DECORRELATION ESTIMATES FOR THE EIGENLEVELS OF THE DISCRETE ANDERSON MODEL IN THE LOCALIZED REGIME

FRÉDÉRIC KLOPP

ABSTRACT. The purpose of the present work is to establish decorrelation estimates for the eigenvalues of the discrete Anderson model localized near two distinct energies inside the localization region. In dimension one, we prove these estimates at all energies. In higher dimensions, the energies are required to be sufficiently far apart from each other. As a consequence of these decorrelation estimates, we obtain the independence of the limits of the local level statistics at two distinct energies.

RÉSUMÉ. Dans ce travail, nous établissons des inégalités de décorrélation pour les valeurs propres proches de deux énergies distinctes. En dimension 1, nous démontrons que ces inégalités sont vraies quel que soit le choix de ces deux énergies. En dimension supérieure, il nous faut supposer que les deux énergies sont suffisamment éloignées l'une de l'autre. Comme conséquence de ces inégalités de décorrélation, nous démontrons que les limites des statistiques locales des valeurs propres sont indépendantes pour deux énergies distinctes.

1. Introduction

On $\ell^2(\mathbb{Z}^d)$, consider the random Anderson model

$$H_\omega = -\Delta + V_\omega$$

where $-\Delta$ is the free discrete Laplace operator

$$(-\Delta u)_n = \sum_{|m-n|=1} u_m \quad \text{for } u = (u_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d) \tag{1.1}$$

and V_ω is the random potential

$$(V_\omega u)_n = \omega_n u_n \quad \text{for } u = (u_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d). \tag{1.2}$$

We assume that the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are independent identically distributed and that their common distribution admits a compactly supported bounded density, say g.
It is then well known (see e.g. [12]) that

- let $\Sigma := [-2d, 2d] + \operatorname{supp} g$ and S_- and S_+ be the infimum and supremum of Σ; for almost every $\omega = (\omega_n)_{n\in\mathbb{Z}^d}$, the spectrum of H_ω is equal to Σ;
- for some $S_- < s_- \leq s_+ < S_+$, the intervals $I_- = [S_-, s_-)$ and $I_+ = (s_+, S_+]$ are contained in the region of localization for H_ω i.e. the region of Σ where the finite volume fractional moment criteria of [1] are verified for restrictions of H_ω to sufficiently large cubes (see also Proposition 2.1). In particular, $I := I_- \cup I_+$ contains only pure point spectrum associated to exponentially decaying eigenfunctions; for the precise meaning of the region of localization, we refer to section 2.1.2; if the disorder is sufficiently large or if the dimension $d = 1$ then, one can pick $I = \Sigma$;

2000 *Mathematics Subject Classification.* 81Q10,47B80,60H25,82D30.
Key words and phrases. random Schrödinger operators, renormalized local eigenvalues, decorrelation estimates.
The author is supported by the grant ANR-08-BLAN-0261-01.

- there exists a bounded density of states, say $\lambda \mapsto \nu(E)$, such that, for any continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, one has

$$\int_{\mathbb{R}} \varphi(E)\nu(E)dE = \mathbb{E}(\langle \delta_0, \varphi(H_\omega)\delta_0\rangle). \tag{1.3}$$

  Here, and in the sequel, $\mathbb{E}(\cdot)$ denotes the expectation with respect to the random parameters, and $\mathbb{P}(\cdot)$ the probability measure they induce.
  Let N be the integrated density of states of H_ω i.e. N is the distribution function of the measure $\nu(E)dE$. The function ν is only defined E-almost everywhere. In the sequel, when we speak of $\nu(E)$ for some E, we mean that the non decreasing function N is differentiable at E and that $\nu(E)$ is its derivative at E.

1.1. **The results.** For $L \in \mathbb{N}$, let $\Lambda = \Lambda_L = [-L, L]^d$ be a large box and $N := \#\Lambda_L = (2L+1)^d$ be its cardinality. Let $H_\omega(\Lambda)$ be the operator H_ω restricted to Λ with periodic boundary conditions. The notation $|\Lambda| \to +\infty$ is a shorthand for considering $\Lambda = \Lambda_L$ in the limit $L \to +\infty$. Let us denote the eigenvalues of $H_\omega(\Lambda)$ ordered increasingly and repeated according to multiplicity by $E_1(\omega,\Lambda) \le E_2(\omega,\Lambda) \le \cdots \le E_N(\omega,\Lambda)$.

Let E be an energy in I such that $\nu(E) > 0$. The local level statistics near E is the point process defined by

$$\Xi(\xi, E, \omega, \Lambda) = \sum_{n=1}^{N} \delta_{\xi_n(E,\omega,\Lambda)}(\xi) \tag{1.4}$$

where

$$\xi_n(E,\omega,\Lambda) = |\Lambda|\, \nu(E)\, (E_n(\omega,\Lambda) - E), \quad 1 \le n \le N. \tag{1.5}$$

One of the most striking results describing the localization regime for the Anderson model is

Theorem 1.1 ([15]). *Assume that $E \in I$ be such that $\nu(E) > 0$.*
When $|\Lambda| \to +\infty$, the point process $\Xi(\cdot, E, \omega, \Lambda)$ converges weakly to a Poisson process on $\mathbb{R}$ with intensity the Lebesgue measure i.e. for $(U_j)_{1\le j\le J}$, $U_j \subset \mathbb{R}$ bounded measurable and $U_{j'} \cap U_j = \emptyset$ if $j \ne j'$ and $(k_j)_{1\le j\le J} \in \mathbb{N}^J$, one has

$$\mathbb{P}\left(\left\{\omega;\ \left\{\begin{array}{ccc} \#\{j; \xi_n(E,\omega,\Lambda) \in U_1\} = k_1 \\ \vdots & \vdots \\ \#\{j; \xi_n(E,\omega,\Lambda) \in U_J\} = k_J \end{array}\right.\right\}\right) \underset{\Lambda\to\mathbb{Z}^d}{\to} \prod_{j=1}^{J} e^{-|U_j|}\frac{|U_j|^{k_j}}{k_j!}.$$

An analogue of Theorem 1.1 was first proved in [17] for a different one-dimensional random operator.
Once Theorem 1.1 is known, a natural question arises:

- for $E \ne E'$, are the limits of $\Xi(\xi, E, \omega, \Lambda)$ and $\Xi(\xi, E', \omega, \Lambda)$ stochastically independent?

This question has arisen and has been answered for other types of random operators like random matrices (see e.g. [14]); in this case, the local statistics are not Poissonian.
For the Anderson model, this question has been open (see e.g. [16, 19]) and to the best of our knowledge, the present paper is the first to bring an answer. The conjecture is also open for the continuous Anderson model and random CMV matrices where the local statistics have also been proved to be Poissonian (see e.g. [4, 7, 19, 20]).
The main result of the present paper is

Theorem 1.2. *Assume that the dimension $d = 1$. Pick $E \in I$ and $E' \in I$ such that $E \neq E'$, $\nu(E) > 0$ and $\nu(E') > 0$.*
When $|\Lambda| \to +\infty$, the point processes $\Xi(E,\omega,\Lambda)$ and $\Xi(E',\omega,\Lambda)$, defined in (1.4), converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity the Lebesgue measure. That is, for $(U_j)_{1\leq j\leq J}$, $U_j \subset \mathbb{R}$ bounded measurable and $U_{j'} \cap U_j = \emptyset$ if $j \neq j'$ and $(k_j)_{1\leq j\leq J} \in \mathbb{N}^J$ and $(U'_j)_{1\leq j\leq J'}$, $U'_j \subset \mathbb{R}$ bounded measurable and $U'_{j'} \cap U'_j = \emptyset$ if $j \neq j'$ and $(k'_j)_{1\leq j\leq J} \in \mathbb{N}^{J'}$ one has

$$\mathbb{P}\left(\left\{\omega;\begin{array}{ccc}\#\{j;\xi_n(E,\omega,\Lambda)\in U_1\} = k_1\\ \vdots & \vdots\\ \#\{j;\xi_n(E,\omega,\Lambda)\in U_J\} = k_J\\ \#\{j;\xi_n(E',\omega,\Lambda)\in U'_1\} = k'_1\\ \vdots & \vdots\\ \#\{j;\xi_n(E',\omega,\Lambda)\in U_{J'}\} = k_{J'}\end{array}\right\}\right) \underset{\Lambda\to\mathbb{Z}^d}{\to} \prod_{j=1}^{J} e^{-|U_j|}\frac{|U_j|^{k_j}}{k_j!}\cdot\prod_{j=1}^{J'} e^{-|U_{j'}|}\frac{|U_{j'}|^{k_{j'}}}{k_{j'}!}. \tag{1.6}$$

When $d \geq 2$, we also prove

Theorem 1.3. *Assume that d is arbitrary. Pick $E \in I$ and $E' \in I$ such that $|E - E'| > 2d$, $\nu(E) > 0$ and $\nu(E') > 0$.*
When $|\Lambda| \to +\infty$, the point processes $\Xi(E,\omega,\Lambda)$ and $\Xi(E',\omega,\Lambda)$, defined in (1.4), converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity the Lebesgue measure.

In section 3, we show that Theorems 1.2 and 1.3 follow from Theorem 1.1 and the decorrelation estimates that we present now. They are the main technical results of the present paper.

Lemma 1.1. *Assume $d = 1$ and pick $\beta \in (1/2, 1)$. For $\alpha \in (0,1)$ and $\{E, E'\} \subset I$ s.t. $E \neq E'$, for any $c > 0$, there exists $C > 0$ such that, for $L \geq 3$ and $cL^\alpha \leq \ell \leq L^\alpha/c$, one has*

$$\mathbb{P}\left(\begin{Bmatrix}\sigma(H_\omega(\Lambda_\ell))\cap(E+L^{-d}(-1,1))\neq\emptyset,\\ \sigma(H_\omega(\Lambda_\ell))\cap(E'+L^{-d}(-1,1))\neq\emptyset\end{Bmatrix}\right) \leq C(\ell/L)^{2d}e^{(\log L)^\beta}. \tag{1.7}$$

This lemma shows that, up to sub-polynomial errors, the probability to obtain simultaneously an eigenvalue near E and another one near E' is bounded by the product of the estimates given for each of these events by Wegner's estimate (see section 2.1.1). In this sense, (1.7) is similar to Minami's estimate for two distinct energies.
Lemma 1.1 proves a result conjectured in [16, 19] in dimension 1.

In arbitrary dimension, we prove (1.7), actually a somewhat stronger estimate, only when the two energies E and E' are sufficiently far apart.

Lemma 1.2. *Assume d is arbitrary. Pick $\beta \in (1/2, 1)$. For $\alpha \in (0,1)$ and $\{E, E'\} \subset I$ s.t. $|E - E'| > 2d$, for any $c > 0$, there exists $C > 0$ such that, for $L \geq 3$ and $cL^\alpha \leq \ell \leq L^\alpha/c$, one has*

$$\mathbb{P}\left(\begin{Bmatrix}\sigma(H_\omega(\Lambda_\ell))\cap(E+L^{-d}(-1,1))\neq\emptyset,\\ \sigma(H_\omega(\Lambda_\ell))\cap(E'+L^{-d}(-1,1))\neq\emptyset\end{Bmatrix}\right) \leq C(\ell/L)^{2d}(\log L)^C. \tag{1.8}$$

This e.g. proves the independence of the processes for energies in opposite edges of the almost sure spectrum.
The estimate (1.8) in Lemma 1.2 is somewhat stronger than (1.7); one can obtain an analogous estimate in dimension 1 if one restricts oneself to energies E and E' such that $E - E'$ does not belong to some set of measure 0 (see Lemma 2.11 in Remark 2.2 at the end of section 2.3).

Remark 1.1. As the proof of Theorems 1.2 and 1.3 shows, the estimates (1.7) are (1.8) are stronger than what it needed. It suffices to show that the probabilities in (1.7) are (1.8) are $o((\ell/L)^d)$.
In [7] (see also [8]), the authors provide another proof of Theorems 1.1 and of Theorems 1.2 and 1.3 under the assumption that the probabilities in (1.7) are (1.8) are $o((\ell/L)^d)$. The analysis done in [8] deals with both discrete and continuous models. It yields a stronger version of Theorem 1.1 and Theorems 1.2 and 1.3 in essentially the same step.
Whereas in the proof of Lemma 1.1, we explicitly use the fact that $H_\omega = H_0 + V_\omega$ where H_0 is the free Laplace operator (1.1), the proof we give of Lemma 1.2 still works if H_0 is any convolution matrix with exponentially decaying off diagonal coefficients if one replaces the condition $|E - E'| > 2d$ with the condition $|E - E'| > \sup \sigma(H_0) - \inf \sigma(H_0)$.

2. Proof of the decorrelation estimates

Before starting with the proofs of Lemma 1.1 and 1.2, let us recall additional properties for the discrete Anderson model known to be true under the assumptions we made on the distribution of the random potential.

2.1. Some facts on the discrete Anderson model.

Basic estimates on the distribution of the eigenvalues of the Anderson model are the Wegner and Minami estimates.

2.1.1. *The Wegner and Minami estimates.* One has

Theorem 2.1 ([22]). *There exists $C > 0$ such that, for $J \subset \mathbb{R}$, and Λ, a cube in $\mathbb{Z}^d$, one has*

$$\mathbb{E}\left[tr(\mathbf{1}_J(H_\omega(\Lambda)))\right] \leq C|J|\,|\Lambda| \tag{2.1}$$

where

- $H_\omega(\Lambda))$ *is the operator* H_ω *restricted to* Λ *with periodic boundary conditions,*
- $\mathbf{1}_J(H)$ *is the spectral projector of the operator* H *on the energy interval* J.

We refer to [10, 13, 21] for simple proofs and more details on the Wegner estimate.
Another crucial estimate is the Minami estimate.

