A priori and a posteriori estimates for three-dimensional Stokes equations with non standard boundary conditions

Hyam Abboud, Fida El Chami, Toni Sayah

A PRIORI AND A POSTERIORI ESTIMATES FOR THREE-DIMENSIONAL STOKES EQUATIONS WITH NON STANDARD BOUNDARY CONDITIONS

HYAM ABBOUD[†], FIDA EL CHAMI[†] AND TONI SAYAH[‡]

ABSTRACT. In this paper we study the Stokes problem with some non standard boundary conditions. The variational formulation decouples into a system velocity and a Poisson equation for the pressure. The continuous and corresponding discrete system do not need an inf-sup condition. Hence, the velocity is approximated with **curl** conforming finite elements and the pressure with standard continuous elements. Next, we establish optimal a priori and a posteriori estimates and we finish this paper with numerical tests.

Keywords Stokes equations, a piori and a posteriori errors.

1. INTRODUCTION.

This paper is devoted to the numerical solution of the Stokes equations for an incompressible fluid

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

with the incompressibility condition

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

with boundary conditions

$$\mathbf{u}\times\mathbf{n} = \mathbf{0}, \quad p = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

or

$$\mathbf{u}.\mathbf{n} = 0, \quad \mathbf{curl}\,\mathbf{u}\times\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{1.4}$$

where Ω is a bounded, simply connected domain of $\mathbb{R}^3$ with a polyhedral connected boundary $\Gamma = \partial\Omega$ and $\mathbf{n}$ the exterior unit normal to Γ, $\mathbf{u}$ the velocity and p the pressure.

These sets of boundary conditions lend themselves readily to a variational formulation where the Laplacian operator is expressed by a $(\mathbf{curl}, \mathbf{curl})$ term and the incompressibility condition by an equation of the form $(\nabla q, \mathbf{v})$. Usually, for the Stokes problem, we use the inf-sup condition to establish the existence and the uniqueness of the theoretical solution; and for the discretization of the pressure and the velocity, we use a set of finite elements which also verifies a discrete inf-sup condition. In our work, by decoupling the variational system in a Poisson equation for the pressure and an other system for the velocity, we prove the existence and the uniqueness of the theoretical solution without the inf-sup condition. Hence, the finite elements used for the discretized system do not need to verify a discrete inf-sup condition and lead to matrix systems with an optimal dimension and optimal time of resolution. We use the non-conforming finite elements method where just the **curl** of the velocity is continuous at interface boundaries whereas the pressure is globally continuous.

The convexity assumption on Ω is a well-known theoretical consequence of the fact that Γ is not smooth. There is no practical evidence that it is necessary and his assumption is disregarded in practice: instead, we can assume that Ω is simply-connected and Γ is connected. A domain with "holes" or a multiply-connected domain can be handled with the techniques of Bendali, Dominguez and Gallic [5]. We refer

September 15, 2010.

[†] Faculté des Sciences II, Université Libanaise, Département de mathématiques, B.P. 90656, Fanar-Maten, Liban.
[‡] Faculté des Sciences, Université Saint-Joseph, B.P 11-514 Riad El Solh, Beyrouth 1107 2050, Liban.

to Dubois [9] for a good treatment of the potential problem on a domain with a curved and multiply-connected boundary. As far as the theory is concerned, the reader will find in Bègue, Conca, Murat and Pironneau [4] a very comprehensive study of the Stokes and the Navier-Stokes equations with non-standard (and non-homogeneous) boundary conditions on a variety of domains. These author include a conforming approximation of the Taylor-Hood type for the velocity (the corresponding theoretical analysis is done by Franca and Hugues [13]). We refer also to Girault's work [16] for a vector potential-vorticity approximation of similar Navier-Stokes type problems and to [15] for the steady-state incompressible Navier-Stokes equations with non standard boundary conditions. For the Vorticity-velocity-pressure formulation for the Stokes problem, we refer to [10], [11] and [28]. We also refer to [24] where Repin establishes a posteriori estimates for the velocity, stress and pressure fields for the stationary Stokes problem and where his approach is based on duality theory of the calculus of variations. A posteriori estimates for the Stokes problem and for some viscous flow problems were studied by a number of authors, [3], [31], [21] and [22]. Typically, they have been obtained in the frame of the so-called "residual method" originally proposed in [1] and [2] for the finite element approximations. This type estimates are crucially based on the Galerkin orthogonality condition. Therefore, they are only valid for exact solutions of the corresponding finite dimensional problem which form a very special subset in the natural set of admissible functions. For the a posteriori estimations of the Stokes problems, we can cite the works of S. Repin [25], [26] and [27].

The remainder of this article is organized as follows: In Section 2, we introduce the problem and we establish a decoupled variational formulation into a system of velocity and a Poisson equation for the pressure. In section 3, we introduce the finite elements and a discrete system using the **curl** conforming finite elements for the velocity and the standard continuous elements for the pressure. In the section 4, we establish an optimal corresponding a priori estimates. In the section 4, we begin by establish an optimal a posteriori estimates for the pressure. Next, by writing the error $\mathbf{u}-\mathbf{u_h}$ with specific decomposition, we establish an optimal a posteriori estimate for the velocity. In the last section, we show numerical results.

2. Description and analysis of the model

We denote by $(Problem1)$ the system of equations (1.1), (1.2) and (1.3), and by $(Problem2)$ the system of equations (1.1), (1.2) and (1.4). In all the paper, we suppose that $\mathbf{f} \in L^2(\Omega)^3$ and we denote by C a generic positive constant.

In order to write the variational formulation of the previous problems, we introduce some spaces:

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega),\, \partial^\alpha v \in L^p(\Omega), \quad \forall \mid \alpha \mid \leq m\},$$
$$H^m(\Omega) = W^{m,2}(\Omega),$$

equipped with the following semi-norm and norm :

$$\mid v \mid_{m,p,\Omega} = \left\{ \sum_{|\alpha|=m} \int_\Omega \mid \partial^\alpha v(x) \mid^p dx \right\}^{1/p} \quad \text{and} \quad \| v \|_{m,p,\Omega} = \left\{ \sum_{k\leq m} \mid v \mid_{k,p,\Omega}^p \right\}^{1/p}.$$

As usual, we shall omit p when $p=2$ and denote by $(\cdot,\cdot)$ the scalar product of $L^2(\Omega)$. Also, recall the familiar notation :

$$H_0^1(\Omega) = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma\},$$

with the Poincaré inequality

$$\forall v \in H_0^1(\Omega); \quad ||v||_{0,\Omega} \leq C|v|_{1,\Omega}. \tag{2.1}$$

Finally, we introduce the spaces :

$$H(\mathrm{div},\Omega) = \{\mathbf{v} \in L^2(\Omega)^3, \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}; \quad H_0(\mathrm{div},\Omega) = \{\mathbf{v} \in H(\mathrm{div},\Omega), \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\};$$
$$H(\mathbf{curl},\Omega) = \{\mathbf{v} \in L^2(\Omega)^3, \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^3\}; \quad H_0(\mathbf{curl},\Omega) = \{\mathbf{v} \in H(\mathbf{curl},\Omega), \mathbf{v}\times\mathbf{n} = \mathbf{0} \text{ on } \Gamma\};$$

normed respectively by :

$$\| \mathbf{v} \|_{H(\mathrm{div},\Omega)} = \left\{ \| \mathbf{v} \|_{0,\Omega}^2 + \| \mathrm{div}\ \mathbf{v} \|_{0,\Omega}^2 \right\}^{1/2},$$

and

$$\| \mathbf{v} \|_{H(\mathbf{curl},\Omega)} = \left\{ \| \mathbf{v} \|_{0,\Omega}^2 + \| \mathbf{curl}\, \mathbf{v} \|_{0,\Omega}^2 \right\}^{1/2}.$$

For the following regularity theorems, we refer to Bernardi [6], Dauge [8], Girault & Raviart [14], Grisvard [17] and Nedelec [20].

