(x) - f(x_0) - (2\varepsilon)^{-1}|x - x_0|^2$ over $\overline{B}_1(x_0)$. It is easy to see that $\lim_{\varepsilon\to 0} x_\varepsilon = x_0$; $x_\varepsilon \in B_1(x_0)$ for small $\varepsilon > 0$.

It follows that we can write the viscosity inequality for the subsolution u of (1.1) at x_ε (see *e.g.* [9]): for any $\varepsilon > 0$, there exists $X_\varepsilon \in S^N$ such that

$$(p_\varepsilon, X_\varepsilon) \in \bar{J}^{2,+}u(x_\varepsilon), \quad \text{with} \quad X_\varepsilon \leq \frac{3}{\varepsilon}I, \tag{4.1}$$

where $p_\varepsilon = \varepsilon^{-1}(x_\varepsilon - x_0)$. We have

$$\lambda u(x_\varepsilon) - \mathcal{P}(x_\varepsilon, X_\varepsilon) - b_0(x_\varepsilon)|p_\varepsilon| + H(x_\varepsilon, p_\varepsilon) \leq f(x_\varepsilon).$$

By $(A3)$ and $(A4)$, we can find some constants $C_{\sigma,1}, C_{b,1}, C_1 > 0$ such that, for ε small enough, we have

$$|\sigma_0(x_\varepsilon)| \leq C_{\sigma,1}|x_\varepsilon - x_0|, \quad |b_0(x_\varepsilon)| \leq C_{b,1}|x_\varepsilon - x_0|,$$
$$\text{and} \quad |H(x_\varepsilon, p_\varepsilon)| \leq C_1|x_\varepsilon - x_0|^q|p_\varepsilon|^q.$$

It follows that there exists $C > 0$ such that

$$\lambda u(x_\varepsilon) - C\left(\varepsilon^{-1}|x_\varepsilon - x_0|^2 + \varepsilon^{-q}|x_\varepsilon - x_0|^{2q}\right) \leq f(x_\varepsilon).$$

Since $\lim_{\varepsilon\to 0} \varepsilon^{-1}|x_\varepsilon - x_0|^2 = 0$ and $\lim_{\varepsilon\to 0} u(x_\varepsilon) = u(x_0)$, letting $\varepsilon \to 0$, we get

$$\lambda u(x_0) \leq f(x_0).$$

Step 2: Comparison on $\Omega^+ \cup \Gamma$.

We can proceed exactly as in the convex case (Step 1 in the proof of Theorem 3.1) to prove that $w_\mu = \mu u - v$ (for $0 < \mu < 1$) is a subsolution of (3.2) in Ω^+. Define $\Phi = (1-\mu)(C_1 + \alpha\langle x\rangle^{q'})$ with the same choice of constant α, C_1 as before. Notice that, with this choice, $\lambda\Phi \geq (\mu - 1)f$ in $\mathbb{R}^N$.

Consider $\sup_{\Omega^+\cup\Gamma}(w_\mu - \Phi_\mu)$. Since $w_\mu \in \mathcal{SSG}^-_{q'}$, this supremum is finite and is achieved at a point $\bar{x}$ which belongs to the closed set $\Omega^+\cup\Gamma$. We distinguish two cases.

At first, if $\bar{x} \in \Omega^+$, then, arguing as in the convex case (Step 2 in the proof of Theorem 3.1) we can write the viscosity inequality for w_μ using Φ as a test-function to show that the supremum is nonpositive.

Now, if $\bar{x} \in \Gamma$, then, from Step 1, we get $u(\bar{x}) \leq f(\bar{x})/\lambda \leq v(\bar{x})$ and therefore $w_\mu(\bar{x}) \leq (\mu - 1)f(\bar{x})/\lambda \leq \Phi(\bar{x})$; thus the supremum is nonpositive. In both case, $w_\mu - \Phi \leq 0$. Letting $\mu \nearrow 1$, we conclude $u \leq v$ in $\Omega^+ \cup \Gamma$.

Step 3: Conclusion.