Theorem 2.2 ([15, 2, 9, 5]). *There exists $C > 0$ such that, for $J \subset K$, and Λ, a cube in $\mathbb{Z}^d$, one has*

$$\mathbb{E}\left[tr(\mathbf{1}_J(H_\omega(\Lambda))) \cdot tr(\mathbf{1}_K(H_\omega(\Lambda)) - 1)\right] \leq C|J|\,|K|\,|\Lambda|^2. \tag{2.2}$$

For $J = K$, the estimate (2.2) was proved in [15, 2, 9, 5]; for $J \neq K$, it can be found in [5]. In their nature, (1.7) or (1.8) and (2.2) are quite similar: the Minami estimate can be interpreted as a decorrelation estimate for close together eigenvalues. It can be used to obtain the counterparts of Theorems 1.2 and 1.3 when E and E' tend to each other as $|\Lambda| \to +\infty$ (see [7]).

Note that the Minami estimate (2.2) has been proved for the discrete Anderson model on intervals I irrelevant of the spectral type of H_ω in I. Our proof of the decorrelation estimates (1.7) and (1.8) makes use of the fact that I lies in the localized region.

2.1.2. *The localized regime.* Let us now give a precise description of what we mean with the region of localization or the localized regime. We prove

Proposition 2.1. *Recall that $I = I_+ \cup I_-$ is the region of Σ where the finite volume fractional moment criteria of* [1] *for $H_\omega(\Lambda)$ are verified for Λ sufficiently large.*
Then,

(Loc): *there exists $\nu > 0$ such that, for any $p > 0$, there exists $q > 0$ and $L_0 > 0$ such that, for $L \geq L_0$, with probability larger than $1 - L^{-p}$, if*

(1) *$\varphi_{n,\omega}$ is a normalized eigenvector of $H_\omega(\Lambda_L)$ associated to an energy $E_{n,\omega} \in I$,*

(2) *$x_{n,\omega} \in \Lambda_L$ is a maximum of $x \mapsto |\varphi_{n,\omega}(x)|$ in Λ_L,*

then, for $x \in \Lambda_L$, one has

$$|\varphi_{n,\omega}(x)| \leq L^q e^{-\nu|x - x_{n,\omega}|}. \tag{2.3}$$

The point $x_{n,\omega}$ is called a localization center for $\varphi_{n,\omega}$ or $E_{n,\omega}$.

Note that, by Minami's estimate, the eigenvalues of $H_\omega(\Lambda)$ are almost surely simple. Thus, we can associate a localization center to an eigenvalue as it is done in Proposition 2.1.
In its spirit, this result is not new (see e.g. [1, 6, 7]). We state it in a form convenient for our purpose. We prove Proposition 2.1 in section 4

2.2. **The proof of Lemmas 1.1 and 1.2.** The basic idea of the proof is to show that, when ω varies, two eigenvalues of $H_\omega(\Lambda)$ cannot vary in a synchronous manner, or, put in another way, locally in ω, if $E(\omega)$ and $E'(\omega)$ denote the two eigenvalues under consideration, for some γ and γ', the mapping $(\omega_\gamma, \omega_{\gamma'}) \mapsto (E(\omega), E'(\omega))$ is a local diffeomorphism when all the other random variables, that is $(\omega_\alpha)_{\alpha \notin \{\gamma,\gamma'\}}$, are fixed.
As we are in the localized regime, we will exploit this by noting that eigenvalues of $H_\omega(\Lambda)$ can only depend significantly of $(\log L)^d$ random variables i.e. we can study what happens in cubes that are of side-length $\log L$ while the energy interval where we want to control things are of size L^{-d}. This is the essence of Lemma 2.1 below. This lemma is proved under the general assumptions (2.1), (2.2) and (Loc). In particular, it is valid for if one replaces the discrete Laplacian with any convolution matrix with exponentially decaying off diagonal coefficients.
The second step consists in analyzing the mapping $(\omega_\gamma, \omega_{\gamma'}) \mapsto (E(\omega), E'(\omega))$ on these smaller cubes. The main technical result is Lemma 2.4 that shows that, under the conditions of Lemmas 1.1 and 1.2, with a large probability, eigenvalues away from each other cannot move synchronously as functions of the random variables. Of course, this will not be correct for all random models: constructing artificial degeneracies, one can easily coin up random models where this is not the case.
Lemmas 1.1 and 1.2 will be proved in essentially the same way; the only difference will be in Lemma 2.4 that controls the joint dependence of two distinct eigenvalues on the random variables.

Let $J_L = E + L^{-d}[-1,1]$ and $J'_L = E' + L^{-d}[-1,1]$. Pick L sufficiently large so that $J_L \subset I$ and $J'_L \subset I$ are contained in I where (Loc) holds true.
Pick $cL^\alpha \leq \ell \leq L^\alpha / c$ where $c > 0$ is fixed. By (2.2), we know that

$$\mathbb{P}\left(\#[\sigma(H_\omega(\Lambda_\ell)) \cap J_L] \geq 2 \text{ or } \#[\sigma(H_\omega(\Lambda_\ell)) \cap J'_L] \geq 2\right) \leq C(\ell/L)^{2d}$$

where $\#[\cdot]$ denotes the cardinality of $\cdot$.
So if we define

$$\mathbb{P}_0 = \mathbb{P}\left(\#[\sigma(H_\omega(\Lambda_\ell)) \cap J_L] = 1, \#[\sigma(H_\omega(\Lambda_\ell)) \cap J'_L] = 1\right),$$

it suffices to show that

$$\mathbb{P}_0 \leq C(\ell/L)^{2d} \cdot \begin{cases} e^{(\log L)^\beta} & \text{if the dimension } d = 1, \\ (\log L)^C & \text{if the dimension } d > 1. \end{cases} \tag{2.4}$$

First, using the assumption (Loc), we are going to reduce the proof of (2.4) to the proof of a similar estimate where the cube Λ_ℓ will be replaced by a much smaller cube, a cube of side length of order $\log L$. We prove

Lemma 2.1. *There exists $C>0$ such that, for L sufficiently large,*

$$\mathbb{P}_0 \le C(\ell/L)^{2d} + C(\ell/\tilde\ell)^d\,\mathbb{P}_1$$

where $\tilde\ell = C\log L$ and

$$\mathbb{P}_1 := \mathbb{P}(\#[\sigma(H_\omega(\Lambda_{\tilde\ell}))\cap \tilde J_L]\ge 1) \text{ and } \#[\sigma(H_\omega(\Lambda_{\tilde\ell}))\cap \tilde J_L']\ge 1)$$

where $\tilde J_L = E + L^{-d}(-2,2)$ and $\tilde J_L' = E' + L^{-d}(-2,2)$.

Proof of Lemma 2.1. Fix $C>0$ large so that $e^{-C\gamma\log L/2}\le L^{-2d-q}$ where q and γ are given by assumption (Loc) where we choose $p=d$. Let Ω_0 be the set of probability $1-L^{-p}$ where (1) and (2) in assumption (Loc) are satisfied. Define $\tilde\ell = C\log L$. We prove

Lemma 2.2. *There exists a covering of Λ_ℓ by cubes, say $\Lambda_\ell = \cup_{\gamma\in\Gamma}[\gamma+\Lambda_{\tilde\ell}]$, such that $\#\Gamma \asymp (\ell/\tilde\ell)^d$, and, if $\omega\in\Omega_0$ is such that $H_\omega(\Lambda_\ell)$ has exactly one eigenvalue in J_L and exactly one eigenvalue in J_L', then*

(1) *either, there exists γ and γ' such that $\gamma+\Lambda_{\tilde\ell}\cap\gamma'+\Lambda_{\tilde\ell}=\emptyset$ and*
- *$H_\omega(\gamma+\Lambda_{\tilde\ell})$ has exactly one e.v. in $\tilde J_L$*
- *$H_\omega(\gamma'+\Lambda_{\tilde\ell})$ has exactly one e.v. in $\tilde J_L'$.*

(2) *or $H_\omega(\Lambda_{5\tilde\ell}(\gamma))$ has exactly one e.v. in $\tilde J_L$ and exactly one e.v. in $\tilde J_L'$.*

We postpone the proof of Lemma 2.2 to complete that of Lemma 2.1. Using the estimate on $\mathbb{P}(\Omega_0)$, the independence of $H_\omega(\gamma+\Lambda_{\tilde\ell})$ and $H_\omega(\gamma'+\Lambda_{\tilde\ell})$ when alternative (1) is the case in Lemma 2.2, Wegner's estimate (2.1) and the fact the random variables are identically distributed, we compute

$$\begin{aligned}\mathbb{P}_0 \le L^{-2d} + C(\ell/\tilde\ell)^d\mathbb{P}&\left(\begin{Bmatrix}\sigma(H_\omega(\Lambda_{3\tilde\ell}(0)))\cap\tilde J_L\neq\emptyset\\ \sigma(H_\omega(\Lambda_{3\tilde\ell}(0)))\cap\tilde J_L'\neq\emptyset\end{Bmatrix}\right)\\ &+C(\ell/\tilde\ell)^{2d}\mathbb{P}(\#[\sigma(H_\omega(\Lambda_{\tilde\ell}(0)))\cap\tilde J_L]\ge1)\mathbb{P}(\#[\sigma(H_\omega(\Lambda_{\tilde\ell}(0)))\cap\tilde J_L']\ge1)\\ \le CL^{-2d}+C(\ell/\tilde\ell)^{2d}(\tilde\ell/L)^{2d}&+C(\ell/\tilde\ell)^d\,\mathbb{P}_1\le C(\ell/L)^{2d}+C(\ell/\tilde\ell)^d\,\mathbb{P}_1\end{aligned}$$

where $\mathbb{P}_1$ is defined in Lemma 2.1 for $5\tilde\ell$ replaced with $\tilde\ell$. This completes the proof of Lemma 2.1. □

Proof of Lemma 2.2. For $\gamma\in\tilde\ell\mathbb{Z}^d\cap\Lambda_\ell$, consider the cubes $(\gamma+\Lambda_{\tilde\ell})_{\gamma\in\tilde\ell\mathbb{Z}^d\cap\Lambda_\ell}$. They cover Λ_ℓ. Recall that we are taking periodic boundary conditions. If the localization centers associated to the two eigenvalues of $H_\omega(\Lambda_\ell)$ assumed to be respectively in $\tilde J_L$ and $\tilde J_L'$ are at a distance less than $3\tilde\ell$ from one another, then we can find $\gamma\in\tilde\ell\mathbb{Z}^d$ such that both localization centers belong $\gamma+\Lambda_{4\tilde\ell}$ (for $\tilde\ell = C\log L$ and $C>0$ sufficiently large). Thus, by the localization property (Loc), we are in case (2).
If the distance is larger than $3\tilde\ell$, we can find $\gamma\in\tilde\ell\mathbb{Z}^d$ and $\gamma'\in\tilde\ell\mathbb{Z}^d$ such that each of the cubes $\gamma+\Lambda_{\tilde\ell/2}$ and $\gamma'+\Lambda_{\tilde\ell/2}$ contains exactly one of the localization centers and $(\gamma+\Lambda_{\tilde\ell/2})\cap(\gamma'+\Lambda_{\tilde\ell/2})=\emptyset$. So for $\tilde\ell = C\,\log L$ and $C>0$ sufficiently large, by the localization property (Loc), we are in case (1).
This completes the proof of Lemma 2.2. □

We now proceed with the proof of (2.4). Therefore, by Lemma 2.1, it suffices to prove that $\mathbb{P}_1$, defined in Lemma 2.1, satisfies, for some $C>0$,

(2.5) $$\mathbb{P}_1 \leq C(\tilde\ell/L)^{2d}\cdot\begin{cases} e^{\tilde\ell^\beta} & \text{if the dimension } d=1,\\ \tilde\ell^C & \text{if the dimension } d>1.\end{cases}$$

Let $(E_j(\omega,\tilde\ell))_{1\leq j\leq(2\tilde\ell+1)^d}$ be the eigenvalues of $H_\omega(\Lambda_{\tilde\ell})$ ordered in an increasing way and repeated according to multiplicity.
Assume that $\omega\mapsto E(\omega)$ is the only eigenvalue of $H_\omega(\Lambda_{\tilde\ell})$ in J_L. In this case, by standard perturbation theory arguments (see e.g. [11, 18]), we know that

(1) $E(\omega)$ being simple, $\omega\mapsto E(\omega)$ is real analytic, and if $\omega\mapsto\varphi(\omega)=(\varphi(\omega;\gamma))_{\gamma\in\Lambda_{\tilde\ell}}$ denotes the associated normalized real eigenvector, it is also real analytic in ω;
(2) one has $\partial_{\omega_\gamma}E(\omega)=\varphi^2(\omega;\gamma)\geq 0$ which, in particular, implies that

(2.6) $$\|\nabla_\omega E(\omega)\|_{\ell^1}=1;$$

(3) the Hessian of E is given by $\mathrm{Hess}_\omega E(\omega)=((h_{\gamma\beta}))_{\gamma,\beta}$ where
   - $h_{\gamma,\beta}=-2\mathrm{Re}\langle(H_\omega(\Lambda_{\tilde\ell})-E(\omega))^{-1}\psi_\gamma(\omega),\psi_\beta(\omega)\rangle$,
   - $\psi_\gamma=\varphi(\omega;\gamma)\Pi(\omega)\delta_\gamma$
   - $\Pi(\omega)$ is the orthogonal projector on the orthogonal to $\varphi(\omega)$.

