Vector Addition System Reachability Problem In Less Than 7 Pages

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Abstract

The reachability problem for Vector Addition Systems (VASs) is a central problem of net theory. The general problem is known decidable by algorithms exclusively based on the classical Kosaraju-Lambert-Mayr-Sacerdote-Tenney decomposition. Recently from this decomposition, we deduced that a final configuration is not reachable from an initial one if and only if there exists a Presburger inductive invariant that contains the initial configuration but not the final one. Since we can decide if a Preburger formula denotes an inductive invariant, we deduce from this result that there exist checkable certificates of non-reachability. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by enumerating finite sequences of actions and a second one that tries to prove the non-reachability by enumerating Presburger formulas. In this paper we provide the first proof of the VAS reachability problem that is not based on the classical Kosaraju-Lambert-Mayr-Sacerdote-Tenney decomposition. The new proof is based on the notion of productive sequences inspired from Hauschildt that directly provides the existence of Presburger inductive invariants.

1. Introduction

Vector Addition Systems (VASs) or equivalently Petri Nets are one of the most popular formal methods for the representation and the analysis of parallel processes [EN94]. The reachability problem is central since many computational problems (even outside the parallel processes) reduce to the reachability problem. Sacerdote and Tenney provided in [ST77] a partial proof of decidability of this problem. The proof was completed in 1981 by Mayr [May81] and simplified by Kosaraju [Kos82] from [ST77, May81]. Ten years later [Lam92], Lambert provided a more simplified version based on [Kos82]. This last proof still remains difficult and the upper-bound complexity of the corresponding algorithm is just known non-primitive recursive. Nowadays, the exact complexity of the reachability problem for VASs is still an open-problem. Even an elementary upper-bound complexity is open. In fact, the known general reachability algorithms are exclusively based on the Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition.

Recently [Ler09] we proved thanks to the KLMST decomposition that Parikh images of languages accepted by VASs are semi-pseudo-linear, a class that extends the Presburger sets. An application of this result was provided; we proved that a final configuration is not reachable from an initial one if and only if there exists a forward inductive invariant definable in the Presburger arithmetic that contains the initial configuration but not the final one. Since we can decide if a Presburger formula denotes a forward inductive invariant, we deduce that there exist checkable certificates of non-reachability. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by enumerating finite sequences of actions and a second one that tries to prove the non-reachability by enumerating Presburger formulas.

In this paper we provide a new proof of the reachability problem that is not based on the KLMST decomposition. The proof is based on the *productive sequences* inspired by Hauschildt [Hau90] and it provides directly that reachability sets are *Lambert sets*, a class of sets introduced in this paper that extend the class of Presburger sets and contained in the class of semi-pseudo-linear sets. In particular this paper provides a more precise characterization of the reachability sets of VASs.

Outline of the paper: Section 3 introduces the class of *Lambert sets*. Section 4 provides a sufficient condition on binary relations over $\mathbb{N}^d$ such that there exist invariants definable in the Presburger arithmetic proving the non-reachability. Section 5 introduces the class of *Vector Addition Systems (VASs)*. Section 6 provides the definition of *productive sequences*. Finally, in Section 7 we prove that reachability sets are Lambert sets. We conclude from this result that the non-reachability problem for VAS can be solved with Presburger inductive invariants.

2. Notations

In the sequel $\mathbb{N}, \mathbb{N}_{>0}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}_{>0}$ denotes the set of natural numbers, positive integers, integers, rational numbers, and positive rational numbers. The ith component of a vector $s \in X^d$ is denoted by $s(i)$. The addition function $+$ and the order $\leq$ are extended component-wise over $\mathbb{Q}^d$, i.e $s \leq s'$ if $s(i) \leq s'(i)$ for every $i \in \{1, \dots, d\}$. Given a function $f : S \rightarrow S'$ we denote by $f(X) = \{f(x) \mid x \in X\}$ for every subset $X \subseteq S$. In particular the sum $S_1 + S_2$ is well defined for every $S_1, S_2 \subseteq \mathbb{Q}^d$. In the same way given $T \subseteq \mathbb{Q}$ and $S \subseteq \mathbb{Q}^d$ we let $TS = \{ts \mid (t, s) \in T \times S\}$. We also denote by $s_1 + S_2$ and $S_1 + s_2$ the sets $\{s_1\} + S_2$ and $S_1 + \{s_2\}$, and we denote by tS and Ts the sets $\{t\}S$ and $T\{s\}$.

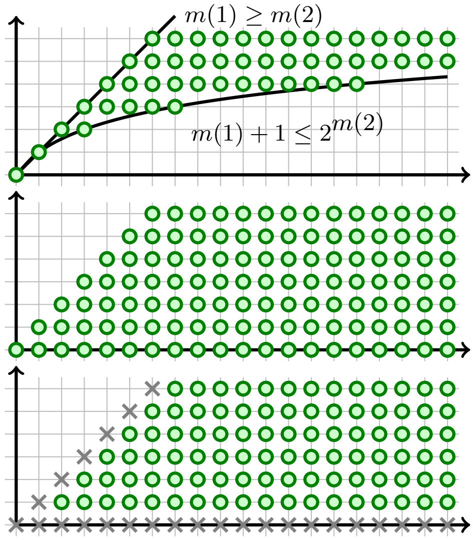

Figure 1. A simple monoid, a linearization P, and the interior of P

Given a *monoid* $(S, \circ)$ we denote s^n where $s \in S$ and $n \in \mathbb{N}$ the element in S defined by s^0 is the neutral element and the induction $s^{n+1} = s^n \circ s$. A subset $R \subseteq S$ is called a *submonoid* of $(S, \circ)$ if R contains the neutral element of $(S, \circ)$ and if $r_1 \circ r_2 \in R$ for every $r_1, r_2 \in R$. Recall that for every $X \subseteq S$ there exists a unique submonoid R of $(S, \circ)$ that is minimal for the inclusion and such that $X \subseteq R$. This monoid is called the submonoid of $(S, \circ)$ *generated* by X. A monoid $(S, \circ)$ is said to be *finitely generated* if there exists a finite set $X \subseteq S$ such that S is the submonoid generated by X. A *morphism* from a monoid $(S, \circ)$ to a monoid $(S', \circ')$ is a total function $f : S \to S'$ such that the neutral elements e, e' of $(S, \circ)$ and $(S, \circ')$ satisfy $f(e) = e'$ and such that $f(s_1 \circ s_2) = f(s_1) \circ' f(s_2)$ for every $s_1, s_2 \in S$.

Given an ordered set $(S, \sqsubseteq)$, an element $s \in S$ is said to be minimal for $\sqsubseteq$ if for every $s' \in S$ such that $s' \sqsubseteq s$ we have $s = s'$. We denote by $\min_{\sqsubseteq}(S)$ the set of minimal elements in S. An order $\sqsubseteq$ over a set S is said to be *well* if for every sequence $(s_n)_{n \in \mathbb{N}}$ of elements $s_n \in S$ we can extract a sub-sequence that is non-decreasing for $\sqsubseteq$. Note that if $(S, \sqsubseteq)$ is well-ordered then $\min_{\sqsubseteq}(S)$ is finite and for every $s' \in S$ there exists a minimal element $s \in S$ such that $s \sqsubseteq s'$.

3. Lambert Sets

A subset $M \subseteq \mathbb{N}^d$ is called a *Presburger set* if it can be denoted by a formula in the Presburger arithmetic FO $(\mathbb{N}, +, \leq)$. Let us recall [GS66] that a subset $M \subseteq \mathbb{N}^d$ is Presburger if and only if it is a finite union of *linear sets*, i.e. sets of the form $b + P$ where $b \in \mathbb{N}^d$ and P is a finitely generated submonoid of $(\mathbb{N}^d, +)$. In this section we introduce the class of *simple submonoids* of $(\mathbb{N}^d, +)$ that extends the finitely generated submonoids, and the class of *Lambert sets* defined as the finite unions of sets $b + S$ where $b \in \mathbb{N}^d$ and S is a simple submonoid $(\mathbb{N}^d, +)$.

