COMBINATORIAL SUBSTITUTIONS AND SOFIC TILINGS

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Abstract. A combinatorial substitution is a map over tilings which allows to define sets of tilings with a strong hierarchical structure. In this paper, we show that such sets of tilings are sofic, that is, can be enforced by finitely many local constraints. This extends some similar previous results (Mozes'90, Goodmann-Strauss'98) in a much shorter presentation.

1. Introduction

Tiling some space with geometrical shapes, or tiles, consists into covering that space with copies of the tiles. When a set of tilings can be characterized by adding finitely many local constraints on tiles so that the set of tilings corresponds exactly to the set of tilings satisfying the constraints, such a set of tilings is called sofic. Soficity corresponds to the interesting idea that the validity of a tiling can be locally proved by decorating tiles with constraints. In this paper, we contribute to a general question: what sets of tilings are sofic?

Substitutions provide a simple way to express how a set of tilings can be obtained by iteratively constructing bigger and bigger aggregates of tiles. The strong hierarchical structure of substitutions tilings permits to enforce global properties, for example aperiodicity. To prove that the set of tilings generated by a substitution is sofic, one has to encode the global hierarchical structure into local constraints on tiles. Such technique is at the root of classical papers on the undecidability of the Domino Problem [2, 8]. In the case of tilings on the square grid, Mozes [5] proved in a seminal paper that the set of tilings generated by rectangular non-deterministic substitutions satisfying a particular property is sofic. Goodman-Strauss [4] proved that such a construction method can be extended to a wide variety of geometrical substitutions. In this paper, based on ideas developed in [6], we further extend the construction to a broader class of substitutions by replacing the geometrical conditions by combinatorial conditions while decreasing the length of the presentation.

Let us sketch some definitions to state our main theorem. A sofic tiling is a valid tiling by a finite set of tiles with decorations. Combinatorial substitutions, introduced by Priebe-Frank in [7], map a tiling by tiles onto a tiling by so-called

Partially supported by the ANR project EMC (ANR-09-BLAN-0164).

macro-tiles (finite tilings, here assumed to be connected), so that the macro-tiles of the latter are arranged as the tiles of the former. The limit set of a substitution is the set of complete tilings that admits preimages of any depth by the substitution. A good substitution is a substitution that is both connecting (a combinatorial condition ensuring that there is enough room for all information to flow) and consistent (a geometrical condition ensuring that the substitution can be correctly iterated).

Theorem 1.1. *The limit set of a good combinatorial substitution is sofic.*

The paper is organized as follows. Sections 2 and 3 formally define main notions, in particular sofic tilings and good combinatorial substitutions. Section 4 then presents *self-simulation*, which plays a central role in the constructive proof of Theorem 1.1, which is given in section 5. Last, section 6 concludes the paper by discussing an important parameter of this proof.

A single example illustrates definitions and results throughout the whole paper: the "Rauzy" example. It relies on the theory of *generalized substitutions* introduced in [1], a self-contained presentation of which is beyond the scope of this paper. Let us just mention that *Rauzy tilings* are digitizations of the planes of the Euclidean space with a specific given irrational normal vector. More details, as well as the results we here implicity rely on, can be found in [3]. We chose this example, not the simplest one, because it is not covered by results in [5, 4].

2. Sofic tilings

Polytopes are here assumed to be homeomorphic to closed balls of $\mathbb{R}^d$, with $d \geq 2$, and to have finitely many faces, with the $(d-1)$-dimensional faces being called *facets*.

A *tile* is a polytope of the Euclidean space $\mathbb{R}^d$. A *tiling* of a *domain* $D \subset \mathbb{R}^d$ is a covering of D by interior-disjoint tiles, with the additional condition that two tiles can intersect (if they do) only along entire faces. A facet of a tiling is said to be *external* if it is on the boundary of the domain, *internal* otherwise. We denote by ∂Q the set of external facets of a tiling Q. If this set is empty, then the tiling is said to be *complete*: its domain is the whole $\mathbb{R}^d$.

A *decorated tile* $\mathcal{T}$ is a tile with a real map defined on its boundaries, called *decoration.* Two decorated tiles are said to *match* if their decorations are equal wherever they intersect. A *decorated tiling* is a tiling by decorated tiles which pairwise match. Decorations are thus local constraints on the way tiles can be arranged in a tiling.

A *tileset* is a set of decorated tiles. It is said to be *finite* if it contains only a finite number of tiles up to direct isometries. A decorated tiling whose tiles belong to a tileset τ is called a τ-tiling; the set of such tilings is denoted by Λ_τ.

One says that a tiling *can be seen as* a decorated tiling if both are equal up to decorations. One denotes by π the map which removes the decorations. One easily checks that any tiling can be seen as a τ-tiling if τ can be infinite. The interesting case is the one of finite tilesets:

Definition 2.1. A set of tilings is said to be *sofic* if it can be seen as the set of τ-tilings of some finite tileset τ.

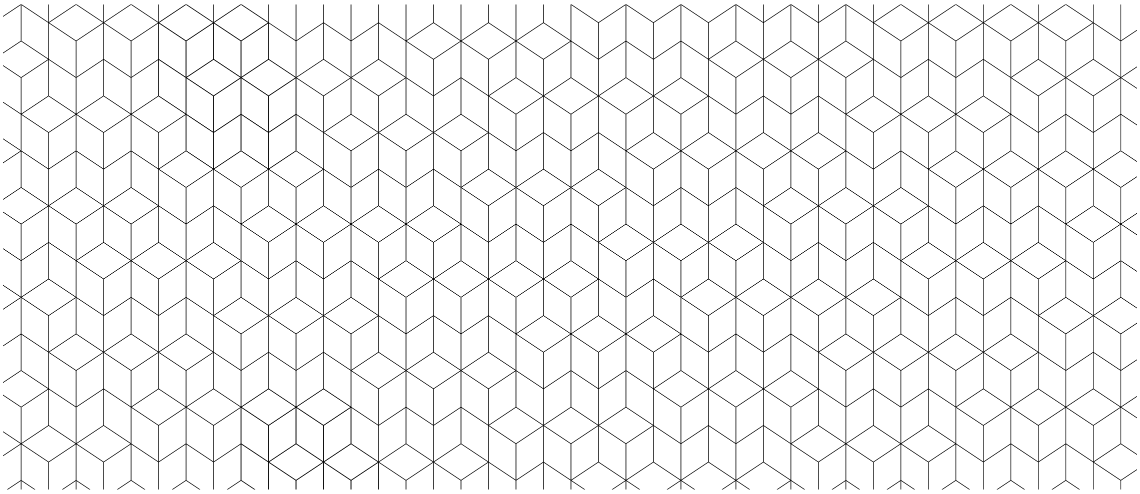

Figure 1: A Rauzy tiling (partial view). Are Rauzy tilings sofic?