We prove

Lemma 2.3. *There exists $C>0$ such that*
$$\|Hess_\omega(E(\omega))\|_{\ell^\infty\to\ell^1}\leq\frac{C}{\mathrm{dist}\,(E(\omega),\sigma(H_\omega(\Lambda_{\tilde\ell}))\setminus\{E(\omega)\})}.$$

Proof of Lemma 2.3. First, note that, by definition, $H_\omega(\Lambda_{\tilde\ell})$ depends on $(2\tilde\ell+1)^d$ random variables so that $\mathrm{Hess}_\omega E(\omega)$ is a $(2\tilde\ell+1)^d\times(2\tilde\ell+1)^d$ matrix. Hence, for $a=(a_\gamma)_{\gamma\in\Lambda_{\tilde\ell}}\in\mathbb{C}^{\Lambda_{\tilde\ell}}$ and $b=(b_\gamma)_{\gamma\in\Lambda_{\tilde\ell}}\in\mathbb{C}^{\Lambda_{\tilde\ell}}$, we compute
$$\langle\mathrm{Hess}_\omega E\,a,b\rangle=-2\langle(H_\omega(\Lambda_{\tilde\ell})-E(\omega))^{-1}\psi_a,\psi_b\rangle$$
where
$$\psi_a=\Pi(\omega)\left(\sum_{\gamma\in\Lambda_{\tilde\ell}}a_\gamma|\delta_\gamma\rangle\langle\delta_\gamma|\right)\varphi(\omega)=\sum_{\gamma\in\Lambda_{\tilde\ell}}a_\gamma\varphi(\omega;\gamma)\Pi(\omega)\delta_\gamma.$$
Hence, $\|\psi_a\|_2\leq C\|a\|_\infty$ and, for some $C>0$,
$$\|\mathrm{Hess}_\omega(E(\omega))\|_{\ell^\infty\to\ell^1}\leq\frac{C}{\mathrm{dist}\,(E(\omega),\sigma(H_\omega(\Lambda_{\tilde\ell}))\setminus\{E(\omega)\})}.$$
This completes the proof of Lemma 2.3. □

Note that, using (2.2), Lemma 2.3 yields, for $\varepsilon\in(4L^{-d},1)$,
$$\mathbb{P}\left(\left\{\omega;\ \begin{matrix}\sigma(H_\omega(\Lambda_{\tilde\ell}))\cap\tilde J_L=\{E(\omega)\}\\ \|\mathrm{Hess}_\omega(E(\omega))\|_{\ell^\infty\to\ell^1}\geq\varepsilon^{-1}\end{matrix}\right\}\right)\leq C\varepsilon\tilde\ell^{2d}L^{-d}.$$
Hence, for $\varepsilon\in(4L^{-d},1)$, one has

(2.7) $$\mathbb{P}_1\leq C\varepsilon\tilde\ell^{2d}L^{-d}+\mathbb{P}_\varepsilon$$

where

(2.8) $$\mathbb{P}_\varepsilon=\mathbb{P}(\Omega_0(\varepsilon))$$

and

$$\Omega_0(\varepsilon)=\left\{\omega;\ \begin{matrix}\sigma(H_\omega(\Lambda_{\tilde\ell}))\cap\tilde J_L=\{E(\omega)\}\\ \{E(\omega)\}=\sigma(H_\omega(\Lambda_{\tilde\ell}))\cap(E-C\varepsilon,E+C\varepsilon),\\ \sigma(H_\omega(\Lambda_{\tilde\ell}))\cap\tilde J'_L=\{E'(\omega)\}\\ \{E'(\omega)\}=\sigma(H_\omega(\Lambda_{\tilde\ell}))\cap(E'-C\varepsilon,E'+C\varepsilon)\end{matrix}\right\} \tag{2.9}$$

We will now estimate $\mathbb{P}_\varepsilon$. The basic idea is to prove that the eigenvalues $E(\omega)$ and $E'(\omega)$ depend effectively on at least two independent random variables. A simple way to guarantee this is to ensure that their gradients with respect to ω are not co-linear. In the present case, the gradients have non negative components and their ℓ^1-norm is 1; hence, it suffices to prove that they are different to ensure that they are not co-linear.
We prove

Lemma 2.4. *Let $L\geq 1$. For the discrete Anderson model, one has*

(1) *in any dimension d: for $\Delta E>2d$, if the random variables $(\omega_\gamma)_{\gamma\in\Lambda_L}$ are bounded by K, for $E_j(\omega)$ and $E_k(\omega)$, simple eigenvalues of $H_\omega(\Lambda_L)$ such that $|E_k(\omega)-E_j(\omega)|\geq\Delta E$, one has*

$$\|\nabla_\omega(E_j(\omega)-E_k(\omega))\|_2\geq\frac{\Delta E-2d}{K}(2L+1)^{-d/2}; \tag{2.10}$$

(2) *in dimension 1: fix $E<E'$ and $\beta>1/2$; let $\mathbb{P}$ denote the probability that there exists $E_j(\omega)$ and $E_k(\omega)$, simple eigenvalues of $H_\omega(\Lambda_L)$ such that $|E_k(\omega)-E|+|E_j(\omega)-E'|\leq e^{-L^\beta}$ and such that*

$$\|\nabla_\omega(E_j(\omega)-E_k(\omega))\|_1\leq e^{-L^\beta}; \tag{2.11}$$

then, there exists $c>0$ such that

$$\mathbb{P}\leq e^{-cL^{2\beta}}. \tag{2.12}$$

We postpone the proof of Lemma 2.4 for a while to estimate $\mathbb{P}_\varepsilon$. Set

$$\lambda=\lambda_L=\begin{cases}e^{-\tilde\ell^\beta} & \text{if the dimension } d=1,\\ \frac{\Delta E-2d}{K}\tilde\ell^{-d/2} & \text{if the dimension } d>1.\end{cases} \tag{2.13}$$

For γ and γ' in $\Lambda_{\tilde\ell}$, define

$$\Omega_{0,\beta}^{\gamma,\gamma'}(\varepsilon)=\Omega_0(\varepsilon)\cap\left\{\omega;\ |J_{\gamma,\gamma'}(E(\omega),E'(\omega))|\geq\lambda\right\} \tag{2.14}$$

where $J_{\gamma,\gamma'}(E(\omega),E'(\omega))$ is the Jacobian of the mapping $(\omega_\gamma,\omega_{\gamma'})\mapsto(E(\omega),E'(\omega))$ i.e.

$$J_{\gamma,\gamma'}(E(\omega),E'(\omega))=\begin{vmatrix}\partial_{\omega_\gamma}E(\omega) & \partial_{\omega_{\gamma'}}E(\omega)\\ \partial_{\omega_\gamma}E'(\omega) & \partial_{\omega_{\gamma'}}E'(\omega)\end{vmatrix}.$$

In section 2.4, we prove

Lemma 2.5. *Pick $(u,v)\in(\mathbb{R}^+)^{2n}$ such that $\|u\|_1=\|v\|_1=1$. Then*

$$\max_{j\neq k}\begin{vmatrix}u_j & u_k\\ v_j & v_k\end{vmatrix}^2\geq\frac{1}{4n^5}\|u-v\|_1^2.$$

We apply Lemma 2.4 with $L=\tilde\ell$ and Lemma 2.5 to obtain that

$$\mathbb{P}_\varepsilon\leq\sum_{\gamma\neq\gamma'}\mathbb{P}(\Omega_{0,\beta}^{\gamma,\gamma'}(\varepsilon))+\mathbb{P}_r \tag{2.15}$$

where

(1) in dimension 1, we have $\mathbb{P}_r \leq C\tilde{\ell}^{2d}e^{-c\tilde{\ell}^{2\beta'}}$ for any $1/2 < \beta' < \beta$; thus, for L sufficiently large, as $\tilde{\ell} \geq c\log L$ and $\beta > 1/2$, we have

$$\mathbb{P}_r \leq L^{-2d}. \tag{2.16}$$

(2) in dimension d, as by assumption $\Delta E > 2d$, one has $\mathbb{P}_r = 0$, thus, (2.16) still holds.

In the sequel, we will write $\omega = (\omega_\gamma, \omega_{\gamma'}, \omega_{\gamma,\gamma'})$ where $\omega_{\gamma,\gamma'} = (\omega_\beta)_{\beta\notin\{\gamma,\gamma'\}}$.

To estimate $\mathbb{P}(\Omega_{0,\beta}^{\gamma,\gamma'}(\varepsilon))$, we use

Lemma 2.6. *Pick $\varepsilon = L^{-d}\lambda^{-3}$. For any $\omega_{\gamma,\gamma'}$, if there exists $(\omega^0_\gamma, \omega^0_{\gamma'}) \in \mathbb{R}^2$ such that $(\omega^0_\gamma, \omega^0_{\gamma'}, \omega_{\gamma,\gamma'}) \in \Omega_{0,\beta}^{\gamma,\gamma'}(\varepsilon)$, then, for $(\omega_\gamma, \omega_{\gamma'}) \in \mathbb{R}^2$ such that $|(\omega_\gamma, \omega_{\gamma'}) - (\omega^0_\gamma, \omega^0_{\gamma'})|_\infty \geq L^{-d}\lambda^{-2}$, one has $(E_j(\omega), E_{j'}(\omega)) \notin \tilde{J}_L \times \tilde{J}'_L$.*

Recall that g is the density of the random variables $(\omega_\gamma)_\gamma$; it is assumed to be bounded and compactly supported. Hence, the probability $\mathbb{P}(\Omega_{0,\beta}^{\gamma,\gamma'}(\varepsilon))$ is estimated as follows

$$\begin{aligned}\mathbb{P}(\Omega_{0,\beta}^{\gamma,\gamma'}(\varepsilon)) &= \mathbb{E}_{\gamma,\gamma'}\left(\int_{\mathbb{R}^2} \mathbf{1}_{\Omega_{0,\beta}^{\gamma,\gamma'}(\varepsilon)}(\omega) g(\omega_\gamma) g(\omega_{\gamma'}) d\omega_\gamma d\omega_{\gamma'}\right) \\ &\leq \mathbb{E}_{\gamma,\gamma'}\left(\int_{|(\omega_\gamma,\omega_{\gamma'})-(\omega^0_\gamma,\omega^0_{\gamma'})|_\infty < L^{-d}\lambda^{-2}} g(\omega_\gamma) g(\omega_{\gamma'}) d\omega_\gamma d\omega_{\gamma'}\right) \\ &\leq CL^{-2d}\lambda^{-4}\end{aligned} \tag{2.17}$$

where $\mathbb{E}_{\gamma,\gamma'}$ denotes the expectation with respect to all the random variables except ω_γ and $\omega_{\gamma'}$.

Summing (2.17) over $(\gamma,\gamma') \in \Lambda^2_{\tilde{\ell}}$, using (2.15) and (2.16), we obtain

$$\mathbb{P}_\varepsilon \leq CL^{-2d}\lambda^{-4}.$$

We now plug this into (2.7) and use the fact that $\varepsilon = L^{-d}\lambda^{-3}$ to complete the proof of (2.5). This completes the proofs of Lemmas 1.1 and 1.2. □

Proof of Lemma 2.6. Recall that, for any γ, $\omega_\gamma \mapsto E_j(\omega)$ and $\omega_\gamma \mapsto E_{j'}(\omega)$ are non decreasing. Hence, to prove Lemma 2.6, it suffices to prove that, for $|(\omega_\gamma, \omega_{\gamma'}) - (\omega^0_\gamma, \omega^0_{\gamma'})|_\infty = L^{-d}\lambda^{-2}$, one has $(E_j(\omega), E_{j'}(\omega)) \notin \tilde{J}_L \times \tilde{J}'_L$.

Let $\mathcal{S}_\beta$ denote the square $\mathcal{S}_\beta = \{|(\omega_\gamma, \omega_{\gamma'}) - (\omega^0_\gamma, \omega^0_{\gamma'})|_\infty \leq L^{-d}\lambda^{-2}\}$.

Recall that $\varepsilon = L^{-d}\lambda^{-3}$. Pick $\omega_{\gamma,\gamma'}$ such that there exists $(\omega^0_\gamma, \omega^0_{\gamma'}) \in \mathbb{R}^2$ for which one has $(\omega^0_\gamma, \omega^0_{\gamma'}, \omega_{\gamma,\gamma'}) \in \Omega_{0,\beta}^{\gamma,\gamma'}(\varepsilon)$. To shorten the notations, in the sequel, we write only the variables $(\omega_\gamma, \omega_{\gamma'})$ as $\omega_{\gamma,\gamma'}$ stays fixed throughout the proof; e.g. we write $E((\omega_\gamma, \omega_{\gamma'}))$ instead of $E((\omega_\gamma, \omega_{\gamma'}, \omega_{\gamma,\gamma'}))$.

Consider the mapping $(\omega_\gamma, \omega_{\gamma'}) \mapsto \varphi(\omega_\gamma, \omega_{\gamma'}) := (E(\omega), E'(\omega))$. We will show that φ defines an analytic diffeomorphism form $\mathcal{S}_\beta$ to $\varphi(\mathcal{S}_\beta)$.