Lemma 2.1. *There exists a unique solution q in $H^1(\Omega)/\mathbb{R}$ (resp. $H_0^1(\Omega)$) such that :*

$$(\nabla q, \nabla \xi = (f, \nabla \xi) \quad \forall\, \xi \in H^1(\Omega)/\mathbb{R} \quad (resp\ H_0^1(\Omega)),$$

and there exists a positive constant C such that :

$$||q||_{1,\Omega} \leq C||f||_{0,\Omega}.$$

Theorem 2.2. *Let Ω be convex. All functions $\mathbf{v} \in L^2(\Omega)^3$ satisfying :*

$$\mathrm{div}\, \mathbf{v} = 0, \quad \mathbf{curl}\, \mathbf{v} \in L^2(\Omega)^3, \quad \mathbf{v}.\mathbf{n} = 0 \ \ or\ \ \mathbf{v} \times \mathbf{n} = 0 \ \ on\ \Gamma,$$

belong to $H^1(\Omega)^3$ and we have

$$||\mathbf{v}||_{1,\Omega} \leq C||\mathbf{curl}\, \mathbf{v}||_{0,\Omega}.$$

In view of the relation :

$$-\Delta \mathbf{u} = \mathbf{curl}\ \mathbf{curl}\ \mathbf{u} \quad (\text{as we have } \mathrm{div}\ \mathbf{u} = 0),$$

we can establish the next theorem.

Theorem 2.3. $(Problem1)$ *has the following weak variational formulation :*

Find $\mathbf{u} \in H_0(\mathbf{curl}, \Omega)$ and $p \in H_0^1(\Omega)$ such that:

$$\nu\ (\mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v}) + (\nabla p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall\ \mathbf{v} \in H_0(\mathbf{curl}, \Omega), \tag{2.2}$$

$$(\nabla q, \mathbf{u}) = 0 \quad \forall\ q \in H_0^1(\Omega), \tag{2.3}$$

and $(Problem2)$ has the following weak variational formulation:

Find $\mathbf{u} \in H(\mathbf{curl}, \Omega)$ and $p \in H^1(\Omega)/\mathbb{R}$ such that:

$$\nu\ (\mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v}) + (\nabla p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall\ \mathbf{v} \in H(\mathbf{curl}, \Omega), \tag{2.4}$$

$$(\nabla q, \mathbf{u}) = 0 \quad \forall\ q \in H^1(\Omega). \tag{2.5}$$

Proof: First, let $(\mathbf{u}, p)$ is the solution of the problem $(Problem1)$. The density of $\mathcal{D}(\Omega)^3$ in $H_0(\mathbf{curl}, \Omega)$ (see [15] Chap. 1 or [29] Chap. 1) and the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$ give that $(\mathbf{u}, p)$ is also solution of the problem (2.2) and (2.3). Conversely, let $(\mathbf{u}, p)$ be the solution of (2.2) and (2.3), the equations (1.1) and (1.2) are satisfied in the distribution sense, and the third equation (1.3) becomes from the definition of the spaces $H_0(\mathbf{curl}, \Omega)$ and $H_0^1(\Omega)$.
For the second problem, we proceed in the same way. In fact, let $(\mathbf{u}, p)$ a solution of (2.4) and (2.5), the equations (1.1) and (1.2) are satisfied in the distribution sense. The first boundary condition of (1.4) is then derived by integrating by parts the equation:

$$\int_\Omega \mathrm{div}\ \mathbf{u}\ q = 0 \quad \forall q \in H^1(\Omega),$$

(note that it is satisfied in the dual space of $H^{\frac{1}{2}}(\partial\Omega)$). The second boundary condition of (1.4) can be obtaining as follow: The equation (1.1) and (2.4) give

$$(\mathbf{curl}\, \mathbf{curl}\, \mathbf{u}, \mathbf{v}) = (\mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v}) \quad \forall \mathbf{v} \in H(\mathbf{curl}, \Omega),$$

which leads to $\mathbf{curl}\, \mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$. $\square$

Each variational formulation is split into a system for the velocity and a Poisson equation for the pressure. Let us introduce the spaces:

$$V_0 = \{\mathbf{v} \in H_0(\mathbf{curl}, \Omega);\ (\nabla q, \mathbf{v}) = 0\ \ \forall q \in H_0^1(\Omega)\},$$

and

$$U = \{\mathbf{v} \in H(\mathbf{curl}, \Omega);\ (\nabla q, \mathbf{v}) = 0\ \ \forall q \in H^1(\Omega)\}.$$

For every $\mathbf{v} \in H_0(\mathbf{curl}, \Omega)$ (rep. $H(\mathbf{curl}, \Omega)$), the lemma (2.1) with $\mathbf{f} = \mathbf{v}$ gives that, there exists a unique q such that:

$$(\mathbf{v} - \nabla q, \nabla \xi) = 0 \quad \forall \xi \in H_0^1(\Omega)\ (\text{resp. } H^1(\Omega)/\mathbb{R})$$

We deduce that every $\mathbf{v} \in H_0(\mathbf{curl}, \Omega)$ (rep. $H(curl, \Omega)$) can be decomposed as $\mathbf{v} = \mathbf{w} + \nabla q$ where $\mathbf{w} \in V_0$ (res. U) and $q \in H_0^1(\Omega)$ (rep. $H^1(\Omega)/\mathbb{R}$). This fact and the theorems 2.2 and 2.3 (which allow us to use $(\mathbf{curl}., \mathbf{curl}.)$ as scalar product in V_0 and U) allow us to establish the following theorem:

Theorem 2.4. *The problem (2.2)-(2.3) is equivalent to the problem:*

Find $\mathbf{u} \in V_0$ such that:

$$\nu(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall\ \mathbf{v} \in V_0. \tag{2.6}$$

Find $p \in H_0^1(\Omega)$ such that:

$$(\nabla p, \nabla q) = (\mathbf{f}, \nabla q) \quad \forall\ q \in H_0^1(\Omega). \tag{2.7}$$

The problem (2.4)-(2.5) is equivalent to the problem :

Find $\mathbf{u} \in U$ such that

$$\nu(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall\ \mathbf{v} \in U. \tag{2.8}$$

Find $p \in H^1(\Omega)/\mathbb{R}$ such that

$$(\nabla p, \nabla q) = (\mathbf{f}, \nabla q) \quad \forall\ q \in H^1(\Omega). \tag{2.9}$$

In both cases, if Ω is convex, we using the Lax-Milgram theorem to proove that there exists a unique solution and we have the following bounds :

$$|p|_{1,\Omega} \le ||\mathbf{f}||_{0,\Omega}; \quad ||\mathbf{curl}\,\mathbf{u}||_{0,\Omega} \le \frac{C_1}{\nu}||\mathbf{f}||_{0,\Omega} \quad \textit{and} \quad ||\mathbf{u}||_{1,\Omega} \le \frac{C_2}{\nu}||\mathbf{f}||_{0,\Omega}.$$

Consequently, the pressure in the problem (2.9) verify the Neumann boundary condition

$$\frac{\partial p}{\partial \mathbf{n}} = f.\mathbf{n} \quad \textit{on } \partial\Omega$$

3. Finite element discretization

We introduce a regular family of triangulations $(\tau_h)_h$ in the sens that :

- for each h, $\bar{\Omega}$ is the union of all elements of τ_h;
- for each h, the intersection of two different elements of τ_h, if not empty, is a node, a whole edge or a whole face of both of them;
- the ratio of the diameter h_κ of an element κ in τ_h to the diameter of its inscribed sphere is bounded by a constant independent of κ and h;

As usual, h denotes the maximum of the diameters of the elements of τ_h.

