Spin and Clifford algebras, an introduction

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July 15, 2010

Abstract In this short pedagogical presentation, we introduce the spin groups and the spinors from the point of view of group theory. We also present, independently, the construction of the low dimensional Clifford algebras. And we establish the link between the two approaches. Finally, we give some notions of the generalisations to arbitrary space-times, by the introduction of the spin and spinor bundles.

Contents

1 Clifford algebras

A Clifford algebra is canonically associated to any vector space (V, g) with a quadratic form g (a scalar product). This algebra, compatible with the quadratic form, extends the capacities of calculations on V. One distinguishes real and complex Clifford algebras, which extend real and complex vector spaces.

1.1 Preliminaries: tensor algebra and exterior algebra over a vector space

1.1.1 Tensor algebra over a vector space

Let us consider a vector space V (no scalar product is assumed). From V and its dual V^* (space of one-forms), are constructed new objects called tensors, which form an algebra.

Among the tensors, we will select the completely antisymmetric ones, called *multivectors* or *multiforms*. Multivectors form the exterior algebra of V. Multiforms form the exterior algebra of the dual vector space V^*.

We recall the definition of the the vector space of tensors of type (s, r):

$$\tau^{(s,r)}V \equiv \bigotimes^{s} V^* \bigotimes^{r} V,$$

with s and r factors respectively. Vectors are (0,1) tensors; one-forms are (1,0) tensors. An (s, r) tensor is a linear operator on $V^s \otimes (V^*)^r$, and this may be taken as the definition.

A basis (frame) (e_A) for V induces canonically a reciprocal basis (coframe) (e^A) for V^*. Their tensor products provide a canonical basis $(e_{A_1}) \otimes (e_{A_2}) \otimes ...(e_{A_s}) \otimes (e^B_1) \otimes (e^{B_2}) \otimes ...(e^{B_r})$ for tensors. This extends the isomorphism between V and V^* to an isomorphisms between $\tau^{(r,s)}$ and $\tau^{(s,r)}$.

By the direct sum operation, one defines the vector space of all tensors

$$\tau V = \bigoplus_{s=0,r=0}^{\infty} \tau^{(s,r)}V.$$

It has a (non commutative) algebra structure with respect to the tensor product.

The sets of all tensors of $(s, 0)$ type, $\forall s$ (called covariant), and of all tensors of $(0, r)$ type, $\forall r$ (called contravariant) have similar vector space structures. These definitions of tensors require no other structure than that of the vector space.

Now we will define antisymmetrisation properties of tensors: An antisymmetric [contravariant]tensor of type $(0; p)$ will be called a *p-vector*, more generally a *multivector*. An antisymmetric [covariant] tensor of type $(p; 0)$ defines a *p-form*, more generally a *multiform* (more simply, a form).

1.1.2 Antisymmetry and the wedge product

Given a vector space V, the (normalized) antisymmetric part of the tensor product of two vectors is defined as

$$v \wedge w = \frac{1}{2} \ (v \otimes w - w \otimes v).$$

(care must be taken that one often finds the definition without the normalizing factor). This is an antisymmetric tensor of rank (0,2), also called a *bivector*. The goal of this section is to extend this definition, i.e., to define the antisymmetric part of a tensor product of an arbitrary number of vectors. This defines a new product, the *wedge product*. The wedge product of two vectors defines a bivector. Its generalization will lead to consider new objects called *multivectors* (= skew contravariant tensors).

The wedge product is also defined for the dual V^*. In the same way that the vectors of V^* are called the one-forms of V, the multivectors of V^* are called the *multi-forms* (= skew covariant tensors) of V, which are usually be simply called *forms*.

With the wedge product, multivectors form an algebra, the *exterior algebra* [of multivectors] $\bigwedge V$of V. Multiforms form the exterior algebra of multiforms $\bigwedge V^*$ on V. These algebras are defined in the absence of any inner product or metric in the initial vector space. However, an inner product will allow us to define additional structures:

- a canonical (musical) isomorphism between $V$and its dual V^*, which extends to the exterior algebras $\bigwedge V$ and $\bigwedge V^*$;
- an (Hodge) duality (1.2.4) in the exterior algebras;
- an additional algebra structure: that of Clifford algebra, which result from the definition of new products on $\bigwedge V$ and $\bigwedge V^*$, the Clifford products.

The antisymmetric symbol

To define properly the wedge product, we introduce the antisymmetric symbol. Let us consider the set $\{1,2,...,n\}$ of the n first integers. We recall that a *permutation* is an ordered version of this set, $(i_1,i_2,...,i_n)$, where each $i_k \in \{1,2,...,n\}$. Its parity is defined as the number of pair exchanges necessary to reach it from the permutation $1,2,...n$.

We define the completely antisymmetric symbol $[i_1,i_2,...,i_n]$, which takes the value 1,-1, or 0, according to the parity of the permutation $(i_1,i_2,...,i_n)$. We have for instance

$$[1,2,3,...,n]=1,\ [2,1,3,...,n]=-1,\ [1,1,3,...,n]=0.$$

Note that the number of non zero permutation is $n!$. One often writes $[\alpha,\beta,\gamma,...]$ under the form $\epsilon_{\alpha\beta\gamma...}$.

1.1.3 The operator of antisymmetry and the wedge product

If V is a vector space of dimension d, the tensor product

$$\bigotimes^p V = V \otimes V \otimes ... \otimes V$$

has also a vector space structure. Its elements, the tensors of type (0,p), are sums of elements of the form $v_1 \otimes v_2 \otimes ... \otimes v_p$.

To such a tensor, we associate its (normalized) completely antisymmetric part

$$v_1 \wedge v_2 \wedge ... \wedge v_p \equiv \mathrm{Skew}[v_1 \otimes v_2 \otimes ... \otimes v_p] \equiv \frac{\sum [i_1,i_2,...,i_p]\ v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_p}}{p!}. \quad (1)$$

The sum extends over all permutations (we recall that $p! = \sum [i_1,i_2,...,i_p]$). It is called the *wedge (or external) product*.

Such an external product is a skew $(0,p)$ tensor called a $p-$multivector (or $p-$vector). The definition is extended by linearity : the sum of two $p-$multivectors is a $p-$multivector:

the $p-$multivectors form the vector space $\bigwedge^p V$, of dimension $\begin{pmatrix} d \\ p \end{pmatrix}$ (the binomial coefficient).

If V_p and V_q are a $p-$vector and a $q-$vector, we have

$$V_p \wedge V_q = (-1)^{pq}\, V_q \wedge V_p.$$

[Note that the wedge product is often defined without normalization. In this case, many formulas differ by the factor $p!$].

For instance, the external product of two vectors is the antisymmetrical part of their tensor product:

$$v \wedge w \equiv \frac{v \otimes w - w \otimes v}{2}.$$

It results that $v \wedge v = 0$.

The wedge product of vectors is distributive, associative and completely antisymmetric. The wedge product of a number p of vectors is zero iff the vectors are linearly dependent. This implies that the maximum order of a multivector is d, the dimension of the original vector space, and also that there is only one multivector of order d, up to a multiplicative constant.

A scalar is identified as a multivector of order zero. An usual vector is a multivector of order one. The wedge product of a $p-$mutivector by itself, $M \wedge M$, is always 0 when p is odd. This is not true when p is even.

We will give to multivectors an algebraic structure by extending the external product to them.

1.1.4 The exterior algebra of multivectors

The bivectors form the vector space $\wedge^2(M)$, of dimension $\frac{d(d-1)}{2}$. A general bivector cannot necessarily be decomposed as a wedge product. When this is possible, the bivector is called *simple*. A simple bivector $B = a \wedge b$ can be considered as the oriented triangle with the vectors a and b as sides. Then $\langle B \mid B \rangle$ (see below) is the oriented area of the triangle.

Now we extend the sum to multivectors of different orders, up to d, like

$$A_0 + A_1 + A_2 + ... + A_d,$$

where A_p is a p-vector (the expansion stops at d). Such multivectors have not a definite order. They belong to the vector space

$$\bigwedge V \equiv \bigoplus_{p=0}^{d} \bigwedge^{p} V$$

of all multivectors, of dimension 2^d.

The wedge product is easily extended to all multivectors by linearity, associativity, distributivity and anticommutativity for the 1-vectors. This provides an an algebra structure to $\bigwedge V$: the *exterior algebra of multivectors*.

This also allows the practical calculations of wedge products. For instance,

$$v \wedge w \wedge (v + w + x) = v \wedge w \wedge v + v \wedge w \wedge w + v \wedge w \wedge x$$

$$= -(v \wedge v) \wedge w + v \wedge (w \wedge w) + v \wedge w \wedge x = v \wedge w \wedge x,$$

which is a trivector if we assume v, w and x linearly independent.

A scalar product on V allows us to define an other algebra structure for the multivectors. This results from the introduction of new product, the Clifford product, which unifies the wedge product and the scalar product : the Clifford algebra of multivectors, presented in (1.2). Contrarily to the wedge product, the Clifford product is, in some cases, invertible.

1.2 Clifford algebras

1.2.1 The Clifford product

The definitions of the wedge product, and of the multivectors do not depend on any inner product. Now, we will assume an inner product in V:

$$g: \ u, v \mapsto g(u, v) \equiv u \cdot v.$$

One defines the *Clifford (or geometrical) product* of two vectors as

$$u \ v \equiv u \cdot v + u \wedge v.$$

In general, this appears as the sum of a scalar (polyvector of grade zero) plus a bivector (polyvector of grade 2), thus a *non homogeneous* multivector. The scalar product $u \cdot v = \frac{u \ v + v \ u}{2}$ and the wedge product $u \wedge v = \frac{u \ v - v \ u}{2}$ appear as the symmetrical and antisymmetrical parts of the Clifford product.

The Clifford product is by definition associative and distributive. These properties allow us to extend it to all multivectors. To illustrate:

$$aba = a(ba) = a \ (b \cdot a + b \wedge a) = a \ (a \cdot b - a \wedge b) = a \ (2a \cdot b - ab) = 2a \ (a \cdot b) - aab = 2 \ (a \cdot b) \ a - (a \cdot a)b.$$

or

$$v \ w \ (v + w + x) = v \ w \ v + v \ w \ w + v \ w \ x = 2 \ (v \cdot w) \ v - (v \cdot v)w + v \ (w \ w) + v \ w \ x$$

$$= v \ (w \cdot v + w \wedge v) + v \ (w \cdot w + w \wedge w) + v \ w \ x$$

$$= v \ (v \cdot w - v \wedge w) + v \ (w \cdot w) + v \ w \ x = v \ (v \ w) + v \ (w \cdot w) + v \ w \ x$$

$$= (v \ v) \ w) + v \ (w \cdot w) + v \ w \ x = (v \cdot v) \ w + v \ (w \cdot w) + v \ w \ x.$$

This polyvector is a sum of vectors (polyvector of grade 1) and trivectors (polyvector of grade 3).

1.2.2 The Clifford algebra

The *Clifford algebra* $C\ell(V)$ is defined as $\bigwedge V$, with the Clifford product $v, w \mapsto v\ w$. As a vector space (but not as an algebra), $C\ell(V)$ is isomorphic to the exterior algebra $\bigwedge V$. Thus, its elements are the multivectors defined over V, although with a different multiplication law which takes into account the properties of the metric. It provides an extension of V, and of the the calculation possibilities. (In the special case where the metric is zero, $C\ell(V) = \bigwedge V$.)

More formally, one may define $C\ell(V)$ as the quotient of the tensor algebra $T(V)$ over V by the (bilateral) ideal generated by the set $\{x \in V;\ x \otimes x - g(x,x)\ \mathbb{I}\}$.

Note that the Clifford algebra structure may be defined in an abstract way, with a Clifford product. In this case, the vector space of multivectors is simply a peculiar representation. Here we will present the Clifford algebra structure through this representation. Other representations also exist.