By (2.14) and (2.9), the definitions of $\Omega_{0,\beta}^{\gamma,\gamma'}(\varepsilon)$ and $\Omega_0(\varepsilon)$, we know that

$$\begin{aligned}&\sigma(H_{(\omega^0_\gamma,\omega^0_{\gamma'})}(\Lambda_{\tilde{\ell}})) \cap (E - C\varepsilon, E + C\varepsilon) = \{E(\omega)\} \subset (E - CL^{-d}, E + CL^{-d}), \\ &\sigma(H_{(\omega^0_\gamma,\omega^0_{\gamma'})}(\Lambda_{\tilde{\ell}})) \cap [(E - C\varepsilon, E - C\varepsilon/2) \cup (E + C\varepsilon/2, E + C\varepsilon)] = \emptyset, \\ &\sigma(H_{(\omega^0_\gamma,\omega^0_{\gamma'})}(\Lambda_{\tilde{\ell}})) \cap (E' - C\varepsilon, E' + C\varepsilon) = \{E'(\omega)\} \subset (E' - CL^{-d}, E' + CL^{-d}), \\ &\sigma(H_{(\omega^0_\gamma,\omega^0_{\gamma'})}(\Lambda_{\tilde{\ell}})) \cap [(E' - C\varepsilon, E' - C\varepsilon/2) \cup (E' + C\varepsilon/2, E' + C\varepsilon)] = \emptyset.\end{aligned}$$

By (2.6), as $L^{-d}\lambda^{-2} \le \lambda\varepsilon$, for $(\omega_\gamma, \omega_{\gamma'}) \in \mathcal{S}_\beta$, one has

$$\begin{aligned}
&\sigma(H_\omega(\Lambda_{\tilde\ell})) \cap (E - C\varepsilon/2, E + C\varepsilon/2) = \{E(\omega)\} \subset (E - C\varepsilon/4, E + C\varepsilon/4),\\
&\sigma(H_\omega(\Lambda_{\tilde\ell})) \cap [(E - C\varepsilon/2, E - C\varepsilon/4) \cup (E + C\varepsilon/4, E + C\varepsilon/2)] = \emptyset,\\
&\sigma(H_\omega(\Lambda_{\tilde\ell})) \cap (E' - C\varepsilon/2, E' + C\varepsilon/2) = \{E(\omega)\} \subset (E' - C\varepsilon/4, E' + C\varepsilon/4),\\
&\sigma(H_\omega(\Lambda_{\tilde\ell})) \cap [(E' - C\varepsilon/2, E' - C\varepsilon/4) \cup (E' + C\varepsilon/4, E' + C\varepsilon/2)] = \emptyset.
\end{aligned}$$

Hence, by Lemma 2.3, for $(\omega_\gamma, \omega_{\gamma'}) \in \mathcal{S}_\beta$, one has

$$\|\mathrm{Hess}_\omega(E(\omega))\|_{\ell^\infty\to\ell^1} + \|\mathrm{Hess}_\omega(E'(\omega))\|_{\ell^\infty\to\ell^1} \le C\varepsilon^{-1} \le CL^d\lambda^3.$$

By (2.6) and the Fundamental Theorem of Calculus, for $(\omega_\gamma, \omega_{\gamma'}) \in \mathcal{S}_\beta$, we get that,

$$(2.18)\qquad \begin{aligned}
&\|\nabla\varphi(\omega_\gamma, \omega_{\gamma'}) - \nabla\varphi(\omega^0_\gamma, \omega^0_{\gamma'})\|\\
&\le \left(\|\mathrm{Hess}_\omega(E(\omega))\|_{\ell^\infty\to\ell^1} + \|\mathrm{Hess}_\omega(E'(\omega))\|_{\ell^\infty\to\ell^1}\right) L^{-d}\lambda^{-1} \le C\lambda^2.
\end{aligned}$$

Let us show that φ is one-to-one on the square $\mathcal{S}_\beta$. Using (2.18), we compute

$$\left|\varphi(\omega'_\gamma, \omega'_{\gamma'}) - \varphi(\omega_\gamma, \omega_{\gamma'}) - \nabla\varphi(\omega^0_\gamma, \omega^0_{\gamma'}) \cdot \begin{pmatrix} \omega'_\gamma - \omega_\gamma \\ \omega'_{\gamma'} - \omega_{\gamma'} \end{pmatrix}\right| \le \lambda^2 \left\|\begin{pmatrix} \omega'_\gamma - \omega_\gamma \\ \omega'_{\gamma'} - \omega_{\gamma'} \end{pmatrix}\right\|$$

As $(\omega^0_\gamma, \omega^0_{\gamma'}, \omega_{\gamma,\gamma'}) \in \Omega^{\gamma,\gamma'}_{0,\beta}(\varepsilon)$, we have

$$\left|\mathrm{Jac}\,\varphi(\omega^0_\gamma, \omega^0_{\gamma'})\right| \ge \lambda.$$

Hence, for $\tilde\ell$ large, we have

$$\left|\varphi(\omega'_\gamma, \omega'_{\gamma'}) - \varphi(\omega_\gamma, \omega_{\gamma'})\right| \ge \frac{1}{2}\lambda \left\|\begin{pmatrix} \omega'_\gamma - \omega_\gamma \\ \omega'_{\gamma'} - \omega_{\gamma'} \end{pmatrix}\right\|$$

so φ is one-to-one. The estimate (2.18) yields

$$|\mathrm{Jac}\,\varphi(\omega_\gamma, \omega_{\gamma'}) - \mathrm{Jac}\,\varphi(\omega^0_\gamma, \omega^0_{\gamma'})| \le \lambda^2$$

As $(\omega^0_\gamma, \omega^0_{\gamma'}, \omega_{\gamma,\gamma'}) \in \Omega^{\gamma,\gamma'}_{0,\beta}(\varepsilon)$, for L sufficiently large, this implies that

$$(2.19)\qquad \forall(\omega_\gamma, \omega_{\gamma'}) \in \mathcal{S}_\beta, \quad |J_{\gamma,\gamma'}(E(\omega), E'(\omega))| \ge \frac{1}{2}\lambda.$$

The Local Inversion Theorem then guarantees that φ is an analytic diffeomorphism from $\mathcal{S}_\beta$ onto $\varphi(\mathcal{S}_\beta)$. By (2.19), the Jacobian matrix of its inverse is bounded by $C\tilde\ell^\beta$ for some $C > 0$ independent of L. Hence, if for some $|(\omega_\gamma, \omega_{\gamma'}) - (\omega^0_\gamma, \omega^0_{\gamma'})|_\infty = L^{-d}\lambda^{-2}$, one has $(E(\omega), E'(\omega)) \in \tilde J_L \times \tilde J'_L$, then

$$L^{-d}\lambda^{-2} = |(\omega_\gamma, \omega_{\gamma'}) - (\omega^0_\gamma, \omega^0_{\gamma'})|_\infty = |\varphi^{-1}(E(\omega), E'(\omega)) - \varphi^{-1}(E, E')|_\infty \le CL^{-d}\lambda^{-1}$$

which is absurd when $L \to +\infty$ as $\lambda = \lambda_L \to 0$ (see (2.13)). This completes the proof of Lemma 2.6. □

2.3. **Proof of Lemma 2.4.** A fundamental difference between the points (1) and (2) in Lemma 2.4 is that to prove point (2), we will the fact that H_0 is the discrete Laplacian. In the proof of point (1), we can take H_0 to be any convolution matrix with exponentially decaying off diagonal coefficients if one replaces the condition $|E-E'| > 2d$ with the condition $|E - E'| > \sup\sigma(H_0) - \inf\sigma(H_0)$.
As it is simpler, we start with the proof of point (1).

2.3.1. *The proof of point (1).* Let $E_j(\omega)$ and $E_k(\omega)$ be simple eigenvalues of $H_\omega(\Lambda_L)$ such that $|E_k(\omega) - E_j(\omega)| \geq \Delta E > 2d$. Then, $\omega \mapsto E_j(\omega)$ and $\omega \mapsto E_k(\omega)$ are real analytic functions. Let $\omega \mapsto \varphi_j(\omega)$ and $\omega \mapsto \varphi_k(\omega)$ be normalized eigenvectors associated respectively to $E_j(\omega)$ and $E_k(\omega)$. Differentiating the eigenvalue equation in ω, one computes

$$\begin{aligned}\omega \cdot \nabla_\omega(E_j(\omega) - E_k(\omega)) &= \langle V_\omega \varphi_j(\omega), \varphi_j(\omega)\rangle - \langle V_\omega \varphi_k(\omega), \varphi_k(\omega)\rangle \\ &= E_j(\omega) - E_k(\omega) + \langle -\Delta\varphi_k(\omega), \varphi_k(\omega)\rangle - \langle -\Delta\varphi_j(\omega), \varphi_j(\omega)\rangle.\end{aligned}$$

As $0 \leq -\Delta \leq 2d$ and as $\varphi_j(\omega)$ and $\varphi_k(\omega)$ are normalized, we get that

$$\Delta E - 2d \leq |E_j(\omega) - E_k(\omega)| - 2d \leq |\omega \cdot \nabla_\omega(E_j(\omega) - E_k(\omega))|.$$

Hence, as the random variables $(\omega_\gamma)_{\gamma\in\Lambda}$ are bounded, the Cauchy Schwartz inequality yields

$$\|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_2 \geq \frac{\Delta E - 2d}{K}(2L+1)^{-d/2}.$$

which completes the proof of (2.10).

2.3.2. *The proof of point (2).* Let us now assume $d = 1$. Fix $E < E'$. Pick $E_j(\omega)$ and $E_k(\omega)$, simple eigenvalues of $H_\omega(\Lambda_L)$ such that $|E_k(\omega) - E| + |E_j(\omega) - E'| \leq e^{-L^\beta}$. Then, $\omega \mapsto E_j(\omega)$ and $\omega \mapsto E_k(\omega)$ are real analytic functions. Let $\omega \mapsto \varphi^j(\omega)$ and $\omega \mapsto \varphi^k(\omega)$ be normalized eigenvectors associated respectively to $E_j(\omega)$ and $E_k(\omega)$. One computes

$$\nabla_\omega E_j(\omega) = ([\varphi^j(\omega;\gamma)]^2)_{\gamma\in\Lambda_L} \quad \text{and} \quad \nabla_\omega E_k(\omega) = ([\varphi^k(\omega;\gamma)]^2)_{\gamma\in\Lambda_L}.$$

Hence, if

$$e^{-L^\beta} \geq \|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_1 = \sum_{\gamma\in\Lambda_L} |\varphi^j(\omega;\gamma) - \varphi^k(\omega;\gamma)| \cdot |\varphi^j(\omega;\gamma) + \varphi^k(\omega;\gamma)| \tag{2.20}$$

as $\|\nabla_\omega E_j(\omega)\| = \|\nabla_\omega E_k(\omega)\| = 1$, there exists a partition of $\Lambda_L = \{-L, \cdots, L\}$, say $\mathcal{P} \subset \Lambda_L$ and $\mathcal{Q} \subset \Lambda_L$ such that $\mathcal{P} \cup \mathcal{Q} = \Lambda_L$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$ and such that

- for $\gamma \in \mathcal{P}$, $|\varphi^j(\omega;\gamma) - \varphi^k(\omega;\gamma)| \leq e^{-L^\beta/2}$;
- for $\gamma \in \mathcal{Q}$, $|\varphi^j(\omega;\gamma) + \varphi^k(\omega;\gamma)| \leq e^{-L^\beta/2}$.

Introduce the orthogonal projectors P and Q defined by

$$P = \sum_{\gamma\in\mathcal{P}} |\gamma\rangle\langle\gamma| \quad \text{and} \quad Q = \sum_{\gamma\in\mathcal{Q}} |\gamma\rangle\langle\gamma|.$$

One has

$$\|P\varphi^j - P\varphi^k\|_2 \leq \sqrt{L}\, e^{-L^\beta/2} \quad \text{and} \quad \|Q\varphi^j + Q\varphi^k\|_2 \leq \sqrt{L}\, e^{-L^\beta/2}.$$

Clearly $\|P\varphi^j\|^2 + \|Q\varphi^j\|^2 = \|\varphi^j\|^2 = 1$. As $\langle\varphi^j, \varphi^k\rangle = 0$, one has

$$\begin{aligned}0 = \langle (P+Q)\varphi^j, (P+Q)\varphi^k\rangle &= \langle P\varphi^j, P\varphi^k\rangle + \langle Q\varphi^j, Q\varphi^k\rangle \\ &= \|P\varphi^j\|^2 - \|Q\varphi^j\|^2 + O\left(\sqrt{L}\, e^{-L^\beta/2}\right).\end{aligned}$$

Hence

$$\|P\varphi^j\|^2 = \frac{1}{2} + O(\sqrt{L}\, e^{-L^\beta/2}) \text{ and } \|Q\varphi^j\|^2 = \frac{1}{2} + O(\sqrt{L}\, e^{-L^\beta/2}).$$

This implies that

$$\mathcal{P} \neq \emptyset \text{ and } \mathcal{Q} \neq \emptyset. \tag{2.21}$$

We set $h_- = P\varphi^j - P\varphi^k$ and $h_+ = Q\varphi^j + Q\varphi^k$. The eigenvalue equations for $E_j(\omega)$ and $E_k(\omega)$ yields

$$(-\Delta + W_\omega)\varphi^j = \Delta E(\omega)\varphi^j \text{ and } (-\Delta + W_\omega)\varphi^k = -\Delta E(\omega)\varphi^k$$

where

$$\Delta E(\omega) = (E_j(\omega) - E_k(\omega))/2, \quad W_\omega = V_\omega - \overline{E}(\omega), \quad \overline{E}(\omega) = (E_j(\omega) + E_k(\omega))/2.$$

To simplify the notation, from now on, we write $u = \varphi^j$; then, one has $\varphi^k = Pu - Qu + O(\sqrt{L}\, e^{-L^\beta/2})$. This yields

$$\begin{cases} (-\Delta + W_\omega)(Pu + Qu) & = \Delta E(\omega)(Pu + Qu), \\ (-\Delta + W_\omega)(Pu - Qu + h_- - h_+) & = -\Delta E(\omega)(Pu - Qu + h_- - h_+) \end{cases}$$

that is

$$\begin{cases} (-\Delta + W_\omega)(Pu) & = \Delta E(\omega)Qu - h, \\ (-\Delta + W_\omega)(Qu) & = \Delta E(\omega)Pu + h \end{cases}$$

where $h := (-\Delta + W_\omega - \Delta E(\omega))(h_- - h_+)/2$. As $PW_\omega Q = 0$, this can also be written as

$$\begin{cases} [-(P\Delta Q + Q\Delta P) - \Delta E]u & = h_1, \\ [-(P\Delta P + Q\Delta Q) + V_\omega - \overline{E}]u & = h_2. \end{cases} \tag{2.22}$$

where

$$h_1 := (P - Q)h + (\Delta E(\omega) - \Delta E)u, \quad h_2 := (Q - P)h + (\overline{E}(\omega) - \overline{E})u,$$
$$\Delta E = (E' - E)/2, \qquad \overline{E} = (E + E')/2.$$

By our assumption on $E_j(\omega)$ and $E_k(\omega)$, we know that

$$|\Delta E(\omega) - \Delta E| \le 2e^{-L^\beta}, \quad |\overline{E}(\omega) - \overline{E}| \le e^{-L^\beta}, \quad \|h\| \le C\sqrt{L}\, e^{-L^\beta/2}.$$

Hence, we get that

$$\|h_1\| + \|h_2\| \le C\sqrt{L}\, e^{-L^\beta/2}. \tag{2.23}$$

So the above equations imply that

- ΔE is at a distance at most $\sqrt{L}\, e^{-L^\beta/2}$ to the spectrum of the deterministic operator $-(P\Delta Q + Q\Delta P)$,
- u is close to being in the eigenspace associated to the eigenvalues close to ΔE,
- finally, u is close to being in the kernel of the random operator $-(P\Delta P + Q\Delta Q) + V_\omega - \overline{E}$.

