Path Separability of Graphs

Emilie Diot, Cyril Gavoille

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Abstract

In this paper we investigate the structural properties of k-path separable graphs, that are the graphs that can be separated by a set of k shortest paths. We identify several graph families having such path separability, and we show that this property is closed under minor taking. In particular we establish a list of forbidden minors for 1-path separable graphs.

1 Introduction

"Divide and Conquer" is an common technique in computer science to solve problems. The whole data is separated into different small parts to find a solution in these parts, and then to merge the solutions to obtain the result on the input graph.

A wide theory has been developed for graphs that can be decomposed into small pieces. Such graphs, a.k.a. bounded treewidth graphs, supports polynomial algorithms for many class of problems, whereas no algorithms of complexity better than exponential complexity are known for the general case. This has contributed to new insights into Fixed Parameterized Tractable theory whose consequences for practical algorithms are effective improvements on the running time [AKCF+04, KBvH01].

There are however problems that can be efficient solved even for graphs without small separators (or equivalently of large treewidth). Large separators but "well shaped" reveal very useful for approximation algorithms. For instance, if the separator has a small diameter, or a small dominating set, then distances between vertices can be computed efficiently up to some small additive errors (see [DG07, CDE+08, UY09] for works about the treelength of a graph).

An important observation due to Thorup [Tho04] is that separators consisting of a set of shortest paths are also very useful for the design of compact routing scheme, distance and reachability oracles. More precisely, he used the fact that every weighted planar graph with n vertices has a set of three shortest paths whose deletion split the graph into connected components of at most $n/2$ vertices (a decomposition into components of at most $2n/3$ vertices using two shortest paths was early proved in [LT79]). Using a recursive decomposition, and sampling each such shortest path, he showed that distances between any pair of vertices can

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be approximated up to a factor of $1+\varepsilon$ in polylogarithmic time, for every $\varepsilon > 0$, with an oracle of size[1] $O(\epsilon^{-1} n \log n)$.

This notion of "shortest path" separator has been extended in [AG06]. Roughly speaking, a k-path separator is the union of k shortest paths whose removal halve the graph. The formal definition is actually slightly more complicated and is described in Section 2. The same authors have showed that k-path separable graphs have efficient solutions for several "Object Location Problems" including compact routing schemes, distance oracles, and small-world navigability. Based on the deep Robserton-Seymour's decomposition [RS03], they show in particular that every weighted graph excluding a minor H has a $(1+\varepsilon)$-approximate distance oracle of size $O(\epsilon^{-1} kn \log n)$, where $k = k(H)$ depending only of H. Actually, the oracle can be distributed into balanced labels, each of size $O(\epsilon^{-1} k \log n)$ such that distance queries can be answered from given the source-destination labels only. The graphs excluding a fixed minor is a huge family of graphs including (and not restricted to) bounded treewidth graphs, planar graphs, and graphs of bounded genus.

1.1 Our results

An approximate distance oracle for a graph G is a data-structure that quickly returns, for any source-destination pair, an approximation on cost of a shortest path connecting them. Such data-structures are obtained by preprocessing G where each edge has a weight corresponding to the cost of traversing this edge (or length). However, in practice, the number n of vertices of G is large whereas the number of interesting vertices for which we want approximate the distance is small (say t). Current solutions [AG06, Tho04] provide oracles of size $O(t \log n)$ whereas a space independent of n would be preferable.

Such a compression can be achieved by adding weights on the vertices of the input graph. Typically, interesting vertices receive a weight 1 whereas the others receive a weight 0. A k-path separator on such vertex- and edge-weighted graph is then defined as previously, excepted that the removal of the separator must leave connected components of size at most half the total vertex-weight of the graph. The size of the oracles is improved since $\log t$ recursion levels suffice instead of $\log n$ in the initial formulation.

In this paper we extend the classical notion of k-path separability to edge- and vertex-weighted graphs. In particular, we prove that previous results (e.g., the 3-path separability of planar graphs) still hold in this new framework.

We establish a connection between separators corresponding to the border of a face and k-path separability, and we identify several families of graphs that are 1-path and 2-path separable. We note that most of our proofs are constructive, and lead to polynomial and even linear algorithms.

More interestingly, we show that the family of graphs that are k-path separable for any weight function is minor-closed. Combined with the Graph Minor Theory of Robertson and Seymour [RS04], it follows that the k-path separability can be theoretically tested in cubic time [RS95] for each fixed k, although no algorithm is currently known. Finally, we provide a first step towards the characterization of 1-path separable graphs.

[1] The *size* is actually the number of "distance items" stored in the oracle.

2 Preliminaries

A *minor* of a graph G is a subgraph of a graph obtained from G by edge contraction. We denote by K_r the complete graph (or clique) on r vertices, and $K_{p,q}$ the complete bipartite graph. For convenience, the term *component* is a short for connected component.

A *vertex-weight function* (resp. *edge-weight function*) is a non-negative real function defined on the vertices (resp. edges) of a graph. A non-negative real function applying on both vertices and edges is simply called *weight function*. A weighted graph is graph G having a weight function ω, that we denote also by (G,ω). The *weight* of a subgraph H of G, denoted by $\omega(H)$, is the sum of the weights over the vertices of H.

A *half-separator* for a graph G with vertex-weight function ω is a subset of vertices S such that each component of $G \setminus S$ has weight at most $\omega(G)/2$. Observe that the deletion of a half-separator does not necessarily disconnect the graph.

A *k-path separator* of a weighted graph G is a subgraph $P_0 \cup P_1 \cup \ldots$ where each P_i is a subgraph composed of the union of k_i minimum cost paths in $G \setminus \bigcup_{j<i} P_j$, and where $\sum_i k_i \leqslant k$. A k-path separator is said *strong* if it consists of P_0 only, i.e., composed of the union of k minimum cost paths in G. A weighted graph is (strongly) k-path separable if every induced subgraph has a (strong) k-path separator.

A *tree-decomposition* of a graph G is a tree T whose vertices, called *bags*, are subsets of vertices of G such that:

1. for every vertex u of G, there exists a bag X of T such that $u \in X$;
2. for every edge $\{u,v\}$ of G, there exists a bag X of T such that $u,v \in X$; and
3. for every vertex u of G, the set of bags containing u induces a subtree of T.

An important property following from the last two points is that every path between $u \in X$ and $v \in Y$ in G has to intersect all the bags on the path from X and Y in T. Therefore, the deletion of every bag X disconnects G provided that $T \setminus X$ is composed of more than one subtree and that no bags $Y \subseteq X$.

The *width* of a tree-decomposition T is $\max_{X \in T}\{|X|-1\}$. A *treewidth-t* graph is a graph having a tree-decomposition of width t, and the treewidth of G is the minimum t such that G is a treewidth-t graph.

We will use several times the following basic result.