A polyvector of definite order is called homogeneous. In general, this is not the case, and we define the projectors $< \cdot >_r$ which project a polyvector onto its homogeneous part of grade r.

We call $C\ell^k(V)$ the vector space of polyvectors of grade k. As a vector space, we have

$$C\ell(V) = \bigoplus_{k=0}^{d} C\ell^k(V).$$

As vector spaces, we have $C\ell^0(V) \equiv \mathbb{R}$, which is thus seen as embedded in $C\ell(V)$, as the multivectors of grade 0 (0-vectors). The vector space V itself may be seen as embedded in $C\ell(V)$, as $C\ell^1(V)$: its elements are the multivectors of grade 1 (or 1-vectors).

Paravectors

The addition of a scalar plus a grade one vector is called a *paravector*. It can be expanded as $A = A^0 + A^i\ e_i$, where $A^0 =< A >_0$ and $A^i\ e_i =< A >_1$. The vector space of paravectors is thus $\mathbb{R} \oplus V = C\ell^0(V) \oplus C\ell^1(V) \subset C\ell(V)$.

We define also the even and odd subspaces of a Clifford algebra C as the direct sums

$$C^{even} = \bigoplus_{k\ even} C^k \text{ and } C^{odd} = \bigoplus_{k\ odd} C^k.$$

Both have dimension 2^{d-1} and C^{even} is a subalgebra of C.

The pseudoscalars

Up to a multiplicative scalar, there is a unique d-multivector. To normalise, we choose an oriented ON basis for V, and define $\mathcal{I} = e_1 \ldots e_d = e_1 \wedge \ldots \wedge e_d$ as the *orientation operator*. It verifies

$$\mathcal{I}^2 = (-1)^{\frac{d(d-1)}{2}+s},$$

depending on the dimension and on the signature of the vector space (V, g). The multiples of $\mathcal{I}$ are called the *pseudoscalars*.

When the dimension is odd, $\mathcal{I}$ commutes with all multivectors. When the dimension is even, it commutes with even grade multivectors, and anti-commutes with odd grade ones:

$$\mathcal{I}\ P_r = (-1)^{r(d-1)}\ P_r\ \mathcal{I}.$$

The *center* (the set of elements commuting with all elements) of $\mathrm{C}\ell(V)$ is $\mathrm{C}\ell^0(V)$ for d even, or $\mathrm{C}\ell^0(V) \oplus \mathrm{C}\ell^d(V)$ for d odd.

The multiplication rules imply that the multiplication by $\mathcal{I}$ transforms a grade r polyvector P_r into the grade $d-r$ polyvector $\mathcal{I}P_r$, called the *orthogonal complement* of P_r.

Bivectors

After the scalars and the 1-vectors, the bivectors are the simplest polyvectors. The wedge product of two bivectors is zero or a quadrivector.

A bivector is called " simple " (or decomposable) if it can be written as a wedge product. Not all bivectors are simple, and one defines the rank of a bivector B in the following equivalent ways:

- i) The minimum integer r such that $\wedge^r B = 0$.
- ii) The minimum number of non-zero vectors whose exterior products can add up to form B.
- iii) The number r such that the space $\{X \in V; X \wedge B = 0\}$ has dimension r.
- iv) The rank of the component matrix of B, in any frame on V.

A bivector B of rank 2 (minimum value for a non zero bivector) is simple. The *simplicity condition* is expressed as $B \wedge B = 0$ or, in tensorial components,

$$B_{[\mu\nu}\ B_{\rho\sigma]} = 0. \tag{2}$$

For instance, the Plebanski formulation of general relativity [7] considers a bivector as the dynamical variable, to which is imposed a simplicity constraint.

To each simple bivector, one may associate uniquely, up to a scalar multiplication, the [two-]plane $\mathrm{span}(v, w)$ through the origin, subtended by two vectors v and w. This establishes a one to one correspondence between the planes through the origin and the projective simple bivectors $[v \wedge w]$, where $[v \wedge w]$ is defined as the set of bivectors proportional to the bivector $v \wedge w$. The projective simple bivector $[A \wedge B]$ belongs to the projective space $P \bigwedge^2(V)$, the set of equivalence classes of bivectors under the scalar multiplication.

1.2.3 Automorphisms in Clifford algebras

There are three important automorphisms canonically defined on a Clifford algebra C.

- **The reversion**:
  the *reversion*, or *principal anti-automorphism* is defined as the transformation $C \mapsto C$ which reverses the order of the factors in any polyvector:

  $$R \equiv v_1 \ldots v_k \mapsto R^T \equiv v_k \ldots v_1.$$

  It is trivially extended by linearity. Scalars and vectors remain unchanged. Bivectors change their sign. For an homogeneous multivector, we have

  $$(A_r)^T = (-1)^{r(r-1)/2}\ A_r,\ r \geq 1.$$

- **Main involution**:
  the *main involution* (or grade involution) $a \mapsto a^*$ is defined through its action $e_i \mapsto -e_i$ on the vectors of $\mathrm{C}\ell^1$. It may also be written

  $$a \mapsto (-1)^d\ \mathcal{I}\ a\ \mathcal{I}^{-1}.$$

  Even or odd grade elements of C form the two eigenspaces C^{even} and C^{odd}, with eigenvalues 1 and -1, of the grade involution. Noting that

  $$C = C^{even} \oplus C^{odd},$$

  $$C^{even}\ C^{even} = C^{odd}\ C^{odd} = C^{even};\ C^{even}\ C^{odd} = C^{odd}\ C^{even} = C^{odd}$$

  makes C a Z_2-graded algebra.

- **[Clifford] conjugation**:
  the *conjugation*, or antiautomorphism, is the composition of both:

  $$\bar{R} = (R^*)^T.$$

1.2.4 Scalar product of multivectors and Hodge duality

The scalar product of V is extended to $\mathrm{C}\ell(V)$ as

$$g(A, B) = A \cdot B =< A^T\ B >_0,$$

where $< \cdot >_0$ denotes the scalar part.

It is bilinear. It reduces to zero for homogeneous multivectors of different grades. It reduces to the usual product for scalars (grade 0), and to the metric product for 1-vectors (grade 1). In general, we have the decomposition

$$A \cdot B =< A >_0 \cdot < B >_0 + < A >_1 \cdot < B >_1 + ... + < A >_n \cdot < B >_n .$$

The Hodge duality

The *Hodge duality* is defined as the operator

$$\begin{aligned} {}^\star: \quad \wedge^p &\mapsto \wedge^{n-p} && (3)\\ A_p &\mapsto {}^\star A_p && (4) \end{aligned}$$

such that

$$B_p \wedge ({}^\star A_p) = (B_p \cdot A_p)\ \mathcal{I},\ \forall B_p \in \wedge^p.$$

It may be checked that, for p-forms, it coincides with the usual Hodge duality of forms defined from the metric.

The simplicity condition for a bivector can be written as $B \wedge B = 0 \Leftrightarrow < B, {}^\star B >= 0$, implying that ${}^\star B$ is also simple.

In 4 dimensons, the Hodge duality transforms a bivector into a bivector. Any bivector can be decomposed in a *self-dual* and an *anti-self-dual* part:

$$B = B^+ + B^-,\ {}^\star B = B^+ - B^-.$$

1.2.5 Frames

A frame $(e_i)_{i=1\cdots n}$ for V defines a natural frame for $\bigwedge V$. To define it, we consider all the finite sets of the form

$$I \equiv \{i_1, ..., i_k\} \subset \{1, ..., n\},\ \text{with } i_1 < i_2 < ... < i_k.$$

We define the multivectors $e_I = e_{i_1} \wedge ... \wedge e_{i_k}$, and $e_\emptyset = e_0 = 1$ (ordered sequences only). The multivectors e_I provide a basis for the vector space $\bigwedge V$, and thus for C, with the " orthographic " index I going from 1 to 2^n.

A multivector is expanded in this basis as

$$A = A^I\ e_I \equiv A_0 + A^i\ e_i + A^{ij}\ e_{\{ij\}} + ... + A^{1,2,...,n}\ e_{\{1,2,...,n\}}.$$

Its components A^I may be seen as coordinates in C. Thus, functions on C may be considered as functions of the coordinates, and this allows us to define a differential structure, with a basis for one-forms given by the $\mathrm{d}X^A$.

When the basis (e_i) is ON ($e_i \cdot e_j = \eta_{ij} = \pm\delta_{ij}$), it is so for the basis (e_I) of $\mathrm{C}\ell(V)$, and we may define extended metric coefficients $\eta_{IJ} \equiv e_I \cdot e_J$. In such an ON basis, the scalar product of arbitrary multivectors expands as

$$A \cdot B = \eta_{IJ}\ A^I\ B^J \equiv A^0\ B^0 \pm A^i\ B^i \pm A^{ij}\ B^{ij} \pm ... \pm A^{1,2,...,n}\ B^{1,2,...,n}.$$

Summation is assumed over all orthographic indices, and the $\pm$ signs depend on the signature.

1.3 Vectors and forms

Given a vector space V, we recall that its *dual* V^* (the set of linear forms on V) is a vector space isomorphic to V. A Clifford algebra may be similarly constructed from V^*. Thus, to (V, g) one associates its Clifford algebra of multivectors $\mathrm{C}\ell(V)$, and its Clifford algebra of [multi-]forms $\mathrm{C}\ell(V^*)$. They are isomorphic. A bivector of $\mathrm{C}\ell(V^*)$ is called a 2-form of V, etc.

The scalar product induces the *canonical (or musical) isomorphism* between V and V^*. It is easily extended to an isomorphism between $\mathrm{C}\ell(V)$ and $\mathrm{C}\ell(V^*)$. The scalar product g of V induces the scalar product on V^* (also written g)

$$g(\alpha, \alpha) = g(\sharp\alpha, \sharp\alpha).$$

It is extended to $\mathrm{C}\ell(V^*)$ as above. The pseudoscalar $\mathcal{I}$ of $\mathrm{C}\ell(V^*)$ identifies with the volume form Vol associated to the metric. The Hodge duality (1.2.4) in $\mathrm{C}\ell(V^*)$ identify with its usual definition for forms.

In a [pseudo-]Riemannian manifold $\mathcal{M}$, the tangent spaces $\mathrm{T}_m\mathcal{M}$, and their duals $\mathrm{T}^*_m\mathcal{M}$, at all points $m \in \mathcal{M}$ define the tangent and the cotangent bundles. Similarly, the reunions of their Clifford algebras define the *Clifford bundles* of multivectors and multiforms on $\mathcal{M}$, respectively (see below).

1.4 Complex Clifford algebra

When the vector space is a complex vector space, its Clifford algebra is also complex.

Given a real vector space V, we note $\mathbb{C}\ell(V)$ the complexified Clifford algebra $\mathbb{C} \otimes \mathrm{C}\ell(V)$. A case of interest for physics is when $V = \mathbb{R}^{1,3} = \mathbb{M}$, the Minkowski vector space, and we study below the *space-time-algebra* $\mathrm{C}\ell(\mathbb{M})$. Its complexification $\mathbb{C}\ell(\mathbb{M}) \equiv \mathbb{C}\ell(\mathbb{R}^{1,3})$ is called the *Dirac algebra*. It is isometric to $\mathrm{C}\ell(\mathbb{R}^{2,3})$, and $\mathrm{C}\ell(\mathbb{R}^{1,3})$ appears as a subalgebra of $\mathrm{C}\ell(\mathbb{R}^{2,3})$.