For example, one can wonder whether the *Rauzy tilings* are sofic (Fig. 1).

3. Combinatorial substitutions

Definition 3.1. A *combinatorial substitution* is a finite set of *rules* (P, Q, γ), where P is a tile, Q is a finite connected tiling, and $\gamma : \partial P \to \partial Q$ maps distinct facets on disjoint sets of facets. The tiling Q is called a *macro-tile*, and if f is the k-th facet of P, then $\gamma(f)$ is called the k-th *macro-facet* of Q.

Fig. 2 illustrates this definition.

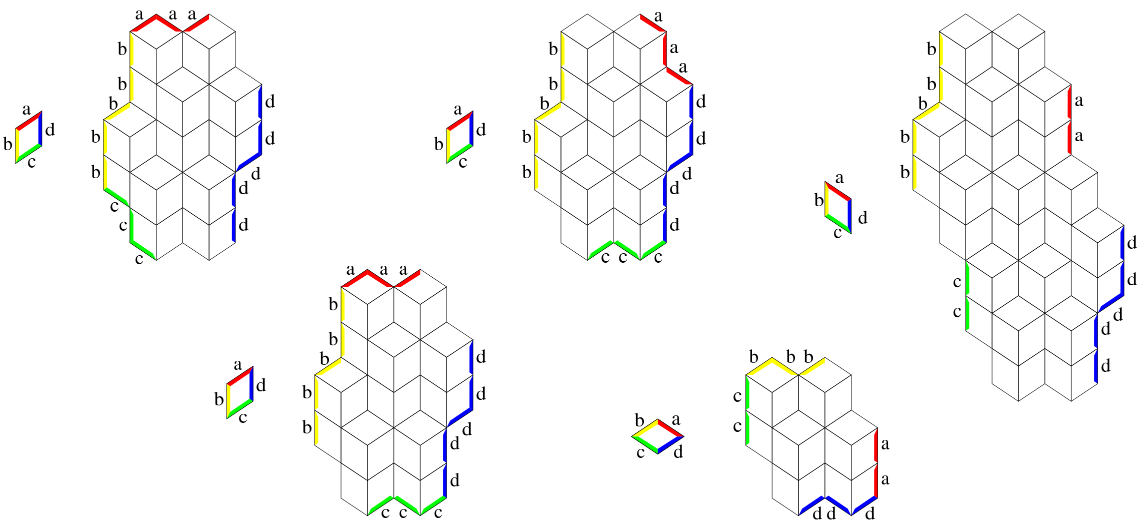

Figure 2: These five rules define the so-called Rauzy combinatorial substitution (a facet f and the corresponding macro-facet $\gamma(f)$ are similarly marked).

We call *tiling by macro-tiles* a tiling whose tiles can be partitioned into macro-tiles, with each macro-facet belonging to the intersection of two macro-tiles. We can associate with each combinatorial substitution a binary relation over tilings:

Definition 3.2. Let σ be a combinatorial substitution. A tiling T by tiles of σ and a tiling T' by macro-tiles of σ are said to be *σ-related* if there is a bijection between the tiles of T and the macro-tiles of T' which preserves the combinatorial structure[1]. More precisely, T is the *preimage* of T', and T' is the *image* of T.

For example, a Rauzy tiling can be uniquely seen as a tiling by Rauzy macro-tiles (Fig. 3), and the corresponding preimage is itself a Rauzy tiling.

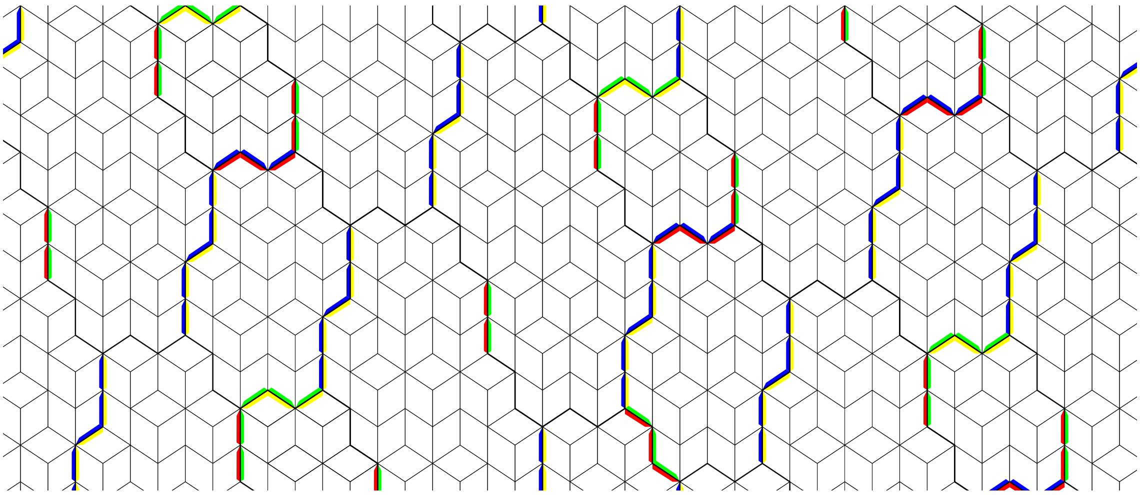

Figure 3: A Rauzy tiling can be uniquely seen as a tiling by Rauzy macro-tiles.

In particular, the relations associated with combinatorial substitutions yield a strong hierarchical structure on so-called limit-sets:

Definition 3.3. The *limit set* of a combinatorial substitution σ, denoted by Λ_σ, is the set of complete tilings which admit an infinite sequence of preimages.

For example, the limit set of the Rauzy combinatorial substitution is exactly the set of Rauzy tilings.