Lemma 1 *Every tree-decomposition of a vertex-weighted graph has a bag that is a half-separator of the graph. Such a bag is called* center *of the tree-decomposition.*

Proof. Let T be a tree-decomposition of a graph G with vertex-weight function ω. We choose an arbitrary bag R to be the root of T, and we denote by $\pi(X)$ the parent of the bag X in T. For a bag X, we denote by T_X the subtree of T rooted at X. By abuse of notation, T_X denotes also the subgraph induced by all vertices contained in the bags of T_X.

For every bag X we define the value:

$$\rho(X) \;=\; \sum_{v \in T_X \setminus \pi(X)} \omega(v)$$

as the weight of the vertices that are in $T_X \setminus \pi(X)$, with the convention that $\pi(R) = \varnothing$. Note that $\rho(R) = \omega(G)$. Consider a component H of $G \setminus X$. If H belongs to $T \setminus T_X$, then

$\omega(H) \leqslant \rho(R) - \rho(X) = \omega(G) - \rho(X)$. If H belongs to T_F for some child F of X, then $\omega(H) \leqslant \rho(F)$, because the vertices of $\pi(F) = X$ are not in H.

The center C can be found, traversing T from its root R, as follows:

1. If $\rho(F) < \omega(G)/2$ for all the children F of the current bag X, then set $C = X$ and stop.
2. Otherwise, choose a child F of X with $\rho(F) \geqslant \omega(G)/2$, set $X = F$ and go to Step 1.

The above procedure stops after at most the depth of T steps. Indeed, at any time along the traversal $\rho(X) \geqslant \omega(G)/2$, and $\rho(X)$ is non-increasing ($\rho(X) \leqslant \rho(\pi(X))$ provided $X \neq R$). Moreover, whenever it stops at C, $\rho(C) \geqslant \omega(G)/2$ and $\rho(F) < \omega(G)/2$ for all the children of C.

Consider a component H of $G \setminus C$. Either H is contained in $T \setminus T_C$, or it is contained in T_F for some child F of C. In the former case, $\omega(H) \leqslant \omega(G) - \rho(C) \leqslant \omega(G)/2$. In the latter case, $\omega(H) \leqslant \rho(F) < \omega(G)/2$. Hence, C is the center as claimed. □

For the sake of presentation, we prove the following folklore results.

Proposition 1

1. *Every weighted treewidth-t graph is strongly $\lceil (t+1)/2 \rceil$-path separable.*
2. *Every weighted planar graph is strongly 3-path separable.*
3. *Every weighted n-vertex graph is strongly $\lceil n/4 \rceil$-path separable*
4. *The uniform[2] weighted clique K_{4k+1} is not k-path separable.*

Proof.

1. Consider any subgraph H of a weighted graph G. The treewidth of H is at most the treewidth of G. So H has a tree-decomposition of width $\leqslant t$, so with bags consisting of at most $t+1$ vertices. By Lemma 1, the center C of the tree-decomposition is a half-separator. It can be covered by at most $\lceil |C|/2 \rceil \leqslant \lceil (t+1)/2 \rceil$ shortest paths. Therefore, H has a strong $\lceil (t+1)/2 \rceil$-path separator, and thus G is strongly $\lceil (t+1)/2 \rceil$-path separable.

2. It is well-known that every planar graph has a tree-decomposition such that every bag consists of at most three shortest paths. This comes from the well-known fact that every planar graph having a depth-h rooted tree has a tree-decomposition where each bag consists of 3 paths of the tree starting from the root (see [FG06][pp. 305]). By Lemma 1, the center C of the tree-decomposition is a half-separator. So, C forms a strong 3-path separator.

3. Consider any subgraph H of a weighted graph (G, ω) with n vertices. Let W be the smallest set of vertices in H such that $\omega(W) \geqslant \omega(H)/2$. Thus, the components of $H \setminus W$ have weight $\leqslant \omega(H)/2$. It is clear that W contains at most half the vertices of H, i.e., $|W| \leqslant \lceil |V(H)|/2 \rceil$. A set of at most $\lceil |W|/2 \rceil$ shortest paths suffices to cover W. Therefore, H has a strong k-path separator with $k = \lceil \lceil |V(H)|/2 \rceil /2 \rceil \leqslant \lceil n/4 \rceil$, completing the proof of Point 3.

4. Let us show that the uniform weighted K_{4k+1} has no k-path separator. Indeed, every k-path separator S consists of at most $2k$ vertices since every shortest path in a clique consists of an edge. $K_{4k+1} \setminus S$ is a clique on at least $2k+1$ vertices, so S is not a half-separator. □

[2]That is with a unit weight for all vertices and edges.

As remark in [AG06], the k-path separability of minor-free graphs holds also for vertex-weighted graphs. However the formal proof of this result cannot be considered as folklore, and its self-contained proof is currently more than the page limitation of this paper.

One the of unresolved problem we left open is to know whether they are planar graphs that are not 2-path separable.

3 Face-Separable Graphs

As we will see later in Section 4, graphs that are k-path separable have strong structural properties. In particular, planarity plays an important role, at least for $k = 1$ in the light of Theorem 3: all 1-path separable graphs are planar but $K_{3,3}$. In this section we will see that a half-separator of special "shape" implies a low path separability of the graph. Interestingly, this half-separator is defined independently of the shortest path metric of the graph. It only depends on the vertex-weight function.

A half-separator S of a weighted graph G is a *face-separator* if G has a plane embedding such that S is the border of a face. A weighted graph is *face-separable* if every induced subgraph has a face-separator.

By definition, outerplanar graphs are face-separable, since the outerface contains all vertices of the graph. We will see that the family of face-separable graphs includes more general graphs, like the series-parallel graphs, the subdivisions of a K_4 (Proposition 2), and even includes some unbounded treewidth planar graphs (Proposition 3).

The main result of this section is:

Theorem 1 *Every face-separable weighted graph is strongly* 2*-path separable.*

Proof. To prove the result it suffices to show that if a weighted graph has a face-separator, then it has a strong 2-path separator. So, consider any weighted graph (G_0, ω) having a face-separator B_0.

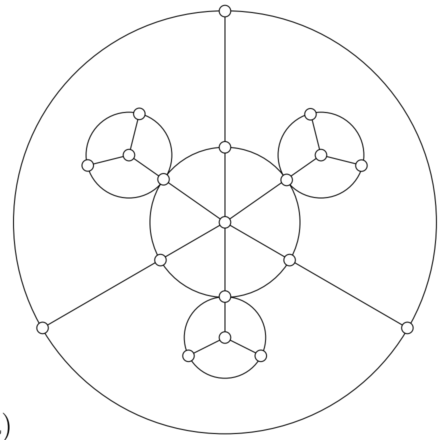
Observe that B_0 is not necessarily a cycle. (See for instance face F in the graph depicted on Fig. 1(b)). It might even be not connected, e.g. if B_0 is the border of the outerface and G_0 is not connected. For technical reason, we will need later in the proof that the face-separator is biconnected. Note also that it might be possible that all bicomponents of G_0 are face-separable but not G_0, as suggested by Fig. 1(a). So, face-separability of a graph does not reduce to face-separability of its bicomponents.