More generally, from the complex algebra $\mathbb{C}\ell(n)$ it is possible to extract the real Clifford algebra $\mathrm{C}\ell(p, q)$ with $p + q = n$. To do so, we extract $\mathbb{R}^{p,q}$ from $\mathbb{C}^n$: as a complex space, $\mathbb{C}$ admits the basis $e_1, ..., e_n$. We may see $\mathbb{C}$ as a real vector space with the basis $e_1, ..., e_p, ie_{p+1}, ..., ie_{p+q}$. Chosing n vectors in this list, we construct the real subvector space $\mathbb{R}^{p,q}$, which heritates from the quadratic form. It follows that any element $a \in \mathbb{C}\ell(n)$ may be decomposed as $a = a_r + ia_c,\ a_r, a_c \in \mathrm{C}\ell(p, q)$ (see more details in, e.g., [12]).

Matrix representations

There are natural representations of $\mathbb{C}\ell(d)$ on a (complex) vector space of dimension 2^k, with $k \equiv [d/2]$ (integer part) [9]. Its elements are called Dirac spinors, see below. Elements of $\mathbb{C}\ell(d)$ are represented by matrices of order 2^k, i.e., elements of the algebra $\mathrm{Mat}_{2^k}(\mathbb{C})$, acting as endomorphisms.

This representation is faithful when d is even and non-faithful when d is odd.

1.5 The simplest Clifford algebras

The structure of a real Clifford algebra is determined by the dimension of the vector space and the signature of the metric, so that it is written $C\ell_{p,q}(\mathbb{R})$. It is expressed by its multiplication table. A *matrix representation* of a Clifford algebra is an isomorphic algebra of matrices, which thus obeys the same multiplication table. (Such matrix representations lead to the construction of spinors, see below).

The table (1) gives the matrix representations of the lower dimensional Clifford algebras. It is extracted from [9], who gives its extension up to $d = 8$. Note the links with complex numbers and quaternions.

Periodicity theorems allow to explore the Clifford algebras beyond dimension 8. They obey the following algebra isomorphisms

$$C\ell(p+1, q+1) \approx C\ell(1,1) \otimes C\ell(p,q), \tag{5}$$

$$C\ell(p+2, q) \approx C\ell(2,0) \otimes C\ell(p,q),$$

$$C\ell(p, q+2) \approx C\ell(0,2) \otimes C\ell(p,q).$$

We will pay special attention to

- the algebra of the plane $C\ell(\mathbb{R}^2) = C\ell(2)$;
- the space algebra, or *Pauli algebra* $C\ell(\mathbb{R}^3) = C\ell(3)$;
- The space-time algebra $C\ell(1,3)$, the algebra of [Minkowski] space-time, that we describe below in (3.4). Note the difference between $C\ell(1,3)$ and $C\ell(3,1)$ which may indicate a non complete equivalence between the two signatures for Minkowski space-time.

1.6 The geometric algebra of the plane

The Clifford algebra of the plane, $C\ell(\mathbb{R}^2) \equiv C\ell(2)$ extends the two-dimensional plane $(\mathbb{R}^2, g)$, with the Euclidean scalar product $g(u,v) = u \cdot v$. Let us use an ON basis ($e_i \cdot e_j = \delta_{ij}$) for $\mathbb{R}^2$.

Antisymmetry implies that the only bivector (up to a scalar) is

$$e_1\ e_2 = e_1 \wedge e_2 = -e_2\ e_1 \equiv \mathcal{I}_{C\ell(2)} = \mathcal{I}.$$

The rules above imply $\mathcal{I}^2 = -1$. We may check that $C\ell(2)$ is closed for multiplication, and admits the basis $(1,\ e_1,\ e_2, \mathcal{I})$, as indicated in the table (2).

The general polyvector expands as

$$A = A^0\ 1 + A^1\ e_1 + A^2\ e_2 + A^3\ \mathcal{I}, \tag{6}$$

$C\ell(0,0)$	$\mathbb{R}$
$C\ell(1,0)$	$\mathbb{R} \oplus \mathbb{R}$
$C\ell(0,1)$	$\mathbb{C}$
$C\ell(2,0)$	$\mathrm{Mat}_2(\mathbb{R})$
$C\ell(1,1)$	$\mathrm{Mat}_2(\mathbb{R})$
$C\ell(0,2)$	$\mathbb{H}$ (quaternions)
$C\ell(3,0)$	$\mathrm{Mat}_2(\mathbb{C})$
$C\ell(2,1)$	$\mathrm{Mat}_2(\mathbb{R}) \oplus M_2(\mathbb{R})$
$C\ell(1,2)$	$\mathrm{Mat}_2(\mathbb{C})$
$C\ell(0,3)$	$\mathbb{H} \oplus \mathbb{H}$
$C\ell(4,0)$	$\mathrm{Mat}_2(\mathbb{H})$
$C\ell(3,1)$	$\mathrm{Mat}_4(\mathbb{R})$
$C\ell(2,2)$	$\mathrm{Mat}_4(\mathbb{R})$
$C\ell(1,3)$	$\mathrm{Mat}_2(\mathbb{H})$
$C\ell(0,4)$	$\mathrm{Mat}_2(\mathbb{H})$

Table 1: The first low dimensional Clifford algebras (from [9]). $\mathrm{Mat}_n(K)$ denotes the algebra of $n \times n$ matrices with elements in K.

$j_0 \equiv 1$	$j_1 \equiv e_1$, $j_2 \equiv e_2$	$j_3 \equiv \mathcal{I} \equiv e_1 e_2$
one scalar	2 vectors	one bivector

Table 2: The basis of the algebra of the plane

1	e_1	e_2	$\mathcal{I}$
e_1	1	$\mathcal{I}$	e_2
e_2	- $\mathcal{I}$	1	$-e_1$
$\mathcal{I}$	$-e_2$	e_1	-1

Table 3: The multiplication table for the algebra of the plane

$e_0 := 1$	$e_1, \; e_2, \; e_3$	$\mathcal{I}\ e_1, \mathcal{I}\ e_2, \mathcal{I}\ e_3$	$\mathcal{I} := \mathcal{I}\ e_0$
one scalar	3 vectors	3 pseudo-vectors	one pseudo-scalar

Table 4: The basis of the Pauli algebra

so that the four numbers $A^i \in \mathbb{R}$ play the role of coordinates for $C\ell(2)$. The full multiplication rules (see table 3) follow from associativity, symmetry and antisymmetry of the different parts.

The algebra of the plane and complex numbers

The Euclidean plane $\mathbb{R}^2$ is naturally embedded (as a vector space) in $C\ell(2)$ as $C\ell^1(2)$, the set of 1-vectors. We have the embedding isomorphism

$$\mathbb{R}^2 \quad \mapsto C\ell^1(2) \tag{7}$$

$$(x, y) \quad \mapsto x\ e_1 + y\ e_2. \tag{8}$$

Since $\mathbb{R}^2$ may be seen as $\mathbb{C}$, this may also be written

$$\mathbb{C} \quad \mapsto C\ell^1(2) \tag{9}$$

$$x + i\ y \quad \mapsto x\ e_1 + y\ e_2. \tag{10}$$

The right multiplication of such a 1-vector by $\mathcal{I}$ gives another 1-vector: $(x\ e_1 + y\ e_2)\ \mathcal{I} = x\ e_2 - y\ e_1$. We recognize a rotation by $\pi/2$ in $\mathbb{R}^2$. The geometrical role of the Clifford bivectors as rotation operators will be emphasized below.

1.7 The (Pauli) algebra of space

From the usual space $V = \mathbb{R}^3$, with an ON basis $(e_i)_{i=1,2,3}$, we construct $C\ell(\mathbb{R}^3) \equiv C\ell(3)$, the (Pauli) algebra of space. Its elements are sometimes called the *Pauli numbers*.

The orientation operator

The antisymmetrical products of two vectors give three bivectors (see the table 4). The trivector $e_1\ e_2\ e_3 \equiv \mathcal{I}$ with $\mathcal{I}^2 = -1$, the *orientation operator*, closes the multiplication law. The trivectors, also called the *pseudoscalars*, are all proportional to $\mathcal{I}$. They commute with all elements of $C\ell(3)$.

The *center* of $C\ell(3)$, i.e., the set of elements which commute with all elements, is

$$C\ell^0(3) \oplus C\ell^3(3) = \mathrm{span}(1, \mathcal{I}).$$

The similar algebraic properties of $\mathcal{I}$ and of the complex pure imaginary i allow us to define an algebra isomorphism between $\mathrm{span}(1, \mathcal{I}) \subset C\ell(3)$ and $\mathbb{C} \equiv \mathrm{span}(1, i)$.

We may write any bivector

$$e_\mu\ e_\nu = e_\mu\ e_\nu e_\rho\ e_\rho = \mathcal{I}\ e_\rho,$$

1	e_1	e_2	e_3	$\mathcal{I}\ e_1$	$\mathcal{I}\ e_2$	$\mathcal{I}\ e_3$	$\mathcal{I}$
e_1	1	$\mathcal{I}\ e_3$	$-\mathcal{I}\ e_2$	$\mathcal{I}$	$-e_3$	e_2	$\mathcal{I}\ e_1$
e_2	$-\mathcal{I}\ e_3$	1	$\mathcal{I}\ e_1$	e_3	$\mathcal{I}$	$-e_1$	$\mathcal{I}\ e_2$
e_3	$\mathcal{I}\ e_2$	$-\mathcal{I}\ e_1$	1	$-e_2$	e_1	$\mathcal{I}$	$\mathcal{I}\ e_3$
$\mathcal{I}\ e_1$	$\mathcal{I}$	$-e_3$	e_2	-1	$-\mathcal{I}\ e_3$	$\mathcal{I}\ e_2$	$-e_1$
$\mathcal{I}\ e_2$	e_3	$\mathcal{I}$	$-e_1$	$\mathcal{I}\ e_3$	-1	$-\mathcal{I}\ e_1$	$-e_2$
$\mathcal{I}\ e_3$	$-e_2$	e_1	$\mathcal{I}$	$-\mathcal{I}\ e_2$	$\mathcal{I}\ e_1$	-1	$-e_3$
$\mathcal{I}$	$\mathcal{I}\ e_1$	$\mathcal{I}\ e_2$	$\mathcal{I}\ e_3$	$-e_1$	$-e_2$	$-e_3$	-1

Table 5: The multiplication table for the Pauli algebra

where the index ρ is defined through $\epsilon_{\mu\nu\rho} = 1$. This allows us to rewrite the basis of $C\ell(3)$ under the form

$$1, (e_i), (\mathcal{I}\ e_i), \mathcal{I}$$

(see table 4). This divides $C\ell(3)$ into

- a " real " part $C\ell^0(3) \oplus C\ell^1(3) \equiv \{paravectors\} \equiv \mathrm{span}(1, e_i)$: the real multivectors identify to the paravectors.

- and an " imaginary " part: $C\ell^2(3) \oplus C\ell^3(3) \equiv \mathrm{span}(\mathcal{I}\ e_i, \mathcal{I})$. An imaginary multivector is the sum of a bi-vector and a tri-vector.

Thus, any Pauli number may be seen as a complex paravector.

Pauli algebra and matrices

With the identification above (of $\mathcal{I}$ by i), the restriction of the multiplication table (5) to the four paravectors $(1, e_i)$ identifies with that of the four Pauli matrices $(\mathbb{I}, \sigma_i)_{i=1,2,3} \equiv (\sigma_\mu)_{\mu=0,1,2,3}$. Thus, the real part $C\ell^{real}(3)$ is isomorphic (as an vector space) to Herm(2), the set of Hermitian complex matrices of order 2. This isomorphism extends to an algebra isomorphism between the complete algebra $C\ell(3)$ and the algebra of complex matrices of order 2, $M_2(\mathbb{C})$, explicited as

$$1,\ e_i \quad \mathcal{I}\ e_i,\quad \mathcal{I} \tag{11}$$

$$1,\ \sigma_i \quad i\ \sigma_i,\quad i \tag{12}$$

The three grade 1 vectors e_i identify with the three traceless Hermitian matrices σ_i, which span $\mathrm{Herm}_0(2)$ (traceless Hermitian matrices).