Let us now turn to the *good combinatorial substitutions* to which Theorem 1.1 applies. First, a good combinatorial substitution must be *connecting*:

Definition 3.4. A combinatorial substitution σ is *connecting* if, for each rule (P, Q, γ), the dual graph[2] of Q has a subgraph N, called its *network*, such that

(1) N is a star with one branch for each macro-facet, and the leaf of the k-th branch is a tile with a facet, called k-th *port*, in the k-th macro-facet of Q;
(2) Each macro-facet has non-port facets, and removing the edges of N and its central vertex yields a connected graph which connects[3] all these facets;
(3) the center of N corresponds to a tile in the interior of Q, called *central tile*.

We also assume that two macro-tiles which match along a port also match along the corresponding macro-facet.

[1]That is, the a-th facet of a first tile of T matches the b-th facet of a second tile of T if and only if the a-th macro-facet of the first corresponding macro-tile of T' matches the b-th macro-facet of the second corresponding macro-tile of T'.

[2]The dual graph of a tiling is the graph whose vertices correspond to tiles of the tiling and whose edges connect vertices corresponding to adjacent tiles.

[3]One says that a subgraph *connects* a set of facets if these facets all belong to tiles which correspond to vertices of this subgraph.

Informally, the two first conditions ensure that the macro-facets are big enough to transfer all the informations (encoded by decorations) that we need to enforce the hierarchical structure of the limit set, while the last condition ensure that, by iteratively considering macro-tiles of macro-tiles (when moving in the hierarchy), we get tilings covering arbitrarily big balls. One can note that the last assumption is equivalent to the "sibling-edge-to-edge" assumption of [4].

Second, a good combinatorial substitution must be *consistent*:

Definition 3.5. A combinatorial substitution σ is said to be *consistent* if any tiling by macro-tiles of σ admits a preimage under σ.

Intuitively, consistency ensures that, if a tiling meets all the combinatorial conditions to have a preimage, then there is no geometrical obstruction to the existence of such a preimage.

Open Problem 3.6. Characterize consistent combinatorial substitutions.

For example, the Rauzy combinatorial substitution is connecting (one easily finds a suitable network for each rule, see Fig. 4) and consistent. It is thus a good combinatorial substitution, and Theorem 1.1 yields that its limit set, that is the set of Rauzy tilings, is sofic.

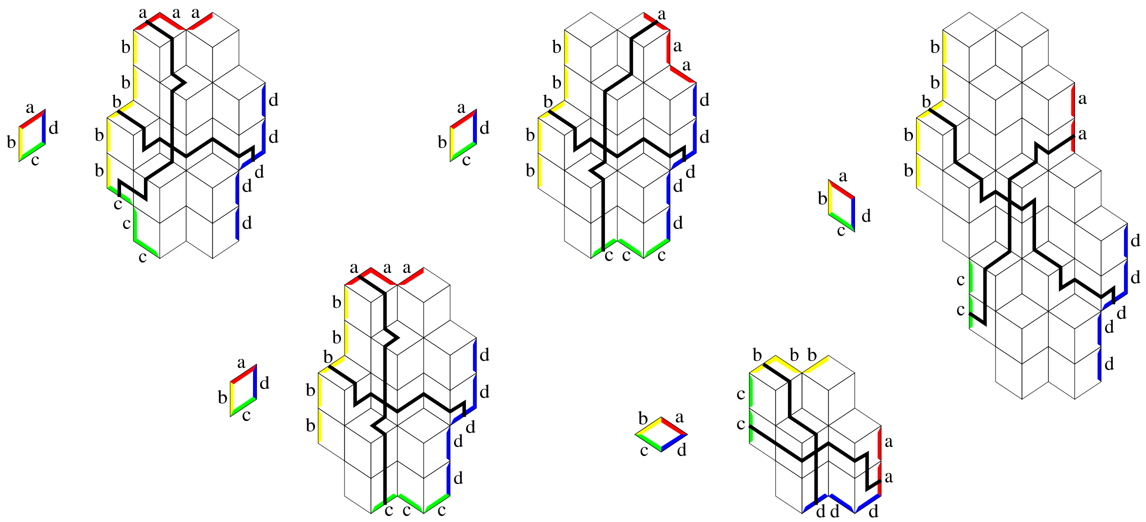

Figure 4: The Rauzy combinatorial substitution is connecting.

4. Self-simulation

Definition 4.1. Let σ be a combinatorial substitution with rules $\{(P_i, Q_i, \gamma_i)\}_i$. A tileset τ is said to σ-*self-simulates* if there is a set of τ-tilings, called τ-macro-tiles, and a map ϕ from these τ-macro-tiles into τ such that

(a) for any τ-macro-tile $\mathcal{Q}$, there is i such that $\pi(\mathcal{Q}) = Q_i$ and $\pi(\phi(\mathcal{Q})) = P_i$;
(b) any complete τ-tiling can be seen as a tiling by τ-macro-tiles;
(c) the a-th macro-facet of a τ-macro-tile $\mathcal{Q}$ can match the b-th macro-facet of a τ-macro-tile $\mathcal{Q}'$ if and only if the a-th facet of the τ-tile $\phi(\mathcal{Q})$ can match the b-th facet of the τ-tile $\phi(\mathcal{Q}')$.

Proposition 4.2. *If a tileset σ-self-simulates for a consistent combinatorial substitution σ, then its tilings are, up to decorations, in the limit set of σ.*

Proof. Consider a τ-tiling $\mathcal{T}$. Conditions (a) and (b) ensure that removing the decorations of $\mathcal{T}$ yields a tiling by macro-tiles of σ, say T. The consistency of σ then ensures that T admits a preimage under σ, say R. Let us show that R can be endowed by decorations to get a τ-tiling. Consider a tile P of R. This tile corresponds (via the one-to-one correspondence in Def. 3.2) to a macro-tile Q of T, which is itself the image under π of a τ-macro-tile $\mathcal{Q}$ of $\mathcal{T}$. We associate with this tile P the τ-tile $\phi(\mathcal{Q})$. Now, condition (c) ensures that replacing tiles in R by their associated τ-tiles yields a complete τ-tiling, say $\mathcal{R}$. We can thus repeat all the above process, with $\mathcal{R}$ instead of $\mathcal{T}$. By induction, we get an infinite sequence of complete tilings, with each one being the preimage under σ of the previous one. The result follows. ∎

5. Constructive proof of Theorem 1.1

Let σ be a consistent and bi-connecting combinatorial substitution with rules $(P_i, Q_i, \gamma_i)_i$. We construct a tileset τ and show that it σ-self-simulates. Prop. 4.2 thus ensures that $\pi(\Lambda_\tau) \subseteq \Lambda_\sigma$ holds. We also show that the converse inclusion holds. This constructively proves Theorem 1.1.