The firsts conditions will be used to describe u. The last condition will be interpreted as a condition determining the random variables ω_γ for sites γ such that $|u_\gamma|$ is not too small. We will show that the number of these sites is of size the volume of the cube Λ_L; so, the probability that the second equation in (2.22) be satisfied should be very small.
To proceed, we first study the operator $-P\Delta Q - Q\Delta P$. As we consider periodic boundary conditions, we compute

$$-P\Delta Q - Q\Delta P = \sum_{\gamma\in\partial\mathcal{P}} (|\gamma+1\rangle\langle\gamma| + |\gamma\rangle\langle\gamma+1|) + \sum_{\gamma\in\partial\mathcal{Q}} (|\gamma+1\rangle\langle\gamma| + |\gamma\rangle\langle\gamma+1|) \tag{2.24}$$

where $\partial\mathcal{P} = \{\gamma \in \mathcal{P};\ \gamma + 1 \in \mathcal{Q}\} \subset \mathcal{P}$ and $\partial\mathcal{Q} = \{\gamma \in \mathcal{Q};\ \gamma + 1 \in \mathcal{P}\} \subset \mathcal{Q}$. By (2.21), we know that $\partial\mathcal{P} \neq \emptyset$ and $\partial\mathcal{Q} \neq \emptyset$.
We first note that $\partial\mathcal{P} \cap \partial\mathcal{Q} = \emptyset$. Here, as we are considering the operators with periodic boundary conditions on Λ_L, we identify Λ_L with $\mathbb{Z}/L\mathbb{Z}$.
For $\mathcal{A} \subset \Lambda_L$ we define $\mathcal{A} + 1 = \{p + 1;\ p \in \mathcal{A}\}$ to be the shift by one of $\mathcal{A}$. By definition, $(\partial\mathcal{P} + 1) \subset \mathcal{Q}$ and $(\partial\mathcal{Q} + 1) \subset \mathcal{P}$. Hence, $(\partial\mathcal{P} + 1) \cap \partial\mathcal{P} = \emptyset$ and $(\partial\mathcal{Q} + 1) \cap \partial\mathcal{Q} = \emptyset$.
Consider the set $\mathcal{C} := \partial\mathcal{P} \cup \partial\mathcal{Q}$. We can partition it into its “connected components” i.e.

$\mathcal{C}$ can be written as a disjoint union of intervals of integers, say $\mathcal{C} = \cup_{l=1}^{l_0} \mathcal{C}_l^c$. Then, by the definition of $\partial\mathcal{P}$ and $\partial\mathcal{Q}$, for $l \neq l'$, one has,

$$\mathcal{C}_l^c \cap \mathcal{C}_{l'}^c = \mathcal{C}_l^c \cap (\mathcal{C}_{l'}^c + 1) = \emptyset. \tag{2.25}$$

Define $\mathcal{C}_l = \mathcal{C}_l^c \cup (\mathcal{C}_l^c + 1)$. (2.25) implies that, for $l \neq l'$,

$$\mathcal{C}_l \cap \mathcal{C}_{l'} = \emptyset. \tag{2.26}$$

Note that one may have $\cup_{l=1}^{l_0} \mathcal{C}_l = \Lambda_L$. The representation (2.24) then implies that the following block decomposition

$$-P\Delta Q - Q\Delta P = -\sum_{l=1}^{l_0} C_l \Delta C_l \tag{2.27}$$

where C_l is the projector $C_l = \sum_{\gamma \in \mathcal{C}_j} |\gamma\rangle\langle\gamma|$.

Note that, by (2.26), the projectors C_l and $C_{l'}$ are orthogonal to each other for $l \neq l'$. So the spectrum of the operator $-P\Delta Q - Q\Delta P$ is given by the union of the spectra of $(C_l \Delta C_l)_{1 \leq l \leq l_0}$. Each of these operators is the Dirichlet Laplacian on an interval of length $\#\mathcal{C}_l$. Its spectral decomposition can be computed explicitly. We will use some facts from this decomposition that we state now.

Lemma 2.7. *On a segment of length n, the Dirichlet Laplacian Δ_n i.e. the $n \times n$ matrix*

$$\Delta_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 0 & & \\ 0 & 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}$$

satisfies

- *its eigenvalues are simple and are given by* $(2\cos(k\pi/(n+1)))_{1 \leq k \leq n}$*;*
- *for $k \in \{1, \cdots, n\}$, the eigenspace associated to $2\cos(k\pi/(n+1))$ is generated by the vector* $(\sin[kj\pi/(n+1)])_{1 \leq j \leq n}$*.*

Moreover, there exists $K_1 > 0$ such that, for any $n \geq 1$, one has

$$\inf_{1 \leq k < k' \leq n} \left| 2\cos\left(\frac{k\pi}{n+1}\right) - 2\cos\left(\frac{k'\pi}{n+1}\right) \right| \geq \frac{1}{K_1 n^2}. \tag{2.28}$$

Proof of Lemma 2.7. The first statement follows immediately from the identity

$$\sin\left(\frac{k(j+1)\pi}{n+1}\right) + \sin\left(\frac{k(j-1)\pi}{n+1}\right) = 2\cos\left(\frac{k\pi}{n+1}\right)\sin\left(\frac{kj\pi}{n+1}\right).$$

The estimate (2.28) is an immediate consequence of

$$\cos\left(\frac{k\pi}{n+1}\right) - \cos\left(\frac{k'\pi}{n+1}\right) = -2\sin\left(\frac{(k+k')\pi}{2(n+1)}\right)\sin\left(\frac{(k-k')\pi}{2(n+1)}\right).$$

□

We now solve the first equation in (2.22) that is describe u solution to this equation.

Lemma 2.8. *Let u be a solution to (2.22) such that $\|u\|=1$. Then, for L sufficiently large, one has*

$$(2.29)\qquad \left\| u-\sum_{l=1}^{l_0} C_l u\right\| \le e^{-L^\beta/3}$$

where, if for $1\le l\le l_0$, we write $\mathcal{C}_l=\{\gamma_l^-,\cdots,\gamma_l^+\}$ ($n_l=\gamma_l^+-\gamma_l^-+1$), then,

- *either there exists a unique $k_l\in\{1,\cdots,n_l\}$ satisfying*

$$(2.30)\qquad \left|2\cos\left(\frac{k_l\pi}{n_l+1}\right)-\Delta E\right| < \frac{1}{K_1 n^2}$$

  and $\alpha^l\in\mathbb{R}$ such that

$$(2.31)\qquad \|C_l u-\alpha^l u^l\|\le e^{-L^\beta/3}$$

  where

$$u^l_\gamma=\begin{cases}\sin\left(\frac{k_l(\gamma-\gamma_l^-+1)\pi}{n_l+1}\right) & \text{if } \gamma\in C_l,\\ 0 & \text{if } \gamma\notin C_l.\end{cases}$$

- *there exists no $k_l\in\{1,\cdots,n_l\}$ satisfying (2.30) then*

$$\|C_l u\|\le e^{-L^\beta/3}.$$

Proof of Lemma 2.8. By Lemma 2.7, the spacing between consecutive eigenvalues of $-C_l\Delta C_l$ is bounded below by $1/(K_1n^2)$.

Let $C^\perp=1-\sum_{l=1}^{l_0}C_l$. Hence, $u=\sum_{l=1}^{l_0}C_l u+C^\perp u$, the terms in this sums being two by two orthogonal to each other. As $\Delta E>0$, the first equation in (2.22) then yields

$$(2.32)\quad \forall 1\le l\le l_0,\quad \|-C_l\Delta C_l u-\Delta E\, C_l u\|\le C\sqrt{L}\,e^{-L^\beta/2}\quad\text{and}\quad \|C^\perp u\|\le C\sqrt{L}\,e^{-L^\beta/2}.$$

Write $\mathcal{C}_l=\{\gamma_l^-,\gamma_l^-+1,\cdots,\gamma_l^+\}$ where one may have $\gamma_l^-=\gamma_l^+$. We assume that the $(\mathcal{C}_l)_{1\le l\le l_0}$ are ordered so that $\gamma_l^+<\gamma_{l+1}^-$.
By the characterization of the spectrum of $-C_l\Delta C_l$,

- if $2\cos(k_l\pi/(n+1))$ is an eigenvalue of $-C_l\Delta C_l$ closer to ΔE than a distance $L^{-2}/4K_1$ (by the remark made above, such an eigenvalue is unique), then, for some α^l real, one has
$$\|C_l u-\alpha^l u^l\|\le CL^{5/2}\,e^{-L^\beta/2}.$$
- if there is no such eigenvalue, then

$$(2.33)\qquad \|C_l u\|\le CL^{5/2}\,e^{-L^\beta/2}.$$

This completes the proof of Lemma 2.8. □

We now prove that $|u_\gamma|$ cannot be really small for too many γ.

Lemma 2.9. *There exists $c>0$ such that, for L sufficiently large,*

1. *either $\#\mathcal{C}\ge L/3$ and, for $\gamma\in\mathcal{C}$, $|u_\gamma|\ge e^{-L^\beta/6}$,*
2. *or $l_0\ge 2cL^\beta$ and there exists $l^*\in\{1,\cdots,l_0\}$ such that, for $|l-l^*|\le cL^\beta$, and $\gamma\in\mathcal{C}_l$, one has $|u_\gamma|\ge e^{-L^\beta/6}$.*

Proof of Lemma 2.9. To prove Lemma 2.9, we compare the values of u on $\mathcal{C}_l$ and $\mathcal{C}_{l+1}$, that is, the vectors $C_l u$ and $C_{l+1}u$ given by Lemma 2.8.
First, notice that up to an error of size at most $e^{-L^\beta/3}$, u on $\mathcal{C}_l$ is determined by its coefficient $u_{\gamma_l^+}$, or equivalently, by its coefficient $u_{\gamma_l^-}$; in particular as $\sin(k_l\pi/(n_l+1)) = (-1)^{k_l-1}\sin(k_l n_l\pi/(n_l+1))$, the representations (2.29) and (2.31) yields

$$\left||u_{\gamma_l^-}|-|u_{\gamma_l^+}|\right|+\left|u_{\gamma_l^+}-\alpha^l\sin(k_l\pi/(n_l+1))\right|\leq Ce^{-L^\beta/3}. \tag{2.34}$$

Notice also that, as $2\leq n_l\leq 2L+1$ is fixed, for $\rho^*:=\sqrt{\dfrac{n_l}{2}+\dfrac{1}{2}\cos\left(\dfrac{2k_l\pi}{n_l+1}\right)}$, one has

$$\sup_{1\leq l\leq l_0}\left|\|C_l u\|-\rho_l|\alpha^l|\right|\leq Ce^{-L^\beta/3}. \tag{2.35}$$

To compare the values of u on $\mathcal{C}_l$ and $\mathcal{C}_{l+1}$, we use the second equation of (2.22) or, equivalently, the eigenvalue equation for u that reads (see (2.22))

$$(-\Delta+V_\omega-\overline{E})u=\Delta E\,u+e \tag{2.36}$$

where $e=h_1+h_2$ (see (2.22)); hence, $\|e\|\leq C\sqrt{L}\,e^{-L^\beta/2}$.
We will discuss three cases depending on how far γ_l^+ and γ_{l+1}^- are from one another:

(1) if $\text{dist}(\mathcal{C}_l,\mathcal{C}_{l+1})\geq 3$, that is, if $\gamma_l^+<\gamma_l^++1<\gamma_{l+1}^--1<\gamma_{l+1}^-$: as $\{\gamma_l^++1,\cdots,\gamma_{l+1}^--1\}\cap[\cup_{l=1}^{l_0}\mathcal{C}_l]=\emptyset$, by (2.29), we know that $|u_n|\leq CL^{2-\alpha}$ for $n\in\{\gamma_l^++1,\cdots,\gamma_{l+1}^--1\}$. The eigenvalue equation (2.36) at the points γ_l^++1 and γ_{l+1}^--1 then tells us that

$$|u_{\gamma_l^+}|+|u_{\gamma_{l+1}^-}|\leq Ce^{-L^\beta/3}.$$

Thus, by (2.34) and (2.35)

$$\|C_l u\|+\|C_{l+1}u\|\leq Ce^{-L^\beta/4}. \tag{2.37}$$

(2) if $\text{dist}(\mathcal{C}_l,\mathcal{C}_{l+1})=2$, that is, if $\gamma_l^+<\gamma_l^++1=\gamma_{l+1}^--1<\gamma_{l+1}^-$: as $\gamma_l^++1\not\in\cup_{l=1}^{l_0}\mathcal{C}_l$, by (2.29), we know that $|u_{\gamma_l^++1}|\leq CL^{2-\alpha}$. Hence, in the same way as above, the eigenvalue equation (2.36) at the point γ_l^++1 tells us that

$$|u_{\gamma_l^+}+u_{\gamma_{l+1}^-}|\leq Ce^{-L^\beta/3}.$$

Thus, by (2.34) and (2.35)

$$|\,\|C_l u\|-\|C_{l+1}u\|\,|\leq Ce^{-L^\beta/4}. \tag{2.38}$$

(3) if $\text{dist}(\mathcal{C}_l,\mathcal{C}_{l+1})=1$, that is, if $\gamma_l^++1=\gamma_{l+1}^-$: then, the first equation in (2.22) and the decomposition (2.27) yield

$$|u_{\gamma_l^+-1}-\Delta E\,u_{\gamma_l^+}|+|u_{\gamma_{l+1}^-+1}-\Delta E\,u_{\gamma_{l+1}^-}|\leq Ce^{-L^\beta/3}.$$