Quaternions in $C\ell(3)$

It is easy to check, from the multiplication table (5), the algebra isomorphism

$$C\ell(3)^{even} \equiv \mathbb{H},$$

the algebra of quaternions. Here $C\ell^{even}(3)$ is the algebra of even elements, scalars and bivectors. (However, the odd elements do not form a sub-algebra.) The isomorphism is realized through

$$1 \rightsquigarrow j_0,\ \mathcal{I}\ e_1 = e_2 e_3\ \rightsquigarrow j_1,\ \ -\mathcal{I}\ e_2 = -e_3 e_1 \rightsquigarrow j_2,\ \mathcal{I}\ e_3 = e_1 e_2 \rightsquigarrow j_3.$$

We may extend the isomorphism with

$$\mathcal{I} \rightsquigarrow i$$

(the pure imaginary i, with the usual rule $i^2 = -1$), with the prescription that i commutes with the four j_μ. This allows us to the $C\ell(3)$ as the set of *complex quaternions*, $\mathbb{H}\times\mathbb{C}$. The complex conjugation $i \mapsto -i$ is distinguished from the quaternionic conjugation $j_i \mapsto -j_i$. It does not reverse the order of the product.

We have also an isomorphism between $C\ell^{even}(3)$ and $C\ell(0,2)$, resulting from the trivial identification of the latter with the agebra $\mathbb{H}$.

Paravectors span Minkowski spacetime

The paravectors are the Clifford numbers of the form $x = x^0\ 1 + x^i\ e_i = x^\mu\ e_\mu$ (summation is assumed over $i = 1, 2, 3$, and $\mu = 0, 1, 2, 3$). This allows us to see the Minkowski spacetime as naturally embedded in the Clifford algebra of $\mathbb{R}^3$ [2], as the vector space of paravectors, $C\ell_0(3) \oplus C\ell_1(3)$.

The [Clifford] conjugation

$$x = (x^\mu) = (x^0, x^i) \mapsto \bar{x} = (\bar{x}^\mu) \equiv (x^0, -x^i)$$

allows us to define a quadratic form for the paravectors (which differs, however, from the Clifford scalar product defined above)

$$Q(x, y) \equiv \frac{1}{2}\ \ (\bar{x}y + \bar{y}x) = \eta_{\mu\nu}\ x^\mu\ y^\nu,$$

where η is the Minkowski norm.

This provides the vector space isomorphisms:

$$\begin{array}{rcl}
C\ell_0(3) \oplus C\ell_1(3) & \approx M & \approx \mathrm{Herm}(2) \\
(\text{paravectors}) & \approx (\text{Minkowski spacetime}) & \approx \mathrm{Herm}(2) \\
c = x^0 + x^i\ e_i & \approx (x^\mu) = (x^0, x^i) & \approx m = x^\mu\ \sigma_\mu \\
Q(c, c) & = \eta(x, x) & = \det\ m \\
x^0 & = x^0 & = \frac{1}{2}\ \ \mathrm{Tr}\ m.
\end{array}$$

(We have included the isomorphism between Minkowski spacetime and Hermitian matrices.) The three grade 1 vectors e_i identify with the three ON basis vectors of $\mathbb{R}^3 \subset M$. Incidently, this suggests that the choice of signature (1,-1,-1,-1) for Minkowski spacetime may be more natural that (-1, 1, 1, 1). For a development of this approach, see [2].

1.8 Spinors

In the next section, we will introduce the spinors of space-time, and later we will link them with the space-time algebra $\mathrm{C}\ell(1,3)$. Here we give some preliminary insights, to show how spinors appear from a purely algebraic point of view (we follow [3]).

First we remark that, in $\mathrm{C}\ell(3)$, the two elements (among others)

$$e_\pm := \frac{1}{2}(1 \pm e_3)$$

are idempotent, i.e., $e_\pm^2 = e_\pm$. Further, the sets $\mathrm{C}\ell(3)\ e_\pm$ and $e_\pm\ \mathrm{C}\ell(3)$ are left and right ideals of $\mathrm{C}\ell(3)$, respectively. They are vector spaces of (complex) dimension 2, and the identification of $\mathcal{I}$ to the complex imaginary i makes each of them identical to $\mathbb{C}^2$. As we will see, a spinor is precisely an element of a two dimensional representation space for the group $\mathrm{SL}(2,\mathbb{C})$, which is $\mathbb{C}^2$.

Let us first consider $\mathrm{C}\ell(3)\ e_+$. If we chose an arbitrary frame (for instance $\begin{pmatrix}1\\0\end{pmatrix} = e_+$, $\begin{pmatrix}0\\1\end{pmatrix} = e_1\ e_+$), we may decompose an arbitrary element

$$\forall \varphi \in \mathrm{C}\ell(3)\ e_+,\ \varphi = \begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix} \in \mathbb{C}^2.$$

We write $\varphi = (\varphi^A)_{A=1,2}$ and $\mathrm{C}\ell(3)\ e_+ = \mathcal{O}^A$. Such elements constitute a representation, called $D^{(1/2,0)}$, of the special linear group $\mathrm{SL}(2,\mathbb{C})$. It corresponds to the so called *Weyl spinors*, that we will study in more details below.

A similar procedure applies to $e_+\ \mathrm{C}\ell(3)$. Choosing a basis (e.g., $(1,0) = e_+, (0,1) = e_+\ e_1$), we write its vectors with covariant (rather than contravariant) and pointed indices:

$$e_+\ \mathrm{C}\ell(3) \equiv \mathcal{O}_{\dot{A}} = \{\xi \equiv (\xi_{\dot{A}}) \equiv (\xi_{\dot{1}}, \xi_{\dot{2}})\}. \tag{13}$$

We have the very important mapping

$$\begin{aligned} \mathcal{O}^A \times \mathcal{O}_{\dot{A}} \quad &\mapsto \mathrm{C}\ell(3) \qquad (14)\\ \varphi = (\varphi^A), \dot{\xi} = (\dot{\xi}_{\dot{A}}) \quad &\mapsto \varphi\ \dot{\xi} = \Big((\varphi\ \dot{\xi})^A_{\dot{A}} = \varphi^A\ \dot{\xi}_{\dot{A}} \Big). \qquad (15) \end{aligned}$$

The last relation is a matrix product. It provides a complex matrix of order 2, identified to a point of $\mathrm{C}\ell(3)$ as indicated above.

2 Spinors in Minkowski spacetime

In the following section, we will study the *space-time algebra*, which is the Clifford algebra of the Minkowski spacetime. This will provide a natural way to consider the group Spin and the spinors, with the main advantage to allow generalization in any number of dimensions. Before turning to the study of the space-time algebra, this section presents an introduction to the spinors and the Spin group without reference to the Clifford algebras. The link will be made in the next section.

We introduce spinors in Minkowski spacetime $\mathbb{M} = \mathbb{R}^{1,3}$, from group theoretical considerations. We recall that the isotropy group of Minkowski spacetime is the *orthogonal group* $\mathrm{O}(1,3)$, with four connected components, and which may be seen as a matrix group in its fundamental representation. The restriction to matrices with determinant 1 leads to the *special orthogonal group* $\mathrm{SO}(1,3)$, with 2 connected components. Finally, the component of $\mathrm{SO}(1,3)$ connected to the unity is the *proper* Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$. None of these groups is singly connected. Their (1-2) universal coverings are respectively the groups $\mathrm{Pin}(1,3)$, $\mathrm{Spin}(1,3)$ and $\mathrm{Spin}^{\uparrow}(1,3)$ (see, e.g., [12]). The groups O, SO and $\mathrm{SO}^{\uparrow}$ act on Minkowski spacetime through the fundamental representation. The construction of spinors is based on the group isomorphisms

$$\mathrm{Spin}^{\uparrow}(1,3) = \mathrm{SL}(2,\mathbb{C}) = \mathrm{Sp}(2,\mathbb{C}).$$

(Note also the group isomorphism $\mathrm{SO}^{\uparrow}(1,3) = \mathrm{SO}(3,\mathbb{C})$).

2.1 Spinorial coordinates in Minkowski spacetime

There is a one to one correspondence between the (real) Minkowski spacetime $\mathbb{M}$ and the set $\mathrm{Herm}(2) \subset \mathrm{Mat}_2(\mathbb{C})$ of *Hermitian* matrices: to any point $x = (x^\mu)$ of $\mathbb{M}$ is associated the *Hermitian* matrix

$$X := x^\mu \ \sigma_\mu := \begin{pmatrix} X^{1\dot{1}} & X^{1\dot{2}} \\ X^{2\dot{1}} & X^{2\dot{2}} \end{pmatrix} := \begin{pmatrix} x^0 + x^1 & x^2 + i\ x^3 \\ x^2 - i\ x^3 & x^0 - x^1 \end{pmatrix}, \tag{16}$$

where the σ_μ are the Pauli matrices. The matrix coefficients $X^{A\dot{A}}$, with $A = 1, 2$, $\dot{A} = \dot{1}, \dot{2}$, are the *spinorial coordinates*. The reason for this appellation will appear below. We have:

$$x \cdot x = \det X, \ 2\ x^0 = \mathrm{Tr} X \tag{17}$$

(the dot denotes the scalar product in Minkowski spacetime). In the following, we will distinguish usual (x^μ) and spinorial $(x^{A\dot{A}})$ coordinates only by the indices. Hermiticity reads $X^{A\dot{B}} = \overline{X^{B\dot{A}}}$.

An element of the Lorentz group acts on the Minkowski vector space $\mathbb{M}$ as a matrix $L: \ x \mapsto L\ x$. The *same* action is expressed in $\mathrm{Herm}(2)$ through a matrix Λ as

$$X \mapsto \Lambda \ X \ \Lambda^\dagger. \tag{18}$$

Here, Λ is a matrix of the group Spin, the universal covering of the Lorentz group SO. (When there is no risk of confusion, I will write, e.g., SO for SO(1,3). This will help to recall that most of the derivations below hold in any even dimension, for the Lorentzian case ; odd dimension or other signatures allow similar, although non identical treatments). We have the group homomorphism

$$\varphi: \ \text{Spin} \ \mapsto \text{SO} \tag{19}$$

$$\Lambda \ \mapsto L. \tag{20}$$

To insure the action (18), we may chose the matrix Λ such that

$$\Lambda \ \sigma_\nu \ \Lambda^\dagger = L^\mu{}_\nu \ \sigma_\mu,$$

which implies

$$L^\mu{}_\nu = \tfrac{1}{2} \ \text{Tr}(\Lambda \ \sigma_\nu \ \Lambda^\dagger \ \sigma_\mu).$$

Note that Λ and $-\Lambda$ correspond to the same element of the Lorentz group, which reflect the fact that Spin is the 1-2 universal covering of SO.

*** At the infinitesimal level, $\Lambda \sim \mathbb{I} + \lambda, \ L^\mu{}_\nu \sim \delta^\mu{}_\nu + \ell^\mu{}_\nu$, so that

$$\lambda \ \sigma_\nu \ + \ \sigma_\nu \ \lambda^\dagger \sim \ell^\mu{}_\nu \ \sigma_\mu,$$

which implies $\lambda = A\ell^\mu{}_\nu \ \sigma_\mu \ \sigma_\nu$.

$$A \ (2 \ \ell^0{}_i \ \sigma_i + 2\ell^j{}_0 \ \sigma_j \ ) \sim \ell^i{}_0 \ \sigma_i,$$

which implies $\lambda = A\ell^\mu{}_\nu \ \sigma_\mu \ \sigma_\nu$. ***

2.1.1 The complex Minkowski spacetime

The *complex Minkowski spacetime* $\mathbb{M}_\mathbb{C}$ is defined by extending the coordinates to complex numbers, and extending the Minkowski metric to the corresponding bilinear (not Hermitian) form $g(z, z') \equiv \eta_{\mu\nu} \ z^\mu \ z'^\nu$. The same spinorial correspondance as above leads to identify $\mathbb{M}_\mathbb{C}$ with the set $\text{Mat}_2(\mathbb{C})$ of all (not necessarly Hermitian) *complex matrices* $Z = [Z^{A\dot{A}}]$:

$$\mathbb{C}^4 \ni z = (z^\mu) \mapsto Z \equiv \begin{pmatrix} Z^{1\dot{1}} & Z^{1\dot{2}} \\ Z^{2\dot{1}} & Z^{2\dot{2}} \end{pmatrix} \equiv \begin{pmatrix} z^0 + z^1 & z^2 + i \ z^3 \\ z^2 - i \ z^3 & z^0 - z^1 \end{pmatrix}. \tag{21}$$

Any 2×2 matrix with complex coefficients admits a unique decomposition that we write $Z = X + i \ Y$, where X and Y are both Hermitian:

$$X = \frac{Z + Z^\dagger}{2}, \qquad Y = i \ \frac{Z - Z^\dagger}{2}, \tag{22}$$

and † denotes the conjugate transposed. Hereafter, iY will be called the anti-Hermitian part.