However, we will show in Claim 1 that B_0 contains a cycle which is a face-separator of G_0, as suggested by Fig. 1(b).

Claim 1 *Separator B_0 either contains a vertex or an edge that is a half-separator of G_0, or contains a cycle that is a face-separator of G_0.*

Proof. We concentrate our attention on the heaviest component G of G_0 intersecting B_0. Denote by $B = B_0 \cap G$, the part of the face-separator in component G. Note that B is connected. Moreover, all components of $G_0 \setminus B$ (those of $G \setminus B$ and of $G_0 \setminus G$) have weight $\leqslant \omega(G_0)/2$.

We consider a plane embedding of G such that B is the border of its outerface. Note that every vertex of $G \setminus B$ is drawn in a region delimited by a cycle of B. The bicomponents of



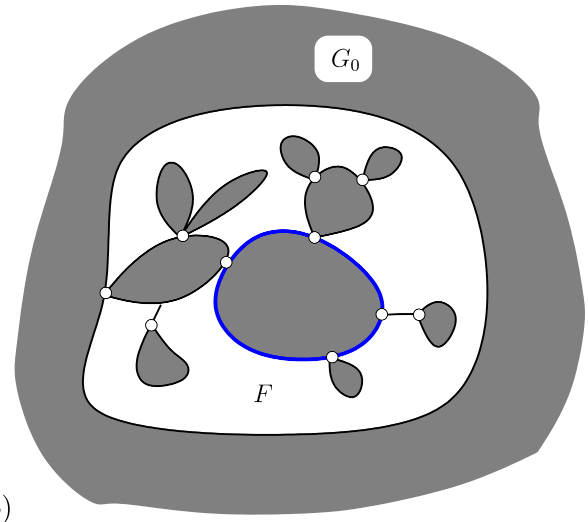


Figure 1: (a) A uniform weighted graph with 19 vertices that is not face-separable whereas all its bicomponents are. (b) Graph G_0 is depicted rayed, and face-separator B_0, which is not biconnected, is the border of face F. The face-separator provided by Claim 1 is drawn blue. To make B the border of a face, the embedding of G_0 has be to changed.

B form a tree of cycles, connected together by cut-vertices or bridges. We consider a tree-decomposition T of B where each bag corresponds to a bicomponent or a bridge of B. One can check that T is indeed a tree-decomposition of B. We fix a bag R as the root of T. Denote by $\pi(X)$ the parent bag of X, and set $\pi(R) = \varnothing$. We denote by $G(X)$ the subgraph induced by the vertices of G that are either in $X \setminus \pi(X)$ or inside the region delimited by X. Note that $\{G(X)\}_X$ forms a partition of the vertices of G, whereas the set of bags of T, $\{X\}_X$, is not a partition of the vertices of B (because of the cut-vertices).

We now define a vertex-weight function ω_B on B in order to find out a suitable half-separator for G. Roughly speaking, we uniformly distribute on the vertices of any bag X the weight (under ω) of the vertices of $G(X)$. More precisely, for every bag X of T, and every vertex $u \in X \setminus \pi(X)$, we set $\omega_B(u) = \omega(G(X))/|V(G(X))|$. Clearly, $\sum_{u \in X \setminus \pi(X)} \omega_B(u) = \omega(G(X))$, and thus $\omega_B(B) = \omega(G)$ by partitioning.

From Lemma 1, T has a center C that is a half-separator of the weighted graph (B, ω_B). The components of $G \setminus (C \cup G(C))$ have weights no more than the weights (under ω_B) of the components of $B \setminus C$, which is $\leqslant \omega_B(B)/2 = \omega(G)/2$. Therefore, $C \cup G(C)$ is a half-separator of G. We observe that the weight of each component of $G(C) \setminus C$ is $\leqslant \omega(G)/2$, since B is a half-separator of G and the vertices of $G(C) \setminus C$ cannot be adjacent to any vertex outside C. Thus C is in fact a half-separator of G. This also a half-separator of G_0 because we have seen that $\omega(G) \leqslant \omega(G_0)$, and all components of $G_0 \setminus G$ have weight $\leqslant \omega(G_0)/2$.

Center C is a bag of the tree-decomposition of B. So, it is composed either of a vertex (if $|B| = 1$), of an edge, or of a cycle. We are done in the first two cases. So, assume that the center C in a cycle. We remark that it is possible to embed component G such that C is the border of the outerface of G. This is due to the fact that all bicomponents attached to C are attached by cut-vertices or bridges. It is possible to redraw each such components inside C by preserving planarity of the embedding. It follows that C is a face-separator of G, and also of G_0. This concludes the proof of the claim. □

If B_0 contains a vertex or an edge that is a half-separator of G_0, then it forms a 1-path separator for G_0 and we are done. So, assume we are in the second case of Claim 1, and let us denote by B the cycle of B_0 that is a face-separator for G_0. We denote by G the component of G_0 containing B. We now redraw graph G on the plane such that the face (say F) whose cycle B is the border is not the outerface.

Let T be any shortest-path spanning tree of G rooted at some vertex r of B. For every vertex u of T, we denote by T_u the path from r to u in T. We shall prove there exist an edge uv of B such that $T_u \cup T_v$ is a half-separator of G_0.

Given two vertices u, v of B, possibly with $u = v$, we denote by $X_{u,v}$ the heaviest component of $G \setminus (T_u \cup T_v)$, and by $B_{u,v} = B \cap X_{u,v}$. We check that $B_{u,v}$ is connected because otherwise by planarity $X_{u,v}$ would intersect T_u or T_v.

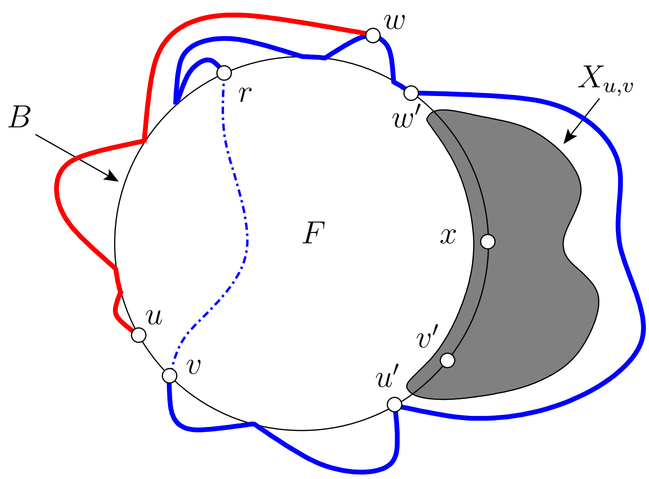


Figure 2: Illustration for the proof of Theorem 1. Path T_v is in blue, path T_u becomes red after its blue common prefix with T_v (w is the nearest common ancestor of u, v). Component $B_{u,v}$ is the segment from u' to w' when traversing B counter-clockwise, u' and w' excluded. Note that the subpath of T_v from u' to w' may be drawn differently (e.g. with $X_{u,v}$ touching the outerface).