To get the comparison in $\Omega^- \cup \Gamma$, we use the fact that we are in the concave case in Ω^-. As noticed before, we can prove $u \leq v$ in $\Omega^- \cup \Gamma$.

□

In Introduction, we give a nonconvex $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$H(x,\xi) = \min\{H_k(x,\xi) \mid k = 1,2,\dots,m\}, \tag{4.2}$$

where H_k is convex in ξ and $m \in \mathbb{N}$. We shall denote by A the set $\{1,2,\dots,m\}$.

Theorem 4.2. *Assume that* $(F1-4)$ *holds, that* H *in* (1.1) *is given by* (4.2) *and that* $(H1-4)$ *holds for each* H_k *with common* $\delta \in C(\mathbb{R}^N)$, $C_0 > 0$ *and* $\omega_R \in \mathcal{M}$ *for* $k \in A$. *Let* $u \in USC(\mathbb{R}^N) \cap \mathcal{SSG}^-_{q'}$ *and* $v \in LSC(\mathbb{R}^N) \cap \mathcal{SSG}^+_{q'}$ *be, respectively, a viscosity subsolution and a viscosity supersolution of* (1.1). *If* $f \in \mathcal{SSG}^+_{q'}$, *then for any* $\lambda > 0$, *we have* $u \le v$ *in* $\mathbb{R}^N$.

Proof. It is enough to verify Step 1 in the proof of Theorem 3.1. More precisely, we only need to check if (3.3) holds. We shall use the same notation in the proof of Theorem 3.1. For any $\varepsilon > 0$, we can choose $k_\varepsilon \in A$ such that

$$H(x_\varepsilon, p_\varepsilon) = H_{k_\varepsilon}(x_\varepsilon, p_\varepsilon).$$

Hence, we have

$$\begin{aligned}
& H(y_\varepsilon, p_\varepsilon - D\phi(y_\varepsilon)) - \mu^{1-q} H(x_\varepsilon, p_\varepsilon) \\
\le\ & H_{k_\varepsilon}(y_\varepsilon, p_\varepsilon - D\phi(y_\varepsilon)) - \mu^{1-q} H_{k_\varepsilon}(x_\varepsilon, p_\varepsilon) \\
\le\ & -(\mu^{1-q} - (\frac{1+\mu}{2})^{1-q})\delta(y_\varepsilon)|p_\varepsilon|^q + \mu^{1-q}\omega_R(|x_\varepsilon - y_\varepsilon|)|p_\varepsilon|^q \\
& +(\frac{1-\mu}{2})^{1-q} C_0 |D\phi(y_\varepsilon)|.
\end{aligned}$$

Therefore, since the remaining proof is the same as in the proof of Theorem 3.1, we conclude the proof. □

We shall generalize the above H.

Let $\mathcal{A}$ and $\mathcal{B}$ be compact metric spaces. For $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, we consider continuous functions $\sigma, \tau : \mathbb{R}^N \times \mathcal{A} \times \mathcal{B} \to M(N,n)$, where $M(N,n)$ denotes the set of $N \times n$ real-valued matrices. For $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, $a, b \in \mathbb{R}^n$, $x, \xi \in \mathbb{R}^N$, we define $H^{\alpha,a}_{\beta,b} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$H^{\alpha,a}_{\beta,b}(x,\xi) = 2\langle \sigma(x,\alpha,\beta)a - \tau(x,\alpha,\beta)b, \xi\rangle - |a|^2 + |b|^2.$$