This spinorial notation identifies $\mathbb{M}_{\mathbb{C}}$ with $\mathrm{Mat}_2(\mathbb{C})$, the set of complex $2{\times}2$ matrices, and $\mathbb{M}$ with Herm(2), the set of complex $2{\times}2$ Hermitian matrices.

We have defined the isomorphism $\mathbb{M}_{\mathbb{C}} \mapsto \mathrm{Mat}_2(\mathbb{C})$ through the Pauli matrices. This is a peculiar choice. More generally, it may be expressed by the *Infeld-van der Waerden symbols*. We will however consider here only this representation.

2.2 The Weyl Spinor Space

The Lorentz group plays a fundamental role in relativistic physics. According to spinorial or twistorial formalisms, even more fundamental is its universal covering, the group

$$\mathrm{Spin}^{\uparrow}(1,3) \equiv \mathrm{SL}(2,\mathbb{C}) = \mathrm{Sp}(2,\mathbb{C}),$$

that we will hereafter simply write $\mathrm{Spin}^{\uparrow}$, when no confusion is possible. It is at the basis of the spinor formalism.

In its fundamental representation, $\mathrm{SL}(2,\mathbb{C})$ is the subgroup of $\mathrm{GL}(2,\mathbb{C})$ (the general linear transformations acting on $\mathbb{C}^2$) of those matrices whose determinant $=1$. It has complex dimension 3 ($\mathrm{GL}(2,\mathbb{C})$ has complex dimension 4).

Thus, $\mathrm{Spin}^{\uparrow}{=}\mathrm{SL}(2,\mathbb{C})$ acts naturally on the vectors of $\mathbb{C}^2$, which are called *Weyl spinors*, or *chiral spinors*. This is the so called $D^{(0,1/2)}$, or *left*, or *negative helicity*, representation.

As a vector of the vector space $\mathbb{C}^2$, a Weyl spinor expands as $\xi = \xi^A\ \mathsf{o}_A$ (index summation) in a basis $(\mathsf{o}_A) = (\mathsf{o}_1, \mathsf{o}_2)$.

Thus it appears as a two-component column vector $\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$ and, by definition, an element of the group Spin acts linearly on it, as a 2×2 matrix Λ :

$$\begin{aligned} \mathrm{Spin}: \quad & \mathbb{C}^2 \ \mapsto \mathbb{C}^2 \\ \Lambda: \quad & \xi \quad \mapsto \Lambda\, \xi. \end{aligned} \tag{23}$$

The set of Weyl spinors, with this group action, is written $\mathcal{O}^A$. A Weyl spinor is written ξ^A.

2.3 Symplectic form and duality

Since $\mathrm{Spin}^{\uparrow}{=}\mathrm{Sp}(2,\mathbb{C})$, it may also be seen as the group of transformation of $\mathrm{GL}(2,\mathbb{C})$ which preserve a *symplectic form* ϵ of $\mathbb{C}^2$:

$$\epsilon: \quad \mathcal{O}^A \times \mathcal{O}^A \ \mapsto \mathbb{C} \tag{24}$$

$$\xi, \zeta \quad \mapsto \epsilon(\xi, \zeta). \tag{25}$$

This gives to the Weyl-spinor space $\mathcal{O}^A$ a *symplectic structure* $(\mathbb{C}^2, \epsilon)$. Thus, Spin appears as the symmetry group of the symplectic space $\mathcal{O}^A$.

A frame of $\mathcal{O}^A$ is *symplectic* iff the symplectic form is represented by the matrix

$$\varepsilon_{AB} = \epsilon(\mu_A, \mu_B) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This justifies the notation since, in a symplectic basis, the component $\epsilon_{AB} =$ identifies with the familiar Levi-Civita symbol. In vector notation,

$$\epsilon(\xi, \zeta) = \xi^T \ \epsilon \ \zeta = \varepsilon_{AB} \ \xi^A \zeta^B = \xi^1 \zeta^2 - \xi^2 \zeta^1. \tag{26}$$

The antisymmetric form ϵ defines an antisymmetric Spin-invariant scalar product, called the *symplectic scalar product*. Antisymmetry implies that the " symplectic norm " of any spinor is zero: $\epsilon(\xi, \xi) = 0$. Note that $\epsilon(\xi, \zeta) = 0$ implies that ξ is proportional to ζ.

From now, we will only consider *symplectic frames*, that we also call ON frames,

The matrix ϵ is called the *Levi-Civita spinor* (although it is not a Weyl spinor, but a spinor in a more general sense that will appear below). Later, we will consider ϵ as the spinorial expression of [the square root of] the Minkowski metric.

2.3.1 Duality and the dual representation

The dual $\mathcal{O}_A = (\mathcal{O}^A)^*$ of the vector space $\mathcal{O}^A$ is the space of one-forms on it. They are isomorphic. The symplectic form ϵ on $\mathcal{O}^A$ provides a *duality isomorphism* between both spaces:

$$\epsilon: \quad \mathcal{O}^A \quad \mapsto \mathcal{O}_A \equiv (\mathcal{O}^A)^* \tag{27}$$

$$\xi \quad \mapsto \xi^* = \epsilon(\xi, \cdot) \tag{28}$$

$$\mu_A \quad \mapsto \mu^A = \epsilon(\mu_A, \cdot). \tag{29}$$

To the frame o_A is associated the co-frame o^A. An element of $\mathcal{O}_A$ expands as $\eta = \eta_A \ \mathsf{o}^A$, and the symplectic isomorphism is written as a raising or lowering of the spinor-indices. Hence the abstract index notation $\mathcal{O}_A \equiv (\mathcal{O}^A)^*$.

This is in complete analogy with the metric (musical) isomorphism defined by a metric in a [pseudo-]Riemannian manifold. Care must be taken however that the calculations differ because of the antisymmetry of ϵ. For instance, we have $u^A \ v_A = -u_A \ v^A$ (sum on indices).

The naturally induced (dual) action of an element of the Spin group,

$$\Lambda: \ \eta \mapsto \eta \ \Lambda^{-1}; \ \eta_A \mapsto \eta_B \ (\Lambda^{-1})^B_{\ A},$$

defines the dual representation, that we note Spin*.

2.4 Dotted spinors and the conjugation isomorphism

Complex conjugation

On the other hand, the (complex) conjugate representation $\overline{\mathrm{Spin}}$ of the group Spin on $\mathbb{C}^2$ is defined as

$$\Lambda : \eta \mapsto \bar{\Lambda}\ \eta,\ \ \eta \in \mathbb{C}^2, \tag{30}$$

instead of (23), where the bar denotes the complex conjugate. It preserves also the symplectic form ϵ on $\mathbb{C}^2$. We note $\overline{\mathcal{O}^A} \equiv \mathcal{O}^{\dot{A}}$ this representation vector space. An element is written with *dotted indices*, as $\eta = (\eta^{\dot{A}}) = \begin{pmatrix} \eta^{\dot{1}} \\ \eta^{\dot{2}} \end{pmatrix}$, where the index $\dot{A}$ takes the values $\dot{1}, \dot{2}$. We call $\overline{\mathrm{Spin}}$ the group acting in this representation, the $D^{(1/2,0)}$, or *right representation*.

Complex conjugation defines the isomorphism (called *anti-isomorphism*)

$$\mathcal{O}^A \quad \mapsto \quad \mathcal{O}^{\dot{A}} \tag{31}$$

$$\xi = \xi^A = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \mapsto \quad \bar{\xi} = \bar{\xi}^{\dot{A}} = \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}. \tag{32}$$

We write $\bar{\xi}$ with dotted indices since it belongs to $\mathcal{O}^{\dot{A}}$. Since notation may appear confusing, it is important to remark that ξ and $\bar{\xi}$ are considered as members of two different (dotted and undotted) spinor spaces.

The symplectic structure ϵ being preserved by the anti-isomorphism, $\varepsilon_{AB} = \varepsilon_{\dot{A}\dot{B}}$. It also allows to raise or lower the dotted indices:

$$\epsilon : \quad \mathcal{O}^{\dot{A}} \quad \mapsto \mathcal{O}_{\dot{A}} \tag{33}$$

$$\zeta \quad \mapsto \epsilon(\zeta, \cdot) \tag{34}$$

$$\zeta^{\dot{A}} \quad \mapsto \zeta_{\dot{A}} = \zeta^{\dot{B}}\ \varepsilon_{\dot{B}\dot{A}}. \tag{35}$$

The symplectic form is also preserved:

$$\Lambda^{-1}\ \epsilon\ \Lambda = \epsilon = \bar{\epsilon} = \bar{\Lambda}^{-1}\ \epsilon\ \bar{\Lambda}.$$

2.5 Spinor-tensors and the Minkowski vector space

The general element Z of the tensor product

$$\mathcal{O}^{A\dot{A}} \equiv \mathcal{O}^A \otimes \mathcal{O}^{\dot{A}}$$

is called a *mixed spinor-tensor* of rank 2. In a symplectic basis, it expands as

$$Z = Z^{A\dot{A}} \ \mathsf{o}_A \otimes \bar{\mathsf{o}}_{\dot{A}},$$

and so is represented by the complex 2×2 matrix $Z \in \mathrm{Mat}_2(\mathbb{C})$ with components $Z^{A\dot{A}}$. Using the Pauli matrices as a (complex) basis of $\mathrm{Mat}_2(\mathbb{C})$, it expands in turn as $Z = Z^\mu \ \sigma_\mu, Z^\mu \in \mathbb{C}$. It identifies with the (complex) vector $z \in \mathbb{M}_\mathbb{C}$ with components $z^\mu = Z^\mu = Z^{A\dot{A}} \ (\sigma^\mu)_{A\dot{A}}$.

The elements of the form $Z = \xi \ \otimes \ \overline{\zeta} = \xi^A \ \overline{\zeta}^{\dot{A}} \ \mathsf{o}_A \ \otimes \bar{\mathsf{o}}_{\dot{A}}$ are called *decomposable*. In matrix notations,

$$Z \equiv \xi \ \zeta^T \ : \ Z^{A\dot{A}} = \xi^A \zeta^{\dot{A}},$$

where the subscript T indicates matrix (or vector) transposition.

This establishes a one-to-one correspondence between

- vectors z in *complex* Minkowski vector space $\mathbb{M}_\mathbb{C}$, $z^\mu = Z^{A\dot{A}} \ (\sigma^\mu)_{A\dot{A}}$;
- complex 2×2 matrix $Z \in \mathrm{Mat}_2(\mathbb{C})$ with components $Z^{A\dot{A}}$;
- *mixed spinor-tensor* of rank 2, $Z^{A\dot{A}} \ \mathsf{o}_A \otimes \bar{\mathsf{o}}_{\dot{A}}$.