We now assume that u, v are chosen neighbors in B such that $|V(B_{u,v})|$ is minimum, i.e., such that the number of vertices of B belonging to the heaviest component of $G \setminus (T_u \cup T_v)$ is minimum.

If $\omega(X_{u,v}) \leqslant \omega(G_0)/2$, then $T_u \cup T_v$ is the willing half-separator for G_0 since $X_{u,v}$ is the heaviest component of $G \setminus (T_u \cup T_v)$ and G is also the heaviest component of G_0, so all the other components of G_0 have weight at most $\omega(G_0)/2$. If $|V(B_{u,v})| = 0$, then $T_u \cup T_v$ is also an half-separator for G_0 because in this case $X_{u,v}$ contains no vertices of B, and B is a half-separator of G_0.

So, assume that $\omega(X_{u,v}) > \omega(G_0)/2$, and $|V(B_{u,v})| \geqslant 1$. Let x be any vertex of $B_{u,v}$, and assume that r, u, v, x, r are encountered in this order when traversing B counter-clockwise. If not, we consider a reverse drawing of G (like in a mirror). This is possible because u, v are neighbors in B, so x is not between u and v. Note that $u = r$ is possible, however $v \neq r$. An important observation is that path T_u is useless to minimize $|V(B_{u,v})|$, path T_v suffices. More formally,

Claim 2 *$X_{v,v} = X_{u,v}$, and thus $B_{v,v} = B_{u,v}$.*

Proof. This is due to the fact that path T_v "separates" u from $X_{u,v}$, i.e., every path from u to any vertex of $X_{u,v}$ has to meet T_v. Indeed, this is trivially true if $u = r$, and otherwise, u and $X_{u,v}$ falls into different regions of the plane delimited by the closed curve defined by T_v and an extra curve in F joining r to v (see the dashed curve on Fig. 2). Thus, no vertex of the subpath $T_u \setminus T_v$ can be adjacent to any vertex of $X_{u,v}$. In other words, in $G \setminus T_v$ there is a component that consists exactly of $X_{u,v}$. So, the heaviest component of $G \setminus T_v$, i.e., $X_{v,v}$, has weight at least $\omega(X_{u,v})$. But only one component of $G \setminus T_v$ can have a weight larger than half the weight of the graph, so $X_{v,v} = X_{u,v}$. □

According to Claim 2, the two vertices delimiting $B_{u,v}$ belong necessarily to T_v. Among these two vertices of T_v, define u' as the closest (on T_v) from v (see Fig. 2), noting that $u' = v$ is possible. In fact, Claim 2 holds not only for v, but also for its ancestor u': $X_{u',u'} = X_{u,v}$ and $B_{u',u'} = B_{u,v}$. The subpath $T_v \setminus T_{u'}$ is useless for minimizing $|V(B_{u,v})|$, path $T_{u'}$ suffices.

Let v' be the neighbor of u' in $B_{u,v}$. Note that $v' \notin T_v$ because $B_{u,v}$ contains at least one vertex. Consider component $X_{u',v'}$, the heaviest component in $G \setminus (T_{u'} \cup T_{v'})$. Either $\omega(X_{u',v'}) \leqslant \omega(G_0)/2$, and $T_{u'} \cup T_{v'}$ is the willing half-separator, or $\omega(X_{u',v'}) > \omega(G_0)/2$. In the latter case, $X_{u',v'}$ is included in $X_{u,v}$ because $X_{u',u'} = X_{u,v}$, because there is only one component of weight larger than half the total weight. It follows that $|V(B_{u',v'})| < |V(B_{u,v})|$ because of v'. This contradict the fact that u, v was selected as an edge of B minimizing $|V(B_{u,v})|$.

This completes the proof of Theorem 1. □

The bound given by Theorem 1 is best possible because there are face-separable graphs that are not 1-path separable. This can be proved by combining Proposition 2 and the fact there are treewidth-2 graphs and subdivisions of K_4 that are not 1-path separable – see Fig. 6.

Proposition 2 *Every weighted treewidth-2 graph or weighted subdivision of K_4 is face-separable.*

Proof. Let (G, ω) be any weighted treewidth-2 graph. It is known that every treewidth-2 graph is a subgraph of a series-parallel graph, and in particular a planar graph. As any subgraph of G is also a treewidth-2 graph, it is sufficient to prove that G has a face-separator.

We consider the graph H obtained from G by adding as many edges as possible while preserving a treewidth-2 graph. Let T be a tree-decomposition of H of width 2, and let C be the center of T. Bag C is composed of a K_3. We embed H in the plane such that C is the border of a face of this embedding. This is possible by moving some subgraph from inside the K_3 to outside. If not, H would contain a K_4-minor, contradicting the fact that H has treewidth 2. We can now remove the edges that have been added to H in order to obtain G, and we consider the border S of the face containing the three vertices of C. Such a face exists since deleting edges can only enlarge the existing faces of a plane embedding. We have

$S \subseteq C$, and C is a half-separator of H (Lemma 1). Note that H has the same total weight of G (we have added only edges). It follows that S is a half-separator for G. This completes the first part of the statement of the proposition.

Consider now a subdivision G of K_4 having a vertex-weight function ω, and H be an induced subgraph of G. If H is a proper subgraph of G (i.e., $H \neq G$), then H is outerplanar and thus has a face-separator. So, assume that $H = G$.

We assume given a plane embedding of H. We denote by $v_1, \ldots, v_4$ the four degree-3 vertices of H, and by $P_{i,j}$ the path between v_i and v_j, for all $i, j \in \{1, \ldots, 4\}$. Let $w_i = \omega(v_i)$, and let $p_{i,j} = \omega(P_{i,j} \setminus \{v_i, v_j\})$ be the sum of the weights of vertices on $P_{i,j}$ excluding its extremities.

Let assume that H has no face-separator. There are four possible faces $F_1, \ldots, F_4$, each one bordered by three paths. Faces are ordered such that whenever the border of F_i is removed, the remaining component is composed of three paths sharing vertex v_i. The total weight of this component is $w_i + \sum_{j \neq i} p_{i,j}$. As the border of F_i is not a face-separator, we have $w_i + \sum_{j \neq i} p_{i,j} > \omega(H)/2$. This holds for each $i \in \{1, \ldots, 4\}$. Summing these four equations, it turns out (observe that each path $p_{i,j}$ occurs twice in this sum):

$$\sum_i w_i + 2\sum_{i \neq j} p_{i,j} \; > \; 2\omega(H) = 2\left(\sum_i w_i + \sum_{i \neq j} p_{i,j}\right) .$$

It implies $0 > \sum_i w_i$: a contradiction, by definition $\omega(v) \geqslant 0$ for each vertex v. Therefore, one of the face F_i is a face-separator for H, that completes the proof. □

Up to now, the graphs we have proved to be face-separable are all of treewidth $\leqslant 3$. From Proposition 1 (Point 1), they are 2-path separable. It is natural to ask whether all face-separable graphs have such a low treewidth property. We answer negatively to this question.