Let P be a finitely generated submonoid of $(\mathbb{N}^d, +)$. A vector $v \in P$ is said to be in the *interior* of P if for any $p \in P$ there exists $n \in \mathbb{N}_{>0}$ such that $nv \in p + P$. The set of interior vectors of $(P, +)$ is denoted by $\text{int}(P)$. Let S be a submonoid of $(\mathbb{N}^d, +)$. A *direction* of S is a vector $v \in \mathbb{N}^d$ such that there exists $s \in S$ satisfying $s + \mathbb{N}v \subseteq S$. The set of directions of S forms a submonoid of $(\mathbb{N}^d, +)$ denoted by $\text{dir}(S)$. A submonoid S of $(\mathbb{N}^d, +)$ is said to be *simple* if there exists a finitely generated submonoid P of $(\mathbb{N}^d, +)$ such that $S \subseteq P$ and $\text{int}(P) \subseteq \text{dir}(S)$. In this case P is called a *linearization of* S. A *Lambert set* is a finite union of sets of the form $b + S$ where $b \in \mathbb{N}^d$ and S is a simple submonoid of $(\mathbb{N}^d, +)$.

Example 3.1. *The set* $S = \{m \in \mathbb{N}^2 \mid m(2) \leq m(1) \leq 2^{m(2)} - 1\}$, *the finitely generated submonoid* $P = \{m \in \mathbb{N}^2 \mid m(2) \leq m(1)\}$ *and the interior* $\text{int}(P) = \{m \in \mathbb{N}^2 \mid 1 < m(2) < m(1)\}$ *are represented in Figure 1. Since* $S \subseteq P$ *and* $\text{int}(P) \subseteq \text{dir}(S)$ *we deduce that* S *is a simple submonoid and* P *is a linearization.*

The following lemma provides a characterization of $\text{int}(P)$.

Lemma 3.2. *Let* $P = \mathbb{N}p_1 + \cdots + \mathbb{N}p_t$ *with* $t \in \mathbb{N}_{>0}$ *and* $p_j \in \mathbb{N}^d$. *We have* $\text{int}(P) = P \cap (\mathbb{Q}_{>0}p_1 + \cdots + \mathbb{Q}_{>0}p_t)$.

Proof. Let $v \in P$. Observe that if v is in the interior of P, as $p = \sum_{j=1}^{t} p_j$ is in P, there exists $n \in \mathbb{N}_{>0}$ such that $nv \in p + P$. Thus, there exists a sequence $(n_j)_{1 \leq j \leq t}$ with $n_j \in \mathbb{N}$ such that $nv = p + \sum_{j=1}^{t} n_j p_j$. We deduce that $v = \sum_{j=1}^{t} \lambda_j p_j$ with $\lambda_j = \frac{n_j + 1}{n}$. Conversely, let us assume that $v = \sum_{j=1}^{t} \lambda_j p_j$ with $\lambda_j \in \mathbb{Q}_{>0}$. Let $p \in P$. There exists a sequence $(n_j)_{1 \leq j \leq t}$ with $n_j \in \mathbb{N}$ such that $p = \sum_{j=1}^{t} n_j p_j$. As $\lambda_j > 0$, there exists an integer $n \in \mathbb{N}_{>0}$ such that $n\lambda_j$ is an integer satisfying $n\lambda_j \geq n_j$ for every $j \in \{1, \ldots, t\}$. We deduce that $nv \in p + P$. Thus v is in the interior of P. □

In the sequel, we used the following lemma.

Lemma 3.3. *Let* P *be a linearization of a simple submonoid* S *of* $(\mathbb{N}^d, +)$ *and let* f *be a morphism from* $(P, +)$ *to* $(\mathbb{N}^{d'}, +)$. *Then* $S' = f(S)$ *is a simple submonoid of* $(\mathbb{N}^{d'}, +)$ *and* $P' = f(P)$ *is a linearization of* S'.

Proof. Since P is finitely generated, there exists a finite sequence $(p_i)_{1 \leq i \leq t}$ of vectors $p_i \in P$ such that P is generated by $\{p_1, \ldots, p_t\}$. As P' is generated by $\{f(p_1), \ldots, f(p_t)\}$ we deduce that P' is finitely generated. Observe that $S' \subseteq P'$. In order to prove that S' is a simple submonoid and P' is a linearization of S', it is sufficient to prove that $\text{int}(P') \subseteq \text{dir}(S')$. So, let us consider an interior vector $v' \in \text{int}(P')$.

We first prove that there exists an interior vector v of $(P, +)$ and an integer $h \in \mathbb{N}_{>0}$ such that $f(v) = hv'$. Lemma 3.2 shows that there exists a sequence $(\lambda_j)_{1 \leq j \leq t}$ with $\lambda_j \in \mathbb{Q}_{>0}$ such that $v' = \lambda_1 f(p_1) + \cdots + \lambda_t f(p_t)$. Let us consider $h \in \mathbb{N}_{>0}$ such that $h_j = h\lambda_j$ is in $\mathbb{N}_{>0}$ for every j. Observe that $hv' = f(v)$ with $v = h_1 p_1 + \cdots + h_t p_t$. Lemma 3.2 shows that v is in the interior of $(P, +)$.

As $v' \in P'$ and $P' = f(P)$ there exists $p \in P$ such that $v' = f(p)$. As v is in the interior of P and $p \in P$ we deduce that $v_n = v + np$ is in the interior of P for every $n \in \mathbb{N}$. As P is a linearization of S, there exists $s_n \in S$ such that $s_n + \mathbb{N}(v + np) \subseteq S$. Hence $s'_n + \mathbb{N}(h + n)v' \subseteq S'$ with $s'_n = f(s_n)$. We deduce that $s' + \sum_{n=0}^{h-1} \mathbb{N}(h + n)v' \subseteq S'$ with $s' = \sum_{n=0}^{h-1} s'_n$. As $\sum_{n=0}^{h-1} \mathbb{N}(h + n) = \{0\} \cup (h + \mathbb{N})$ we have proved that $s' + hv' + \mathbb{N}v' \subseteq S'$. Therefore $v' \in \text{dir}(S')$ and we have proved that $\text{int}(P') \subseteq \text{dir}(S')$. We deduce that S' is simple and P' is a linearization of S'.

□

4. Presburger Invariants

Let $\to$ be a binary relation over $\mathbb{N}^d$ denoted by a Presburger formula and let $\xrightarrow{*}$ be its reflexive and transitive closure. The reachability problem consists to decide if a pair (m, m') of vectors in

$\mathbb{N}^d$ satisfies $m \xrightarrow{*} m'$. In general this problem is undecidable since the semantics of every Minsky machine can be denoted by a Presburger binary relation $\rightarrow$. We introduce in this section a sufficient condition on $\rightarrow$ such that if m' is not reachable from m, there exists a Presburger inductive invariant that contains m but not m'. In particular the reachability problem for this class is decidable.

Let $X \subseteq \mathbb{N}^d$. We introduce the *forward/backward one-step reachability sets* $\mathrm{post}(X)$ and $\mathrm{pre}(X)$, and the *forward/backward reachability sets* $\mathrm{post}^*(X)$ and $\mathrm{pre}^*(X)$ as follows:

$$\begin{cases} \mathrm{post}(X) = \{m' \in \mathbb{N}^d \mid \exists x \in X \; x \rightarrow m'\} \\ \mathrm{post}^*(X) = \{m' \in \mathbb{N}^d \mid \exists x \in X \; x \xrightarrow{*} m'\} \\ \mathrm{pre}(X) = \{m \in \mathbb{N}^d \mid \exists x \in X \; m \rightarrow x\} \\ \mathrm{pre}^*(X) = \{m \in \mathbb{N}^d \mid \exists x \in X \; m \xrightarrow{*} x\} \end{cases}$$

The set X is said to be a *forward invariant* if $\mathrm{post}(X) \subseteq X$ and it is said to be a *backward invariant* if $\mathrm{pre}(X) \subseteq X$.