We next set

$$\begin{aligned}
H_{\beta,b}(x,\xi) &= \sup_{\alpha\in\mathcal{A}, a\in\mathbb{R}^n} H^{\alpha,a}_{\beta,b}(x,\xi) \\
&= \sup_{\alpha\in\mathcal{A}} \{|\sigma^T(x,\alpha,\beta)\xi|^2 - 2\langle \tau(x,\alpha,\beta)b, \xi\rangle\} + |b|^2
\end{aligned}$$

for $\beta \in \mathcal{B}$, $b \in \mathbb{R}^n$ and $x, \xi \in \mathbb{R}^N$. Finally, set

$$\begin{aligned} H(x,\xi) &= \inf_{\beta\in\mathcal{B}, b\in\mathbb{R}^n} H_{\beta,b}(x,\xi) \\ &= \inf_{\beta\in\mathcal{B}} \sup_{\alpha\in\mathcal{A}} \{|\sigma^T(x,\alpha,\beta)\xi|^2 - |\tau^T(x,\alpha,\beta)\xi|^2\}. \end{aligned} \tag{4.3}$$

Defining $S(x,\alpha,\beta) = \sigma(x,\alpha,\beta)\sigma^T(x,\alpha,\beta), T(x,\alpha,\beta) = \tau(x,\alpha,\beta)\tau^T(x,\alpha,\beta) \in S^N$, for $x \in \mathbb{R}^N$ and $(\alpha,\beta) \in \mathcal{A}\times\mathcal{B}$, we give a condition on S, T so that H satisfies $(H2)$.

$$(H2') \quad \begin{cases} & \text{There are } \delta \in C(\mathbb{R}^N) \text{ and } C_0 > 0 \text{ such that} \\ \text{(i)} & \delta(x) > 0 \text{ for } x \in \mathbb{R}^N, \\ \text{(ii)} & \text{for any } x \in \mathbb{R}^N \text{ and } \beta \in \mathcal{B}, \text{ there exists } \alpha_{\beta,x} \in \mathcal{A} \\ & \text{satisfying } S(x,\alpha_{\beta,x},\beta) - T(x,\alpha_{\beta,x},\beta) \geq \delta(x) I, \\ \text{(iii)} & \text{for any } x \in \mathbb{R}^N, \text{ there exists } \beta_x \in \mathcal{B} \text{ satisfying} \\ & \sup_{\alpha\in\mathcal{A}} |S(x,\alpha,\beta_x)| \leq C_0. \end{cases}$$

Assuming that $S, T : \mathbb{R}^N \times \mathcal{A} \times \mathcal{B} \to S^N$ satisfy $(H2')$, we easily verify that the above H satisfies $(H2)$ and $(H3)$ with $q = 2$. In fact, for $x, \xi \in \mathbb{R}^N$, we choose $\beta_x = \beta_{x,\xi} \in \mathcal{B}$ such that $H(x,\xi) = \sup_{\alpha\in A}\{|\sigma^T(x,\alpha,\beta_x)\xi|^2 - |\tau^T(x,\alpha,\beta_x)\xi|^2\}$. Thus, by $(H2')$, we can find $\alpha_x = \alpha_{x,\xi} \in \mathcal{A}$ such that

$$\begin{aligned} H(x,\xi) &\geq |\sigma^T(x,\alpha_x,\beta_x)\xi|^2 - |\tau^T(x,\alpha_x,\beta_x)\xi|^2 \\ &= \langle (S(x,\alpha_x,\beta_x) - T(x,\alpha_x,\beta_x))\xi, \xi\rangle \\ &\geq \delta(x)|\xi|^2. \end{aligned}$$

The other inequality is trivial by (iii) of $(H2')$. Furthermore, assuming that

$$(H4') \quad \begin{cases} & \text{for } R > 0, \text{ there are } C_R > 0 \text{ and } \hat{\omega}_R \in \mathcal{M} \text{ such that} \\ \text{(i)} & |\sigma(x,\alpha,\beta)| + |\tau(x,\alpha,\beta)| \leq C_R \text{ for } x \in B_R \text{ and} \\ & (\alpha,\beta) \in \mathcal{A}\times\mathcal{B}, \\ \text{(ii)} & |\sigma(x,\alpha,\beta) - \sigma(y,\alpha,\beta)| + |\tau(x,\alpha,\beta) - \tau(y,\alpha,\beta)| \\ & \leq \hat{\omega}_R(|x-y|) \text{ for } x, y \in B_R \text{ and } (\alpha,\beta) \in \mathcal{A}\times\mathcal{B}, \end{cases}$$

we can show that $(H4)$ holds with some $\omega_R \in \mathcal{M}$.