The eigenvalue equation (2.36) at the points γ_l^+ and γ_{l+1}^- yields

$$\begin{aligned}|u_{\gamma_l^+-1}+u_{\gamma_{l+1}^-}+(\omega_{\gamma_l^+}-\overline{E}-\Delta E)u_{\gamma_l^+}|&\\ +|u_{\gamma_l^+}+u_{\gamma_{l+1}^-+1}+(\omega_{\gamma_{l+1}^-}-\overline{E}-\Delta E)u_{\gamma_{l+1}^-}|&\leq Ce^{-L^\beta/3}.\end{aligned}$$

Summing these two equations, we obtain

$$|u_{\gamma_{l+1}^-}+(\omega_{\gamma_l^+}-\overline{E})u_{\gamma_l^+}|+|u_{\gamma_l^+}+(\omega_{\gamma_{l+1}^-}-\overline{E})u_{\gamma_{l+1}^-}|\leq Ce^{-L^\beta/3}.$$

Then, as the random variables $(\omega_n)_{n\in\mathbb{Z}}$ are bounded, using (2.34) and (2.35), there exists $C>1$ such that

$$\frac{1}{C}(\|C_l u\| - Ce^{-L^\beta/4}) \leq \|C_{l+1}u\| \leq C(\|C_l u\| + e^{-L^\beta/4}). \tag{2.39}$$

Notice that (2.38) and (2.37) also imply that (2.39) (at the expense of possibly changing the constant C) also holds in case (1) and case (2). Hence, for $1\leq l,l'\leq l_0$, we have

$$C^{-|l'-l|}\|C_{l'}u\| - C^{|l'-l|}e^{-L^\beta/4} \leq \|C_l u\| \leq C^{|l'-l|}\|C_{l'}u\| + C^{|l'-l|}e^{-L^\beta/4} \tag{2.40}$$

If case (1) in the above alternative never holds i.e. if for $1\leq l\leq l_0$, one has $\text{dist}(\mathcal{C}_l,\mathcal{C}_{l+1})\leq 2$, then, one has $\#\mathcal{C}\geq L/3$.
We know that $\|Cu\| = 1+O(e^{-L^\beta/3})$. So, for L sufficiently large, there exists $1\leq l^*\leq l_0$ such that

$$\|C_{l^*}u\| \geq (2\sqrt{\ell_0})^{-1} \geq (4\sqrt{L})^{-1}.$$

Hence, by (2.40), either of two things occur

- for some l, one has $\|C_l u\|\leq e^{-L^\beta/5}$, then $|l-l^*|\geq \tilde{c}L^\beta$ for some $\tilde{c}>0$; thus, $l_0\geq 2\tilde{c}L^\beta$; and for some $0<c<\tilde{c}$, for $|l-l^*|\leq cL^\beta$, one has $\|C_l u\|\geq e^{-L^\beta/5}$.
- for $1\leq l\leq l_0$, one has $\|C_l u\|\geq e^{-L^\beta/5}$; then, case (1) never occurs, thus, by the observation made above, $\#\mathcal{C}\geq L/3$

Finally, notice that, by (2.35), (2.34) and the form of u^l (see Lemma 2.8), $\|C_l u\|\geq e^{-L^\beta/5}$ implies that $|u_n|\geq e^{-L^\beta/6}$ for $n\in\mathcal{C}_l$.
This completes the proof of Lemma 2.9. □

We now show that our characterization of u, a solution of (2.22), imposes very restrictive conditions on the random variables $(\omega_\gamma)_{-L\leq\gamma\leq L}$.
If γ is inside one of the connected components of $\mathcal{C}$, say $\mathcal{C}_l$, that is, if $\{\gamma-1,\gamma,\gamma+1\}\subset\mathcal{C}_l$, then, by the first equation in (2.22), we know that

$$|u_{\gamma+1}+u_{\gamma-1}-\Delta E u_\gamma| \leq Ce^{-L^\beta/3}.$$

Plugging this into (2.36), the eigenvalue equation for u, we get

$$|(\omega_\gamma-\overline{E})u_\gamma| \leq e^{-L^\beta/4}.$$

Hence, if γ belongs to one of the $(\mathcal{C}_l)_l$ singled out in Lemma 2.9, the lower bound for $|u_\gamma|$ given in Lemma 2.9 yields

$$|\omega_\gamma-\overline{E}| \leq Ce^{-L^\beta/12}. \tag{2.41}$$

Now, if $n_l>2$, there exists $\gamma\in\mathcal{C}_l$ such that $\{\gamma-1,\gamma,\gamma+1\}\subset\mathcal{C}_l$. On the other hand, if $n_l=2$, then, the approximate eigenvalue equation on $\mathcal{C}_l$ reads

$$\left\|\begin{pmatrix} E & \omega_{\gamma_l^-} \\ \omega_{\gamma_l^+} & E\end{pmatrix}\begin{pmatrix} u_{\gamma_l^+} \\ u_{\gamma_l^-}\end{pmatrix}\right\| \leq C\sqrt{L}\,e^{-L^\beta/2}.$$

So, if $\|C_l u\|\geq e^{-L^\beta/6}$, one has

$$|1-(\omega_{\gamma_l^-}-E)(\omega_{\gamma_l^+}-E)| \leq Ce^{-L^\beta/3}. \tag{2.42}$$

Hence, we see that the random variables must satisfy at least cL^β distinct conditions of the type (2.41) or (2.42). As the random variables are supposed to be independent, identically distributed with a bounded density, these condition imply that (2.20) can occur with a given partition $\mathcal{P}$ and $\mathcal{Q}$ with a probability at most, $e^{-cL^{2\beta}}$ for some $c>0$. As the total number of partitions is bounded by 2^L and as $\beta > 1/2$, we obtain that, $\mathbb{P}$, the probability that (2.20) holds, is bounded by (2.12). This completes the proof of Lemma 2.4. □

Remark 2.1. The estimate (2.12) can be improved as, actually, not all partitions are allowed as we saw in the course of the proof. Moreover, it is sufficient to assume that the distribution function of the random variables be Hölder continuous for the method to work.

Remark 2.2. We now present a natural weaker analogue of point (2) in Lemma 2.4. Fix $\rho > 0$ and define

$$\Delta\mathcal{E}_L^c = \bigcup_{l=0}^{L} \sigma(-C_l \Delta C_l) + [-L^{-\rho}, L^{-\rho}].$$

then, for $\rho > 3$, one has $|\Delta\mathcal{E}_L^c| \leq 2L^{2-\rho}$, thus,

$$\left| \bigcap_{n\geq 1} \bigcup_{L\geq n} \Delta\mathcal{E}_L^c \right| = 0$$

Define the set of total measure

$$\Delta\mathcal{E} = \mathbb{R} \setminus \left(\bigcap_{N\geq 1} \bigcup_{L\geq N} \Delta\mathcal{E}_L^c \right).$$

Hence, if $E - E' = \Delta E \in \Delta\mathcal{E}$, for L sufficiently large, as

$$\inf_{1\leq l\leq L} \mathrm{dist}(\Delta E, \sigma(-C_l \Delta C_l)) \geq L^{-\rho},$$

by the decomposition (2.27), a solution u to the first equation in (2.22) must satisfy $\|u\| \leq L^{-(\nu-\rho)}$ if $\|h_1\| \leq L^{-\nu}$. Hence, we obtain

Lemma 2.10. *Fix $\nu > 4$. For the discrete Anderson model in dimension 1, for $E - E' \in \Delta\mathcal{E}$, for L sufficiently large, if $E_j(\omega)$ and $E_k(\omega)$ are simple eigenvalues of $H_\omega(\Lambda_L)$ such that $|E_k(\omega) - E| + |E_j(\omega) - E'| \leq L^{-\nu}$ then $\|\nabla_\omega(E_j(\omega) - E_k(\omega))\|_1 \geq L^{-\nu}$.*

This can then be used as Lemma 2.4 is used in the proof of Lemma 1.1 to prove the following variant of the decorrelation estimates in dimension 1

Lemma 2.11. *Assume $d = 1$. For $\alpha \in (0, 1)$ and $E - E' \in \Delta\mathcal{E}$ s.t. $\{E, E'\} \subset I$, for any $c > 0$, there exists $C > 0$ such that, for $L \geq 3$ and $cL^\alpha \leq \ell \leq L^\alpha/c$, one has*

$$\mathbb{P}\left(\begin{Bmatrix} \sigma(H_\omega(\Lambda_\ell)) \cap (E + L^{-d}(-1, 1)) \neq \emptyset, \\ \sigma(H_\omega(\Lambda_\ell)) \cap (E' + L^{-d}(-1, 1)) \neq \emptyset \end{Bmatrix} \right) \leq C(\ell/L)^{2d}(\log L)^C.$$

Comparing with Lemma 1.1, we improved the bound on the probability at the expense of reducing the set of validity in (E, E').

2.4. **Proof of Lemma 2.5.** Pick $(u, v) \in (\mathbb{R}^+)^{2n}$ such that $\|u\|_1 = \|v\|_1 = 1$. At the expense of exchanging u and v, we may assume that $\|v\|_2 \geq \|u\|_2$. Write $u = \alpha v + v^\perp$ where $\langle v, v^\perp \rangle = 0$. Note that, as all the coefficient of both u and v are non negative, $v^\perp = 0$ is equivalent $u = v$. Let us now assume $u \neq v$ that is $v^\perp \neq 0$. One computes

$$\|u\|_2^2 = \alpha^2 \|v\|_2^2 + \|v^\perp\|_2^2 \text{ and } \|u - v\|_2^2 = (\alpha - 1)^2 \|v\|_2^2 + \|v^\perp\|_2^2. \tag{2.43}$$

Moreover, as all the coefficients of v are non negative, $v^\perp$ admits at least one negative coefficient. As all the coefficients of u are non negative, the decomposition $u = \alpha v + v^\perp$

implies that $\alpha > 0$. The first equation in (2.43) and the condition $\|v\|_2 \geq \|u\|_2$ then imply $\alpha \in (0,1)$. Combining this with $u = \alpha v + v^\perp$ and $\|u\|_1 = \|v\|_1 = 1$ yields

$$0 < 1 - \alpha \leq \|v^\perp\|_1.$$

Hence, by the second equation in (2.43) and the Cauchy-Schwartz inequality, we get

$$\frac{1}{\sqrt{n}}\|u-v\|_1 \leq \|u-v\|_2 \leq \|v\|_2\|v^\perp\|_1 + \|v^\perp\|_2 \leq 2\sqrt{n}\|v^\perp\|_2. \tag{2.44}$$

For any (j,k), one has

$$\begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix} = \begin{vmatrix} v_j^\perp & v_k^\perp \\ v_j & v_k \end{vmatrix}.$$

As $\langle v, v^\perp \rangle = 0$, one computes

$$\begin{aligned}\sum_{j,k} \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix}^2 &= \sum_{j,k} \left((v_j v_k^\perp)^2 + (v_k v_j^\perp)^2 - 2 v_j v_k^\perp v_k v_j^\perp \right) \\ &= 2\left(\sum_j v_j^2\right)\left(\sum_k (v_k^\perp)^2\right) - 2\left(\sum_j v_j v_j^\perp\right)\left(\sum_k v_k v_k^\perp\right) \\ &= 2\|v\|_2^2 \|v^\perp\|_2^2 \geq \frac{1}{2n^3}\|u-v\|_1^2.\end{aligned}$$

Thus,

$$\max_{j\neq k} \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix}^2 \geq \frac{1}{4n^5}\|u-v\|_1^2$$

which completes the proof of Lemma 2.5. □

3. The proofs of Theorems 1.2 and 1.3

In [7], the authors extensively study the distribution of the energy levels of random systems in the localized phase. Their results apply also to the discrete Anderson model; in particular, they provide a proof of Theorems 1.2 and 1.3 once the decorrelation estimates obtained in Lemmas 1.1 and 1.2 are known. We provide an alternate proof. The proof in [7] relies on a construction that also proves Theorem 1.1 (actually a stronger uniform result). Here, we only prove Theorems 1.2 and 1.3 independently of the values of the limits in Theorem 1.1.
The localization centers of Proposition 2.1 are not defined uniquely. One can easily check that, under the assumptions of Proposition 2.1, all the localization centers for a given eigenvalue or eigenfunction are contained in a disk of radius at most $C \log L$ (for some $C > 0$). To define a unique localization center, we order the centers lexicographically and let the localization center associated to the eigenvalue or eigenfunction be the largest one (i.e. the one most upper left in dimension 2).
We prove

Lemma 3.1. *Pick $\alpha \in (0,1)$ and $c > 0$. Let ν be defined by (Loc). Assume $\ell = \ell(L)$ satisfies $cL^\alpha \leq \ell \leq L^\alpha/c$.*
If (Loc) (see Proposition 2.1) is satisfied then, for any $p > 0$ and $\varepsilon > 0$, there exists $L_0 > 0$ such that, for $L \geq L_0$, with probability larger than $1 - L^{-p}$,

(1) *if $(E_j)_{1\leq j\leq J} \in I^J$ are eigenvalues of $H_\omega(\Lambda_L)$ with localization center in $\gamma + \Lambda_\ell$, then the operator $H_\omega(\gamma + \Lambda_{\ell(1+\varepsilon)})$ has J eigenvalues, say $(\tilde{E}_j)_{1\leq j\leq J}$, with localization center in $\gamma + \Lambda_{\ell(1+\varepsilon/2)}$ and such that $\sup_{1\leq j\leq J} |E_j - \tilde{E}_j| \leq e^{-\nu\varepsilon\ell/4}$.*

(2) *if* $(E_j)_{1\leq j\leq J}\in I^J$ *are eigenvalues of* $H_\omega(\gamma+\Lambda_{\ell(1+\varepsilon)})$ *with localization center in* $\gamma+\Lambda_\ell$, *then the operator* $H_\omega(\Lambda_L)$ *has* J *eigenvalues, say* $(\tilde E_j)_{1\leq j\leq J}$, *with localization center in* $\gamma+\Lambda_{\ell(1+\varepsilon/2)}$ *and such that* $\sup_{1\leq j\leq J}|E_j-\tilde E_j|\leq e^{-\nu\varepsilon\ell/4}$.