In [Ler09], the reachability problem for VAS was proved by introducing the class of *semi-pseudo-linear sets*. Many geometrical results was proved in this paper that are *independent of the reachability problem*. Since every Lambert set is semi-pseudo-linear, from [Ler09], we deduce the following theorem.

Theorem 4.1. *Assume that for every Presburger set $X \subseteq \mathbb{N}^d$ the following sets are Lambert sets:*

$$\mathrm{post}^*(X) \backslash X \qquad \mathrm{pre}^*(X) \backslash X$$

For every pair (M, M') of Presburger sets such that $\mathrm{post}^*(M) \cap \mathrm{pre}^*(M') = \emptyset$, there exists a partition of $\mathbb{N}^d$ into a Presburger forward invariant $I \supseteq M$ and a Presburger backward invariant $I' \supseteq M'$.

Now let us consider a pair (m, m') of vectors $m, m' \in \mathbb{N}^d$ and observe that if (m, m') is not in $\xrightarrow{*}$ then $\mathrm{post}^*(M) \cap \mathrm{pre}^*(M') = \emptyset$ with $M = \{m\}$ and $M' = \{m'\}$. Since M, M' are Presburger sets, from the previous theorem we deduce that there exists a Presburger forward invariant I such that $m \in I$ and $m' \notin I$.

5. Vector Addition Systems

A *vector addition system* is a couple $\mathcal{V} = (A, d, \delta)$ where A is a non-empty finite set, $d \in \mathbb{N}_{>0}$ is the *dimension*, $\delta : A \rightarrow \mathbb{Z}^d$ is a total function called the *displacement function*. A vector in $\mathbb{N}^d$ is called a *marking*. The displacement function δ is extended over words in A^* by $\delta(\sigma) = \sum_{j=1}^{k} \delta(a_j)$ for every word $\sigma = a_1 \ldots a_k$ with $a_j \in A$ and $k \in \mathbb{N}$. A word σ is said to be *fireable from a marking* m if for every prefix w of σ the vector $m + \delta(w)$ is non-negative. In this case we write $m \xrightarrow{\sigma} m'$ where $m' = m + \delta(\sigma)$. The *one-step reachability binary relation* $\rightarrow$ over $\mathbb{N}^d$ is defined by $m \rightarrow m'$ if there exists $a \in A$ such that $m \xrightarrow{a} m'$.

The following proposition shows that Theorem 4.1 can be applied if we prove that $\mathrm{post}^*(\{m_0\}) \cap (m + R)$ is a Lambert set for every $m_0, m \in \mathbb{N}^d$ and for every finitely generated submonoids R (see Figure 2).

Proposition 5.1. *Assume that for every VAS $\mathcal{V}$, for every markings m_0, m, and for every finitely generated submonoid R the following set is a Lambert set:*

$$\mathrm{post}^*(\{m_0\}) \cap (m + R)$$

Then for every VAS $\mathcal{V}$ and for every Presburger sets X, Y the following sets are Lambert sets:

$$\mathrm{post}^*(X) \cap Y \qquad \mathrm{pre}^*(X) \cap Y$$

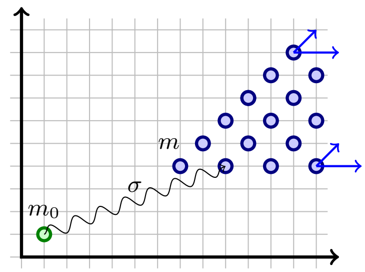

Figure 2. A word σ fireable from m_0 that reaches a marking in $m + R$.

Proof. In order to clarify the notations, let us denote by $\mathrm{post}^*_{\mathcal{V}}(X)$ and $\mathrm{pre}^*_{\mathcal{V}}(X)$ the sets $\mathrm{post}^*(X)$ and $\mathrm{pre}^*(X)$ associated to the one-step binary relation of a VAS $\mathcal{V}$. Since $\mathrm{pre}^*_{\mathcal{V}}(X)$ is equal to $\mathrm{post}^*_{-\mathcal{V}}(X)$ where $-\mathcal{V} = (A, d, -\delta)$, we just have to prove that $\mathrm{post}^*_{\mathcal{V}}(X) \backslash Y$ is a Lambert set for every Presburger sets $X, Y \subseteq \mathbb{N}^d$ and for every VAS $\mathcal{V}$. As a Presburger set is a finite union of linear sets, it is sufficient to prove that $\mathrm{post}^*_{\mathcal{V}}(m_0 + R_0) \cap (m + R)$ is a Lambert set for every $m_0, m \in \mathbb{N}^d$ and for every finitely generated submonoids R_0, R of $(\mathbb{N}^d, +)$. Since R_0 is finitely generated, there exists a finite sequence $r_1, \ldots, r_k$ of vectors $r_j \in \mathbb{N}^d$ such that R_0 is the submonoid generated by $\{r_1, \ldots, r_k\}$. Let us consider a sequence $(b_1, \ldots, b_k)$ of distinct elements disjoint of A and let $A' = A \cup \{b_1, \ldots, b_k\}$. We introduce the extension δ' of δ over A' defined by $\delta'(b_j) = r_j$. Now just observe that $\mathrm{post}^*_{\mathcal{V}}(m_0 + R_0) = \mathrm{post}^*_{\mathcal{V}'}(\{m_0\})$ where $\mathcal{V}' = (A', d, \delta')$. $\square$

6. Finite Decomposition

Let us consider two markings $m_0, m \in \mathbb{N}^d$ and a finitely generated submonoid R (see Figure 2). We introduce the set $\Sigma_{m_0,m,R}$ of words $\sigma \in A^*$ fireable from m_0 and such that $m_0 + \delta(\sigma) \in m + R$.

Definition 6.1 (Inspired by Hauschildt [Hau90])**.** *Let $\sigma = a_1 \ldots a_k$ be a word in $\Sigma_{m_0,m,R}$ with $a_j \in A$ for every j. A sequence $\pi = (w_j)_{0 \leq j \leq k}$ of words $w_j \in A^*$ is said to be* productive *for (m_0, σ, R) if the following conditions hold:*

- *The word denoted by $\sigma^\pi = w_0 a_1 w_1 \ldots a_k w_k$ is fireable from m_0, and*
- *The partial sums $\delta(w_0) + \cdots + \delta(w_j)$ are non-negative for every $j \in \{0, \ldots, k\}$, and*
- *The total sum denoted by $\delta(\pi) = \sum_{j=0}^{k} \delta(w_j)$ is in R.*

We denote by $\Pi_{m_0,\sigma,R}$ the set of productive sequences.

In this section we prove that $\delta(\Pi_{m_0,\sigma,R}) = \{\delta(\pi) \mid \pi \in \Pi_{m_0,\sigma,R}\}$ is a submonoid of $(R, +)$ and there exists a finite set $\Sigma \subseteq \Sigma_{m_0,m,R}$ such that the following *finite decomposition* holds:

$$\mathrm{post}^*(\{m_0\}) \cap (m + R) = \bigcup_{\sigma \in \Sigma} m_0 + \delta(\sigma) + \delta(\Pi_{m_0,\sigma,R})$$

In particular we reduce the problem of proving that $\mathrm{post}^*(\{m_0\}) \cap (m + R)$ is a Lambert set to prove that $\delta(\Pi_{m_0,\sigma,R})$ is a simple submonoid.

We consider the total function $\circ$ defined over $\Pi_{m_0,\sigma,R} \times \Pi_{m_0,\sigma,R}$ by $\pi \circ \pi' = (w_j w'_j)_{0 \leq j \leq k}$ where $\pi = (w_j)_{0 \leq j \leq k}$ and $\pi' = (w'_j)_{0 \leq j \leq k}$. The following lemma shows that $(\Pi_{m_0,\sigma,R}, \circ)$ is a monoid with the neutral element $(\epsilon)_{0 \leq j \leq k}$. Since $\delta(\pi \circ \pi') = \delta(\pi) + \delta(\pi')$ we deduce that $\delta(\Pi_{m_0,\sigma,R})$ is a submonoid of $(R, +)$.