Next, for each κ in τ_h, we introduce the spaces $\mathbb{P}_0(\kappa)$ of the restrictions to κ of constant functions on $\mathbb{R}^3$, $\mathbb{P}_1(\kappa)$ of the restrictions to κ of affine function on $\mathbb{R}$ and the space $\mathbb{P}_K(\kappa)$ of the restrictions to κ of polynomials $\mathbf{v}$ of the form :

$$\mathbf{v}(x) = \mathbf{a} + \mathbf{b} \times \mathbf{x}, \quad \mathbf{a} \in \mathbb{R}^3, \mathbf{b} \in \mathbb{R}^3.$$

The space $\mathbb{P}_K(\kappa)$ and the corresponding finite elements are studied in [19].

Their degrees of freedom are the average flux along the edges $\int_l (\mathbf{v}.\mathbf{t})dl$, for the six edges l of κ, $\mathbf{t}$ is the

direction vector of l.
Hence, we associate the operator r_κ where $r_\kappa(\mathbf{u})$ is the unique polynomial of $\mathbb{P}_K$ that has the same flux along the edges as $\mathbf{u}$. We define also the operator I_κ where $I_\kappa(q)$ is the unique polynomial of $\mathbb{P}_1(\kappa)$ that has the same values on the vertex of κ as q.

Next, let us introduce the discrete spaces :

$$\begin{aligned} M_h &= \{\mathbf{u}_h \in H(\mathbf{curl},\Omega); \mathbf{u}_h|_\kappa \in \mathbb{P}_K(\kappa),\ \forall\, \kappa \in \tau_h\}, && (3.1)\\ M_{0h} &= M_h \cap H_0(\mathbf{curl},\Omega), && (3.2)\\ Q_h &= \{q_h \in C^0(\overline{\Omega}); q_h|_\kappa \in \mathbb{P}_1(\kappa),\ \forall\, \kappa \in \tau_h\}, && (3.3)\\ Q_{0h} &= Q_h \cap H_0^1(\Omega). && (3.4)\end{aligned}$$

With these spaces, the finite dimensional analogues of V_0 and U are :

$$V_{0h} = \{\mathbf{v}_h \in M_{0h}\ ;\ (\nabla q_h, \mathbf{v}_h) = 0, \quad \forall\, q_h \in Q_{0h}\},$$

and

$$U_h = \{\mathbf{v}_h \in M_h\ ;\ (\nabla q_h, \mathbf{v}_h) = 0 \quad \forall\, q_h \in Q_h\}.$$

We define the interpolation operators r_h from $H^1(\Omega)^3$ onto M_h, I_h from $H^2(\Omega)$ onto Q_h by

$$r_h u = r_\kappa(u) \text{ on } \kappa,\ \ \forall \kappa \in \tau_h \quad (\text{similarly for } I_h\ ).$$

Theorem 3.1. *Assume that the triangulation τ_h is regular. For all $k \geq 1$ we have :*

$$\| \mathbf{u} - r_h\mathbf{u} \|_{0,\Omega} + h \| \mathbf{curl}(\mathbf{u} - r_h\mathbf{u}) \|_{0,\Omega} \leq Ch \mid \mathbf{u} \mid_{1,t,\Omega},\ \forall\, \mathbf{u} \in W^{1,t}(\Omega)^3, \ \ \textit{for some } t > 2.$$

Moreover, when $\mathbf{u} \in (H^k(\Omega))^3$ we have :

$$\| \mathbf{u} - r_h\mathbf{u} \|_{0,\Omega} \leq Ch^k \mid \mathbf{u} \mid_{k,\Omega},$$

and, when $\mathbf{u} \in (H^{k+1}(\Omega))^3$ we have :

$$\| \mathbf{curl}(\mathbf{u} - r_h\mathbf{u}) \|_{0,\Omega} \leq Ch^k \mid \mathbf{u} \mid_{k+1,\Omega}.$$

There is also an important result given by V. Girault [15] for the imbedding between the spaces V_{0h} or U_h and $L^4(\Omega)^3$ (or $L^2(\Omega)$):

Theorem 3.2. *Let Ω be a convex polyhedron and τ_h a uniformly regular family of triangulations of Ω. For each space V_{0h} and U_h, there exists constants C and C', independent of h, such that*

$$||\mathbf{u}_h||_{0,\Omega} \leq C||\mathbf{u}_h||_{0,4,\Omega} \leq C'||\mathbf{curl}\ \mathbf{u}_h||_{0,\Omega} \quad \forall \mathbf{u}_h \in V_{0h} \textit{ or } U_h. \tag{3.5}$$

We discretize $(Problem1)$ by :

Find $\mathbf{u}_h \in V_{0h}$ and $p_h \in Q_{0h}$ such that

$$\nu(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + (\nabla p_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in M_{0h}. \tag{3.6}$$

Similarly, we discretize $(Problem2)$ by :

Find $\mathbf{u}_h \in U_h$ and $p_h \in Q_h/\mathbb{R}$ such that

$$\nu(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + (\nabla p_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in M_h. \tag{3.7}$$

As in the continuous way, the problem (3.6) can be split into

Find $\mathbf{u}_h \in V_{0h}$ such that

$$\nu(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in V_{0h}, \tag{3.8}$$

Find $p_h \in Q_{0h}$ such that

$$(\nabla p_h, \nabla q_h) = (\mathbf{f}, \nabla q_h), \quad \forall\, q_h \in Q_{0h}. \tag{3.9}$$

And the problem (3.7) can be splited to

Find $\mathbf{u}_h \in U_h$ such that

$$\nu(\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in U_h, \tag{3.10}$$

Find $p_h \in Q_h/\mathbb{R}$ such that

$$(\nabla p_h, \nabla q_h) = (\mathbf{f}, \nabla q_h), \quad \forall\, q_h \in Q_h. \tag{3.11}$$

It is easy to show, using theorem (3.2), that these two last discrete problems have a unique solution. The pressure is entirely dissociated from the velocity, i.e. can be computed without knowing the velocity. We have also for both discrete problems :

$$||\mathbf{curl}\,\mathbf{u}_h||_{0,\Omega} \le \frac{C}{\nu}||\mathbf{f}||_{0,\Omega},$$

and

$$|p_h|_{1,\Omega} \le ||\mathbf{f}||_{0,\Omega}.$$

4. A priori error analysis

In this section, we will establish the error estimates for the pressure and the velocity.

