(VERY) SHORT PROOF OF RAYLEIGH'S THEOREM (AND EXTENSIONS)

OLIVIER BERNARDI

Abstract. Consider a walk in the plane made of n steps of length 1, with directions chosen independently and uniformly at random at each step. Rayleigh's Theorem asserts that the probability for such a walk to end at a distance less than 1 from its starting point is $1/(n+1)$. We give an elementary proof of this result. We also prove the following generalization valid for any probability distribution μ on the positive real numbers: if two walkers start at the same point and make respectively i and j independent steps with uniformly random directions and with lengths chosen according to μ, then the probability that the first walker ends farther than the second is $i/(i+j)$.

We consider a random walk in the plane starting from the origin and made of independent steps $s_1, s_2, \dots, s_n$ with lengths given by some real positive random variables $X_1, X_2, \dots, X_n$ and with directions chosen uniformly at random. We denote by $X_1 * X_2 * \dots * X_n$ the random variable corresponding to the distance between the origin and the end of the walk. Rayleigh's Theorem asserts that if each step has length $X_i = 1$, then this random variable satisfies

$$\mathbb{P}(X_1 * X_2 * \dots * X_n < 1) = \frac{1}{n+1}$$

for all $n > 1$. This theorem was first derived from Rayleigh's investigation of "random flights" in connection with Bessel functions (see [3]) and appears as an exercise in [2, p.104][1]. A simpler proof was given by Kenyon and Winkler as a corollary of their investigation of branched polymers [1]. The goal of this note is to give an elementary proof of the following generalization of Rayleigh's Theorem:

Theorem 1. *Let X be a random variable taking real positive values, and let i, j be non-negative integers satisfying $i + j > 2$. If X^{*i} and X^{*j} denote independent random variables distributed respectively like $X_1 * \dots * X_i$ and $X_1 * \dots * X_j$ where the X_k's are independent copies of X, then*

$$\mathbb{P}(X^{*i} > X^{*j}) = \frac{i}{i+j}.$$

In words, if two random walkers start at the same point and take respectively i and j independent steps with uniformly random directions and with lengths chosen according to the distribution of X, then the probability that the first walker ends farther than the second is $i/(i+j)$

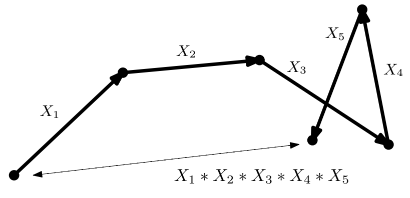

Figure 1. Distance $X_1 * X_2 * X_3 * X_4 * X_5$ achieved after 5 steps.

Date: May 20, 2010.

[1]The exercise calls for developing the requisite Fourier analysis for spherically symmetric functions in order to obtain an identity involving Bessel functions.

Our proof starts with a lemma based on the fact that the angles of a triangle sum to π.

Lemma 2. *For any random variables A, B, C taking positive values,*

$$\mathbb{P}(A > B * C) + \mathbb{P}(B > A * C) + \mathbb{P}(C > A * B) = 1.$$

Proof. We condition on the values a, b, c of A, B, C and show that $\mathbb{P}(a > b*c) + \mathbb{P}(b > a*c) + \mathbb{P}(c > a * b) = 1$ (the probability being taken on the angles). If one of the lengths a, b, c is greater than the sum of the others, then one of the probabilities is 1 and the others are 0. In the opposite case, we consider the angles of the triangle with side lengths a, b, c. The probability $\mathbb{P}(a > b * c)$ is equal to θ_a/π, where θ_a is angle between the edges of length b and c (because $a > b * c$ if and only if the angle between the step of length b and the step of length c is less than θ_a in absolute value). Summing this relation for the three probabilities gives

$$\mathbb{P}(a > b * c) + \mathbb{P}(b > a * c) + \mathbb{P}(c > a * b) = \frac{\theta_a + \theta_b + \theta_c}{\pi} = 1.$$

□

We consider independent copies $X_i, i \in \mathbb{N}$ of the random variable X, and independent random variables $X^{*i}, i \in \mathbb{N}$ distributed like $X_1 * \cdots * X_i$. For all integers $i \leq n$, we denote by $P_X(i, n)$ the probability $\mathbb{P}(X^{*i} > X^{*(n-i)})$ and want to prove $P_X(i, n) = i/n$. Let i, j, k be non-negative integers summing to n. Applying Lemma 2 to $A = X^{*i}$, $B = X^{*j}$, $C = X^{*k}$ gives

$$P_X(i, n) + P_X(j, n) + P_X(k, n) = 1.$$

Moreover, since $P_X(k, n) = 1 - P_X(n - k, n)$ for all $n > 2$, one gets

$$P_X(i, n) + P_X(j, n) = P(i + j, n),$$

for all i, j such that $i + j \leq n$. In particular, $n\, P_X(1, n) = P_X(n, n) = 1$, hence $P_X(1, n) = 1/n$. And more generally, $P_X(i, n) = i\, P_X(1, n) = i/n$. This concludes the proof of Theorem 1. □

Acknowledgments: I thank Peter Winkler for extremely stimulating discussions.

References

[1] R. Kenyon and P. Winkler. Branched polymers. *Am. Math. Monthly*, 7:612–628, 2009.
[2] F. Spitzer. *Principle of random walk.* Van Nostrand, 1964.
[3] G.N. Watson. *A treatise on the Theory of Bessel functions.* Cambridge U. Press, 1944.