Lemma 6.2. *The sequence* $\pi \circ \pi'$ *is in* $\Pi_{m_0,\sigma,R}$ *for every* $\pi, \pi' \in \Pi_{m_0,\sigma,R}$.

Proof. Let us consider $\pi = (w_j)_{0\leq j\leq k}$ and $\pi' = (w'_j)_{0\leq j\leq k}$ in $\Pi_{m_0,\sigma,R}$ and let $\pi'' = (w''_j)_{0\leq j\leq k}$ with $w''_j = w_j w'_j$ for every j. We introduce the partial sums $r_j = \delta(w_0) + \dots + \delta(w_j)$, $r'_j = \delta(w'_0) + \dots + \delta(w'_j)$, and $r''_j = \delta(w''_0) + \dots + \delta(w''_j)$. Observe that $r''_j = r_j + r'_j$ which is a non-negative vector and $r''_k = r_k + r'_k$ which is a vector in R since $r_k, r'_k \in R$. Let us prove that $\sigma^{\pi''}$ is fireable from m_0. A prefix u of this word has the form $w_0 w'_0 a_1 \dots w_{j-1} w'_{j-1} a_j w$ where w is either a prefix of w_j or w is a word of the form $w = w_j w'$ where w' is a prefix of w'_j. In the first case $m_0 + \delta(u) = r'_{j-1} + m_0 + \delta(v)$ with $v = w_0 a_1 \dots w_{j-1} a_j w$. Since v is a prefix of σ^π that is fireable from m_0 we deduce that $m_0 + \delta(v)$ is non-negative. Therefore $m_0 + \delta(u)$ is non-negative. In the second case $m_0 + \delta(u) = r_j + m_0 + \delta(v')$ with $v' = w'_0 a_1 \dots w'_{j-1} a_j w'$. Since v' is a prefix of $\sigma^{\pi'}$ that is fireable from m_0 we deduce that $m_0 + \delta(v')$ is non-negative. Therefore $m_0 + \delta(u)$ is non-negative. We have proved that $\sigma^{\pi''}$ is fireable from m_0. Thus $\pi'' \in \Pi_{m_0,\sigma,R}$. □

We introduce the order $\leq_R$ over R defined by $r \leq_R r'$ if $r' \in r + R$.

Lemma 6.3. $(R, \leq_R)$ *is a well-ordered set.*

Proof. Since R is finitely generated, there exists a finite sequence $(r_j)_{1\leq j\leq t}$ of vectors $r_j \in R$ such that $R = \mathbb{N}r_1 + \dots + \mathbb{N}r_t$. In particular there exists a total function $f : R \to \mathbb{N}^t$ that maps every vector $r \in R$ onto a sequence $f(r) = (n_j)_{1\leq j\leq t}$ such that $r = \sum_{j=1}^t n_j r_j$. From the Dickson lemma, $(\mathbb{N}^t, \leq)$ is a well-order. We deduce that $(R, \leq_R)$ is well-ordered since for every $r, r' \in R$ if $f(r) \leq f(r')$ then $r \leq_R r'$. □

We introduce a well-order $\sqsubseteq_{m_0,m,R}$ over $\Sigma_{m_0,m,R}$ as follows. We first consider the infinite set $H = A \times \mathbb{N}^d$ ordered by $(a, x) \preceq (a', x')$ if and only if $a = a'$ and $x \leq x'$. From the Dickson lemma, the order $\leq$ over $\mathbb{N}^d$ is well. As A is finite, we deduce that $(H, \preceq)$ is a well-ordered set. We introduce the order $\preceq^*$ over the words in H^* defined by $u \preceq^* u'$ where $u = h_1 \dots h_k$ with $h_j \in H$ if there exists a sequence $(h'_j)_{1\leq j\leq k}$ with $h_j \preceq h'_j$ and a sequence $(u_j)_{0\leq j\leq k}$ of words $u_j \in H^*$ such that $u' = u_0 h'_1 u_1 \dots k'_k u_k$. From the Higman lemma, the order $\preceq^*$ is a well-order over H^*. Next, we associate to every couple (m_0, σ) where $m_0 \in \mathbb{N}^d$ and $\sigma \in A^*$ is a word fireable from m_0, the word $u_{m_0,\sigma} \in H^*$ defined by the following equality where $m_j = m_0 + \delta(a_1 \dots a_j)$:

$$u_{m_0,\sigma} = (a_1, m_1) \dots (a_k, m_k)$$

The set $\Sigma_{m_0,m,R}$ is equipped with the order $\sqsubseteq_{m_0,m,R}$ defined by $\sigma \sqsubseteq_{m_0,m,R} \sigma'$ if and only if $u_{m_0,\sigma} \preceq^* u_{m_0,\sigma'}$ and $(m_0 + \delta(\sigma) - m) \leq_R (m_0 + \delta(\sigma') - m)$. Since $(H^*, \preceq^*)$ and $(R, \leq_R)$ are well-ordered sets, we deduce that $(\Sigma_{m_0,m,R}, \sqsubseteq_{m_0,m,R})$ is also a well-ordered set.

Lemma 6.4. *For every* $\sigma, \sigma' \in \Sigma_{m_0,m,R}$ *we have* $\sigma \sqsubseteq_{m_0,m,R} \sigma'$ *if and only if there exists* $\pi \in \Pi_{m_0,\sigma,R}$ *such that* $\sigma' = \sigma^\pi$.

Proof. We associate to every word $u = (a_1, m_1) \dots (a_k, m_k)$ with $(a_j, m_j) \in H$ the word $w = a_1 \dots a_k$ called the *label of* u. Let $\sigma, \sigma' \in \Sigma_{m_0,m,R}$. We introduce the sequence $(a_j)_{1\leq j\leq k}$ such that $\sigma = a_1 \dots a_k$ and the marking $m_j = m_0 + \delta(a_1 \dots a_j)$.

Let us first assume that there exists $\pi = (w_j)_{0\leq j\leq k}$ in $\Pi_{m_0,\sigma,R}$ such that $\sigma' = \sigma^\pi$ and let us prove that $\sigma \sqsubseteq_{m_0,m,R} \sigma'$. We introduce the partial sum $r_j = \delta(w_0) + \dots + \delta(w_j)$ where $j \in \{-1, \dots, k\}$. We also introduce the vector $m'_j = m_j + r_{j-1}$ for every $j \in \{0, \dots, k\}$. Since σ^π is fireable from m_0, we have:

$$m'_0 \xrightarrow{w_0 a_1} m'_1 \cdots m'_{k-1} \xrightarrow{w_{k-1} a_k} m'_k \xrightarrow{w_k} (m_k + \delta(\pi))$$

We deduce that $u_{m_0,\sigma'} = h_0 (a_1, m'_1) h_1 \dots (a_k, m'_k) h_k$ where h_j is obtained from the intermediate markings in $m'_j \xrightarrow{w_j} (m'_j + \delta(w_j)$. As $m'_j \geq m_j$ we deduce that $u_{m_0,\sigma} \preceq^* u_{m_0,\sigma'}$. Moreover, as $\delta(\pi) \in R$ we have proved that $m_0 + \delta(\sigma') \in m_0 + \delta(\sigma) + R$. Therefore $\sigma \sqsubseteq_{m_0,m,R} \sigma'$.