Theorem 4.1. *The exact solution $(\mathbf{u}, p)$ of the problem (2.6)-(2.7) (resp. (2.8)-(2.9)) and the numerical solution $(\mathbf{u}_h, p_h)$ of the problem (3.8)-(3.9) (resp. (3.10)-(3.11)) verify the error estimes :*

$$|p - p_h|_{1,\Omega} = \inf_{q_h \in Q_{0,h}} |p - q_h|_{1,\Omega} \qquad \textit{(resp. } Q_h\textit{)}\ , \tag{4.1}$$

$$||\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)||_{0,\Omega} \le C\left(\inf_{\mathbf{v}_h \in M_{0h}} ||\mathbf{curl}(\mathbf{u} - \mathbf{v}_h)||_{0,\Omega} + \inf_{q_h \in Q_{0h}} |p - p_h|_{1,\Omega}\right) \qquad \textit{(resp. } M_h \textit{ and } Q_h\textit{)}. \tag{4.2}$$

Proof:
For the pressure, let us choose $q = q_h$. The difference between (2.7) and (3.9) (resp. (2.9) and (3.11)) gives:

$$(\nabla(p - p_h), \nabla q_h) = 0, \quad \forall q_h \in Q_{0h} \quad \text{(resp. } Q_h), \tag{4.3}$$

then

$$|p - p_h|^2_{1,\Omega} = (\nabla(p - p_h), \nabla p) = (\nabla(p - p_h), \nabla(p - q_h)) \le |p - p_h|_{1,\Omega}|p - q_h|_{1,\Omega},$$

and we obtain (4.1).

For the velocity, by taking $\mathbf{v} = \mathbf{v}_h$, the difference between (2.6) and (3.8) (resp. (2.8) and (3.10)) gives:

$$\nu(\mathbf{curl}(\mathbf{u} - \mathbf{u}_h), \mathbf{curl}\mathbf{v}_h) + (\nabla(p - p_h), \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in M_{0h} \qquad \text{(resp. } M_h)\ .$$

Then for all $\mathbf{w}_h$ in V_{0h} (resp. U_h) we have

$$\nu(\mathbf{curl}(\mathbf{u} - \mathbf{w}_h), \mathbf{curl}\mathbf{v}_h) + \nu(\mathbf{curl}(\mathbf{w}_h - \mathbf{u}_h), \mathbf{curl}\mathbf{v}_h) + (\nabla(p - p_h), \mathbf{v}_h) = 0.$$

By choosing $\mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h \in V_{0h}$ and using the relation (3.5), we obtain

$$||\mathbf{curl}(\mathbf{u}_h - \mathbf{w}_h)||_{0,\Omega} \le ||\mathbf{curl}(\mathbf{u} - \mathbf{w}_h)||_{0,\Omega} + C|p - p_h|_{1,\Omega}.$$

Now we extend this last inequality to all functions $\mathbf{v}_h$ of M_{0h} (resp M_h): Define q_h in Q_{0h} (resp. Q_h) by

$$(\nabla q_h, \nabla \mu_h) = (\mathbf{v}_h, \nabla \mu_h) \quad \forall \mu_h \in Q_{0h} \qquad \text{(resp. } Q_h)\ ,$$

and set $\mathbf{w}_h = \mathbf{v}_h - \nabla q_h$. Then $\mathbf{w}_h$ belongs to V_{0h} (resp. U_h) and $\mathbf{curl}\mathbf{w}_h = \mathbf{curl}\mathbf{v}_h$ and we obtain

$$||\mathbf{curl}(\mathbf{u}_h - \mathbf{v}_h)||_{0,\Omega} = ||\mathbf{curl}(\mathbf{u}_h - \mathbf{w}_h)||_{0,\Omega}.$$

Moreover

$$\begin{aligned} ||\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)||_{0,\Omega} &\leq ||\mathbf{curl}(\mathbf{u} - \mathbf{v}_h)||_{0,\Omega} + ||\mathbf{curl}(\mathbf{u}_h - \mathbf{v}_h)||_{0,\Omega} \\ &\leq ||\mathbf{curl}(\mathbf{u} - \mathbf{v}_h)||_{0,\Omega} + ||\mathbf{curl}(\mathbf{u}_h - \mathbf{w}_h)||_{0,\Omega} \\ &\leq 2||\mathbf{curl}(\mathbf{u} - \mathbf{v}_h)||_{0,\Omega} + C|p - p_h|_{1,\Omega}, \end{aligned}$$

and we obtain (4.2). □

Corollary 4.2. *Under the assumption of Theorem 4.1 and when the solution is sufficiently smooth we have*

$$|p - p_h|_{1,\Omega} \leq Ch|p|_{2,\Omega}, \tag{4.4}$$

and

$$||\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)||_{0,\Omega} \leq Ch(|p|_{2,\Omega} + |\mathbf{u}|_{2,\Omega}). \tag{4.5}$$

5. A posteriori error analysis

We now intend to prove a posteriori error estimates between the exact solution $(\mathbf{u}, p)$ of the problem (2.6)-(2.7) and the numerical solution $(\mathbf{u}_h, p_h)$ of the problem (3.8)-(3.9). By the same way, we can prove a posteriori error estimates between the solution $(\mathbf{u}, p)$ of the exact problem (2.8)-(2.9) and $(\mathbf{u}_h, p_h)$ of the numerical problem (3.10)-(3.11). In all the rest of the paper, we suppose that $\mathbf{f} \in H^1(\Omega)^3$.

We first introduce the space

$$Z_h = \{\mathbf{g}_h \in L^2(\Omega)^3;\ \forall \kappa \in \tau_h,\ \mathbf{g}_h|_\kappa \in \mathbb{P}_0(\kappa)\},$$

and we fix an approximation $\mathbf{f}_h$ of the data $\mathbf{f}$ in Z_h.

Next, we denote by ε_h the set of all faces of the elements of τ_h that are not contained in $\partial\Omega$. For every element κ in τ_h, we denote by ε_κ the set of faces of κ that are not contained in Γ, Δ_κ the set of union of elements of τ_h that intersect κ, Δ_e the union of elements of τ_h that intersect the face e, h_κ the diameter of κ and h_e the diameter of the face e. Also, $\mathbf{n}_\kappa$ stands for the unit outward normal vector to κ on $\partial\kappa$ and $[\cdot]_e$ the jump through the face e of κ.

For the proof of the next theorems, we introduce for an element κ of τ_h, the bubble l function ψ_κ (resp. ψ_e of the face e) which is equal to the product of the $d+1$ barycentric coordinates associated with the vertices of κ (resp. of e) and $\mathcal{L}_e$ the lifting operator from polynomials defined on e into polynomials defined on the elements κ and κ' contained e, which is constructed by affine transformations from a fixed operator on the reference element.

Property 5.1. *Denoting by $P_r(e)$ the polynomial of degrees r on e, we have*

$$\forall\ v \text{ polynomial of } P_r(e) \qquad c \parallel v \parallel_{L^2(e)} \leq \parallel v\psi_e^{1/2} \parallel_{L^2(e)} \leq c' \parallel v \parallel_{L^2(e)},$$

and $\forall\ v$ polynomial of $P_r(e)$ which vannishes on ∂e, we have

$$\parallel \mathcal{L}_e v \parallel_{L^2(\kappa)} + h_e \mid \mathcal{L}_e v \mid_{H^1(\kappa)} \leq c h_e^{1/2} \parallel v \parallel_{L^2(e)}.$$

We denote by R_h the Clément operator [7]. We have for all function $q \in H_0^1(\Omega)$, $R_h q \in Q_{0h}$ verifies

$$\begin{aligned} &\parallel q - R_h q \parallel_{L^2(\kappa)} \leq c h_\kappa \parallel q \parallel_{H^1(\Delta\kappa)}, \\ &\parallel q - R_h q \parallel_{L^2(e)} \leq c h_e^{1/2} \parallel q \parallel_{H^1(\Delta e)}, \end{aligned} \tag{5.1}$$

and $\mathcal{R}_h$ the Raviart-Thomas operator: for any smooth enough vectorial function $\mathbf{v}$ which is divergence-free in Ω, $\mathcal{R}_h\mathbf{v}$ belongs to M_{0h} and satisfies

$$\forall e \in \varepsilon_h, \quad \int_e (\mathbf{v} - \mathcal{R}_{\mathbf{h}}\mathbf{v}).\mathbf{n} d\tau = 0.$$