Now, we can state the comparison principle for the above H in (1.1). Since we can prove it with the same argument as in the proof of Theorem 4.2, we leave it to the readers.

Corollary 4.3. *Assume that* $(F1-4)$ *holds, that* H *in* (1.1) *is given by* (4.3) *and that* $(H2'), (H4')$ *hold. Let* $u \in USC(\mathbb{R}^N) \cap \mathcal{SSG}_2^-$ *and* $v \in LSC(\mathbb{R}^N) \cap \mathcal{SSG}_2^+$ *be, respectively, a viscosity subsolution and a viscosity supersolution of* (1.1). *If* $f \in \mathcal{SSG}_2^+$, *then for any* $\lambda > 0$, *we have* $u \leq v$ *in* $\mathbb{R}^N$.

In particular, we shall suppose that σ and τ are, respectively, independent of α and β. Then, it is easy to see

$$H(x,\xi) = \min_{\beta\in\mathcal{B}} |\sigma^T(x,\beta)\xi|^2 - \min_{\alpha\in\mathcal{A}} |\tau^T(x,\alpha)\xi|^2.$$

Since it is straightforward to restate the hypothesis $(H2')$ and $(H4')$ in this case, we leave it to the readers.

Remark 4.4. We may give some generalizations of Theorems 4.1 and 4.2 to PDEs with coefficients in $\mathcal{SG}$ instead of $\mathcal{SSG}$ as it was done at the end of Section 3.

5. Monotone systems

In this section, we establish the comparison principle to monotone systems of elliptic PDEs, which were introduced in [15].

For a given integer $m \geq 2$, we set $A = \{1, 2, \dots, m\}$. We consider systems of PDEs: for $k \in A$,

$$F_k(x, u, Du_k, D^2u_k) + H_k(x, Du_k) = f_k(x) \quad \text{in } \mathbb{R}^N, \tag{5.1}$$

where $u = (u_1, u_2, \dots, u_m) : \mathbb{R}^N \to \mathbb{R}^m$ is an unknown function, and $F_k : \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^N \times S^N \to \mathbb{R}$, $H_k : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $f_k : \mathbb{R}^N \to \mathbb{R}$ $(k \in A)$ are given functions.

First of all, we recall the definition of viscosity solutions of general systems of PDEs: for $k \in A$,

$$G_k(x, u, Du_k, D^2u_k) = 0 \quad \text{in } \mathbb{R}^N, \tag{5.2}$$

where $G_k : \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^N \times S^N \to \mathbb{R}$ is continuous.

Definition 5.1. *We call $u = (u_k) : \mathbb{R}^N \to \mathbb{R}^m$ a viscosity subsolution (resp., supersolution) of* (5.2) *if for $\phi \in C^2(\mathbb{R}^N)$ and $k \in A$,*

$$G_k(\hat{x}, u^*(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \leq 0$$

$$(\textit{resp., } G_k(\hat{x}, u_*(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \geq 0)$$

provided $(u_k)^* - \phi$ (resp., $(u_k)_* - \phi$) attains its local maximum (resp., minimum) at $\hat{x} \in \mathbb{R}^N$.

We also call u a viscosity solution of (5.2) *if it is both a viscosity sub- and supersolution of* (5.2).

We will suppose that $F := (F_1, F_2, \dots, F_m)$ is monotone as in [15]:

$$(M) \quad \left\{ \begin{array}{c} \text{There exists } \lambda > 0 \text{ such that} \\ \text{if } r = (r_k), s = (s_k) \in \mathbb{R}^m, \ (x, \xi, X) \in \mathbb{R}^N \times \mathbb{R}^N \times S^N \text{ and} \\ \max_{k\in A}(r_k - s_k) = r_j - s_j \geq 0 \text{ for } j = j(r, s, x, \xi, X) \in A, \\ \text{then } F_j(x, r, \xi, X) - F_j(x, s, \xi, X) \geq \lambda(r_j - s_j). \end{array} \right.$$