Conversely, let us assume that $\sigma \sqsubseteq_{m_0,m,R} \sigma'$ and let us prove that there exists $\pi \in \Pi_{m_0,\sigma,R}$ such that $\sigma' = \sigma^\pi$. Observe that $u_{m_0,\sigma} = (a_1, m_1) \dots (a_k, m_k)$. Since $u_{m_0,\sigma} \preceq u_{m_0,\sigma'}$, there exists a sequence $(m'_j)_{1\leq j\leq k}$ of standard markings $m'_j \geq m_j$ and a sequence $(h_j)_{0\leq j\leq k}$ of words in H^* such that:

$$u_{m_0,\sigma'} = h_0 (a_1, m'_1) h_1 \dots (a_k, m_k) h_k$$

Let us consider the sequence $\pi = (w_j)_{0\leq j\leq k}$ where w_j is the label of h_j. By definition of $u_{m_0,\sigma'}$, we have the following relations where $\delta(\pi) = \sum_{j=0}^k \delta(w_j)$:

$$m'_0 \xrightarrow{w_0 a_1} m'_1 \cdots m'_{k-1} \xrightarrow{w_{k-1} a_k} m'_k \xrightarrow{w_k} (m_k + \delta(\pi))$$

As $m'_j \geq m_j$ we deduce that $\delta(w_0) + \dots + \delta(w_{j-1})$ is non-negative for every $j \in \{1, \dots k\}$. Moreover as $m_0 + \delta(\sigma') \in m_0 + \delta(\sigma) + R$, we get $\delta(\pi) \in R$. We have proved that $\pi \in \Pi_{m_0,\sigma,R}$ and $\sigma' = \sigma^\pi$. □

Now, we can prove the main theorem of this section.

Theorem 6.5. *The set* $\Sigma = \min_{\sqsubseteq_{m_0,m,R}}(\Sigma_{m_0,m,R})$ *is finite and it satisfies:*

$$\text{post}^*(\{m_0\}) \cap (m + R) = \bigcup_{\sigma \in \Sigma} m_0 + \delta(\sigma) + \delta(\Pi_{m_0,\sigma,R})$$

Proof. As $(\Sigma_{m_0,m,R}, \sqsubseteq_{m_0,m,R})$ is a well-ordered set, we deduce that Σ is finite and we have the following equality:

$$\Sigma_{m_0,m,R} = \bigcup_{\sigma \in \Sigma} \{\sigma' \in \Sigma_{m_0,m,R} \mid \sigma \sqsubseteq_{m_0,m,R} \sigma'\}$$

From Lemma 6.4 we get the following equality:

$$\Sigma_{m_0,m,R} = \bigcup_{\sigma \in \Sigma} \{\sigma^\pi \mid \pi \in \Pi_{m_0,\sigma,R}\}$$

In particular by applying δ, since $m_0 + \delta(\Sigma_{m_0,m,R})$ is equal to $\text{post}^*(\{m_0\}) \cap (m + R)$ and $\delta(\sigma^\pi) = \delta(\sigma) + \delta(\pi)$, we get the theorem. □

7. Simple Monoids

In this section we prove that the monoid $(\delta(\Pi_{m_0,\sigma,R}), +)$ is simple and it admits a linearization included in R.

The set $\mathbb{Z}$ is extended with an additional element $\infty \notin \mathbb{Z}$ and we denote by $\mathbb{Z}_\infty$ and $\mathbb{N}_\infty$, respectively, the sets $\mathbb{Z} \cup \{\infty\}$ and $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. The total-order $\leq$ over $\mathbb{Z}$ is extended over $\mathbb{Z}_\infty$ by $z \leq \infty$ for every $z \in \mathbb{Z}_\infty$. The total-order $\leq$ is extended into an order $\leq$ over $\mathbb{Z}_\infty^d$ defined component-wise by $v \leq v'$ if $v(i) \leq v'(i)$ for every $i \in \{1, \dots, d\}$. The addition function is extended into a total function $+ : \mathbb{Z}_\infty \times \mathbb{Z}_\infty \to \mathbb{Z}_\infty$ defined by $z + \infty = \infty = \infty + z$ for every $z \in \mathbb{Z}_\infty$. This function is also extended component-wise. The multiplication is extended over $\mathbb{N}_\infty$ by $\infty n = \infty = n\infty$ if $n \neq 0$ and $\infty 0 = 0 = 0\infty$.

A vector in $\mathbb{N}_\infty^d$ is called an *extended marking*. A word σ is said to be *fireable from an extended marking* m if for every prefix w of σ the vector $m + \delta(w)$ is non-negative. In this case we write $m \xrightarrow{\sigma} m'$ where $m' = m + \delta(\sigma)$.

A graph (labelled by A) is a couple $G = (Q, E)$ where Q is a non-empty set of *states* and $E \subseteq Q \times A \times Q$ is a set of *edges*. The graph G is said to be *finite* if Q and E are both finite. A *path in* G is a word $p = e_1 \dots e_k$ of $k \in \mathbb{N}$ edges $e_j \in E$ such that there exists a sequence $(q_j)_{0\leq j\leq k}$ of states $q_j \in Q$ and a sequence $(a_j)_{1\leq j\leq k}$ of element $a_j \in A$ such that $e_j = (q_{j-1}, a_j, q_j)$ for every $j \in \{1, \dots, k\}$. In this case the word $\sigma = a_1 \dots a_k$ is unique and it is called the label of p. The path p is also denoted by $q_0 \xrightarrow{\sigma}_G q_k$ and it is called a path from q_0 to q_k labelled by σ. When $q_0 = q_k$, the path is called a *cycle*. The graph G is said to be strongly connected if for every pair $(q, q') \in Q \times Q$ there exists a path from q to q'. We say that a total function $f : E \to \mathbb{N}$ satisfies the *Kirchhoff laws of* G if for every $q \in Q$ the following equality holds:

$$\sum_{e\in E\cap(\{q\}\times A\times Q)} f(e) = \sum_{e\in E\cap(Q\times A\times\{q\})} f(e)$$

We observe that the Parikh image $||p||$ of a cycle p is a total function that satisfies the Kirchhoff laws. Conversely, let us recall that if G is strongly connected, f satisfies the Kirchhoff laws, and $f(e) > 0$ for every $e \in E$ then there exists a cycle p such that $f = ||p||$. Observe that the set F of total functions $f : E \to \mathbb{N}$ satisfying the Kirchhoff laws of a finite graph $G = (Q, E)$ is the submonoid of $(\mathbb{N}^E, +)$ generated by the finite set $\min_\leq(F\backslash\{0\})$.

The tuple (m_0, σ, R) is fixed in this section and the set $\Pi_{m_0,\sigma,R}$ is simply denoted by Π. We introduce the sequence $(a_j)_{1\leq j\leq k}$ of elements $a_j \in A$ such that $\sigma = a_1 \dots a_k$, and the sequence $(m_j)_{0\leq j\leq k}$ of markings $m_j = m_0 + \delta(a_1 \dots a_j)$. We also consider the set Q_j of markings $q_j \in \mathbb{N}^d$ such that there exists a productive sequence $\pi = (w_j)_{0\leq j\leq k}$ in Π and a prefix w of w_j such that:

$$m_0 \xrightarrow{w_0 a_1 \dots w_{j-1} a_j w} q_j$$

In this case q_j is called a *jth intermediate marking*. We also introduce the set E_j of couples $(q_j, a, q'_j) \in Q_j \times A \times Q_j$ such that there exists a productive sequence $\pi = (w_j)_{0\leq j\leq k}$ in Π and a prefix of w_j of the form wa with $w \in A^*$ and $a \in A$ such that:

$$m_0 \xrightarrow{w_0 a_1 \dots w_{j-1} a_j w} q_j \xrightarrow{a} q'_j$$

We introduce the (infinite) graph $G_j = (Q_j, E_j)$.

We are interested in folding G_j into a finite graph. We introduce for each $j \in \{0, \dots, k\}$ the set I_j of integers $i \in \{1, \dots, d\}$ such that $\{q_j(i) \mid q_j \in Q_j\}$ is infinite. We introduce the vector $v_j \in \{0, \infty\}^d$ such that $v_j(i) = \infty$ if and only if $i \in I_j$. By construction, the set $\tilde{Q}_j = Q_j + v_j$ is a finite set of extended markings. In the sequel we denote by $\tilde{q}_j$ the extended marking $q_j + v_j$. We also introduce the set $\tilde{E}_j = \{(\tilde{q}_j, a, \tilde{q}'_j) \mid (q_j, a, q'_j) \in E_j\}$. We introduce the finite graph $\tilde{G}_j = (\tilde{Q}_j, \tilde{E}_j)$.