5.1. Settings

Let $T_1, \ldots, T_n$ and $f_1, \ldots, f_m$ be numberings of, respectively, the tiles and the internal facets of all the Q_i's. Given the k-th facet of the tile T_i, $N_\sigma(i,k)$ stands either for its index if it is an internal facet, or for a special value "port", "macro-facet" or "boundary" otherwise (depending whether it is a port, a non-port facet in a macro-facet or another external facet).

Each tile of τ is a T_i endowed with a decoration which encodes on each facet a triple (f,j,g), where f and g are either facet indices or special values "port", "macro-facet" or "boundary", and j is either zero or a tile index. We call f the *macro-index*, j the *parent-index* and g the *neighbor-index*. This clearly allows only a finite number of different tiles.

We skip here the technical details concerning the way these triples are encoded by decorations, but let us stress that it is possible only if we tile a space of dimension $d \geq 2$. We assume that two decorated tiles match along a facet if and only if the same triple is encoded on both facets (and the undecorated tiles match). We also assume that the only direct isometry which leaves invariant the decoration of a facet is the identity (so that a decorated tile cannot trivially match with a translated/rotated copy of itself).

5.2. Decorations

The five following steps completely define a tileset τ.

1. The macro-index of the k-th facet of any decorated T_j is $N_\sigma(j,k)$. This step ensures that any complete τ-tiling can be uniquely seen as a tiling by τ-macro-tiles.

2. Consider a decorated non-central tile of Q_i. Its facets which are internal and not crossed by the network have all the same parent-index, also called parent-index of the tile, which can be any j such that $T_j = P_i$. Its facets which are external, port excluded, have parent-index 0. This step ensure that the τ-tiles of a τ-macro-tile $\mathcal{Q}$ share a common parent-index j; the tile T_j is called the *parent-tile* of $\mathcal{Q}$.

3. Consider, in a non-central τ-tile with parent-index j, a facet which is neither a port nor crossed by the network. Its neighbor-index is either $N_\sigma(j,k)$ if it is in the k-th macro-facet, or equal to its macro-index otherwise. This step ensures that the macro-facets (ports excepted) of a τ-macro-tile are equivalent to the macro-indices of its parent tile (once decorated).

4. Consider a non-central τ-tile with parent-index j. Its facets which are either k-th port or crossed by the k-th branch of the network have all the same pair of parent/neighbor indices, which can be any of those allowed on the k-th facet of T_j. This definition is recursive since the k-th facet of T_j can be crossed by a network. It may even be circular (*e.g.*, if the considered τ-tile is itself a decorated T_j), in which case we allow pair of parent/neighbor indices defined in Steps 1–3. This step ensures that each port of a τ-macro-tile is equivalent to the pair of parent/neighbor indices of a decorated parent-tile[4].

5. Whenever a non-central τ-tile $\mathcal{T}$ has as many facets as a central tile T_j, we define a central τ-tile $\mathcal{T}'$ by endowing each k-th facet of T_j with the pair of parent/neighbor indices on the k-th facet of $\mathcal{T}$ (the macro-indices are defined as usual, see Step 1). One says that $\mathcal{T}'$ *derives* from $\mathcal{T}$.

5.3. First inclusion

Let us show that the above defined tileset σ-self-simulates. Given a τ-macro-tile $\mathcal{Q}$ with parent-index j and central τ-tile $\mathcal{T}'$, let $\phi(\mathcal{Q})$ be the decorated tile obtained by endowing the k-th facet of T_j with the parent/neighbor indices on the k-th facet of $\mathcal{T}'$ (the macro-indices are defined as for any tile, see Step 1). One easily checks that ϕ is a map satisfying conditions (a)–(c) of Def. 4.1. It remains to check that $\phi(\mathcal{Q})$ is in τ.

Let $\mathcal{T}$ be a non-central τ-tile from which derives $\mathcal{T}'$. If T_j is central, then $\phi(\mathcal{Q})$ also derives from $\mathcal{T}$, hence is in τ. Otherwise, Step 4 ensures, for each k, that the parent/neighbor indices on the k-th facet of $\mathcal{T}'$ appear on the k-th facet of a decorated T_j. In particular, this holds on facets of T_j which are not crossed by a network. Since the neighbor-indices of these facets can only[5] appear on a T_j (see Steps 1 and 3), $\mathcal{T}$ is a decorated T_j. This yields $\phi(\mathcal{Q}) = \mathcal{T}$, and thus $\phi(\mathcal{Q})$ is in τ.

[4]With the problem that a parent-tile can be decorated in different ways, and nothing yet prevents the ports from mixing these decorations.

[5]Actually, a facet-index appears on the two tiles of a macro-tile which share the corresponding facet. We thus need, in order to completely characterize T_j, either to assume that there is at least two such facets (this is a rather mild assumption), or to endow facets with an *orientation* and to allow two facets to match if and only if they have opposite orientations.

Prop. 4.2 then applies and yields the first inclusion $\pi(\Lambda_\tau) \subseteq \Lambda_\sigma$.

5.4. Second inclusion

Let us extend τ in a tileset τ' as follow. For each τ-tile $\mathcal{T}$ and each subset S of its facets, we define a τ'-tile $\mathcal{T}_S$ by replacing the decorations of the facets in S by a special decoration "undefined".

Now, let P be a tiling in Λ_σ. Consider an infinite sequence $(P_n)_{n\geq 0}$ of successive preimages of P. Given $n > 0$, P_n can be seen as a τ'-tiling: it suffices to endow any facet with "undefined". Then, P_{n-1} can be seen as a tiling by τ'-macro-tiles, with "undefined" decorations appearing only on the network. Indeed, consider a macro-tile of T_{n-1} which corresponds (via the one-to-one correspondence in Def. 3.2) to a tile T_j in P_n: it suffices to endow its tiles as in steps 1–3 of the definition of τ, with T_j being the parent-tile, and with the decoration "undefined" on the facets crossed by the network. This can be iterated up to $P = P_0$, and the third condition of Def. 3.4 ensures that the decorations "undefined" appear only on sort of grids whose cells have bigger and bigger size.

Thus, by making n tend to infinity, one can see T as a τ'-tiling whose "undefined" decorations, if any, form either a star with infinite branches, or a single biinfinite branch. In the first case, we can replace the central τ'-tile by any τ-tile with a suitable number of facets, and then the tiles on branches by τ-tiles which carry decorations to infinity. In the second case, we can replace the τ'-tiles by τ-tiles which carry any decoration on the whole branch. In any case, we can thus see P as a τ-tiling.