Spinor-tensors associated to Hermitian matrices are called *Hermitian* also. They span the *real* Minkowski vector space $\mathbb{M}$. This is the vector, or $(\frac{1}{2}\ ,\frac{1}{2}\ )$, representation.

For decomposable spinor-tensors, the *scalar product* is defined through the symplectic form, as

$$\eta(\xi \ \otimes \ \overline{\zeta}, \xi' \ \otimes \ \overline{\zeta}') = \epsilon(\xi, \xi') \ \epsilon(\zeta, \zeta'),$$

and extended by linearity. It is easy to check that it coincides with the Minkowski norm for the Hermitian spinors.

A *decomposable* spinor-tensor $Z^{A\dot{A}} = \xi^A \ \bar{\zeta}^{\dot{A}}$ corresponds to a *null* vector (of zero norm) in $\mathbb{M}_\mathbb{C}$, not necessarily real. Those of the form $Z^{A\dot{A}} = \xi^A \ \bar{\xi}^{\dot{A}} =$ are Hermitian ($Z^{A\dot{A}} = \overline{Z}^{\dot{A}A}$) and, thus, correspond to null vectors in the *real* Minkowski spacetime $\mathbb{M}$: they belong to its null cone.

To any Weyl spinor ξ is associated the null vector $\xi \otimes \bar{\xi}$ in real Minkowski spacetime, called its *flagpole*. Changing the spinor phase (multiplying it by a complex unit number) does not change the null vector. Multiplying the spinor by a real number multiplies the null vector by the same number squared. Note that a null vector of Minkowski spacetime may be seen as the momentum of a zero mass particle.

The table (6) summarizes the properties of the spinor representations.

Two-component spinor calculus in Minkowski spacetime

The use of spinorial indices in Minkowski spacetime may be seen as a simple change of notation: each tensorial index is replaced by a pair $A\dot{A}$ of spinorial indices and all usual

Space		indexed spinor	Representation	form
Weyl spinors ($d_{\mathbb{C}}=2$)	$\mathcal{O}^A$	f^A	$f^A \mapsto \Lambda^A{}_B\ f^B$	ϵ_{AB}
dual Weyl spinors	$(\mathcal{O}^A)^* = \mathcal{O}_A$	f_A	$f_A \mapsto (\Lambda^{-1})^B{}_A\ f_B$	$\epsilon^{\dot{A}\dot{B}}$
dotted Weyl spinors	$\overline{\mathcal{O}^A} = \mathcal{O}^{\dot{A}}$	$f^{\dot{A}}$	$f^{\dot{A}} \mapsto \bar{\Lambda}^{\dot{A}}{}_{\dot{B}}\ f^{\dot{B}}$	$\epsilon_{\dot{A}\dot{B}}$
dual dotted Weyl spinors ($d_{\mathbb{C}}=2$)	$\overline{\mathcal{O}^A}^* = \mathcal{O}_{\dot{A}}$	$f_{\dot{A}}$	$f_{\dot{A}} \mapsto (\bar{\Lambda}^{-1})^{\dot{B}}{}_{\dot{A}}\ f_{\dot{B}}$	ϵ^{AB}
Complex Minkowski space-time ($d_{\mathbb{C}}=4$)	$\mathcal{O}^{A\dot{A}} = \mathcal{O}^A \otimes \mathcal{O}^{\dot{A}}$ $\simeq \mathbb{M}_{\mathbb{C}}$	$f^{A\dot{A}}$ $\simeq f^a$	$f^{A\dot{A}} \mapsto \Lambda^A{}_B\ \bar{\Lambda}^{\dot{A}}{}_{\dot{B}}\ f^{B\dot{B}}$ $f^a \mapsto L^a{}_b\ f^b$	$\epsilon_{AB}\ \epsilon_{\dot{A}\dot{B}}$ $\simeq \eta_{ab}$
dual	$\mathcal{O}_{A\dot{A}} = \mathcal{O}_A \times \mathcal{O}_{\dot{A}}$ $\simeq {\mathbb{M}_{\mathbb{C}}}^*$	$f_{A\dot{A}}$ $\simeq f_a$	$f_{A\dot{A}} \mapsto$ $(\Lambda^{-1})^B{}_A\ (\bar{\Lambda}^{-1})^{\dot{B}}{}_{\dot{A}}\ f_{B\dot{B}}$ $f_a \mapsto L_a{}^b\ f^b$	$\epsilon^{AB}\ \epsilon^{\dot{A}\dot{B}}$ $\simeq \eta^{ab}$
real Minkowski space-time ($d_{\mathbb{R}}=4$)	$(\mathcal{O}^A \otimes \mathcal{O}^{\dot{A}})_{Herm.}$ $\simeq \mathbb{M}$	$f^{A\dot{A}}$ Herm. $\simeq f^a$ real	$f^a \mapsto L^a{}_b\ f^b$	η_{ab}
dual ($d_{\mathbb{R}}=4$)	$(\mathcal{O}_A \times \mathcal{O}_{\dot{A}})_{Herm.}$ $\simeq \mathbb{M}^*$	$f_{A\dot{A}}$ Herm. $\simeq f_a$ real	$f_a \mapsto L_a{}^b\ f^b$	η^{ab}

Table 6: Spinor vector spaces with their tensor products, and their links with Minkowski spacetime.

formulae of tensorial calculus hold. For instance, the gradient ∇_μ becomes $\nabla_{A\dot{A}} := \frac{\partial}{\partial X^{A\dot{A}}}$ and, for any function f, $\mathrm{d}f = \nabla_{A\dot{A}} f \ \mathrm{d}X^{A\dot{A}}$.

Similarly, tensors in Minkowski spacetime appear with spinorial indices, like $S^{AB...\dot{A}\dot{B}...}$. When the dotted and undotted indices appear in pairs, the tensor-spinor may be seen also as a tensor over Minkowski spacetime, written in spinorial notations. Quite often, one needs the symmetrized or antisymmetrized combinations of indices, of the types $S^{(AB...)\dot{A}\dot{B}...}$ or $S^{[AB...]\dot{A}\dot{B}...}$, etc. Any form or tensor in Minkowski spacetime can be written in spinorial components. In particular, the Minkowski metric,

$$\eta_{\mu\nu} \ \sigma^\mu_{A\dot{A}} \ \sigma^\nu_{B\dot{B}} = \varepsilon_{AB} \ \varepsilon_{\dot{A}\dot{B}}. \tag{36}$$

One simply writes usually

$$\eta_{\mu\nu} \simeq \varepsilon_{AB} \ \varepsilon_{\dot{A}\dot{B}}$$

Spin group and Lorentz group

The action of the group Spin on $\mathcal{O}^A$ and $\overline{\mathcal{O}^A}$ induces the following action on the tensorial product $\mathcal{O}^{A\dot{A}}$,

$$\begin{array}{rll} \text{Spin}: & \mathcal{O}^{A\dot{A}} = \mathrm{Mat}_2(\mathbb{C}) \mapsto & \mathcal{O}^{A\dot{A}} = \mathrm{Mat}_2(\mathbb{C}) \\ \Lambda: & Z = \xi \ \zeta^T \mapsto & (\Lambda\xi) \ (\bar{\Lambda}\zeta)^T = \Lambda \ Z \ \Lambda^\dagger. \end{array} \tag{37}$$

We recognize the action (18) of the Lorentz group on $\mathrm{Mat}_2(\mathbb{C}) = \mathbb{M}_\mathbb{C}$. Thus the mixed spinor-tensors of $\mathcal{O}^{A\dot{A}}$ are really vectors of the (complex) Minkowski spacetime:

$$(\mathbb{R}^{1,3}, \eta) \equiv \mathbb{M}_\mathbb{C} \equiv (\mathcal{O}^A, \epsilon) \otimes (\overline{\mathcal{O}^A}, \bar{\epsilon}),$$

considered as a representation space for the Lorentz group.

Note that the correspondence between $\mathrm{Mat}_2(\mathbb{C})$ and $\mathbb{M}_\mathbb{C}$ is defined through the Pauli matrices. The reduction to the set $\mathrm{Herm}_2(\mathbb{C}) \subset \mathrm{Mat}_2(\mathbb{C})$ is the *decomplexification* of $\mathbb{M}_\mathbb{C}$ to the usual Minkowski vector space $\mathbb{M}$. The usual action of the Lorentz group results.

The correspondence between spinorial and tensorial indice may be seen very simply as replacing any index μ by a pair $A\dot{A}$, and conversely. More rigorously, it is expressed by the Infeld-van der Waerden symbols.

2.6 Dirac spinors and Dirac matrices

A *Dirac spinor* is constructed as the direct sum of a left Weyl spinor and a right Weyl spinor. It is written as

$$\Psi = \begin{pmatrix} \pi \\ \eta \end{pmatrix} = \begin{pmatrix} \pi^A \\ \eta^{\dot{A}} \end{pmatrix}$$

(here written in the Weyl representation). We write S_{Dirac} the vector space of the Dirac spinors. Thus the action of a Lorentz matrix is defined as :

$$\Lambda : \begin{pmatrix} \pi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \Lambda & 0 \\ 0 & \overline{\Lambda} \end{pmatrix} \begin{pmatrix} \pi \\ \eta \end{pmatrix}.$$

It is a calculatory exercise to show that the symplectic forms of the Weyl spinor spaces induce an Hermitian metric φ of signature 2,2 for the Dirac spinor space. This makes the space of Dirac spinors appear as the fundamental representation of the group $\mathrm{SU}(2,2) = \mathrm{Spin}(2,4)$. Note that this group is embedded in $\mathrm{C}\ell(\mathbb{M})$ and that we will consider below the Dirac vector space as a representation space for $\mathrm{C}\ell(\mathbb{M})$.

Dirac matrices

The *Dirac matrices* are four matrices γ_μ acting on S_{Dirac}, obeying the anticommutation relations (see below for the link with Clifford algebras):

$$[\gamma_\mu, \gamma_\nu]_+ = 2\ \eta_{\mu\nu}. \tag{38}$$

Their indices are lowered or raised with the Minkowski metric. Thus,

$$\gamma^0 = \gamma_0,\ \gamma^i = -\gamma_i.$$

Chirality

One defines the *orientation operator* $\gamma_0\ \gamma_1\ \gamma_2\ \gamma_3$, of square $-\mathbb{I}$, and the *chirality operator*

$$\chi = \frac{i}{4}\ \epsilon_{\mu\nu\rho\sigma}\ \gamma^\mu\ \gamma^\nu\ \gamma^\rho\ \gamma^\sigma = i\ \gamma^0\ \gamma^1\ \gamma^2\ \gamma^3\ .$$

($-\chi$ is also written γ_5.)

The Weyl spinors may be seen as the eigenstates of χ, with eigenvalue ± 1, in S_{Dirac}. The projection operators $\frac{1}{2}\ (\mathbb{I} \pm \chi)$ project a Dirac spinor into a left or right spinor. So that a general Dirac spinor may be written

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix},$$

involving the left and right Weyl representations.

The parity transformation is defined as

$$\Psi \mapsto \gamma^0\ \Psi.$$

The *rotation generators* are

$$\Sigma^{mn} = -\tfrac{1}{4}\ [\gamma^m, \gamma^n].$$

$$\psi^\mu \mapsto -\tfrac{1}{2}\ (\Sigma^{mn})^\mu_\nu\ \psi^\nu.$$

Charge conjugation and Majorana spinors

The *charge conjugation* is defined as the operation

$$\Psi \mapsto \Psi^c = -\gamma^2\ \overline{\Psi},$$

where the bar means complex conjugation. From the physical point of view, the charge conjugation transforms a particle into an antiparticle.

The *Majorana spinors* are defined as those Dirac spinors which are self-conjugate under charge conjugation :

$$\Psi = \Psi^c.$$

The space of Majorana spinors has complex dimension 2.