Lemma 7.1. *The graph $\tilde{G}_j$ is strongly connected and for every productive sequence $\pi = (w_j)_{0\leq j\leq k}$ in Π and for every $j \in \{0, \dots, k\}$, we have:*

$$\tilde{m}_j \xrightarrow{w_j}_{\tilde{G}_j} \tilde{m}_j$$

Proof. Let us consider a productive sequence $\pi = (w_j)_{0\leq j\leq k}$ in Π. We introduce the partial sum $r_j = \delta(w_0) + \dots + \delta(w_j)$. Let $n \in \mathbb{N}$. Since σ^{π^n} is fireable from m we deduce that $m_0 + \delta(w_0^n a_1 \dots w_{j-1}^n a_j)$ and $m_0 + \delta(w_0^n a_1 \dots w_{j-1}^n a_j w_j^n)$ are both non negative. Since these vectors are respectively equal to $m_j + n r_{j-1}$ and $m_j + n r_j$ we deduce that $r_{j-1}(i) > 0$ or $r_j(i) > 0$ implies $i \in I_j$. Therefore $m_j + r_{j-1} + v_j = \tilde{m}_j$ and $m_j + r_j + v_j = \tilde{m}_j$. Since $(m_j + r_{j-1}) \xrightarrow{w_j}_{G_j} (m_j + r_j)$ and immediate induction provides $\tilde{m}_j \xrightarrow{w_j}_{\tilde{G}_j} \tilde{m}_j$. We also deduce that for every $\tilde{q}_j \in \tilde{Q}_j$ there exists a path in $\tilde{G}_j$ from $\tilde{m}_j$ to $\tilde{q}_j$ and a path from $\tilde{q}_j$ to $\tilde{m}_j$. Thus $\tilde{G}_j$ is strongly connected,

□

We consider the set $\tilde{\Pi}$ of sequences $\tilde{\pi} = (\tilde{w}_j)_{0\leq j\leq k}$ of words $\tilde{w}_j \in A^*$ such that $\tilde{m}_j \xrightarrow{\tilde{w}_j}_{\tilde{G}_j} \tilde{m}_j$, the partial sums $\delta(\tilde{w}_0) + \dots + \delta(\tilde{w}_j)$ are non-negative for every $j \in \{0, \dots, k\}$, and the total sum denoted by $\delta(\tilde{\pi}) = \sum_{j=0}^k \delta(\tilde{w}_j)$ is in R. A sequence $\tilde{\pi} \in \tilde{\Pi}$ is called a *weak-productive sequence*. The previous lemma shows that Π is the set of weak-productive sequences $\tilde{\pi} \in \tilde{\Pi}$ such that $\sigma^{\tilde{\pi}}$ is fireable from m_0. In order to transform a weak-productive sequence into a productive sequence, we introduce the mixed-productive sequences.

We consider the set Θ of sequences $\theta = (w_{j\uparrow}, w_{j\downarrow})_{0\leq j\leq k}$ where $w_{j\uparrow}, w_{j\downarrow} \in A^* \times A^*$ are such that $\pi = (w_j)_{0\leq j\leq k}$, with $w_j = w_{j\uparrow} w_{j\downarrow}$, is in Π and such that partial sums $x_j = \delta(w_0) + \dots + \delta(w_{j-1}) + \delta(w_{j\uparrow})$ are non-negative for every $j \in \{0, \dots, k\}$. A sequence $\theta \in \Theta$ is called a *mixed-productive sequence* and the vector x_j is called the jth increase of θ. We introduce the total function $\circ$ defined over $\Theta \times \Theta$ by $\theta \circ \theta' = (w_{j\uparrow} w'_{j\uparrow}, w'_{j\downarrow} w_{j\downarrow})_{0\leq j\leq k}$ where $\theta = (w_{j\uparrow}, w_{j\downarrow})_{0\leq j\leq k}$ and $\theta' = (w'_{j\uparrow}, w'_{j\downarrow})_{0\leq j\leq k}$. The following lemma shows that $(\Theta, \circ)$ is a monoid. In particular the mixed-productive sequence θ^n is well defined for every $n \in \mathbb{N}$.

Lemma 7.2. *We have $\theta \circ \theta' \in \Theta$ for every $\theta, \theta' \in \Theta$.*

Proof. Let $\theta = (w_{j\uparrow}, w_{j\downarrow})_{0\leq j\leq k}$ and $\theta' = (w'_{j\uparrow}, w'_{j\downarrow})_{0\leq j\leq k}$. We introduce $w''_{j\uparrow} = w_{j\uparrow} w'_{j\uparrow}$ and $w''_{j\downarrow} = w'_{j\downarrow} w_{j\downarrow}$. We have $\theta \circ \theta' = \theta''$ with $\theta'' = (w''_{j\uparrow}, w''_{j\downarrow})_{0\leq j\leq k}$. We introduce the words $w_j = w_{j\uparrow} w_{j\downarrow}$, $w'_j = w'_{j\uparrow} w'_{j\downarrow}$, and $w''_j = w''_{j\uparrow} w''_{j\downarrow}$, the sequences $\pi = (w_j)_{0\leq j\leq k}$, $\pi' = (w'_j)_{0\leq j\leq k}$, and $\pi'' = (w''_j)_{0\leq j\leq k}$, the partial sums $r_j = \delta(w_0) + \dots + \delta(w_j)$, $r'_j = \delta(w'_0) + \dots + \delta(w'_j)$, and $r''_j = \delta(w''_0) + \dots + \delta(w''_j)$, and the jth increases $x_j = \delta(w_0) + \dots + \delta(w_{j-1}) + \delta(w_{j\uparrow})$, $x'_j = \delta(w'_0) + \dots + \delta(w'_{j-1}) + \delta(w'_{j\uparrow})$, and $x''_j = \delta(w''_0) + \dots + \delta(w''_{j-1}) + \delta(w''_{j\uparrow})$. We observe that $r''_j = r_j + r'_j$ and $x''_j = x_j + x'_j$ are non-negative for every $j \in \{0, \dots, k\}$, and $r''_k = r_k + r'_k \in R$. Thus, in order to prove that θ is in Θ, it just remains to prove that $\sigma^{\pi''}$ is fireable from m_0. A prefix u of this word has the form $u = w''_0 a_1 w''_1 \dots w''_{j-1} a_j w$ where w is either a prefix of $w_{j\uparrow}$, or a word of the form $w_{j\uparrow} w'$ where w' is a prefix of w'_j, or a word of the form $w_{j\uparrow} w'_j w$ where w is a prefix of $w_{j\downarrow}$. We divide the proof in these three cases. If w is a prefix of $w_{j\uparrow}$ then $m_0 + \delta(u) = r'_{j-1} + m_0 + \delta(v)$ with $v = w_0 a_1 \dots w_{j-1} a_j w$. Since v is a prefix of σ^π that is fireable from m_0 we deduce that $m_0 + \delta(v)$ is non negative. Thus $m_0 + \delta(u)$ is non-negative. If w is a word of the form $w_{j\uparrow} w'$ where w' is a prefix of w'_j then $m_0 + \delta(u) = x_j + m_0 + \delta(v')$ with $v' = w'_0 a_1 \dots w'_{j-1} a_j w'$. Since v' is a prefix of $\sigma^{\pi'}$ that is fireable from m_0 we deduce that $m_0 + \delta(v')$ is non-negative. Thus $m_0 + \delta(u)$ is non-negative. If w is a word of the form $w_{j\uparrow} w'_j w$ where w is a prefix of $w_{j\downarrow}$ then $m_0 + \delta(u) = r'_j + m_0 + \delta(v)$ with $v = w_0 a_1 \dots w_{j-1} a_j w_{j\uparrow} w$. Since v is a prefix of σ^π that is fireable from m_0 we deduce that $m_0 + \delta(v)$ is non-negative. Therefore $m_0 + \delta(u)$ is non-negative. We have proved that $\sigma^{\pi''}$ is fireable from m_0. Therefore $\theta'' \in \Theta$.