Proposition 3 *For every n, there is a uniform weighted face-separable graph with at most n vertices whose treewidth is $\Omega(\log \log n)$.*

Proof. The proof is based on the construction of a graph called G_p, for integral $p \geqslant 1$. It has treewidth at least $k = p - O(\log \log p)$ because we can show it contains a $k \times k$-grid minor, and the number of vertices of G_p is $n < 2^{2^p}$. In other words, the treewidth of G_p is at least $\log \log n - O(\log^{(4)} n)$.

Graph G_p is composed of a tree T_p of depth p where each vertex of depth $i < p$ has exactly $d(i)$ children, for some function d defined later. Furthermore, for each depth i, a path linking all depth-i vertices is added to T_p to form G_p. Let us denote by $L(i)$ the number of depth-i vertices in T_p. The values $L(i)$ and $d(i)$ obey to the following induction: $L(0) = 1$ and $L(i) = L(i-1) \cdot d(i-1)$, where $d(i) = \sum_{j=0}^{i} L(j)$. The first values of $L(i)$ and $d(i)$ are given in the table hereafter, and G_4 is depicted on Fig. 3.

To prove Proposition 3, we show that every subgraph H of G_p contains a face-separator. An important property we use is that in G_p, the number $d(i)$ of children for a vertex of depth i is at least the vertex number of the graph induced by T_{i-1}. The key point is that H is either outerplanar, or there must exist a vertex v of depth i in T_p such that all its children belongs to the border of the outerface of H. In the first case, H is trivially face-separable. In

i	0	1	2	3	4	5	...
L	1	1	2	8	96	10368	...
d	1	2	4	12	108	10464	...

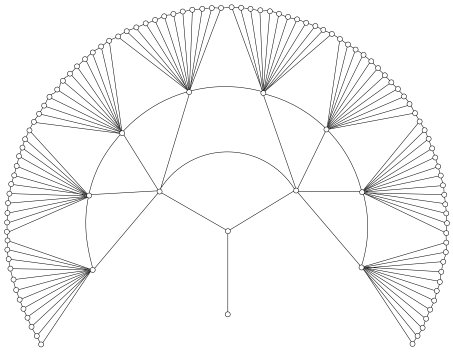


Figure 3: The graph G_4 with 108 vertices.

the second one, and using the property on $d(i)$, we derive that at least half the vertices of H lie on the outerface. □

4 The Hierarchy of Separable Graphs

For every integer $k \geqslant 1$, we denote by PS_k the family of all the graphs G that are k-path separable for every weight function ω. More formally,

$$\mathrm{PS}_k \;=\; \{G \mid \forall\omega, (G,\omega) \text{ is } k\text{-path separable}\}\ .$$

We define similarly the family SPS_k of all the graphs that are strongly k-path separable for every weight function.

We have seen that every weighted planar graph is strongly 3-path separable. In other words, planar graphs are in SPS_3. In Section 3, we have seen that treewidth-2 graphs are face-separable, and thus strongly 2-path separable. Thus this family is in SPS_2. We will show in Proposition 6 that outerplanar graphs are in PS_1. Obviously, families PS_1 and SPS_1 coincide.

Clearly, for each k, $\mathrm{SPS}_k \subset \mathrm{PS}_k$ since a strongly k-path separator is a particular k-path separator. Also, the hierarchies $\mathrm{PS}_1 \subset \cdots \subset \mathrm{PS}_k$ and $\mathrm{SPS}_1 \subset \cdots \subset \mathrm{SPS}_k$ are strict because of the complete graph. By Proposition 1 (points 3 and 4), $K_{4k+1} \in \mathrm{SPS}_{k+1}$ and $K_{4k+1} \notin \mathrm{PS}_k$. The family PS_k is however much larger than SPS_k as suggested by the next proposition.

Proposition 4 *For each $k > 4$, there is a graph A_k with $O(k^2)$ vertices such that $A_k \notin \mathrm{SPS}_k$, but $A_k \in \mathrm{PS}_4$.*

Proof. Consider the graph A_k composed of a $(2k+1) \times (2k+1)$-grid in which a vertex v is connected to all the vertices of the grid. A_k has $O(k^2)$ vertices. Edges incident to v have weights 1/2, whereas all other vertex or edge weights are unitary. Graph A_k has no strong k-path separator. The removal of any set of k shortest paths deletes at most $2k+1$ different vertices: all of them go thru v. However, $2k+2$ vertices are required to halve the graph because its treewidth is $2k+1$. Therefore, $A_k \notin \mathrm{SPS}_k$. However, for every weight function ω, (A_k,ω) has a 4-path separator. The first path consists of the universal vertex v, and the three others are defined as in the planar case (since $A_k \setminus \{v\}$ is planar, and thus 3-path separable).

Therefore, $A_k \in \mathrm{PS}_4$. □

For the study of PS_k and SPS_k graphs families, the next proposition tell us that we can restrict our attention to only k-path separator of biconnected graphs.

Proposition 5 *A graph G belongs to PS_k (resp. SPS_k) if and only if every weighted bicomponent of G has a (resp. strong) k-path separator.*

Proof. Because the proof is similar for PS_k and SPS_k, we denote by $P_k \in \{\mathrm{PS}_k, \mathrm{SPS}_k\}$ any of these families. Assume $G \in P_k$. It is clear from the definitions of k-path separability and of P_k that every induced weighted subgraph of G must have a k-path separator. Let us show the other way.

Assume that every weighted bicomponent of G has a k-path separator. We will show that $G \in P_k$. For that we need to show that any weighted induced subgraph of G, say (H, ω_H), has a k-path separator. We restrict our attention to the case where H is connected (otherwise it suffices to consider the largest component of H). Consider the weight function ω_G for G defined by: for every vertex $x \in H$, we set $\omega_G(x) = \omega_H(x)$, and $\omega_G(x) = 0$ otherwise. Observe that $\omega_G(G) = \omega_H(H)$. For every edge $e \in H$, we set $\omega_G(e) = \omega_H(e)$, and $\omega_G(e) = 1 + \sum_{e' \in H} \omega_H(e')$ for all other edges. Note that any shortest path in G between two vertices of H can use only edges of H.

It is not difficult to see that the decomposition in which each bag consists of a bicomponent of G is a tree-decomposition of G. As a consequence (Lemma 1), the center C of this tree-decomposition is a bicomponent of G, and a half-separator for (G, ω_G). The vertices of $C \cap H$ are also a half-separator for (H, ω_H).