Representations

There are different representations, depending on the basis in which they are written. In the *Weyl representation* (or *chiral representation*), we have

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix},\ \gamma^5 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \tag{39}$$

3 Spin and spinors in Clifford algebras

3.1 Rotations in a vector space

We first recall some properties of the orthogonal groups.

In the real vector space $V = \mathbb{R}^{p,q}$ with a scalar product of signature (p,q), the group of isometries preserving the scalar product is the orthogonal group $\mathrm{O}(p,q)$. For an Euclidean (resp. Lorentzian) signature, it has two (resp. 4) connected components. Given a basis for V, $\mathrm{O}(p,q)$ may be seen as a group of matrices. The subgroup $\mathrm{SO}(p,q)$ of matrices with unit determinant has one (resp. 2) connected components. In the Lorentzian case, one also defines $\mathrm{SO}^{\uparrow}$ (also written $\mathrm{SO}_0(p,q)$), the component of $\mathrm{SO}(p,q)$ connected to the identity. This is the group of *proper orthogonal transformations*. For space-time, $\mathrm{SO}(1,3)$ and $\mathrm{SO}^{\uparrow}(1,3)$ are the Lorentz group and the special Lorentz group, respectively. We will study them in more details below.

As we will see, the (2 to 1) universal covering of O is the group Pin. The (2 to 1) universal covering of SO is the group Spin. The (2 to 1) universal covering of $\mathrm{SO}^{\uparrow}$ is the group $\mathrm{Spin}^{\uparrow}$:

$$\mathrm{O} = \mathrm{Pin}/\mathrm{Z}_2,$$

$$\mathrm{SO} = \mathrm{Spin}/\mathrm{Z}_2,$$

$$\mathrm{SO}^{\uparrow} = \mathrm{Spin}^{\uparrow}/\mathrm{Z}_2.$$

3.2 The Clifford group

A Clifford algebra is not, in general, a multiplicative group, since some elements are not invertible. This is for instance the case of the null vectors of the Minkowski vector space $C\ell^1(1,3) = \mathbb{R}^{1,3}$. However, we will extract some multiplicative groups from a Clifford algebra, after selecting the invertible elements.

Given an invertible element x of a Clifford algebra $C = C\ell(V)$, we define its action on C as

$$\begin{aligned} T_x : \quad C &\mapsto C : \\ c &\mapsto -x\ c\ x^{-1} = \forall c \in C. \end{aligned} \tag{40}$$

It is convenient to demand that this preserves the vector space $V = C^1$, i.e.,

$$v \in V \Rightarrow x\ v\ x^{-1} \in V.$$

This defines the *Clifford group*, also called *Lipschitz group*, of $C\ell(V)$ as its subset

$$\Gamma(V) \equiv \ \{x \in C\ell(V);\ x \text{ inversible };\ v \in V \Rightarrow x\ v\ x^{-1} \in C\ell^1(V) = V\}. \tag{41}$$

The composition law of the Clifford group is the Clifford multiplication.

The subset Γ^{even} (also written Γ^+) of all *even* elements of Γ is called the *special Lipschitz group*; the subset of all odd elements (not a group) is written Γ^{odd} (or Γ^-). One also distinguishes $_{\pm}\Gamma$ the set of elements of norm ± 1.

3.3 Reflections, rotations and Clifford algebras

Let us examine this action when both c and x belong to $V \equiv C^1 \subset C$:

$$v \mapsto -x\ v\ x^{-1} = v - 2\ \frac{v \cdot x}{x \cdot x}\ x,$$

with Clifford products in the LHS and scalar products in the RHS. Geometrically, we recognise a *reflection* in V with respect to the hyperplane orthogonal to x. But we know that any rotation in V can be written as the product of two reflections. Thus, the action of a *rotation* takes the form :

$$T_R \equiv T_v \circ T_w :\ x \mapsto vw\ x\ w^{-1}v^{-1} := R\ x\ R^{-1};\ x \in C^1. \tag{42}$$

Being a product of two 1-vectors, R is the sum of a scalar plus a bivector. The formula implies that the composition of rotations is represented by the Clifford product (a Clifford product of even multivectors is an even multivector.) It is clear, however, that R and $a\ R$, with a a scalar, represent the same rotation. Thus, it is natural to introduce a normalisation. Finally, it can be shown that the general rotation in V is represented, in this way, by an even grade multivector R, such that

$$R\ v\ R^{-1} \in C_1,\ \forall v \in C_1 \text{ and } \bar{R}R = \pm 1.$$

This action works on all objects in C. For instance, their action on the bivectors is given by

$$T_R:\ xy \mapsto R\ x\ R^{-1} R\ y\ R^{-1} = R\ xy\ R^{-1}. \tag{43}$$

If we impose $\bar{R}R = +1$, such an element is called a *rotor*. The rotors form the rotor group $\mathrm{Spin}^{\uparrow}$.

3.3.1 The Pin and Spin groups

In the Clifford algebra $\mathrm{C}\ell(V)$, we define the following subgroups of the Clifford group:

The set of all (Clifford) products of non null normed vectors of V ($\bar{s}s = \pm 1$) form a group for the [Clifford] multiplication: this is the *Pin group* associated to V:

$$\mathrm{Pin}(V) \equiv \{s \in \mathrm{C}\ell(V);\ s = s_1 s_2 ... s_k,\ s_i \in V \equiv \mathrm{C}\ell_1(V),\ \bar{s}s = \pm 1\}. \tag{44}$$

Note that an invertible normalized element of V belongs to Pin.

The restriction to the product of even numbers of vectors gives the *Spin group*

$$\mathrm{Spin}(V) \equiv \{s \in \mathrm{C}\ell(V);\ s = s_1 s_2 ... s_{2k},\ s_i \in V \equiv \mathrm{C}\ell_1(V),\ \bar{s}s = \pm 1\}. \tag{45}$$

Thus, $\mathrm{Spin}(V)$ is a subgroup of $\mathrm{Pin}(V)$, and also of $\mathrm{C}\ell^{even}(V)$. It may also be defined as the group of all elements s of C such that

$$svs^{-1} \in C_1,\ \forall v \in C_1 \text{ and } \bar{s}s = \pm 1.$$

Thus, $\mathrm{Spin}(V) = \mathrm{Pin}(V) \bigcap \mathrm{C}\ell^{even}(V)$.

We give in the table 7 the relations between the different groups introduced.

Pin and Spin are double cover representations (universal coverings) of O and SO:

$$\mathrm{O} = \mathrm{Pin}/\mathrm{Z}_2,$$

$$\mathrm{SO} = \mathrm{Spin}/\mathrm{Z}_2,$$

$$\mathrm{SO}^{\uparrow} = \mathrm{Spin}^{\uparrow}/\mathrm{Z}_2,$$

$\mathrm{Spin}(p,q)$ and $\mathrm{Spin}(q,p)$ are isomorphic but this relation does not hold for the Pin groups. Both are Lie groups. One denotes $\mathrm{Spin}(p) = \mathrm{Spin}(0,p) = \mathrm{Spin}(p,0)$.

Lipschitz group = Clifford group	Γ	
special Lipschitz group	Γ^{even}	
Pin group	$_{\pm}\Gamma$	u.c. of O(3,1)
Spin group	$_{\pm}\Gamma^{even}$ =Pin $\cap\ \Gamma^{+}$	u.c. of SO(3,1)
Rotors group = Spin$^{\uparrow}$	$_{+}\Gamma^{even} = \mathrm{SL}(2,\mathbb{C})$	u.c. of SO$^{\uparrow}$(3,1).

Table 7: The groups included in a Clifford algebra, as universal coverings for orthogonal groups

3.3.2 The Clifford - Lie algebra

Let us define the *Clifford bracket* as the commutator

$$[x,y]_{Clifford} \equiv \tfrac{1}{2}\ (xy - yx),$$

where the products on the RHS are Clifford products. This provides to C a Lie algebra structure (note that its dimension is one unit less that the Clifford algebra).

Although the Clifford product of a bivector by a bivector is not, in general a pure bivector, it turns out that the commutator preserves the set of bivectors. Thus, the set of bivectors, with the Clifford bracket is the sub - Lie algebra

$$\mathrm{span}(e_\mu\ e_\nu)_{1\le\mu<\nu\le n},\ e_\mu \in \mathrm{C}\ell^1.$$

This is the Lie algebra of the rotor group Spin$^{\uparrow}$, also of the groups Spin and SO.

Rotors act on bivectors through the adjoint representation of the rotor group, $T_R:\ B \mapsto RB\ R^T = Ad_R(B)$.

Now we consider all these notions in more details, in the case of the space-time algebra. For their extension to an arbitrary number of dimensions, see [9], [5].

Examples

- For the *plane*, the general rotor is

$$\cos\frac{\theta}{2} + \mathcal{I}_2\ \sin\frac{\theta}{2} := \exp[\mathcal{I}_2\ \frac{\theta}{2}]. \tag{46}$$

  The exponential notation results immediately from its series development, and from the anticommutation properties of the algebra. The rotation angle θ parametrizes the rotor group SU(1) = Spin(1), and u is a spatial unit vector, the axis of the rotation.

- For the *space* $\mathbb{R}^3$, the general rotor is of the form

$$\cos\frac{\theta}{2} + \mathcal{I}_3\ u\ \sin\frac{\theta}{2} := \exp[\mathcal{I}_3\ u\ \frac{\theta}{2}];\ u \in \mathrm{C}\ell(3)^1,\ u\ u = 1. \tag{47}$$

  The unit spatial vector u is the rotation axis and θ the rotation angle. The orientation of u, and the angle θ, parametrize the group SU(2).

- In $C\ell(1,3)$, the Clifford algebra of Minkowski vector space, we have $(\mathcal{I}_4)^T = \mathcal{I}_4$, and $B^T = -B$ for an arbitrary bivector B. The general rotor is of the form $\alpha + \beta\ B + \gamma\ \mathcal{I}$, with B an arbitrary bivector which verifies the condition $\alpha^2 + \beta^2\ B\ B^T - \gamma^2 + 2\alpha\ \gamma\ \mathcal{I} = 1$.

  A space+time splitting allows us to write the bivector basis as $(\Sigma_i, \mathcal{I}\ \Sigma_i)_{i=1..3}$, see below. Then, the general rotor is written under the form

$$(\cosh\frac{\phi}{2} + \sinh\frac{\phi}{2}\ u^i\ \Sigma_i)\ (\cos\frac{\psi}{2} + \sin\frac{\psi}{2}\ n^j\ I\ \Sigma_j) = e^{\frac{\phi}{2}\ u^i\ \Sigma_i}\ e^{\frac{\psi}{2}\ n^j\ I\ \Sigma_j}, \tag{48}$$

  or

$$(\cos\frac{\psi'}{2} + \sin\frac{\psi'}{2}\ n'^j\ \mathcal{I}\ \Sigma_j)(\cosh\frac{\phi'}{2} + \sinh\frac{\phi'}{2}\ u'^i\ \Sigma_i)\ = e^{\frac{\psi'}{2}\ n'^j\ \mathcal{I}\ \Sigma_j}\ e^{\frac{\phi'}{2}\ u'^i\ \Sigma_i}. \tag{49}$$

  Care must be taken of the non commutativity on calculations. A rotor $e^{\frac{\phi}{2}\ u^i\ \Sigma_i}$ corresponds to a spatial rotation. A rotor $e^{\frac{\psi}{2}\ n^j\ \mathcal{I}\ \Sigma_j}$ corresponds to a boost.

3.4 The space-time algebra

The *space-time algebra* $C\ell(\mathbb{M}) = C\ell(1,3)$ is the real Clifford algebra of Minkowski vector space $M \equiv \mathbb{R}^{1,3}$. It has a (real) dimension 16. Although it is sometimes called the Dirac algebra, we reserve here the appellation to the complex Clifford algebra $\mathbb{C}\ell(4)$ (see below). The latter is the common complex extension of $C\ell(1,3) \approx \mathrm{Mat}(2,\mathbb{H})$ and $C\ell(3,1) \approx \mathrm{Mat}(4,\mathbb{R})$.