We thus have the second inclusion $\Lambda_\sigma \subseteq \pi(\Lambda_\tau)$. Both inclusions prove Theorem 1.1.

6. On the number of tiles

Let us conclude this paper by discussing the size $\#\tau$ of the tileset τ defined in the previous section (although its finiteness suffices for Theorem 1.1).

Consider a combinatorial substitution with r rules, n tiles among which p on a network, and whose macro-tiles have m internal facets.

First, the $n-p-r$ tiles not on the network can be decorated in at most n different ways, according to the parent-index they carry. This yields at most $N_0 = (n-p)n$ τ-tiles. Then, each of the $p-r$ non-central tiles on the network can be decorated in at most $(n-p)n+pnm$ different ways: $n-p$ parent-indices which correspond to tiles not on the network, hence allow at most n pairs of parent/neighbor indices carried on the network, and p parent-indices which correspond to tiles on the network, hence allow at most nm pairs of parent/neighbor indices carried on the network. This yields at most $N_p = (p-r)((n-p)n + pmn)$ τ-tiles. Last, the r central tiles can be decorated in at most as many different ways as there is non-central τ-tiles. Finally, this yields

$$\#\tau \leq (r+1)(N_0 + N_p) \leq (r+2)p^2mn.$$

This bound is, for example, about one billion in the Rauzy case (Fig. 2).

In order to reduce this huge number of tiles, it is worth noting that, instead of carrying a parent-index through all the tiles of a macro-tile, it suffices to carry

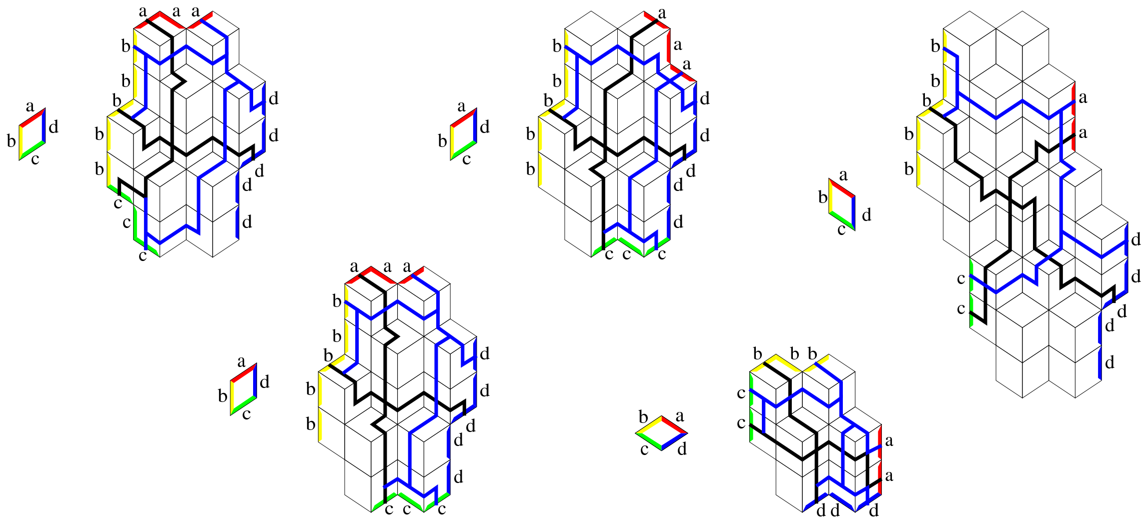

Figure 5: A second network for the Rauzy combinatorial substitution.

it along a *second network* connecting the macro-facets of this macro-tile and intersecting each of the branches of its (first) network (see, *e.g.*, Fig. 5). The "control" described in Step 4 is then performed only on tiles where both networks crosses, while we simply allow any possible pairs of parent/neighbor indices to be carried through the tiles which are only on the first network. If we denote by q the number of tiles on these new second networks and by c be the number of crossings between second and first networks, a similar analysis yields

$$\#\tau \leq (r+1)(N_0' + N_q' + N_p' + N_c') \leq (r+2)cp(m+qn),$$

where:

$$\begin{array}{rclcrcl} N_0' & = & n-p-q+c+2r, & \quad & N_p' & = & (p-r-c)(m+qn), \\ N_q' & = & (q-r-c)n, & & N_c' & = & c(N_0'+N_q'+N_p'). \end{array}$$

This bound is, for example, about 70 millions tiles in the Rauzy case.

This last bound is huge but generic. One can hope to dramatically decrease this bound in specific cases. Indeed, most of the τ-tiles correspond to tiles which simply carry any possible information (the tiles on networks, crossings excepted). Since these tiles all play the same role, it would be worth to replace all the tiles on a network by a single tile (one can thus hope to gain a factor pq in the above bound – this would yield about 25000 tiles in the Rauzy case). This shall however be done carefully, so that tiles still necessarily form macro-tiles. Note that it should be much easier in dimension $d \geq 3$, since the cohesion of macro-tiles can be more easily enforced without relying on tiles on networks.

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Appendix A. More pictures and examples for the referee