Moreover, this operator satisfies, see [23]: $\forall \mathbf{v}$ in $H^1(\Omega)^3$ and $\forall \kappa$ in τ_h,

$$\begin{aligned} \| \mathbf{v} - \mathcal{R}_h\mathbf{v} \|_{L^2(\kappa)^3} &\leq c h_\kappa \| \mathbf{v} \|_{H^1(\kappa)^3} \\ \| \mathbf{v} - \mathcal{R}_h\mathbf{v} \|_{L^2(e)^3} &\leq c h_e^{1/2} \| \mathbf{v} \|_{H^1(\Delta e)^3} \end{aligned} \tag{5.2}$$

Let us begin with a posteriori error for the pressure. The error function $p - p_h$ belongs to $H_0^1(\Omega)$ and satisfies:

$$(\nabla(p - p_h), \nabla q) = \langle F, q\rangle, \quad \forall q \in H_0^1(\Omega),$$

where the "residual" F belongs to the dual space $H^{-1}(\Omega)$ and is defined by:

$$\forall v \in H_0^1(\Omega), \quad \langle F, q\rangle = \int_\Omega \mathbf{f}\nabla q - \int_\Omega \nabla p_h \nabla q. \tag{5.3}$$

We deduce that

$$|p - p_h|_{1,\Omega} \leq ||F||_{H^{-1}(\Omega)}.$$

We define the error indicator by

$$\eta_\kappa = \sum_{e\in\varepsilon_\kappa} h_e^{1/2} \| [(\mathbf{f}_h - \nabla p_h).\mathbf{n}] \|_{L^2(e)} .$$

Lemma 5.2. *The following estimate hold*

$$||F||_{H^{-1}(\Omega)} \leq C \left\{ \sum_{\kappa\in\tau_h} \Big(\eta_\kappa^2 + h_\kappa^2 \| \operatorname{div} \mathbf{f} \|^2_{L^2(\kappa)} + \big(\sum_{e\in\varepsilon_\kappa} h_e^{1/2} \| [(\mathbf{f} - \mathbf{f}_h).\mathbf{n}] \|_{L^2(e)} \big)^2 \Big) \right\}^{1/2}.$$

Proof: For any $q_h \in M_{0h}$, we have

$$\begin{aligned} \langle F, q\rangle &= \int_\Omega \mathbf{f}\nabla(q - q_h) - \int_\Omega \nabla p_h \nabla(q - q_h) \\ &= \sum_{\kappa\in\tau_h} \Big(\int_\kappa (\mathbf{f} - \mathbf{f}_h)\nabla(q - q_h) + \int_\kappa (\mathbf{f}_h - \nabla p_h)\nabla(q - q_h) \Big). \end{aligned} \tag{5.4}$$

By integrating by part, we obtain

$$\langle F, q\rangle = \sum_{\kappa\in\tau_h} \left\{ -\int_\kappa \operatorname{div}\mathbf{f}\ (q - q_h) + \frac{1}{2} \sum_{e\in\varepsilon_\kappa} \int_e \Big([(\mathbf{f} - \mathbf{f}_h).\mathbf{n}](q - q_h) + [(\mathbf{f}_h - \nabla p_h).\mathbf{n}](q - q_h) \Big) \right\}, \tag{5.5}$$

and by taking $q_h = R_h q$, the image of q by the Clément type regularisation operator, we obtain the result.
□

Corollary 5.3. *The following a posteriori estimate holds between the solution p of (2.7) and the solution p_h of (3.9):*

$$|p - p_h|_{1,\Omega} \leq C \left\{ \sum_{\kappa\in\tau_h} \Big(\eta_\kappa^2 + h_\kappa^2 \| \operatorname{div} \mathbf{f} \|^2_{L^2(\kappa)} + \big(\sum_{e\in\varepsilon_\kappa} h_e^{1/2} \| [(\mathbf{f} - \mathbf{f}_h).\mathbf{n}] \|_{L^2(e)} \big)^2 \Big) \right\}^{1/2}.$$

Proposition 5.4. *The error indicators verify the following optimality conditions*

$$\eta_\kappa \leq C \Big(| p - p_h |_{H^1(\Delta_\kappa)} + h_e \| \operatorname{div} \mathbf{f} \|_{L^2(\Delta_\kappa)} + \sum_{e\in\varepsilon_\kappa} h_e^{1/2} \| [(\mathbf{f} - \mathbf{f}_h).\mathbf{n}] \|_{L^2(e)} \Big). \tag{5.6}$$

Proof: We consider the equation (5.5) with $q_h = 0$ and we take $q = q_e = \mathcal{L}_e([(\mathbf{f}_h - \nabla p_h).\mathbf{n}]\psi_e)$:

$$\int_{\kappa\cup\kappa'} \nabla(p - p_h)\nabla q_e = -\int_{\kappa\cup\kappa'} \operatorname{div} \mathbf{f}\ q_e + \frac{1}{2} \int_e \Big([(\mathbf{f} - \mathbf{f}_h).\mathbf{n}] q_e + q_e^2 \Big),$$

then by using the property (5.1)

$$\| [(\mathbf{f}_h - \nabla p_h).\mathbf{n}] \|_{0,e} \leq C\Big(h_e^{-1/2} |p - p_h|_{1,\kappa\cup\kappa'} + h_e^{1/2} \| \operatorname{div} \mathbf{f} \|_{\kappa\cup\kappa'} + \| [(\mathbf{f} - \mathbf{f}_h).\mathbf{n}] \|_{L^2(e)} \Big),$$

multiplying by $h_e^{1/2}$ and summing over ε_κ, we obtain the result. □

Now, let us establish a posteriori error for the velocity. The error function $\mathbf{u} - \mathbf{u}_h$ belongs to $H_0(\mathbf{curl}, \Omega)$, there exists a function $\lambda \in H_0^1(\Omega)$ solution of the problem:

$$\forall \mu \in H_0^1(\Omega), \quad \int_\Omega \nabla\lambda \nabla\mu = \int_\Omega (\mathbf{u} - \mathbf{u}_h)\nabla\mu = -\int_\Omega \mathbf{u}_h \nabla\mu. \tag{5.7}$$

Then the function $\mathbf{w} = (\mathbf{u} - \mathbf{u}_h) - \nabla\lambda$ belongs to V_0 and we have $\mathbf{curl}\,\mathbf{w} = \mathbf{curl}\,(\mathbf{u} - \mathbf{u}_h)$. We obtain

$$\| \mathbf{u} - \mathbf{u}_h \|^2_{H(\mathbf{curl},\Omega)} = \| \nabla\lambda \|^2_{0,\Omega} + \| \mathbf{w} \|^2_{H(\mathbf{curl},\Omega)} \tag{5.8}$$

In order to find the upper and lower bounds of $\| \mathbf{u} - \mathbf{u}_h \|^2_{H(\mathbf{curl},\Omega)}$, we start by finding the upper and lower bounds of the two terms of the left hand side of the last equation.