Let us define the vertex-weight function ω_C for C as follows. For every edge $e \in C \cap H$, we set $\omega_C(e) = \omega_H(e)$, and wet set $\omega_C(e) = 1 + \sum_{e' \in G} \omega_G(e')$ for all the other edges. Note that any shortest path in G between two vertices of $C \cap H$ can use only edges of H. For each vertex x of $C \cap H$, we set $\omega_C(x) = \omega_H(x) + \sum_{v \in H(C,x)} \omega_H(v)$ where $H(C, x)$ is the set of vertices of H that are connected to x by a path of H that does not intersect any vertex of C (excepted x). We set $\omega_C(x) = 0$ for all other vertices. Note that $H(C, x) = \varnothing$ if x is not a cut-vertex of G. Note also that $\omega_C(C) = \omega_H(H)$. Since every weighted bicomponent of G has a k-path separator, then in particular, (C, ω_C) has a k-path separator, say S. We remark that each path composing S intersects vertices of H into a sub-path. This is due to the fact that H is connected and once the path enter a node of H it cannot leave H (edges not in H have too much large weight). So, $S \cap H$ is composed of k paths. We set $S_H = S \cap H$.

It remains to show that S_H is a k-path separator for (H, ω_H). The weights of each component of $C \setminus S$ that appears in $H \setminus S_H$ are the same. Hence, these weights are at most $\omega_H(H)/2$ by construction. However, there are components of $H \setminus S_H$ that are not attached to any component of $C \setminus S$. This occurs each time that such a component of $H \setminus S_H$, say X, is attached by some cut-vertex of G belonging to S. Since C is the center of (G, ω_G), $\omega_G(X) \leqslant \omega_G(G)/2$. However, we have seen that $\omega_G(X) = \omega_H(X)$ and $\omega_G(G) = \omega_H(H)$. Hence $\omega_H(X) \leqslant \omega_H(H)/2$. Therefore, S_H is a half-separator for (H, ω_H) as required, and thus a k-path separator for (H, ω_H). □

4.1 Closed under minor taking

The remarkable property of PS_k and SPS_k families is they are closed under minor taking. From the Graph Minor Theorem, such families can be characterized by a finite list of forbidden minors, and membership of a given graph in one of these families can be done in time $O(n^3)$ for fixed k.

Theorem 2 *For each integer $k \geqslant 1$, the families PS_k and SPS_k are minor-closed.*

Proof. We give the proof for the family SPS_k, the proof for PS_k is similar. Let H be any minor of a graph G. We will prove that if G is k-path separable, then H is k-path separable too. To prove that H is k-path separable, we need to prove that H has a k-path separator for every induced subgraph of H. However, since every subgraph of H is also a minor of G, we simply show that H has a k-path separator.

It is not difficult to see that if H is a minor of G, then with each vertex u of H we can associate a connected subgraph of G, called *super-node* of u, such that if (u, v) is an edge of H, then there exists an edge of G, called *super-edge* of (u, v), connecting a vertex of the super-node of u and a vertex of the super-node of v. (If there are several such edges we select only one.) The super-nodes must be pairwise disjoint (see Fig. 4).

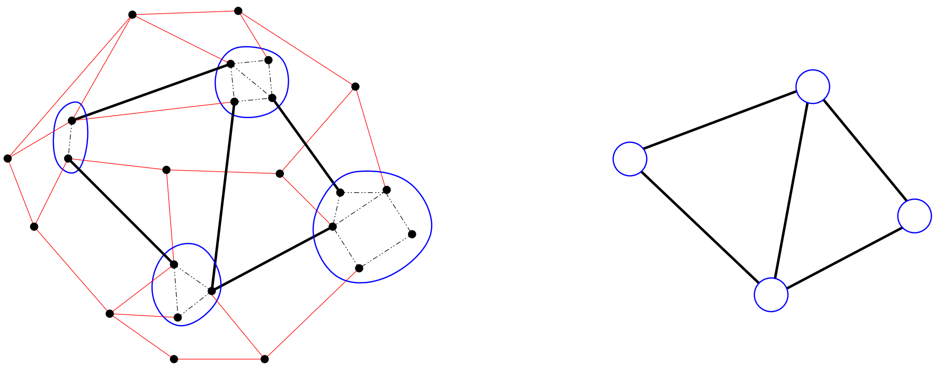


Figure 4: A graph G and a minor H.

Let ω_H be any weight function on H. From ω_H, we construct a weight function ω_G on G as follows. For every edge (x, y) of G that is a super-edge of (u, v) (colored black on Fig. 4), we set $\omega_G(x, y) = \omega_H(u, v)$. For every edge (x, y) of G such that x and y both belongs to the same super-node (called *internal-edge* and dashed on Fig. 4), we set $\omega_G(x, y) = 0$. And, for all other edges (x, y) of G (called *external-edge* and colored red on Fig. 4), we set $\omega_G(x, y) = 1 + \sum_{e \in E(H)} \omega_H(e)$, so that the cost of a path in G using any such edge is strictly larger than the cost of any simple path in H. The weight of a vertex x that belongs to the super-node of u is $\omega_G(x) = \omega_H(u)/t_u$, where t_u is the number of vertices of the super-node of u. Note that the sum of weights of the vertices of the super-node of u is precisely $\omega_H(u)$. The weight of all other vertices is 0. Observe that $\omega_G(G) = \omega_H(H)$.

Since $G \in \mathrm{SPS}_k$, the weighted graph (G, ω_G) has a k-path separator S_G consisting of k shortest paths in G. Let $H_0, H_1, \ldots$ be the components of H, and assume that $\omega_H(H_0)$ is maximum. With each path P of S_G that intersects a super-node of a vertex of H_0, we associate a path Q in H_0 as follows. Let $(U_0, \ldots, U_t)$ be the ordered sequence of all the super-nodes of vertices of H_0 traversed by P. We denote by u_i the vertex of H_0 such U_i is the super-node of u_i. Path Q is obtained by adding an edge between u_{i-1} to u_i, for each $i \in \{1, \ldots, t\}$. We claim that the set composed of each path Q constructed from P as above, and denote by S_H, is a half-separator of H.

First, let us show that Q is a shortest path in H_0 (and thus in H). Path P between the last vertex of U_{i-1} and the first vertex of U_i consists of the super-edge of (u_{i-1}, u_i), because H_0 is connected and the weight of this super-edge is less than the weight of any external-edges. Thus Q is a path in H_0. Now, assume that there exists a path Q' in H_0, from u_0 to u_t, that is shorter than Q. Then, from Q' we can construct a shorter path in G (shorter than P) from the last vertex of U_0 to the first vertex of U_t. This is due to the fact that each super-node is connected and internal-edges have weight 0. This contradicts that P is a shortest path, hence Q is a shortest path in H_0.