A.1. Iterating Rauzy substitution

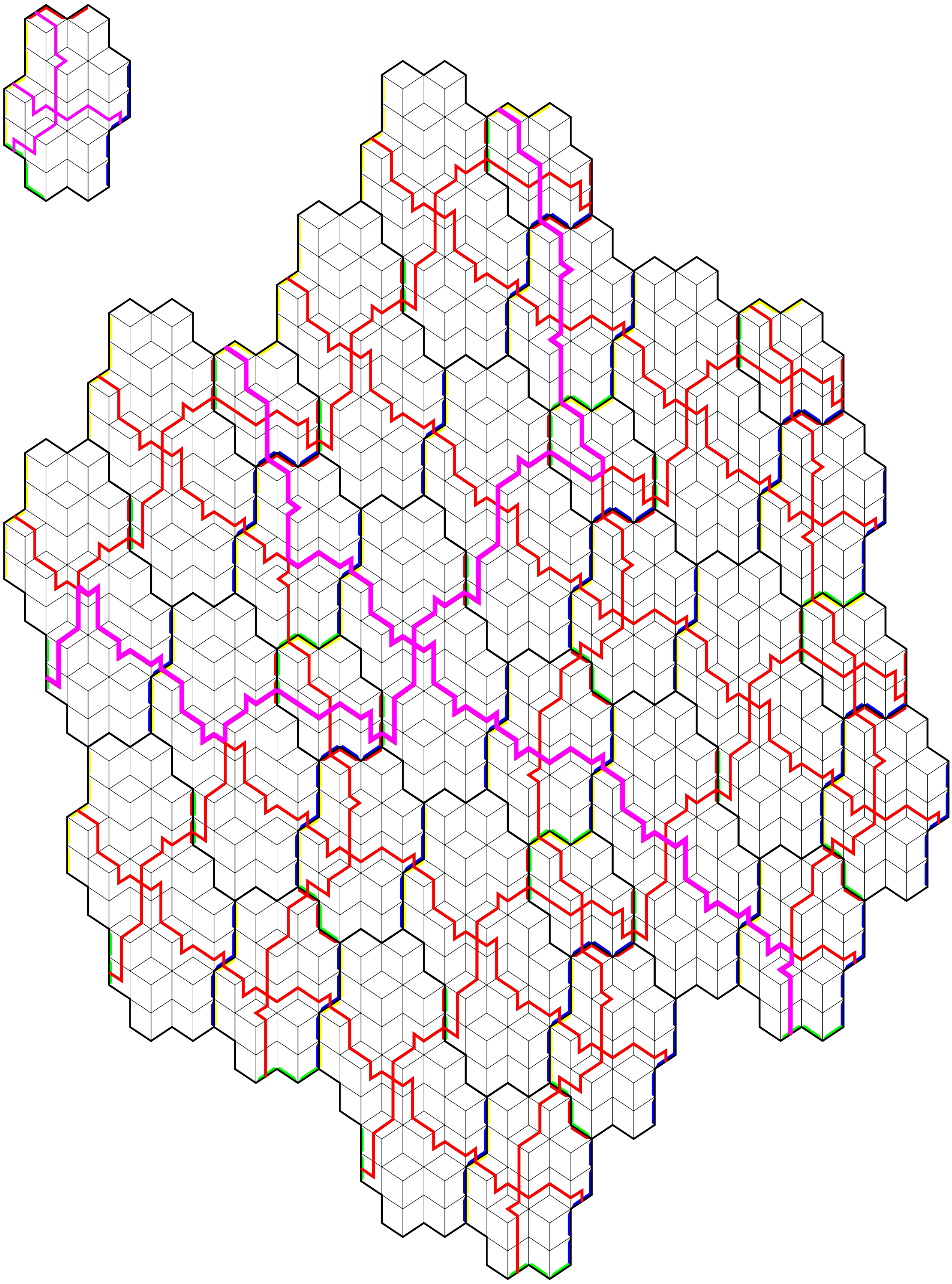


Figure 6: A Rauzy macro-tile (top-left) and an image of it under the Rauzy combinatorial substitution, that one could call Rauzy "macro-macro-tile" (center).