We will suppose that every $F_k = F_k(x, r, \xi, X)$ in $F = (F_k)$ satisfies $(F1)$ with a modulus $m_{R,k}$ uniformly for $|r| \le R$; moreover it satisfies $(F3)$ and $(F4)$ with some $\sigma_k \in \mathcal{SSG}_1$ and $b_k \in \mathcal{SSG}_1$, respectively. Assumption $(F2)$ is replaced with
$(F2')$
$F(x, \theta r, \theta\xi, \theta X) = \theta F(x, r, \xi, X)$ for $\theta \ge 0, x, \xi \in \mathbb{R}^N, r \in \mathbb{R}^m, X \in S^N$.

We set $\mathcal{P}_k(x, X) = -\mathrm{Tr}(\sigma_k(x)\sigma_k^T(x)X)$. In the same way, we will assume that H_k satifies $(H1)$–$(H4)$ with common $\delta \in C(\mathbb{R}^N)$, $q > 1$, and ω_R (though we may allow them to depend on $k \in A$).

Theorem 5.2. *Assume that (M), $(F1, 2', 3, 4)$ hold for F_k and $(H1-4)$ hold for H_k $(k \in A)$.*

Let $u_k \in USC(\mathbb{R}^N) \cap \mathcal{SSG}^-_{q'}$ and $v_k \in LSC(\mathbb{R}^N) \cap \mathcal{SSG}^+_{q'}$, $u = (u_k)$ and $v = (v_k)$ be, respectively, a viscosity subsolution and a viscosity supersolution of (5.1). If $f_k \in \mathcal{SSG}^+_{q'}$ for $k \in A$, then $u_k \le v_k$ in $\mathbb{R}^N$ for $k \in A$.

Proof. First of all, by $(F2)$, $(H1)$ and $(H3)$, we verify that $u_\mu = (u_{\mu,k}) = (\mu u_k)$ $(\mu \in (0, 1))$ is a viscosity subsolution of

$$F_k(x, u_\mu, Du_{\mu,k}, D^2u_{\mu,k}) + \mu^{1-q}H_k(x, Du_{\mu,k}) \le \mu f_k(x) \quad \text{in } \mathbb{R}^N.$$

Step 1: Linearization. Set $w(x) = \max_{k\in A}(u_{\mu,k} - v_k)(x)$ for $x \in \mathbb{R}^N$. We shall verify that w is a viscosity subsolution of

$$\lambda w + \min_{k\in A}\{\mathcal{P}_k(x, D^2w) - b_k(x)|Dw| - \beta_\mu|Dw|^q - (\mu - 1)f_k(x)\} = 0$$

in $\mathbb{R}^N$, where $\beta_\mu = \left(\frac{1-\mu}{2}\right)^{1-q} C_0$. We argue as in the proof of Theorem 3.1 assuming that, for a fixed $\phi \in C^2(\mathbb{R}^N)$, $w - \phi$ attains a strict local maximum at $\hat{x} \in \mathbb{R}^N$. Setting $B := \overline{B}_r(\hat{x}) \times \overline{B}_r(\hat{x})$, up to extract subsequences, we can suppose that

$$\begin{aligned} (5.3) \qquad & \max_{x,y\in B}\max_{k\in A}\{u_{\mu,k}(x) - v_k(y) - (2\varepsilon)^{-1}|x - y|^2 - \phi(y)\} \\ &= u_{\mu,j(\varepsilon)}(x_\varepsilon) - v_{j(\varepsilon)}(y_\varepsilon) - (2\varepsilon)^{-1}|x_\varepsilon - y_\varepsilon|^2 - \phi(y_\varepsilon) \end{aligned}$$

$x_\varepsilon, y_\varepsilon \to \hat{x}$ and $u_{\mu,j(\varepsilon)}(x_\varepsilon) - v_{j(\varepsilon)}(y_\varepsilon) \to w(\hat{x})$. Moreover, since the set A is finite, we may suppose that $j(\varepsilon) = j$ is independent of ε.