□

Let x_j be the jth increase of a mixed-productive sequence $\theta \in \Theta$. The mixed-productive sequence θ^n shows that $m_j + nx_j \in Q_j$ for every $n \in \mathbb{N}$. We deduce that $x_j(i) > 0$ implies $i \in I_j$. A mixed-productive sequence θ is said to be *strong* if for every $j \in \{0, \dots, k\}$, its jth increase x_j satisfies $x_j(i) > 0$ if and only if $i \in I_j$.

Proposition 7.3. *There exists a strong mixed-productive sequence.*

Proof. Since the mixed-productive sequences are composable by $\circ$ and the jth increase of $\theta \circ \theta'$ is equal to $x_j + x'_j$ where x_j, x'_j are the jth increases of θ, θ', it is sufficient to prove that for every $j \in \{0, \dots, k\}$ and for every $i \in I_j$, there exists a mixed-productive sequence θ such that the jth increase x_j satisfies $x_j(i) > 0$. So, let us fix $j \in \{0, \dots, k\}$ and $i \in I_j$.

Since the set $\{q_j(i) \mid q_j \in Q_j\}$ is infinite, there exists an infinite sequence in Q_j such that the ith component is strictly increasing. As $\leq$ is a well order over $\mathbb{N}^d$, we can extract from this sequence a subsequence that is non-decreasing for $\leq$. We have proved that there exist $q_j, q'_j \in Q_j$ such that $q_j \leq q'_j$ and $q_j(i) < q'_j(i)$. By definition of Q_j, there exists a productive sequence $\pi = (w_j)_{0 \leq j \leq k}$ in Π and a decomposition of w_j into $w_j = w_{j,\text{in}} w_{j,\text{out}}$ such that the following relation holds where r_{j-1} and r_j are the partial sums $\delta(w_0) + \dots + \delta(w_{j-1})$ and $\delta(w_0) + \dots + \delta(w_j)$:

$$(m_j + r_{j-1}) \xrightarrow{w_{j,\text{in}}} q_j \xrightarrow{w_{j,\text{out}}} (m_j + r_j)$$

Symmetrically there exists a productive sequence $\pi' = (w'_j)_{0 \leq j \leq k}$ in Π and a decomposition of w'_j into $w'_j = w'_{j,\text{in}} w'_{j,\text{out}}$ such that the following relation holds where r'_{j-1} and r'_j are the partial sums $\delta(w'_0) + \dots + \delta(w'_{j-1})$ and $\delta(w'_0) + \dots + \delta(w'_j)$:

$$(m_j + r'_{j-1}) \xrightarrow{w'_{j,\text{in}}} q'_j \xrightarrow{w'_{j,\text{out}}} (m_j + r'_j)$$

Since $r_j, r'_j, r_{j-1}, r'_{j-1}$ are non negative and $q'_j \geq q_j$, we observe that the following relations hold:

$$\begin{aligned}(m_j + r_{j-1} + r'_{j-1}) &\xrightarrow{w'_{j,\text{in}}} (q'_j + r_{j-1}) \\ &\xrightarrow{w_{j,\text{out}}} (m_j + r_j + r_{j-1} + q'_j - q_j) \\ &\xrightarrow{w_{j,\text{in}}} (q'_j + r_j) \\ &\xrightarrow{w'_{j,\text{out}}} (m_j + r_j + r'_j)\end{aligned}$$

Let us consider the sequence $\theta = (w_{t\uparrow}, w_{t\downarrow})_{0 \leq t \leq k}$ defined by:

$$w_{t\uparrow} = \begin{cases} w_{t,\text{in}} w_{t,\text{out}} & \text{if } t \neq j \\ w'_{j,\text{in}} w_{j,\text{out}} & \text{if } t = j \end{cases}$$

$$w_{t\downarrow} = \begin{cases} w'_{t,\text{in}} w'_{t,\text{out}} & \text{if } t \neq j \\ w_{j,\text{in}} w'_{j,\text{out}} & \text{if } t = j \end{cases}$$

Observe that the vector $x_t = \delta(w_0) + \dots + \delta(w_{t-1}) + \delta(w_{t\uparrow})$ is equal to $r_t + r'_{t-1}$ if $t \neq j$ and it is equal to $r_j + r_{j-1} + (q'_j - q_j)$ if $t = j$. In particular x_t is non-negative for every $t \in \{0, \dots, k\}$ and $x_j(i) > 0$. We have proved that θ is a mixed-productive sequence with a jth increase x_j such that $x_j(i) > 0$.

□

Lemma 7.4. *For every weak-productive sequence $\tilde{\pi} = (\tilde{w}_j)_{0 \leq j \leq k}$ in $\tilde{\Pi}$, there exists a mixed-productive sequence $\theta = (w_{j\uparrow}, w_{j\downarrow})_{0 \leq j \leq k}$ such that for every $n \in \mathbb{N}$ the following sequence π_n is in Π:*

$$\pi_n = (w_{j\uparrow} \tilde{w}_j^n w_{j\downarrow})_{0 \leq j \leq k}$$

Proof. Let us consider a strong mixed-productive sequence $\theta = (w_{j\uparrow}, w_{j\downarrow})_{0 \leq j \leq k}$ and let x_j be the jth increase of θ. Since θ is strong we deduce that $\infty x_j = v_j$ and in particular $\tilde{m}_j = m_j + \infty x_j$. As $\tilde{m}_j \xrightarrow{\tilde{w}_j}_{\tilde{G}_j} \tilde{m}_j$ we deduce that $\tilde{w}_j$ is fireable from $\tilde{m}_j$. Hence, there exists an integer $h_j \in \mathbb{N}$ such that $\tilde{w}_j$ is fireable from $m_j + h_j x_j$. Let us consider $h \in \mathbb{N}$ such that $h_j \leq h$ for every $j \in \{0, \dots, k\}$. By replacing θ by θ^h, we can assume without loss of generality that $h = 1$ and in particular $\tilde{w}_j$ is fireable from $m_j + x_j$ for every $j \in \{0, \dots, k\}$.

We introduce the productive sequence $\pi = (w_j)_{0 \leq j \leq k}$ where $w_j = w_{j\uparrow} w_{j\downarrow}$ for every $j \in \{0, \dots, k\}$ and the sequence $\pi_n = (w_{j\uparrow} \tilde{w}_j^n w_{j\downarrow})_{0 \leq j \leq k}$ where $n \in \mathbb{N}$. We are interested in proving that $\pi_n \in \Pi$. The case $n = 0$ is immediate since $\pi_0 = \pi$. So we can assume that $n \geq 1$. We consider the partial sums $p_j = \delta(\tilde{w}_0) + \dots + \delta(\tilde{w}_j)$ and $r_j = \delta(w_0) + \dots + \delta(w_j)$ associated to the sequences $\tilde{\pi}$ and π. Note that the partial sums of π_n are equal to $r_j + np_j$ which is non-negative. Moreover $\delta(\pi_n) = \delta(\pi) + n\delta(\tilde{\pi})$ which is in R since $(R, +)$ is a monoid and $\delta(\pi), \delta(\tilde{\pi}) \in R$. In order to prove that $\pi_n \in \Pi$ it is sufficient to prove that σ^{π_n} is fireable from m_0. A prefix u of this word is a word of the form $u = w_{0\uparrow} \tilde{w}_0^n w_{0\downarrow} a_1 \dots w_{j-1\uparrow} \tilde{w}_{j-1}^n w_{j-1\downarrow} a_j v$ where v is either a prefix of $w_{j\uparrow}$, or v is a word of the form $v = w_{j\uparrow} \tilde{w}_j^t \tilde{w}$ where $t \in \{0, \dots, n-1\}$ and $\tilde{w}$ is a prefix of $\tilde{w}_j$, or v is a word of the form $v = w_{j\uparrow} \tilde{w}_j^n w$ where w is a prefix of $w_{j\downarrow}$. We prove that $m_0 + \delta(u)$ is non-negative by dividing the proof in three cases. In the first case, if v is a prefix of $w_{j\uparrow}$ then $m_0 + \delta(u) = np_{j-1} + m_0 + \delta(u')$ with $u' = w_0 a_1 w_1 \dots a_j v$. Since u' is a prefix of σ^π that is fireable from m_0, we deduce that $m_0 + \delta(u')$ is non-negative. Thus $m_0 + \delta(u)$ is non-negative. In the second case, we have $v = w_{j\uparrow} \tilde{w}_j^t \tilde{w}$ where $t \in \{0, \dots, n-1\}$ and $\tilde{w}$ is a prefix of $\tilde{w}_j$. We have $m + \delta(u) = tp_j + (n-t)p_{j-1} + m_j + x_j + \delta(\tilde{w})$. As $\tilde{w}_j$ is fireable from $m_j + x_j$, we deduce that $m_j + x_j + \delta(\tilde{w})$ is non-negative. Therefore $m + \delta(u)$ is non-negative. In the last case, we have $v = w_{j\uparrow} \tilde{w}_j^n w$ where w is a prefix of $w_{j\downarrow}$. In this case $m_0 + \delta(u) = np_j + m_0 + \delta(u')$ with $u' = w_0 a_1 \dots w_{j-1} a_j w_{j\uparrow} w$. As u' is a prefix of σ^π that is fireable from m_0, we deduce that $m_0 + \delta(u')$ is non-negative. Therefore $m_0 + \delta(u)$ is non-negative. We have proved that σ^{π_n} is fireable from m_0. Therefore $\pi_n \in \Pi$.