A.2. A simple substitution

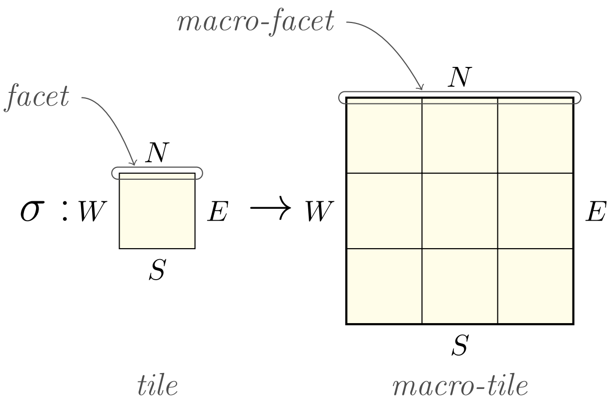


Figure 7: a simple substitution

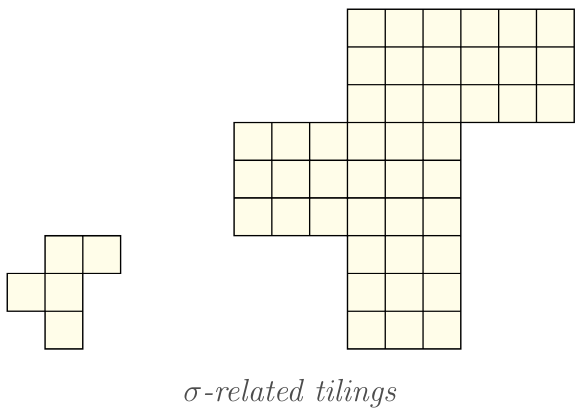


Figure 8: sample σ-related tilings

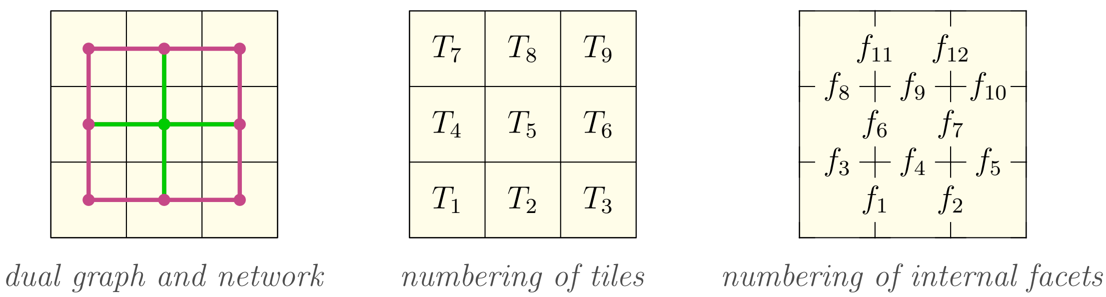


Figure 9: chosen network and numberings

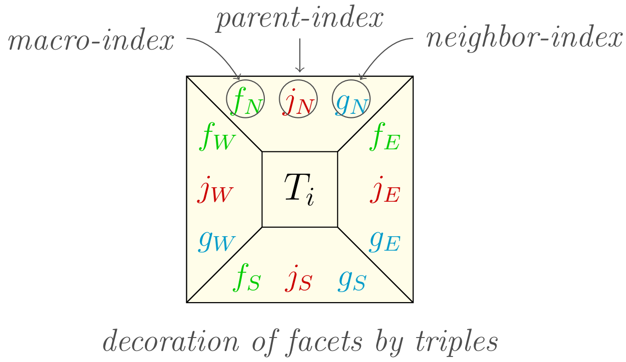


Figure 10: decoration of facets

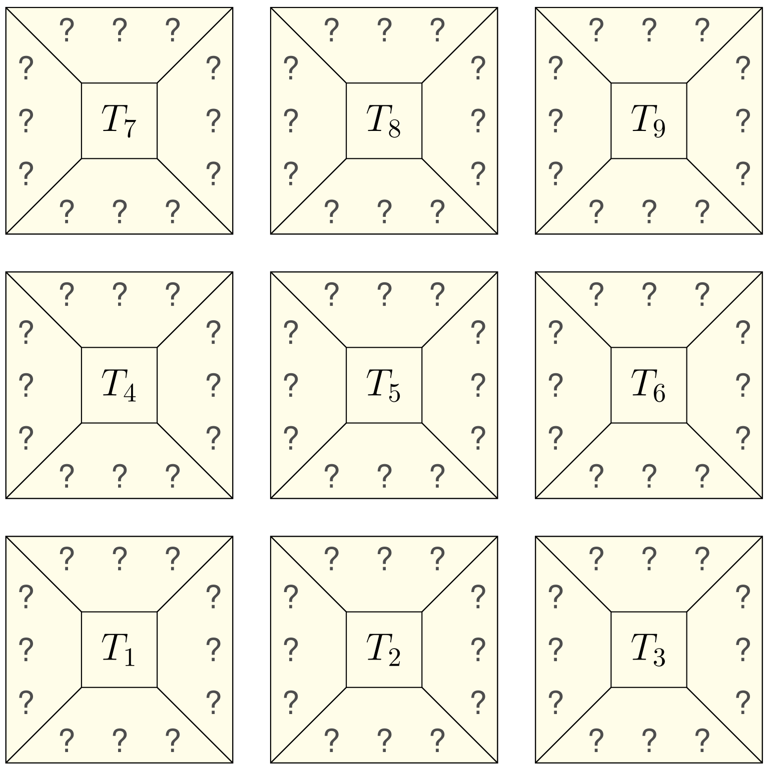


Figure 11: initial decoration scheme

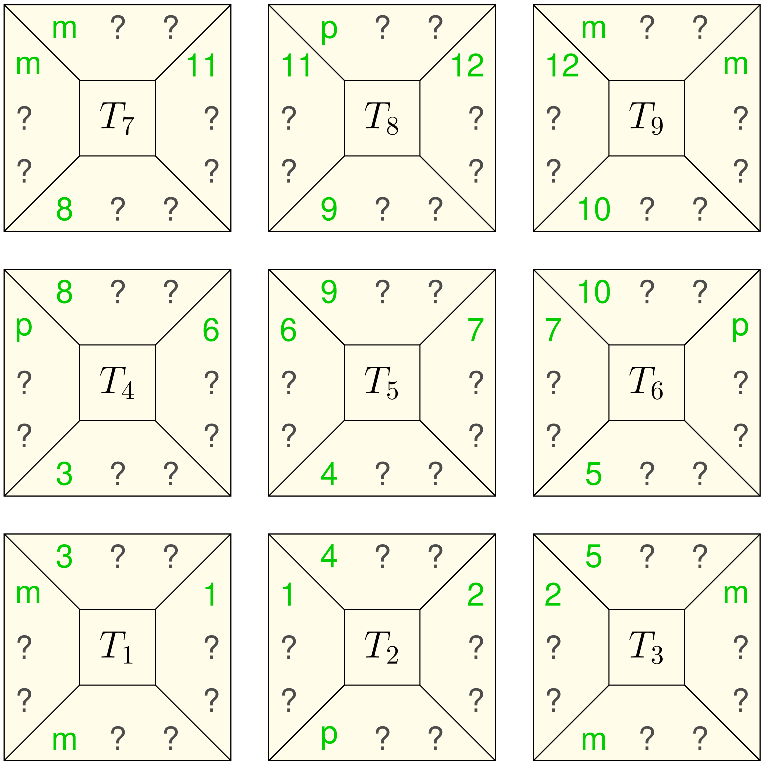


Figure 12: after step 1: macro-indices are fixed

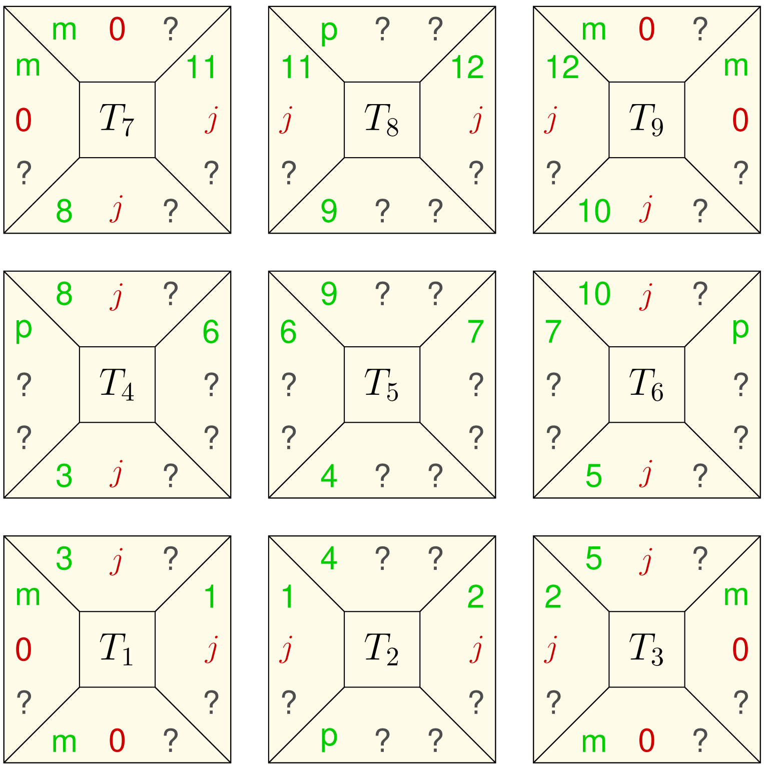
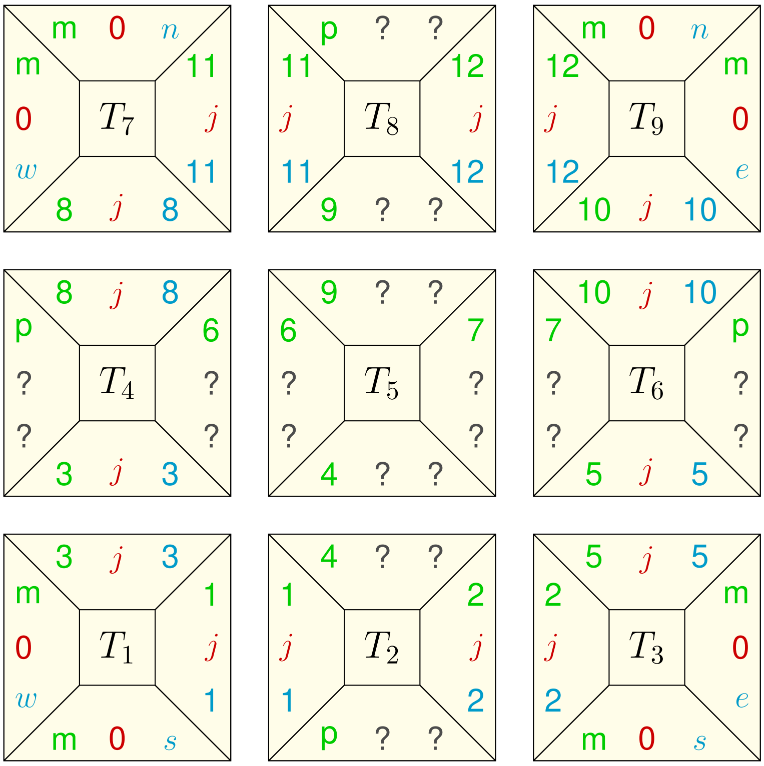
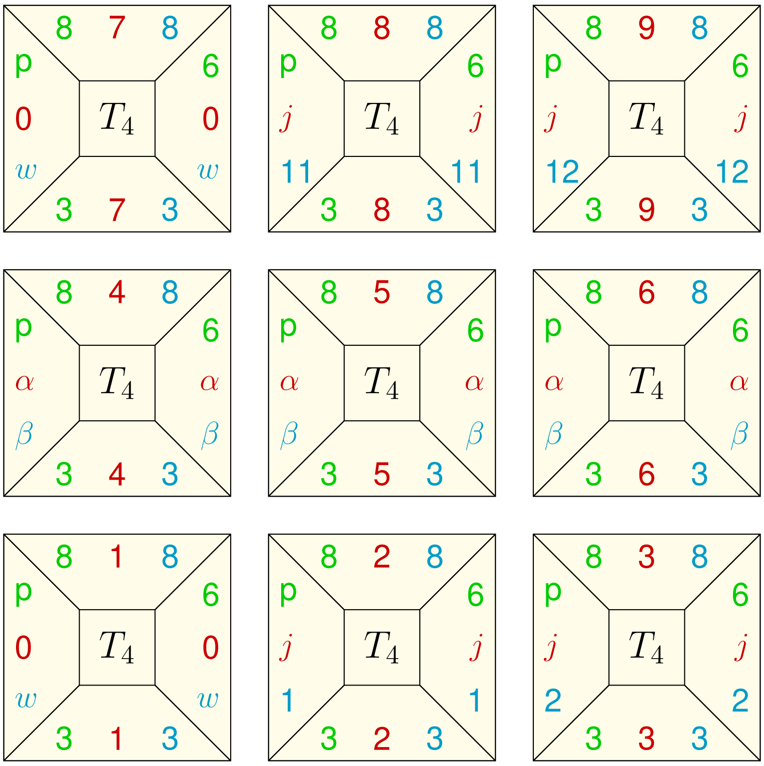