As in the proof of Theorem 3.1, since there are $X_{j,\varepsilon}, Y_{j,\varepsilon} \in S^N$ such that $(p_\varepsilon, X_{j,\varepsilon}) \in \overline{J}^{2,+}u_j(x_\varepsilon)$, $(p_\varepsilon - D\phi(y_\varepsilon), Y_{j,\varepsilon} - D^2\phi(y_\varepsilon)) \in \overline{J}^{2,-}v_j(y_\varepsilon)$, and the matrix inequalities in $(F1)$ hold with $(X_{j,\varepsilon}, Y_{j,\varepsilon})$, we have

$$\begin{aligned} (5.4) \qquad & F_j(y_\varepsilon, u_\mu(x_\varepsilon), p_\varepsilon, Y_{j,\varepsilon}) \\ &\le F_j(x_\varepsilon, u_\mu(x_\varepsilon), p_\varepsilon, X_{j,\varepsilon}) + m_R(|x_\varepsilon - y_\varepsilon| + \varepsilon^{-1}|x_\varepsilon - y_\varepsilon|^2), \end{aligned}$$

where $R = r + |\hat{x}|$.

Moreover, by $(F3)$ and $(F4)$, we have

$$\begin{aligned} (5.5) \qquad & F_j(y_\varepsilon, v(y_\varepsilon), p_\varepsilon - D\phi(y_\varepsilon), Y_{j,\varepsilon} - D^2\phi(y_\varepsilon)) \\ \leq \ & F_j(y_\varepsilon, v(y_\varepsilon), p_\varepsilon, Y_{j,\varepsilon}) - \mathcal{P}_j(y_\varepsilon, D^2\phi(y_\varepsilon)) + b_j(y_\varepsilon)|D\phi(y_\varepsilon)|. \end{aligned}$$

From (5.3), we note that

$$\max_{k\in A}(u_{\mu,k}(x_\varepsilon) - v_k(y_\varepsilon)) \ = \ u_{\mu,j}(x_\varepsilon) - v_j(y_\varepsilon)$$

and therefore, by (M), we have

$$\begin{aligned} (5.6) \qquad & \lambda(u_{\mu,j}(x_\varepsilon) - v_j(y_\varepsilon)) \\ \leq \ & F_j(y_\varepsilon, u_\mu(x_\varepsilon), p_\varepsilon, Y_{j,\varepsilon}) - F_j(y_\varepsilon, v(y_\varepsilon), p_\varepsilon, Y_{j,\varepsilon}). \end{aligned}$$

On the other hand, from the definition, we have

$$F_j(x_\varepsilon, u_\mu(x_\varepsilon), p_\varepsilon, X_{j,\varepsilon}) + \mu^{1-q}H(x_\varepsilon, p_\varepsilon) \leq \mu f_j(x_\varepsilon),$$

and

$$F_j(y_\varepsilon, v(y_\varepsilon), p_\varepsilon - D\phi(y_\varepsilon), Y_{j,\varepsilon} - D^2\phi(y_\varepsilon)) + H(y_\varepsilon, p_\varepsilon) \geq f_j(y_\varepsilon).$$

Thus, following the same calculations for H_j as in Theorem 3.1, by (5.4), (5.5) and (5.6), we have

$$\begin{aligned} (5.7) \qquad & \lambda(u_{\mu,j}(x_\varepsilon) - v_j(y_\varepsilon)) \\ & +\mathcal{P}_j(y_\varepsilon, D^2\phi(y_\varepsilon)) - b_j(y_\varepsilon)|D\phi(y_\varepsilon)| - \beta_\mu|D\phi(y_\varepsilon)|^q \\ & -\mu f_j(x_\varepsilon) + f_j(y_\varepsilon) \\ \leq \ & m_R(|x_\varepsilon - y_\varepsilon| + \varepsilon^{-1}|x_\varepsilon - y_\varepsilon|^2) \end{aligned}$$

for small enough $\varepsilon > 0$. Hence, sending $\varepsilon \to 0$ in (5.7), we obtain the desired extremal PDE

$$\lambda w(\hat{x}) + \min_{k\in A}\{\mathcal{P}_k(\hat{x}, D^2\phi(\hat{x})) - b_k(\hat{x})|D\phi(\hat{x})| - \beta_\mu|D\phi(\hat{x})|^q - (\mu-1)f_k(\hat{x})\} \leq 0.$$