(3) *if* $(E_j)_{1\leq j\leq J}\in I^J$ *are eigenvalues of* $H_\omega(\gamma+\Lambda_{\ell(1+\varepsilon)})$ *with localization center in* $\gamma+(\Lambda_{\ell(1+\varepsilon/2)}\setminus\Lambda_\ell)$, *then there exists* $(\beta_j)_{1\leq j\leq J}$ *such that, for* $1\leq j\leq J$, *one has*
   - $\beta_j\in\dfrac{\varepsilon\ell}{16}\mathbb{Z}^d\cap\left[\gamma+(\Lambda_{\ell(1+\varepsilon/2)}\setminus\Lambda_\ell)\right]$,
   - *the operator* $H_\omega(\beta_j+\Lambda_{\varepsilon\ell/4})$ *has an eigenvalue, say* $\tilde E_j$, *satisfying* $|E_j-\tilde E_j|\leq e^{-\nu\varepsilon\ell/8}$.

The number $\nu>0$ *is given by (Loc).*

Similar results can be found in [7].

Proof. With probability at least $1-L^{-p}$, the conclusions of Proposition 2.1 hold which we assume from now on.
To prove (1), let $(\varphi_j)_{1\leq j\leq J}$ be normalized eigenfunctions associated to $(E_j)_{1\leq j\leq J}$. Then, setting $\tilde\varphi_j=\mathbf{1}_{\gamma+\Lambda_{\ell(1+\varepsilon)}}\varphi_j$ and using (2.3) from (Loc) and the assumption that the localization center are in $\gamma+\Lambda_\ell$, one obtains

$$\left\|\left(\left(\langle\tilde\varphi_j,\tilde\varphi_k\rangle_{\ell^2(\gamma+\Lambda_{\ell(1+\varepsilon)})}\right)\right)_{\substack{1\leq j\leq J\\1\leq k\leq J}}-\mathrm{Id}\right\|\leq J^2e^{-\nu\varepsilon\ell/4},$$
$$\sup_{1\leq j\leq J}\|\mathbf{1}_{\gamma+(\Lambda_{\ell(1+\varepsilon)}\setminus\Lambda_{\ell(1+\varepsilon/2)})}\tilde\varphi_j\|_{\ell^2(\gamma+\Lambda_{\ell(1+\varepsilon)})}\leq e^{-\nu\varepsilon\ell/6},$$
$$\sup_{1\leq j\leq J}\|(H_\omega(\gamma+\Lambda_{\ell(1+\varepsilon)})-E_j)\tilde\varphi_j\|_{\ell^2(\gamma+\Lambda_{\ell(1+\varepsilon)})}\leq e^{-\nu\varepsilon\ell/4}.$$

This immediately yields (1) for L sufficiently large as

- $J\leq(2L+1)^d$ and $cL^\alpha\leq\ell\leq L^\alpha/c$,
- at a localization center, the modulus of an eigenfunction is at least of order $L^{-d/2}$.

Points (2) is proved in the same way. We omit further details.
To prove (3), we set $\tilde\varphi_j=\mathbf{1}_{\beta_j+\Lambda_{\ell\varepsilon/4}}\varphi_j$ where β_j is the point in $\frac{\varepsilon\ell}{16}\mathbb{Z}^d$ closest to the localization center of φ_j. The conclusion then follows from the same reasoning as above.
This completes the proof of Lemma 3.1. □

Pick $(U_j)_{1\leq j\leq J}$, $(k_j)_{1\leq j\leq J}\in\mathbb{N}^J$, $(U'_j)_{1\leq j\leq J'}$ and $(k'_j)_{1\leq j\leq J}\in\mathbb{N}^{J'}$ as in Theorem 1.2. To prove Theorems 1.2 and 1.3, it suffices to prove (1.6) for $(U_j)_{1\leq j\leq J}$ and $(U'_j)_{1\leq j\leq J'}$ non empty compact intervals which we assume from now on.
Pick L and ℓ such that $(2L+1)=(2\ell+1)(2\ell'+1)$, $cL^\alpha\leq\ell\leq L^\alpha/c$ for some $\alpha\in(0,1)$ and $c>0$. Pick $\varepsilon>0$ small. Partition $\Lambda_L=\bigcup_{|\gamma|\leq\ell'}\Lambda_\ell(\gamma)$ where $\Lambda_\ell(\gamma)=(2\ell+1)\gamma+\Lambda_\ell$. For $\Lambda'\subset\Lambda$ and $U\subset\mathbb{R}$, consider the random variables

$$X(E,U,\Lambda,\Lambda'):=\begin{cases}1 & \text{if } H_\omega(\Lambda)\text{ has at least one eigenvalue in}\\ & E+(\nu(E)|\Lambda|)^{-1}U\text{ with localization center in }\Lambda',\\ 0 & \text{if not;}\end{cases}$$

if $\Lambda'=\Lambda$, we write $X(E,U,\Lambda):=X(E,U,\Lambda,\Lambda)$, and

$$\Sigma(E,U):=\sum_{|\gamma|\le\ell'}X(E,U,\Lambda,\Lambda_\ell(\gamma)),\quad \Sigma(E,U,\ell):=\sum_{|\gamma|\le\ell'}X(E,U,\Lambda_\ell(\gamma)).$$

We prove

Lemma 3.2.

$$\left|\mathbb{P}\left(\left\{\omega;\begin{array}{ccc}\#\{j;\xi_n(E,\omega,\Lambda)\in U_1\}=k_1\\ \vdots & \vdots\\ \#\{j;\xi_n(E,\omega,\Lambda)\in U_J\}=k_J\\ \#\{j;\xi_n(E',\omega,\Lambda)\in U'_1\}=k'_1\\ \vdots & \vdots\\ \#\{j;\xi_n(E',\omega,\Lambda)\in U_{J'}\}=k_{J'}\end{array}\right\}\right)-\mathbb{P}\left(\left\{\omega;\begin{array}{ccc}\Sigma(E,U_1,\ell)=k_1\\ \vdots & \vdots\\ \Sigma(E,U_J,\ell)=k_J\\ \Sigma(E',U'_1,\ell)=k'_1\\ \vdots & \vdots\\ \Sigma(E',U_{J'},\ell)=k'_{J'}\end{array}\right\}\right)\right|\underset{L\to+\infty}{\to}0,$$

$$\left|\mathbb{P}\left(\left\{\omega;\begin{array}{ccc}\#\{j;\xi_n(E,\omega,\Lambda)\in U_1\}=k_1\\ \vdots & \vdots\\ \#\{j;\xi_n(E,\omega,\Lambda)\in U_J\}=k_J\end{array}\right\}\right)-\mathbb{P}\left(\left\{\omega;\begin{array}{ccc}\Sigma(E,U_1,\ell)=k_1,\\ \vdots & \vdots\\ \Sigma(E,U_J,\ell)=k_J\end{array}\right\}\right)\right|\underset{L\to+\infty}{\to}0$$

and

$$\left|\mathbb{P}\left(\left\{\omega;\begin{array}{ccc}\#\{j;\xi_n(E',\omega,\Lambda)\in U'_1\}=k'_1\\ \vdots & \vdots\\ \#\{j;\xi_n(E',\omega,\Lambda)\in U_{J'}\}=k_{J'}\end{array}\right\}\right)-\mathbb{P}\left(\left\{\omega;\begin{array}{ccc}\Sigma(E',U'_1,\ell)=k'_1,\\ \vdots & \vdots\\ \Sigma(E',U_{J'},\ell)=k'_{J'}\end{array}\right\}\right)\right|\underset{L\to+\infty}{\to}0.$$

Proof. We first prove

Lemma 3.3. *For any $p>0$ and $\varepsilon>0$, there exists $C>0$ such that, for U a compact interval and L sufficiently large, one has*

$$\mathbb{P}\left(\{\omega;\ \#\{n;\xi_n(E,\omega,\Lambda)\in U\}\neq\Sigma(E,U)\}\right)\le C\ell^dL^{-d}(|U|+1)^2+L^{-p}. \tag{3.1}$$

and

$$\mathbb{P}\left(\{\omega;\ \Sigma(E,U)\neq\Sigma(E,U,\ell)\}\right)\le L^{-p}+C\,\varepsilon\,|U|. \tag{3.2}$$

Proof of Lemma 3.3. As $\Lambda=\bigcup_{|\gamma|\le\ell'}\Lambda_\ell(\gamma)$ and these sets are two by two disjoint, the quantities $\#\{n;\xi_n(E,\omega,\Lambda)\in U\}$ and $\Sigma(E,U)$ differ if and only if, for some $|\gamma|\le\ell'$, $H_\omega(\Lambda)$ has at least two eigenvalues in $E+(\nu(E)|\Lambda|)^{-1}U$ with localization center in $\Lambda_\ell(\gamma)$. By Lemma 3.1, this implies that, except on a set of probability at most L^{-p}, $H_\omega((2\ell+1)\gamma+\Lambda_{2\ell})$ has at least two eigenvalues in $U+[-e^{-\nu\ell/4},e^{-\nu\ell/4}]$. Thus, by Minami's estimate (2.2), this happens with a probability at most $C\ell^{2d}L^{-2d}(|U|+1)^2+L^{-p}$. Summing this estimate over all the possible γ's, we complete the proof of (3.1).
The proof of (3.2) is split into two steps. Define

$$\Sigma(E,U,\varepsilon)=\sum_{|\gamma|\le\ell'}X(E,U,(2\ell+1)\gamma+\Lambda_{\ell(1+\varepsilon)},(2\ell+1)\gamma+\Lambda_{\ell(1-\varepsilon)}).$$

Then, we successively prove

$$\mathbb{P}\left(\{\omega;\ \Sigma(E,U)\neq\Sigma(E,U,\varepsilon)\}\right)\leq L^{-p}+C\,\varepsilon\,|U| \tag{3.3}$$

and

$$\mathbb{P}\left(\{\omega;\ \Sigma(E,U,\varepsilon)\neq\Sigma(E,U,\ell)\}\right)\leq L^{-p}+C\,\varepsilon\,|U| \tag{3.4}$$

which implies (3.2).
To prove (3.3), we note that, by Lemma 3.1, except on a set of probability at most L^{-p}, $\Sigma(E,U)$ and $\Sigma(E,U,\varepsilon)$ differ if and only if, for some $|\gamma|\leq\ell'$, one has

1. either $\sigma(H_\omega(\Lambda))\cap\delta\tilde U\neq\emptyset$,
2. or $\sigma(H_\omega((2\ell+1)\gamma+\Lambda_{\ell(1+\varepsilon)}))\cap\delta\tilde U\neq\emptyset$,
3. or $H_\omega((2\ell+1)\gamma+\Lambda_{\ell(1+\varepsilon)})$ has an eigenvalue in $\tilde U$ with a localization center in the cube $(2\ell+1)\gamma+(\Lambda_{\ell(1+\varepsilon)}\setminus\Lambda_{\ell(1-\varepsilon)})$

where $\tilde U=E+(\nu(E)|\Lambda|)^{-1}U+e^{-\nu\varepsilon\ell/8}[-1,1]$ and $\delta\tilde U=\tilde U\setminus(E+(\nu(E)|\Lambda|)^{-1}U)$.
The probability of alternatives (1) and (2) is estimated using the Wegner estimate (2.1). It is bounded by $2L^de^{-\nu\varepsilon\ell/8}\leq L^{-p}$ for L sufficiently large.
By point (3) of Lemma 3.1, except on a set of probability at most L^{-p}, alternative (3) implies that, for some $\beta\in\gamma+(\Lambda_{\ell(1+\varepsilon/2)}\setminus\Lambda_\ell)$, the operator $H_\omega(\beta+\Lambda_{\varepsilon\ell/4})$ has an eigenvalue in $E+(\nu(E)|\Lambda|)^{-1}U+e^{-\nu\varepsilon\ell/8}[-1,1]$. The number of possible β's is bounded by $C\varepsilon\ell^d\varepsilon^{-d}\ell^{-d}=C\varepsilon^{1-d}$. Using Wegner's estimate (2.1) and summing over the possible β's, this probability is bounded by $C\varepsilon^{1-d}(\varepsilon\ell/L)^d|U|+L^{-p}\leq C\varepsilon(\ell/L)^d|U|+L^{-p}$. Finally, we sum this over all possible γ's to obtain that the probability that alternative (3) holds for some γ is bounded by $C\varepsilon|U|+L^{-p}$. This yields (3.3).
To prove (3.8), the reasoning is similar. By Lemma 3.1, except on a set of probability at most L^{-p}, $\Sigma(E,U,\ell)$ and $\Sigma(E,U,\varepsilon)$ differ if and only if, for some $|\gamma|\leq\ell'$, one has

1. either $\sigma(H_\omega((2\ell+1)\gamma+\Lambda_{\ell(1+\varepsilon)}))\cap\tilde U\neq\emptyset$,
2. or $\sigma(H_\omega(\Lambda_\ell(\gamma)))\cap\tilde U\neq\emptyset$,
3. or $H_\omega((2\ell+1)\gamma+\Lambda_{\ell(1+\varepsilon)})$ has an eigenvalue in $\tilde U$ with localization center in the cube $(2\ell+1)\gamma+(\Lambda_{\ell(1+\varepsilon)}\setminus\Lambda_{\ell(1-\varepsilon)})$.
4. or $H_\omega(\Lambda_\ell(\gamma))$ has an eigenvalue in $\tilde U$ with localization center in $(2\ell+1)\gamma+(\Lambda_\ell\setminus\Lambda_{\ell(1-\varepsilon)})$.