Figure 13: after step 2: parent-indices are fixed outside network



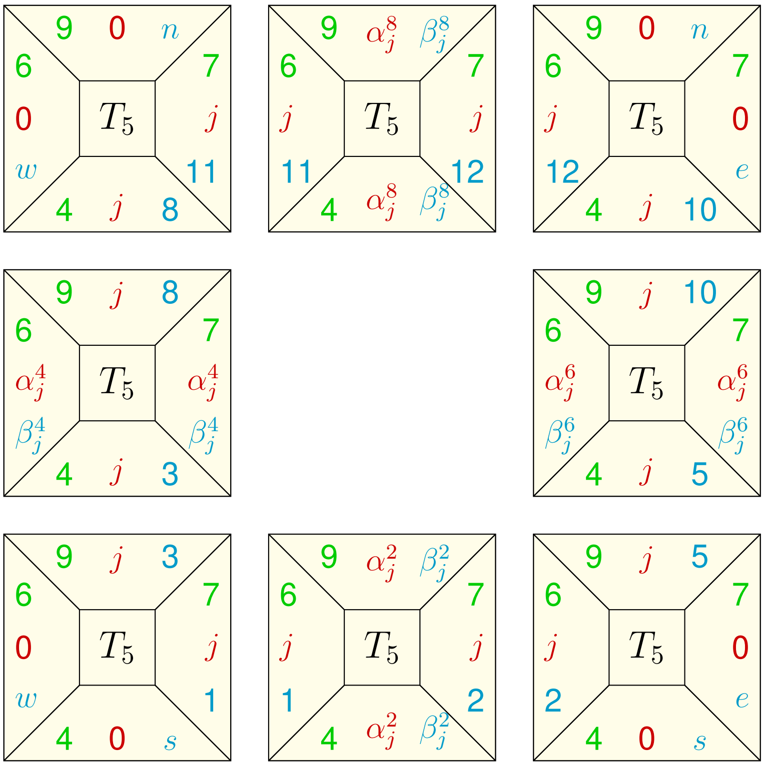
where $n = N_\sigma(j, N)$, $s = N_\sigma(j, S)$, $w = N_\sigma(j, W)$ *and* $e = N_\sigma(j, E)$.

Figure 14: after step 3: all decorations fixed outside network



where $w = N_\sigma(j, W)$, *for any* j *and any horizontal pair* (α, β) *of parent/neighbor indices*

Figure 15: after step 4: all decorations fixed but for central tiles



where, for any j, $n = N_\sigma(j, N)$, $s = N_\sigma(j, S)$, $w = N_\sigma(j, W)$, $e = N_\sigma(j, E)$, (α_j^k, β_j^k) are valid parent/neighbor indices for a tile T_k with parent-index j

Figure 16: after step 5: all decorations are fixed