Step 2: Conclusion. Consider the same function Φ from the proof of Theorem 3.1. We can choose the constant $\alpha, C_0 > 0$ in order that Φ is a strict supersolution of the previous extremal PDE. The conclusion follows. □

Remark 5.3. As in the previous sections, we may give some generalizations of Theorem 4.1 to PDEs with coefficients in $\mathcal{SG}$ instead of $\mathcal{SSG}$ and for nonconvex Hamiltonians H_k satisfying assumptions like (A1)–(A4) on some subsets $\Omega_k^\pm$, Γ_k. The proof combines techniques developed in Section 3 and 4, so we skip it.

References

[1] S. Aizawa and Y. Tomita. On unbounded viscosity solutions of a semilinear second order elliptic equations. *Funkcial. Ekvac.*, 31: 147–160, 1988.

[2] S. Aizawa and Y. Tomita. Unbounded viscosity solutions of nonlinear elliptic equations in $\mathbb{R}^N$. *Adv. Math. Sci. Appl.*, 2: 297–316, 1993.

[3] O. Alvarez. A quasilinear elliptic equation in $\mathbf{R}^N$. *Proc. Roy. Soc. Edinburgh Sect. A*, 126(5):911–921, 1996.

[4] O. Alvarez. Bounded-from-below viscosity solutions of Hamilton-Jacobi equations. *Differential Integral Equations*, 10(3):419–436, 1997.

[5] M. Bardi and I. Capuzzo Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Birkhäuser Boston Inc., Boston, MA, 1997.

[6] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*. Springer-Verlag, Paris, 1994.

[7] G. Barles and A. Porretta. *Personal communication*, 2008.

[8] I. Capuzzo-Dolcetta, F. Leoni, and A. Porretta. Hölder estimate for degenerate elliptic equations with coercive Hamiltonians. *To appear in Trans. Amer. Math. Soc.*, 2008.

[9] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[10] M. G. Crandall, R. Newcomb, and Y. Tomita. Existence and uniqueness of viscosity solutions of quasilinear second order degenerate elliptic equations in $\mathbb{R}^N$. *Appl. Anal.*, 34: 1–23, 1989.

[11] F. Da Lio and O. Ley. Uniqueness results for convex Hamilton-Jacobi equations under $p > 1$ growth conditions on data. *Submitted*.

[12] F. Da Lio and O. Ley. Uniqueness results for second-order Bellman-Isaacs equations under quadratic growth assumptions and applications. *SIAM J. Control Optim.*, 45(1):74–106, 2006.

[13] H. Ishii. Uniqueness of unbounded viscosity solutions of Hamilton-Jacobi equations. *Indiana Univ. J. Math.*, 33: 721–748, 1984.

[14] H. Ishii. Comparison results for Hamilton-Jacobi equations without growth condition on solutions from above. *Appl. Anal.*, 67: 357–372, 1997.

[15] H. Ishii and S. Koike. Viscosity solutions for monotone systems of second-order elliptic PDEs. *Comm. Partial Differential Equations*, 16(6&7): 1095–1128, 1991.

[16] K. Ishii and Y. Tomita. Unbounded viscosity solutions of nonlinear second order PDE's. *Adv. Math. Sci. Appl.*, 10(2): 689–710, 2000.

[17] S. Koike. *A Beginner's Guide to the Theory of Viscosity Solutions*. MSJ Memoirs vol. 13. Math. Soc. Japan, Tokyo, 2004.

[18] H. Nagai. Min-max characterization of a small noise limit on risk-sensitive control. *SIAM J. Control Optim.*, 35(4): 1093-1115, 1997.

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