It remains to show that S_H is a half-separator of H. Observe that for $i \neq 0$, $\omega_H(H_i) \leqslant \omega_H(H)/2$ because $\omega(H_0)$ is maximum. Let X_{H_0} be the set of vertices of any component in $H_0 \setminus S_H$. Then, there must exists a component X_G in $G \setminus S_G$ wholly containing all the super-nodes of the vertices of X_{H_0}. Let v be a vertex of X_{H_0} whose its super-node belongs to none component of $G \setminus S_G$. Then, there exists a vertex of this super-node that is in S_G. From our construction, v belongs to S_H (vertices of Q and super-nodes of P correspond): contradiction. Therefore, $\omega_H(X_{H_0}) \leqslant \omega_G(X_G)$. Moreover, $\omega_G(X_G) \leqslant \omega_G(G)/2 = \omega_H(H)/2$. Thus, S_H is a half-separator of H, completing the proof. □

4.2 One-path separable graphs

In this part, we concentrate our attention to the graphs that belong to PS_1. From Proposition 2, outerplanar graphs and subdivisions of outerplanar graphs are face-separable (they are treewidth-2 graphs), and thus belong to PS_2 (and even to SPS_2). Actually, outerplanar graphs are in PS_1:

Proposition 6 *Every weighted outerplanar graph is* 1*-path separable.*

Proof. Let (G, ω) be any weighted outerplanar graph. Since outerplanar graphs are hereditary family, i.e., closed under induced subgraphs taking, it suffices to show that (G, ω) has a 1-path separator.

Let r be any vertex of G. We assume that a outerplane embedding of G is given. From r, we traverse all the vertices by following the border of the outerface, and we denote by v_i the i-th vertices encountered. We have $v_1 = r$. Let i_0 be the smallest integer such that $\sum_{i=1}^{i_0} \omega(v_i) \geqslant \omega(G)/2$. We consider a shortest path P between r and v_{i_0}.

We shall prove that P is a 1-path separator. Let A be $\{v_i \mid i < i_0\}$ and B be $\{v_i \mid i > i_0\}$. Note that $\omega(A)$ and $\omega(B) \geqslant \omega(G)/2$. So, we just need to prove that there is no edge between vertices of A and vertices of B. We will prove this by contradiction.

Let a be a vertex of A and b a vertex of B such that (a, b) is an edge. The edge (a, b) cannot cross P by planarity. Thus, the edge belongs to the border of the outerface, which implies that either v_{i_0}, or r is not on the border of the outerface: contradiction. □

Unfortunately, Proposition 6 does not generalize to treewidth-2 graphs. As depicted on Fig. 6, there are simple series-parallel graphs and subdivisions of K_4 that are not in PS_1.

The family PS_1 does not reduce to outerplanar graphs, as shown in the next proposition (Proposition 7). A *globe graph* is a subdivision of $K_{2,r}$, for some r, in which the two degree-r vertices may be adjacent.

Proposition 7 *Every weighted globe graph is 1-path separable.*

Proof. Let (G, ω) be any weighted globe graph. According to Proposition 5, it suffices to show that (G, ω) has a 1-path separator to prove that $G \in \mathrm{PS}_1$.

Let $P_1, \ldots, P_r$ be the r paths composing G, and let us denote by x, y the common extremities of all the P_i's. We assume that $\omega(P_1)$ is maximum over all P_i's. Consider a shortest path spanning tree T rooted at x.

If all the edges of P_1 are in T, then we select P_1. It is a shortest path and a half-separator since the components of $G \setminus P_1$ consists of all the $P_i \setminus \{x, y\}$ for $i > 1$, each one having a weight $\leqslant \omega(G)/2$ (otherwise this would contradict that $\omega(P_1)$ is maximum).

Otherwise, let (a, b) be an the edge of P_1 that is not in T. There is only one such edge in P_1, otherwise T would be not connected. When going from x to y on P_1 we assume that a is traversed before b. Denote by A the path in T from x to a, and by B the path in T from x to b. Note that A, B are shortest paths in G.

If $\omega(A) \geqslant \omega(G)/2$, then we select A. Any component of $G \setminus A$ have weight at most $\omega(G) - \omega(A) \leqslant \omega(G)/2$. Finally, if $\omega(A) < \omega(G)/2$, then we select B. The components of $G \setminus B$ are $A \setminus \{x\}$ and $P_i \setminus \{x, y\}$ for all $i > 1$, and thus all of them have weight $\leqslant \omega(G)/2$. In all cases, we have selected a half-separator for G. □

A first attempt to characterize PS_1 is given by Theorem 3.

Theorem 3 *Every biconnected graph of* PS_1 *is either isomorphic to* $K_{3,3}$*, or planar and excludes the list of minors depicted on Fig. 6.*

Proof. Let (G, ω) be a biconnected weighted graph with $G \in \mathrm{PS}_1$. First assume that G is non planar. From Kuratowski's criteria, G contains a subdivision of K_5 or $K_{3,3}$.

The complete graph K_5 is not 1-path separable graph from Proposition 1 (cf. Point 4 with $k = 2$). From Theorem 2, it follows that G cannot contain a subdivision of K_5, so it must contain a subdivision of $K_{3,3}$.

We shall proof that the only non planar graph in PS_1 is $K_{3,3}$. We show that any proper subdivision of $K_{3,3}$ or super-graph of $K_{3,3}$ is not in PS_1.

Denote by M a $K_{3,3}$ in which only one edge is subdivided into two edges. We set unitary all the weights so that the total vertex-weight is 7. The diameter of M is two. The deletion of any shortest path deletes at most three vertices. Moreover, such a deletion cannot disconnect M, and thus leaves a component with at least $4 > 7/2$ vertices. M is not 1-path separable.

Denote by M' a super-graph of $K_{3,3}$ composed of $K_{3,3}$ with only one more edge. We denote by $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ the vertex set of each part of $K_{3,3}$. The weight function ω for M' is set as follows (cf. Fig. 5): $\omega(x_i) = 2$, $\omega(y_2) = \omega(y_3) = 3$, $\omega(x_1, y_2) = \omega(x_2, y_3) = 2$, all the other weights are unitary. The total vertex-weight of M' is $\omega(M') = 13$.

Consider any path P in M'. There are three cases:

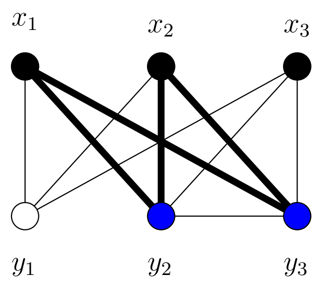


Figure 5: The weighted graph M': a $K_{3,3}$ plus one edge. Black vertices and bold edges have weight 2, blue vertices have weight 3, other vertices and edges have weight 1.