For the first term of the left hand side of (5.8), we have $\forall \mu \in H_0^1(\Omega)$,

$$\int_\Omega \nabla\lambda\nabla\mu = \int_\Omega (\mathbf{w} + \nabla\lambda)\nabla\mu = \int_\Omega (\mathbf{u} - \mathbf{u}_h)\nabla\mu = -\int_\Omega \mathbf{u}_h \nabla\mu = -\int_\Omega \mathbf{u}_h \nabla(\mu - \mu_h) \quad \forall \mu_h \in Q_{0h}.$$

The associate "residual" G of the problem (5.7) belongs to $H^{-1}(\Omega)$ and satisfies

$$\int_\Omega \nabla\lambda\nabla\mu = \langle G, \mu\rangle \quad \forall \mu \in H_0^1(\Omega),$$

then, using the fact that $\operatorname{div} \mathbf{u}_h = 0$ on every element $\kappa \in \tau_h$, G satisfies

$$\langle G, \mu\rangle = -\int_\Omega \mathbf{u}_h\nabla\mu = -\int_\Omega \mathbf{u}_h \nabla(\mu - \mu_h) = -\frac{1}{2}\sum_{\kappa\in\tau_h}\Big(\sum_{e\in\varepsilon_\kappa}\int_e [\mathbf{u}_h.\mathbf{n}](\mu - \mu_h)\Big). \tag{5.9}$$

We introduce the indicators

$$\xi_\kappa = \sum_{e\in\varepsilon_\kappa} h_e^{1/2} \| [\mathbf{u}_h.\mathbf{n}] \|_{0,e}\ . \tag{5.10}$$

Theorem 5.5. *The following bounds hold*

$$|\lambda|_{1,\Omega} \leq C\Big(\sum_{\kappa\in\tau_h}\xi_\kappa^2\Big)^{1/2},$$

and

$$\xi_\kappa \leq C|\lambda|_{1,\Delta_\kappa}. \tag{5.11}$$

Proof: We treat this problem exactly as we did the problem (2.7) and we obtain

$$|\lambda|_{1,\Omega} \leq \| G \|_{-1,\Omega} \leq C\Big(\sum_{\kappa\in\tau_h}\xi_\kappa^2\Big)^{1/2}. \tag{5.12}$$

In order to find the lower bound, we take in the equation

$$\int_\Omega \nabla\lambda\nabla\mu = -\frac{1}{2}\sum_{\kappa\in\tau_h}\Big(\sum_{e\in\varepsilon_\kappa}\int_e [\mathbf{u}_h.\mathbf{n}]\mu\Big),$$

$\mu = \mathcal{L}_e([\mathbf{u}_h.\mathbf{n}]\psi_e)$ and we obtain

$$\| [\mathbf{u}_h.\mathbf{n}] \|_{0,e} \leq C\Big(h_e^{-1/2}|\lambda|_{1,\kappa\cup\kappa'}\Big),$$

which leads to

$$\xi_\kappa \leq 4C\Big(|\lambda|_{1,\Delta_\kappa}\Big). \tag{5.13}$$

□

Now, we take the second term of the left hand side of (5.8). We begin by $\forall \mathbf{v} \in H_0(\mathbf{curl}, \Omega)$

$$\nu \int_\Omega \mathbf{curl}(\mathbf{u}-\mathbf{u}_h)\mathbf{curl}\,\mathbf{v} + \int_\Omega \nabla(p-p_h)\mathbf{v} = \int_\Omega \mathbf{f}\mathbf{v} - \nu \int_\Omega \mathbf{curl}\,\mathbf{u}_h\,\mathbf{curl}\mathbf{v} - \int_\Omega \nabla p_h \mathbf{v}.$$

By replacing $\mathbf{u}-\mathbf{u}_h = \mathbf{w}+\nabla\lambda$ and taking $\mathbf{v} \in V_0$ we obtain

$$\nu \int_\Omega \mathbf{curl}\,\mathbf{w}\;\mathbf{curl}\,\mathbf{v} = \int_\Omega \mathbf{f}\mathbf{v} - \nu \int_\Omega \mathbf{curl}\,\mathbf{u}_h\mathbf{curl}\,\mathbf{v}.$$

The associate "residual" L belongs to V_0' and satisfies

$$\nu \int_\Omega \mathbf{curl}(\mathbf{u}-\mathbf{u}_h)\mathbf{curl}\,\mathbf{v} = \langle L, \mathbf{v}\rangle \quad \forall \mathbf{v} \in V_0,$$

where, $\forall \mathbf{v} \in V_0$, L verifies

$$\begin{aligned}
\langle L, \mathbf{v}\rangle &= \nu \int_\Omega \mathbf{curl}(\mathbf{u}-\mathbf{u}_h)\mathbf{curl}\,\mathbf{v} + \int_\Omega \nabla(p-p_h)\mathbf{v} \\
&= \nu \int_\Omega \mathbf{curl}(\mathbf{u}-\mathbf{u}_h)\mathbf{curl}(\mathbf{v}-\mathbf{v}_h) + \int_\Omega \nabla(p-p_h)(\mathbf{v}-\mathbf{v}_h) \quad \forall \mathbf{v}_h \in M_{0h} \\
&= \int_\Omega \mathbf{f}(\mathbf{v}-\mathbf{v}_h) - \nu \int_\Omega \mathbf{curl}\,\mathbf{u}_h\mathbf{curl}(\mathbf{v}-\mathbf{v}_h) - \int_\Omega \nabla p_h(\mathbf{v}-\mathbf{v}_h) \quad \forall \mathbf{v}_h \in M_{0h} \\
&= \sum_{\kappa\in\tau_h} \Big(\int_\kappa (\mathbf{f}-\mathbf{f}_h)(\mathbf{v}-\mathbf{v}_h) + \int_\kappa (\mathbf{f}_h - \nabla p_h)(\mathbf{v}-\mathbf{v}_h) - \frac{1}{2}\sum_{e\in\varepsilon_\kappa} \int_e ([\mathbf{curl}\mathbf{u}_h \times \mathbf{n}])(\mathbf{v}-\mathbf{v}_h)\Big)
\end{aligned} \tag{5.14}$$

We introduce the indicators

$$\gamma_\kappa = h_\kappa \parallel \mathbf{f}_h - \nabla p_h \parallel_{0,\kappa} + \frac{1}{2}\sum_{e\in\varepsilon_\kappa} h_e^{1/2} \parallel [\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}] \parallel_{0,e}.$$

Theorem 5.6. *Let Ω be convex. The following bounds hold*

$$\parallel \mathbf{w} \parallel_{H(\mathbf{curl},\Omega)} \leq C\Big(\sum_{\kappa\in\tau_h} \Big(h_\kappa^2 \parallel \mathbf{f}-\mathbf{f}_h \parallel_{0,T}^2 + \gamma_\kappa^2 \Big)\Big)^{1/2}, \tag{5.15}$$

and

$$\gamma_\kappa \leq G\Big(\parallel \mathbf{curl}\,\mathbf{w} \parallel_{0,\Delta_\kappa} + (h_\kappa + h_e)\Big(\parallel \mathbf{f}-\mathbf{f}_h \parallel_{0,\Delta_\kappa} + |p-p_h|_{0,\Delta_\kappa}\Big)\Big). \tag{5.16}$$