Following the same steps as in the proof of (3.3), we obtain (3.8). We omit further details.
This completes the proof of Lemma 3.3. □

As $\varepsilon>0$ can be chosen arbitrarily small and J and J' are finite and fixed, Lemma 3.3 clearly implies Lemma 3.2. □

In view of Theorem 1.1 and Lemma 3.2, to prove (1.6), it suffices to prove that, in the limit $L\to+\infty$, the difference between the following quantities vanishes

$$\mathbb{P}\left(\left\{\omega;\ \begin{matrix}\Sigma(E,U_1,\ell)=k_1,\cdots,\Sigma(E,U_J,\ell)=k_J\\ \Sigma(E',U_1',\ell)=k_1',\cdots,\Sigma(E',U_{J'},\ell)=k_{J'}'\end{matrix}\right\}\right) \tag{3.5}$$

and

$$\mathbb{P}\left(\left\{\omega;\ \begin{matrix}\Sigma(E,U_1,\ell)=k_1,\\ \vdots\qquad\qquad\vdots\\ \Sigma(E,U_J,\ell)=k_J\end{matrix}\right\}\right)\mathbb{P}\left(\left\{\omega;\ \begin{matrix}\Sigma(E',U_1',\ell)=k_1',\\ \vdots\qquad\qquad\vdots\\ \Sigma(E',U_{J'},\ell)=k_{J'}'\end{matrix}\right\}\right). \tag{3.6}$$

Both terms in (3.5) and (3.6) define probability measures on $\mathbb{N}^{J+J'}$. By Theorems 1.1 and Lemma 3.2, we know that the limit of the term in (3.6) also defines a probability measure on $\mathbb{N}^{J+J'}$. Thus, by standard results on the convergence of probability measures (see e.g. [3]), the difference of (3.5) and (3.6) vanishes in the limit $L\to+\infty$ if and only if, for any $(t_j)_{1\leq j\leq J}$ and $(t_{j'})_{1\leq j'\leq J'}$ real, in the limit $L\to+\infty$, the following quantity vanishes

$$\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,\Sigma(E,U_j,\ell)-\sum_{j'=1}^{J'} t_{j'}\,\Sigma(E',U_{j'},\ell)}\right) - \mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,\Sigma(E,U_j,\ell)}\right)\mathbb{E}\left(e^{-\sum_{j'=1}^{J'} t_{j'}\,\Sigma(E',U_{j'},\ell)}\right).$$

Note that, as the sets $(\Lambda_\ell(\gamma))_{|\gamma|\leq\ell'}$ are two by two disjoint and translates of each other, for a fixed U, the random variables $(X(E,U,\Lambda_\ell(\gamma))_{|\gamma|\leq\ell'}$ are i.i.d. Bernoulli random variables. Thus,

$$\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,\Sigma(E,U_j,\ell)-\sum_{j'=1}^{J'} t_{j'}\,\Sigma(E',U_{j'},\ell)}\right) = \prod_{|\gamma|\leq\ell'}\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,X(E,U_j,\Lambda_\ell(\gamma))-\sum_{j'=1}^{J'} t_{j'}\,X(E',U_{j'},\Lambda_\ell(\gamma))}\right).$$

The Minami estimate (2.2) and the decorrelation estimates (1.7) and (1.8) of Lemmas 1.1 and 1.2 guarantee that, for any $\rho\in(0,1)$, one has, for some $C>0$ independent of γ,

$$\begin{aligned}\sup_{1\leq j<\tilde j\leq J}\mathbb{P}\begin{pmatrix}X(E,U_j,\Lambda_\ell(\gamma))=1\\ X(E,U_{\tilde j},\Lambda_\ell(\gamma))=1\end{pmatrix} &+ \sup_{1\leq j'<\tilde j'\leq J'}\mathbb{P}\begin{pmatrix}X(E',U_{j'},\Lambda_\ell(\gamma))=1\\ X(E',U_{\tilde j'},\Lambda_\ell(\gamma))=1\end{pmatrix}\\ &+\sup_{\substack{1\leq j\leq J\\1\leq j'\leq J'}}\mathbb{P}\begin{pmatrix}X(E,U_j,\Lambda_\ell(\gamma))=1\\ X(E',U_{j'},\Lambda_\ell(\gamma))=1\end{pmatrix}\leq C\left(\frac{\ell}{L}\right)^{d(1+\rho)}.\end{aligned}\tag{3.7}$$

Using this, we compute

$$\begin{aligned}&\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,X(E,U_j,\Lambda_\ell(\gamma))-\sum_{j'=1}^{J'} t_{j'}\,X(E',U_{j'},\Lambda_\ell(\gamma))}\right)\\ &=1+\sum_{j=1}^J(e^{-t_j}-1)\cdot\mathbb{P}(X(E,U_j,\Lambda_\ell(\gamma))=1)\\ &\quad+\sum_{j'=1}^{J'}(e^{-t_{j'}}-1)\cdot\mathbb{P}(X(E',U_{j'},\Lambda_\ell(\gamma))=1)+O\left(\left(\frac{\ell}{L}\right)^{d(1+\rho)}\right).\end{aligned}\tag{3.8}$$

Here, the term $O((\ell/L)^{d(1+\rho)})$ is uniform in γ.
On the other hand, one has

$$\begin{aligned}\mathbb{E}\left(e^{-t_j\,X(E,U_j,\Lambda_\ell(\gamma))}\right)&=1+(e^{-t_j}-1)\cdot\mathbb{P}(X(E,U_j,\Lambda_\ell(\gamma))=1),\\ \mathbb{E}\left(e^{-t_{j'}\,X(E',U_{j'},\Lambda_\ell(\gamma))}\right)&=1+(e^{-t_{j'}}-1)\cdot\mathbb{P}(X(E,U_j,\Lambda_\ell(\gamma))=1).\end{aligned}\tag{3.9}$$

By the Wegner estimate (2.1), we know that

$$\sup_{\substack{1\le j\le J\\ 1\le j'\le J'}}\left[\mathbb{P}(X(E,U_j,\Lambda_\ell(\gamma))=1)+\mathbb{P}(X(E',U_{j'},\Lambda_\ell(\gamma))=1)\right]\le C\left(\frac{\ell}{L}\right)^d. \tag{3.10}$$

Thus, by (3.8), we have

$$\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,X(E,U_j,\Lambda_\ell(\gamma))-\sum_{j'=1}^{J'} t_{j'}\,X(E',U_{j'},\Lambda_\ell(\gamma))}\right)$$
$$=\prod_{j=1}^J\mathbb{E}\left(e^{-t_j\,X(E,U_j,\Lambda_\ell(\gamma))}\right)\prod_{j'=1}^{J'}\mathbb{E}\left(e^{-t_{j'}\,X(E',U_{j'},\Lambda_\ell(\gamma))}\right)\left[1+O\left(\left(\ell L^{-1}\right)^{d(1+\rho)}\right)\right].$$

In the same way, one proves

$$\begin{aligned}\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,X(E,U_j,\Lambda_\ell(\gamma))}\right)&=\prod_{j=1}^J\mathbb{E}\left(e^{-t_j\,X(E,U_j,\Lambda_\ell(\gamma))}\right)\left[1+O\left(\left(\ell L^{-1}\right)^{d(1+\rho)}\right)\right],\\ \mathbb{E}\left(e^{-\sum_{j'=1}^{J'} t_{j'}\,X(E',U_{j'},\Lambda_\ell(\gamma))}\right)&=\prod_{j'=1}^{J'}\mathbb{E}\left(e^{-t_{j'}\,X(E',U_{j'},\Lambda_\ell(\gamma))}\right)\left[1+O\left(\left(\ell L^{-1}\right)^{d(1+\rho)}\right)\right].\end{aligned} \tag{3.11}$$

As $\#\{|\gamma|\le\ell'\}\le C(L\ell^{-1})^d$, we obtain that

$$\begin{aligned}&\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,\Sigma(E,U_j,\ell)-\sum_{j'=1}^{J'} t_{j'}\,\Sigma(E',U_{j'},\ell)}\right)\\ &\qquad=\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,\Sigma(E,U_j,\ell)}\right)\mathbb{E}\left(e^{-\sum_{j'=1}^{J'} t_{j'}\,\Sigma(E',U_{j'},\ell)}\right)\left[1+O\left(\left(\ell L^{-1}\right)^{d\rho}\right)\right].\end{aligned} \tag{3.12}$$

Finally, note that (3.9), (3.10) and (3.11) imply that, for any $(t_j)_{1\le j\le J}$ and $(t_{j'})_{1\le j'\le J'}$, one has

$$\sup_{L\ge1}\left[\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,\Sigma(E,U_j,\ell)}\right)+\mathbb{E}\left(e^{-\sum_{j'=1}^{J'} t_{j'}\,\Sigma(E',U_{j'},\ell)}\right)\right]<+\infty.$$

Hence, by (3.12), as $cL^\alpha\le\ell\le L^\alpha/c$ for some $\alpha\in(0,1)$, we obtain that

$$\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,\Sigma(E,U_j,\ell)-\sum_{j'=1}^{J'} t_{j'}\,\Sigma(E',U_{j'},\ell)}\right)$$
$$-\,\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\,\Sigma(E,U_j,\ell)}\right)\mathbb{E}\left(e^{-\sum_{j'=1}^{J'} t_{j'}\,\Sigma(E',U_{j'},\ell)}\right)\underset{L\to+\infty}{\to}0.$$

This completes the proof of Theorems 1.2 and 1.3.

Remark 3.1. The basic idea we used here is to split the cube Λ into smaller two-by-two disjoint cubes $(\Lambda_\gamma(\ell))_\gamma$ in such a way that, up to exponentially small errors, the eigenvalues of $H_\omega(\Lambda)$ can be represented as eigenvalues for $H_\omega(\Lambda_\gamma(\ell))$ and that they are independent of each other. In [7] (see also [8] for a review of the results), this idea is exploited thoroughly to study the eigenvalue statistics for random operators in the localized regime.

4. Proof of Proposition 2.1

Let I be a compact subset of the region of localization i.e. the region of Σ where the finite volume fractional moment criteria of [1] for $H_\omega(\Lambda)$ are verified for Λ sufficiently large. Then, by (A.6) of [1], we know that there exists $\alpha > 0$ such that, for any $F \subset I$, $\forall (x,y) \subset \Lambda^2$, one has

$$\mathbb{E}(|\mu^{x,y}_{\omega,\Lambda}|(F)) \leq Ce^{-\alpha|x-y|}. \tag{4.1}$$

where $\mu^{x,y}_{\omega,\Lambda}$ denotes the spectral measures of $H_\omega(\Lambda)$ associated to the vector δ_x and δ_y. In particular, if F contains a single eigenvalue of $H_\omega(\Lambda)$, say E, that is simple and associated to the normalized eigenvector, say, φ then

$$|\mu^{x,y}_{\omega,\Lambda}|(F) = |\varphi(x)|\,|\varphi(y)|. \tag{4.2}$$

Pick ε and δ positive such that $\varepsilon|\Lambda|^2 = \delta/K$ for some large K to be chosen below. Then, partition $I = \cup_{1\leq n\leq N} I_n$ into intervals $(I_n)_n$ of length ε. By Minami's estimate, one has

$$\mathbb{P}(\{\omega;\ \exists n \text{ s.t. } I_n \text{ contains 2 e.v. of } H_\omega(\Lambda)\}) \leq C\delta|I|/K \leq \delta/2$$

if $C|I|/K \leq 1/2$. Pick K so that this be satisfied.
We now apply (4.1) to $F = I_n$ for $1 \leq n \leq N$ and sum the results for $s < \alpha$ to get

$$\forall y \in \Lambda, \quad \mathbb{E}\left(\sum_n \sum_{x\in\Lambda} e^{s|x-y|}|\mu^{x,y}_{\omega,\Lambda}|(I_n)\right) \leq C|I|\varepsilon^{-1}.$$

Hence, by Markov's inequality,

$$\mathbb{P}\left(\sum_n \sum_{(x,y)\in\Lambda^2} e^{s|x-y|}|\mu^{x,y}_{\omega,\Lambda}|(I_n) \geq \frac{\tilde{C}|\Lambda|\,|I|}{\delta\varepsilon}\right) \leq \delta/2.$$

Thus, using the relation between δ and ε, with a probability larger than $1-\delta$, we know that

(1) each interval I_n contain at most a single eigenvalue, say, E_n associated to the normalized eigenfunction, say, φ_n;
(2) by (4.2), one has

$$\forall (x,y) \in \Lambda^2, \quad |\varphi_n(x)|\,|\varphi_n(y)| \leq \frac{C|\Lambda|^3 e^{-s|x-y|}}{\delta^2}.$$

As φ_n is normalized, if x_n is a maximum of $x \mapsto |\varphi_n(x)|$, one has

$$|\varphi_n(x_n)| \geq |\Lambda|^{-1/2},$$

thus,

$$\forall x \in \Lambda, \quad |\varphi_n(x)| \leq C|\Lambda|^{7/2}\delta^{-2}e^{-s|x-x_n|}.$$

This yields Proposition 2.1 if one picks $\delta = L^{-p}$ when $\Lambda = \Lambda_L$. □

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(Frédéric Klopp) LAGA, U.M.R. 7539 C.N.R.S, Institut Galilée, Université Paris-Nord, 99 Avenue J.-B. Clément, F-93430 Villetaneuse, France et Institut Universitaire de France

E-mail address: klopp@math.univ-paris13.fr