1. P connects x_i to x_j: it goes thru y_1 and has length 2, and leaves a component of weight $2+3+3=8 > \omega(M')/2 = 6.5$.
2. P connects two adjacent vertices: it has length 1, of weight at most 6, thus it leaves a component of four vertices of weight $> 7 > \omega(M')/2$.
3. P connects y_1 to y_2 or y_3: it has length 2, and leaves a component of weight $3+2+2 = 7 > \omega(M')/2$.

In all cases, $M' \setminus P$ has a component of weight $> \omega(M')/2$. Thus M' has no 1-path separator. Therefore, if G is not planar, then G can only be isomorphic to $K_{3,3}$.

We prove now that $K_{3,3} \in \mathrm{PS}_1$. According to Proposition 5, it suffices to show that every weighted $K_{3,3}$ has a 1-path separator. W.l.o.g. assume that $\omega(x_1) \geqslant \omega(x_2) \geqslant \omega(x_3)$.

Define P_1 be a shortest path from x_1 to x_2, and assume P_1 contains y_{i_1}. Define P_2 be a shortest path from y_{i_2} to y_{i_3} (with i_1, i_2, i_3 pairwise different indices), and assume it contains x_{j_1} (denote by j_2, j_3 the two other x's indices). We show that P_1 or P_2 is a 1-path separator. By contradiction, if P_1 is not a half-separator, then (and similarly for P_2) $\omega(P_1) < \omega(K_{3,3})/2$ and $\omega(G \setminus P_1) > \omega(K_{3,3})/2$. As $\omega(P_1)$ is lower bounded by $\omega(x_1)+\omega(x_2)+\omega(y_{i_1})$ and $\omega(G \setminus P_1)$ upper bounded by $\omega(x_3)+\omega(y_{i_2})+\omega(y_{i_3})$ (and similarly for P_2), it follows that:

$$\begin{aligned} \omega(x_1)+\omega(x_2)+\omega(y_{i_1}) &< \omega(x_3)+\omega(y_{i_2})+\omega(y_{i_3}) & (1) \\ \omega(x_{j_1})+\omega(y_{i_2})+\omega(y_{i_3}) &< \omega(x_{j_2})+\omega(x_{j_3})+\omega(y_{i_1}) & (2) \end{aligned}$$

By summing these equations, we obtain:

$$\begin{aligned} & \omega(x_1)+\omega(x_2)+\omega(x_{j_1}) < \omega(x_3)+\omega(x_{j_2})+\omega(x_{j_3}) \\ \Rightarrow\quad & \omega(x_1)+\omega(x_2)+\omega(x_3) < \omega(x_3)+\omega(x_{j_2})+\omega(x_{j_3}) \leqslant \omega(x_3)+\omega(x_2)+\omega(x_1) \end{aligned}$$

since, by assumption, $\omega(x_3) \leqslant \omega(x_{j_1})$ and $\omega(x_{j_2})+\omega(x_{j_3}) \leqslant \omega(x_2)+\omega(x_1)$. This leads to a contradiction. Thus, P_1 or P_2 is a 1-path separator for $K_{3,3}$.

For planar graphs, we manage to find forbidden minors represented in Fig. 6. To prove that each minor M of this list is indeed excluded, we exhibit a particular weight function ω for M. Actually, each vertex and edge has weight 1 or 2 as depicted on Fig. 6. We then exhaustively check that, for each pair u, v of vertices of M, the deletion of any shortest path from u to v leaves a component of weight $> \omega(M)/2$.

To illustrate this, consider for instance the "wheel graph", composed of a cycle of length 5 and a degree-5 vertex, called hereafter center, connected of all vertices of the cycle. The total weight of the graph is 7, the center has weight 2. Any shortest path from the center to a non-center vertex consists of one edge. Therefore its deletion leaves a path of 4 vertices, so of weight 4. Any shortest path between two non-center vertices consists of 2 edges at most, so leaving a component with the center and two (or more) non-center vertices, thus of weight at least 4. In both cases, the weight is $> 7/2$. This graph has no half-separator composed of a shortest path, and thus is not in PS_1. □

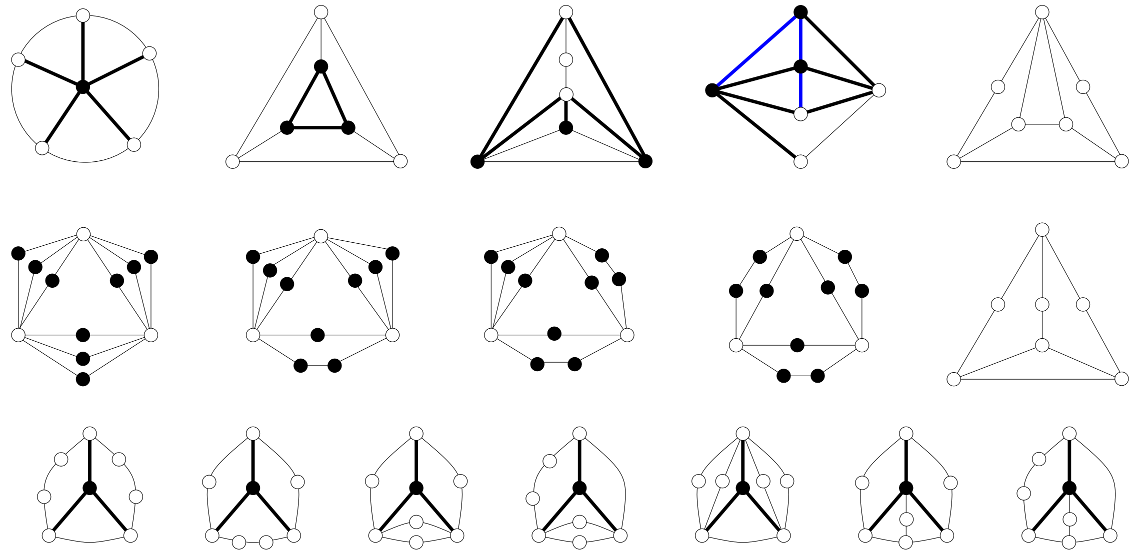


Figure 6: Forbidden minors for planar graphs in PS_1. Black vertices and bold edges have weight 2, blue edges have weight 3, other vertices and edges have weight 1. There are exactly three non-planar forbidden minors for PS_1 (non depicted in the figure): K_5, $K_{3,3}$ plus one edge, and $K_{3,3}$ whose one edge is subdivided into two edges.

5 Conclusion

In this paper we have investigated the family of graphs that are k-path separable. Graph Minor Theory implies that such a family can be characterized by a finite set of forbidden minors that we have started to list for $k = 1$.

We propose here a list of further researches.

1. Determine the full list of forbidden minors for 1-path separable graphs.
2. Find an explicit polynomial time algorithm to determine if a graph is k-path separable, for fixed k.
3. Prove or disprove that planar graphs are 2-path separable.
4. Prove NP-completeness for the problem of determining whether a given *weighted* graph has a k-path separator.
5. Extend the study to more general isometric separators, not only shortest paths.

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