□

By F_j we denote the set of total functions $f_j : \tilde{E}_j \to \mathbb{N}$ that satisfy the Kirchhoff laws of $\tilde{G}_j$. We extend δ over F_j by the following equality:

$$\delta(f_j) = \sum_{\tilde{e} = (\tilde{q}, a, \tilde{q}') \in \tilde{E}_j} f_j(\tilde{e}) \delta(a)$$

We introduce the set F of sequences $f = (f_j)_{0 \leq j \leq k}$ of elements $f_j \in F_j$ such that the partial sums $\delta(f_0) + \dots + \delta(f_j)$ are non-negative for every $j \in \{0, \dots, k\}$ and such that the total sum denoted by $\delta(f) = \sum_{j=0}^k \delta(f_j)$ satisfies $\delta(f) \in R$. Since F is generated by $\min_{\leq}(F \backslash \{0\})$ we deduce that $(F, +)$ is a finitely generated monoid. We associate to every weak-productive sequence $\tilde{\pi} = (\tilde{w}_j)_{0 \leq j \leq k}$ the sequence $f = (f_j)_{0 \leq j \leq k}$ where f_j is the Parikh image of the cycle $\tilde{m}_j \xrightarrow{\tilde{w}_j}_{\tilde{G}_j} \tilde{m}_j$. Observe that $f \in F$ since $\delta(f_j) = \delta(\tilde{w}_j)$. In the sequel, the sequence f is denoted by $||\tilde{\pi}||$ and the total function f_j by $||\tilde{\pi}||_j$.

Lemma 7.5. *For every f interior to $(F, +)$, there exists a weak-productive sequence $\tilde{\pi} \in \tilde{\Pi}$ such that $f = ||\tilde{\pi}||$.*

Proof. Let us consider $f = (f_j)_{0 \leq j \leq k}$ a function interior to $(F, +)$. Let us first prove that $f_j(\tilde{e}) > 0$ for every $\tilde{e} \in \tilde{E}_j$. Let $\tilde{e} \in \tilde{E}_j$. By definition of $\tilde{E}_j$, there exists $\pi \in \Pi$ such that $||\pi||_j(\tilde{e}) > 0$. Since f is in the interior of F and $||\pi|| \in F$, there exists an integer $n \in \mathbb{N}_{>0}$ such that $nf \in ||\pi|| + F$. In particular $nf_j(\tilde{e}) \geq ||\pi||_j(\tilde{e})$. We have proved that $f_j(\tilde{e}) > 0$ for every $\tilde{e} \in \tilde{E}_j$. Since $\tilde{G}_j$ is strongly connected, f_j satisfies the Kirchhoff laws, and $f_j(\tilde{e}) > 0$ for every $\tilde{e} \in \tilde{E}_j$, the function f_j is the Parikh image of a cycle $\tilde{m}_j \xrightarrow{\tilde{w}_j}_{\tilde{G}_j} \tilde{m}_j$. Let $\tilde{\pi} = (\tilde{w}_j)_{0 \leq j \leq k}$. From $\delta(\tilde{w}_j) = \delta(f_j)$ we deduce that $\tilde{\pi} \in \tilde{\Pi}$.

□

Proposition 7.6. *The monoid* $(||\Pi||, +)$ *is simple and* F *is a linearization.*

Proof. Observe that $(||\Pi||, +) \subseteq F$. Next, let us consider an interior function $f \in F$. Lemma 7.5 proves that there exists a weak-productive sequence $\tilde{\pi} = (\tilde{w}_j)_{0 \leq j \leq k}$ such that $f = ||\tilde{\pi}||$. From Lemma 7.4, there exists a mixed-productive sequence $\theta = (w_{j\uparrow}, w_{j\downarrow})_{0 \leq j \leq k}$ such that for every $n \in \mathbb{N}$ the following sequence π_n is in Π:

$$\pi_n = (w_{j\uparrow} \tilde{w}_j^n w_{j\downarrow})_{0 \leq j \leq k}$$

Observe that $||\pi_n|| = ||\pi|| + nf$ where $\pi = (w_{j\uparrow} w_{j\downarrow})_{0 \leq j \leq k}$. We are done.

□

From Lemma 3.3, since δ is a morphism from $(F, +)$ to $(R, +)$, we deduce that $\delta(||\Pi||)$ is simple and $\delta(F)$ is a linearization. As $\delta(||\Pi||) = \delta(\Pi)$ and $\delta(F) \subseteq R$, we get the following theorem:

Theorem 7.7. *The monoid* $(\delta(\Pi_{m_0,\sigma,R}), +)$ *is simple. Moreover it admits a linearization included in* R.

From Theorem 7.7, Theorem 6.5 and Proposition 5.1 we deduce the following main theorem of this paper.

Theorem 7.8. *For every Presburger sets* $X, Y \subseteq \mathbb{N}^d$ *the following sets are Lambert sets:*

$$\text{post}^*(X) \cap Y \quad \text{pre}^*(X) \cap Y$$

8. Conclusion

We deduce that the following algorithm decides the reachability problem.

Reachability(m , $\mathcal{V}$, m')
  $k \leftarrow 0$
  repeat forever
    for each word $\sigma \in A^k$
      if $m \xrightarrow{\sigma} m'$
        return ''reachable''
    for each Presburger formula $\psi(x)$ of length k
      if $\psi(m)$ and $\neg\psi(m')$ are true and
        $\psi(x) \wedge y = x + \delta(a) \wedge \neg\psi(y)$ unsat $\forall a \in \Sigma$
        return ''unreachable''
    $k \leftarrow k + 1$

The correctness is immediate and the termination is guaranteed by Theorem 7.8 and Theorem 4.1. This algorithm is the *very first one* that does not require the classical KLMST decomposition. Even though the termination proof is not based on the KLMST decomposition, the complexity of the algorithm is still open. In fact, the complexity depends on the minimal size of a word $\sigma \in \Sigma^*$ such that $m \xrightarrow{\sigma} m'$ if $m \xrightarrow{*} m'$, and the minimal size of a Presburger formula $\psi(x)$ denoting a forward invariant I such that $m \in I$ and $m' \in I$ otherwise. We left as an open question the problem of computing lower and upper bounds for these sizes. Note that the VAS exhibiting a large (Ackermann size) but finite reachability set given in [MM81] does not directly provide an Ackermann lower-bound for these sizes since inductive separators can over-approximate reachability sets.

We also left as an open question the problem of adapting the *Counter Example Guided Abstract Refinement* approach [CGJ+00] to obtain an algorithm for the VAS reachability problem with termination guarantee. In practice, such an algorithm should be more efficient than the previously given enumeration-based algorithm.

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