Proof: In the equation (5.14), we take $\mathbf{v}_h = \mathcal{R}_h\mathbf{v}$ and use the properties of $\mathcal{R}_h$ and the theorem 2.2 we obtain

$$\parallel \mathcal{L} \parallel_{V_0'} \leq C\Big(\sum_{\kappa\in\tau_h} \Big(h_\kappa^2 \parallel \mathbf{f}-\mathbf{f}_h \parallel_{0,T}^2 + \gamma_\kappa^2 \Big)\Big)^{1/2},$$

which leads to (5.15).
In the other hand, we consider the equation: $\forall \mathbf{v} \in H_0(\mathbf{curl}, \Omega)$

$$\begin{aligned}
&\nu \int_\Omega \mathbf{curl}(\mathbf{u}-\mathbf{u}_h)\mathbf{curl}\;\mathbf{v} + \int_\Omega \nabla(p-p_h)\mathbf{v} = \\
&\qquad \sum_{\kappa\in\tau_h} \Big(\int_\kappa (\mathbf{f}-\mathbf{f}_h)\mathbf{v} + \int_\kappa (\mathbf{f}_h - \nabla p_h)\mathbf{v} - \frac{1}{2}\sum_{e\in\varepsilon_\kappa} \int_e ([\mathbf{curl}\;\mathbf{u}_h \times \mathbf{n}])\mathbf{v}\Big).
\end{aligned}$$

First, we take $\mathbf{v} = (\mathbf{f}_h - \nabla p_h)\psi_\kappa$ to obtain the relation :

$$\parallel \mathbf{f}_h - \nabla p_h \parallel_{0,\kappa} \leq C\Big(h_\kappa^{-1} \parallel \mathbf{curl}\,\mathbf{w} \parallel_{0,\kappa} + |p-p_h|_{1,\kappa} + \parallel \mathbf{f}-\mathbf{f}_h \parallel_{0,\kappa} \Big).$$

Second, we take $\mathbf{v} = \mathcal{L}_e(([\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}])\psi_e)$ to obtain

$$\begin{aligned}
\parallel [\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}] \parallel_{0,e} \leq C\Big\{ & h_e^{-1/2} \parallel \mathbf{curl}\,\mathbf{w} \parallel_{0,\kappa\cup\kappa'} \\
& + h_e^{1/2}\Big(|p-p_h|_{1,\kappa\cup\kappa'} + \parallel \mathbf{f}-\mathbf{f}_h \parallel_{0,\kappa\cup\kappa'} + \parallel \mathbf{f}_h - \nabla p_h \parallel_{0,\kappa\cup\kappa'} \Big)\Big\}.
\end{aligned}$$

Using the definition of γ_κ we obtain the relation (5.16). □

Corollary 5.7. *Let Ω be convex. The optimal a posteriori estimates hold*

$$\begin{aligned} ||\mathbf{u}-\mathbf{u}_h||_{H_0(\mathbf{curl},\Omega)} + |p-p_h|_{1,\Omega} \leq \Big\{ \sum_{\kappa\in\tau_h} \big(\gamma_\kappa^2+\xi_\kappa^2+\eta_\kappa^2+h_\kappa^2(\parallel \mathbf{f}-\mathbf{f}_h \parallel_{0,\kappa}^2) + \parallel \operatorname{div} \mathbf{f} \parallel_{L^2(\kappa)}^2 \big) \\ +\big(\sum_{e\in\varepsilon_\kappa} h_e^{1/2} \parallel [(\mathbf{f}-\mathbf{f}_h).\mathbf{n}] \parallel_{L^2(e)} \big)^2 \Big\}^{1/2}, \end{aligned} \tag{5.17}$$

where γ_κ, ξ_κ and η_κ are given by the formulas (5.6), (5.11) and (5.16).

Conclusion: We observe that estimate (5.17) is optimal: up to the terms involving the data, the full error is bounded by a constant times the sum of all indicators. Estimates (5.6), (5.16) and (5.11) are local, i.e., only involve the error in a neighborhood of K or e. The indicators η_κ, ξ_κ and γ_κ can be viewed as a measure for the error of the space discretization and can be used to adapt the mesh-size in space.

6. Numerical results

In order to confirm these results numerically, we did several experiments by using the FreeFem ++ software (see [18]). On the cubic domain $]0,1[\times]0,1[\times]0,1[$, the numerical velocity and the pressure are taken as $(u,p) = (\text{curl } \psi, p)$, where:

$$\begin{aligned} &\psi = (\phi,\phi,\phi) \quad \text{with} \quad \phi(x,y,z) = x^2y^2z^2(x-1)^2(y-1)^2(z-1)^2 \\ \text{and} \quad &p(x,y,z) = x(x-1)y(y-1)z(z-1). \end{aligned}$$

We take $\nu = 1$ and we denote by Nc the number of the points on edge of the geometry. We take a mesh with 6000 elements. First, we use the equation (3.9) to compute the pressure and second, knowing the pressure, we use the equation (3.8) to compute the velocity. We obtain the following color comparison between the exact and numerical solutions of the velocity and the pressure:

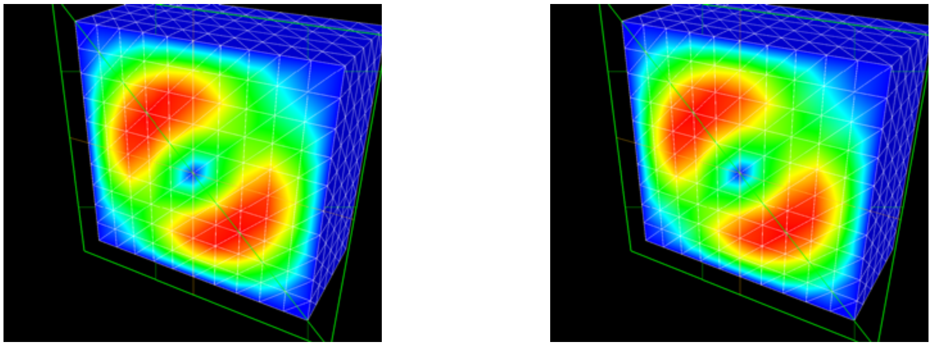

FIGURE 1. The right and left figures represent respectively the numerical and the theoretical velocity

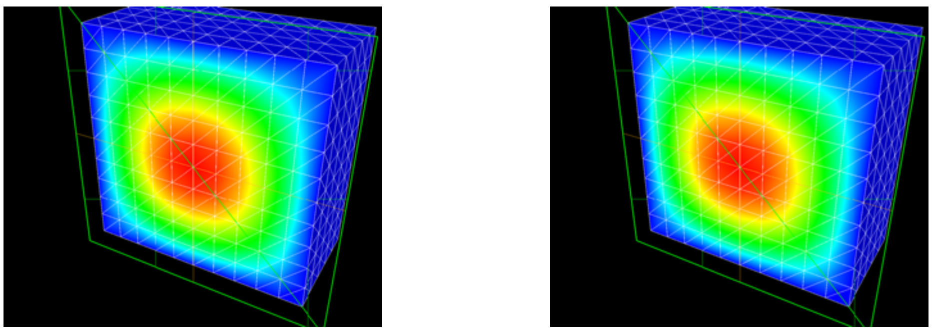

FIGURE 2. The right and left figures represent respectively the numerical and the theoretical pressure

Next, the graphs related to the velocity's and pressure's error estimations have been studied. In logarithmic scale, we represent the errors $\| \mathbf{curl}(\mathbf{u}-\mathbf{u}_h) \|_{0,\Omega}$ and $\| p-p_h \|_{1,\Omega}$ related to the mesh step h in the following figures:

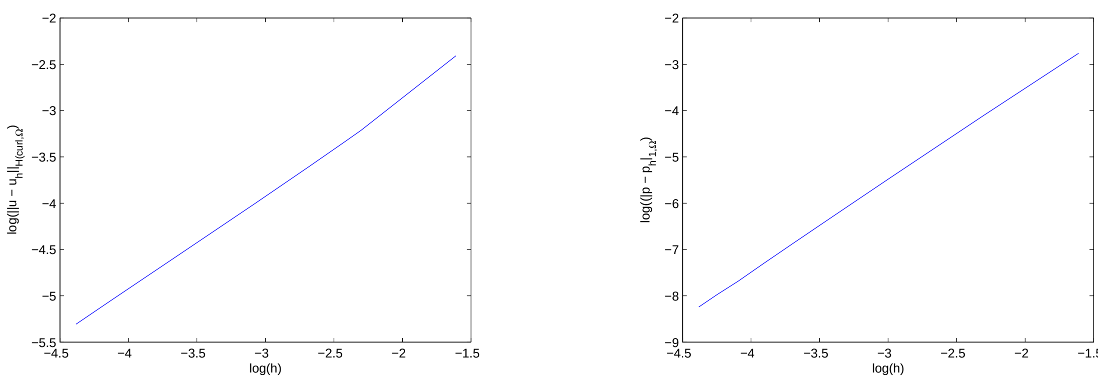

FIGURE 3. The right and left figures represent respectively the error estimation of the velocity and the pressure

We can see that the pressure slope is 1.0454 and velocity slope is 1.965, results that are similar to the theoretical ones.

Remark: FreeFem++ is under development for the three dimensional problems and we cannot yet experiment the adapted mesh with the showed indicators. In fact, it cannot make a local refinement of the mesh in three dimensions.

References

[1] I. Babuška & W.C. Rheinboldt. A posteriori eror estimates for the finite element method, *Internat. J. Numer. Meth. Engrg.,* **12** *(1978), 1597-1615.*

[2] I. Babuška & W.C. Rheinboldt. Error estimates for adaptive finite element computations, *SIAM J. Numer. Anal.,* **15** *(1978), 736-754.*

[3] R.E. Bank & B.D. Welfert. A posteriori eror estimates for the Stokes problem,*SIAM J. Numer. Anal.,* **28** *(1991), 591-623.*

[4] C. Bègue, C. Conca, F. Murat & O. Pironneau. Les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression , *in Nonlinear Partial Differential Equations and their Applications. Collège de France Seminar* **9** *(H. Brézis & J. L. Lions ed). Pitman Research Notes in Math.* **181**. *Longman, Harlow, 179-264 (1988).*

[5] A. Bendali, J. M. Dominguez & S. Gallic. A variational approach for the vector potential formulation of the Stokes and Navier-Stokes problems in three dimensional domains, *J. Math. Anal. Appl.,* **107** *(1985), 537-560.*

[6] C. Bernardi. Méthodes d'éléments finis mixtes pour les équations de Navier-Stokes, *Thèse Univ. Paris VI (1979).*

[7] P. Clément. Approximation by finite element functions using local regularisation.,*R.A.I.R.O. Anal. Numer.,* **9** *(1975), 77-84* .

[8] M. Dauge. Problème de Neumann et de Dirichlet sur un polyèdre dans $\mathbb{R}^3$: régularité dans des espaces de Sobolev L^p,*C.R.A.S. 307,I , pp,27-32 (1988).*

[9] F. Dubois. Discrete vector potential representation of a -free vector field in three dimensional domains: numerical analysis of a model problem, *Report numéro* **163** *(1987), Centre Math. Appl. Ecole Polytechnique.*

[10] F. Dubois. Vorticity-velocity-pressure formulation for the Stokes problem, *Math. Meth. in the Applied Sciences,* **25**. *(2002), 1091-1119.*

[11] F. Dubois, M. Salaün & S. Salmon. Vorticity-velocity-pressure and stream function-vorticity formulations for the Stokes problem, *J. Math. Pures Appl.,* **82** *(2003), 1395-1451..*

[12] G. Duvaut & J.L.Lions. Inequalities in Mechanics and Physics, *Series of Comp. Studies in Math. Springer Verlag (1976).*

[13] L. Franca & T.J. Hugues. A new finite element formulation for computational fluid dynamics: VII. The Stokes Problem with various well-posed boundary conditions : symmetric formulations that converge for all velocity/pressure spaces., *Comp. Meth. in Applied Mech. and Eng.* **65** *(1987), 85-96.*

[14] V. Girault & P.A. Raviart. Finite Element Methods for Navier-Stokes Equations, *Springer Series in Comp. Math.,* **5** *(1986), Springer-Verlag* .

[15] V. Girault. Curl-conforming finite element methods for Navier-Stokes equations with non-standard boundary conditions in $\mathbb{R}^3$, *Publication du laboratoire Jacques Louis Lions, Université Pierre et Marie Curie.*

[16] V. Girault. Incompressible finite element methods for Navier-Stokes equations with nonstandard boundary conditions in $\mathbb{R}^3$, *MAth. of Comp.* **51** *(1988), 55-74* .

[17] P. Grisvard. Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain. Numerical Solution of Partial Differential Equation III. Synspade. (B. Hubbard *ed*). Acadenic Press. (1975).

[18] F. Hecht & O. Pironneau. FreeFem++, *see: http://www.freefem.org.*

[19] J. C. Nedelec. Mixed finite element in $\mathbb{R}^3$. *Numer. Math.,* **35** *(1980), 315-341.*

[20] J. C. Nedelec Eléments finis mixtes incompressibles pour l'équation de Stokes dans $\mathbb{R}^3$. *Numer. Math.,* **39** *(1982), 97-112.*

[21] J.T. Oden, W. Wu & M. Ainthworth An a posteriori eror estimate for finite element approximations of the Navier-Stokes equations, *Comput. Methods Appl. Mech. Engrg.,* **111** *(1994), 185-202.*

[22] C. Padra A posteriori error estimators for nonconforming approximation of some quasi-newtonian flows, *SIAM J. Numer. Anal.,* **34** *(1997), 1600-1615.*

[23] P.-A. Raviart & J.-M. Thomas A mixed finite element method for second order elliptic problems,*Mathematical Aspects of Finite Element Methods. Springer, Berlin. Lect. Notes Math.,* **606** *(1977), 292-315* .

[24] S. Repin A posteriori error estimates for approximate solutions of variational problems , *Proc. 2nd European Conference on Numerical Mathematics and Applications, Heidelberg, Sept. 25-Oct. 3, 1997.*

[25] S. Repin A posteriori estimates for the Stokes problem, *J. Math. Sci. (New York),* **109** *(2002), 1950-1964.*

[26] S. Repin Local a posteriori estimates for the Stokes problem, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI),3***18** *(2004), 233-245.*

[27] S. Repin and R. Stenberg A posteriori error estimates for the generalized Stokes problem, *J. Math. Sci. (New York),* **142** *(2007), 1828-1843.*

[28] S. Salmon Développement numérique de la formulation tourbillon-vitesse-pression pour le problème de Stokes, *Thesis, Université Pierre et Marie Curie, Paris (1999).*

[29] R. R. Temam , Theory and Numerical Analysis of the Navier-Stokes Equations, *North-Holland (1977).*

[30] R. Verfürth Mixed finite element approximation of the vector potential , *Numer. Math.,* **50** *(1987), 685-695.*

[31] R. Verfürth A posteriori eror estimators for the Stokes equations,*Numer. Math.,* **55** *(1989), 309-326.*