The space-time algebra $C\ell(1,3)$ is generated by four vectors e_μ which form an ON basis of $\mathbb{M}$: $e_\mu \cdot e_\nu = \eta_{\mu\nu}$. These four vector, with their [Clifford] products, induce an ON basis of 16 elements for $C\ell(\mathbb{M})$, given in the table (8): unity (scalar of grade 0), 4 vectors (grade 1), 6 bivectors (grade 2), 4 trivectors (grade 3) and the pseudo scalar $\mathcal{I} \equiv e_5 \equiv e_0\ e_1\ e_2\ e_3$ (grade 4).

Duality

Calculations show that $\mathcal{I}^2 = -1$, and that $\mathcal{I}$ anticommutes with the e_μ. The multiplication by $\mathcal{I}$ exchanges the grades r and $4-r$. This allows to chose a basis for the trivectors, under the form of the four $\mathcal{I}\ e_\mu$, that we call *pseudovectors*. This also provides a convenient (altough non covariant) basis for the bivectors: after having selected a time-like (arbitrary) direction e_0: we define the three time-like bivectors $\Sigma_i \overset{def}{=} e_i\ e_0$. Then, the basis is completed by the three $\mathcal{I}\ \Sigma_i = e_j\ e_k$. Note that $\mathcal{I} = e_0\ e_1\ e_2\ e_3 = \Sigma_1\ \Sigma_2\ \Sigma_3$. The basis is given in table (8).

Even part

The even part $C\ell^{even}(1,3)$ is the algebra generated by $1, \mathcal{I}$ and the 6 bivectors $(\Sigma_i,\ \mathcal{I}\ \Sigma_i)$. It is isomorphic to the (Pauli) algebra $C\ell(3)$, generated by the Σ_i. The latter generate the 3-dimensional space orthogonal to the time direction e_0 in Minkowski spacetime. Note that

Table 8: The basis of the space-time algebra, $i = 1, 2, 3,\ \mu = 0, 1, 2, 3$

1	(e_μ)	$(e_\mu\ e_\nu)$	$(e_\mu\ e_\nu\ e_\rho)=$	$(e_\mu\ e_\nu\ e_\rho\ e_\sigma)$
1	(e_μ)	(Σ_i), $(\mathcal{I}\ \Sigma_i)$	$\mathcal{I}\ e_\mu$	$\mathcal{I} = e_5$
one scalar	4 vectors	6 bivectors	4 trivectors	one quadrivector
			= pseudovectors	= pseudoscalar

this space-time splitting, which requires the choice of an arbitrary time direction, is non covariant.

Real matrices

Note that $C\ell(\mathbb{M})$ is isomorphic to $M_4(\mathbb{R})$, the set of real-valued matrices of order 4. An isomorphism may be constructed from an ON basis e of Minkowski spacetime: first one defines $e^{02} \equiv e^0 e^2$ and the elements
$P_1 \equiv \frac{1}{4}\ (1 + e_1)\ (1 + e_0\ e_2)$,
$P_2 \equiv \frac{1}{4}\ (1 + e_1)\ (1 - e_0\ e_2)$,
$P_3 \equiv \frac{1}{4}\ (1 - e_1)\ (1 + e_0\ e_2)$,
$P_4 \equiv \frac{1}{4}\ (1 - e_1)\ (1 - e_0\ e_2)$.
They verify $P_i\ P_j = \delta_{ij}$.

3.5 Rotations in Minkowski spacetime

Given an arbitrary invertible multivector $\lambda \in C\ell(\mathbb{M})$, we define its action T_λ on $C\ell(\mathbb{M})$ as

$$\begin{array}{rrcl} T_\lambda : & C\ell(\mathbb{M}) & \mapsto & C\ell(\mathbb{M}) \\ & v & \mapsto & \lambda\ v\ \lambda^{-1}. \end{array} \tag{50}$$

Those elements which preserve the Minkowski spacetime $\mathbb{M} \equiv C\ell^1(\mathbb{M})$ form the *Clifford group*

$$G_{1,3} = \{\lambda;\ v \in \mathbb{M} \Rightarrow T_\lambda v \equiv \lambda v \lambda^{-1} \in \mathbb{M}\}. \tag{51}$$

To each λ corresponds an element of the group O(1,3) such that $\lambda v \lambda^{-1}$ is the transformed of v by its element. For instance, when $\lambda = \pm e_0$, T_λ represents a space reflection. When $\lambda = \pm e_1\ e_2\ e_3$, T_λ represents a time reflection. Conversely, any Lorentz rotation can be written as T_λ for some (Clifford) product λ of non isotropic vectors. This expresses the fact that any Lorentz rotation can be obtained as a product of reflections.

Pin and Spin

This group action on $\mathbb{M}$ is however not *effective*: the Clifford group is “ too big ”. One considers the subgroup $\mathrm{Pin}(1,3)$ of $G_{1,3}$ as those elements which are products of unit elements of $\mathbb{M}$ only, i.e., such that $v \cdot v = \pm 1$. The even part of $\mathrm{Pin}(1,3)$ is the spin group $\mathrm{Spin}(1,3)$. It is multiconnected; its component of the unity is the (2-fold) universal covering of the *proper* Lorentz group, $\mathrm{Spin}^{\uparrow}(1,3) = \mathrm{Spin}(1,3)/Z_2$.

Finally, any Lorentz rotation is represented as a bivector written $R = A + \mathcal{I}\ B$ in the basis $(\Sigma_i, \mathcal{I}\ \Sigma_i)$ above.

We have the Lie group homomorphism

$$\begin{aligned} \mathcal{H}: \quad \mathrm{Spin}(4) \quad &\mapsto \mathbb{L} = \mathrm{SO} \\ \lambda \quad &\mapsto [a^{\alpha}{}_{\beta}], \end{aligned} \tag{52}$$

such that $\lambda\ e_{\alpha}\ \lambda^{-1} = a^{\beta}{}_{\alpha}\ e_{\beta}$. Since λ and $-\lambda$ correspond to the same rotation, $\mathcal{H}$ is a 2-1 homomorphism.

More generally, in the Lorentzian case, a special orhogonal group is not simply connected and its universal covering group is precisely the spin group. The kernel of the homomorphism $\mathrm{Spin}(1, d-1) \mapsto \mathrm{SO}(1, d-1)$ is isomorphic to Z_2, so that $\mathrm{Spin}(1, d-1)/\mathrm{Z}_2 \approx \mathrm{SO}(1, d-1)$. $\mathrm{Spin}(1, d-1)$ is the universal covering of $\mathrm{SO}(1, d-1)$.

3.6 The Dirac algebra and its matrix representations

3.6.1 Dirac spinors and Dirac matrices

We follow [9] and [5]. The *Dirac algebra* is defined as the complexification $\mathbb{C}\ell(4) = \mathrm{C}\ell(M) \otimes \mathbb{C}$ of the space-time algebra. This is the Clifford algebra of the [complex]vector space $\mathbb{C}^4$ with the quadratic form $g(v, w) = \delta_{\mu\nu}\ v^{\mu}\ w^{\nu}$.

The discussion here is presented for the case of dimension 4. It generalizes to any *even dimension*. In the case of odd dimensions, the things are slightly different, see [1].

There exists [1] a (complex) faithful irreductible representation of the algebra $\mathbb{C}\ell(4)$ as $\mathrm{End}(S_{Dirac})$, the group of *linear endomorphisms* of a vector space $S = S_{Dirac}$, of complex dimension 4, the vector space of *Dirac spinors*. (For a dimension n, the vector space of Dirac spinors is of dimension $2^{[\frac{n}{2}]}$. The representation is faithful when n is even.) This means an *algebra isomorphism*

$$\mathbb{C}\ell(4) \to \mathrm{End}(S_{Dirac}) = \mathrm{Mat}_4(\mathbb{C})$$

$$e_{\mu} \to \Sigma_{\mu}.$$

Since S_{Dirac} is a complex vector space of complex dimension 4, $\mathrm{End}(S_{Dirac})$ is isomorphic to $\mathrm{Mat}_4(\mathbb{C})$.

The unity is represented by identity; the four 1-vectors e_{μ} of $\mathbb{C}\ell^1$ are represented by four complex 4×4 matrices $(\Sigma_{\mu})_{\mu=0,1,2,3}$, which verify the Clifford anticommutation relations

$$[\Sigma_{\mu}, \Sigma_{\nu}]_{+} = 2\ \delta_{\mu\nu}. \tag{53}$$

The *space-time algebra* $\mathrm{C}\ell(1,3)$ is the [real]sub–algebra of $\mathbb{C}\ell(4)$ generated by the four elements e_0, ie_i. Thus, the previous representation of $\mathbb{C}\ell(4)$ provides a matrix representation of $\mathrm{C}\ell(1,3)$, obtained by defining the four *Dirac matrices* $(\gamma_{\mu})_{\mu=0,1,2,3}$:

$$\gamma_0 = \Sigma_0,\ \ \gamma_i = i\ \Sigma_i.$$

They represent the four 1-vectors e_μ of Minkowski vector space. The other representatives are found by explicitation of the products. They verify the Clifford anticommutation relations

$$[\gamma_\mu, \gamma_\nu]_+ = 2\ \eta_{\mu\nu}. \tag{54}$$

These matrices (and their products) act as operators on the vector space S_{Dirac} of Dirac spinors.

In particular, the *spin matrices*

$$\sigma_{\mu\nu} = \sigma_{[\mu\nu]} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu]_-,$$

defined for $\mu \neq \nu$, represent the bivectors, the Lie algebra generators of the Spin group. (An additional factor i is generally introduced in quantum physics).

The *chirality matrix* is defined as

$$\chi \equiv \gamma^5 \equiv \gamma_5 \equiv -i\ \gamma_0\ \gamma_1\ \gamma_2\ \gamma_3, \tag{55}$$

such that $\chi^2 = \mathbb{I}$.

Explicit representations

There are different ways to represent the isomorphism $\mathrm{C}\ell(1,4) \mapsto \mathrm{Mat}_4(\mathbb{C})$. The most familiar one is obtained as the complex 4×4 matrices

$$\gamma_0 = \begin{vmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{vmatrix}, \gamma_i = \begin{vmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{vmatrix}, \tag{56}$$

where the σ_i are the Pauli matrices. Their products provide the rest of the basis with, e.g.,

$$\gamma_5 = i\ \gamma_0\ \gamma_1\ \gamma_2\ \gamma_3 = \begin{vmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{vmatrix}. \tag{57}$$

Variants are found where γ_0 is replaced by $-\gamma_0$, or by $\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$; and where the γ_i are replaced by the $-\gamma_i$.

Among other possibilities [9], one may also define $\gamma_0 = \begin{vmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{vmatrix}$, with $\gamma_5 = \begin{vmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{vmatrix}$.

3.6.2 Klein-Gordon and Dirac equations

The *Klein-Gordon differential operator* $\Box + m^2$ is a second order Lorentz-invariant operator. Historically, the spinors were introduced after the quest for a first order Lorentz-invariant differential operator. This is only possible if the coefficients belong to a non commutative algebraic structure, which will be precisely (modulo isomorphisms) that of the space-time Clifford algebra. The new operator acts on the spinor space both according to the spinor representation (since the spinor space is a representation space for the space-time algebra) and differentially. The last action is to be understood as acting on spinor fields, i.e., space-time functions which take their values in S (equivalently, sections of the *spinor bundle*).

The Klein-Gordon operator is factorized as the product

$$\Box + m^2 = -(i\gamma^\mu \partial_\mu + m)\ (i\gamma^\nu \partial_\nu - m), \tag{58}$$

where the four constant γ^μ verify the conditions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \text{ and } \partial_\mu \gamma^\nu = 0$$

(indices are lowered or raised with the Minkowski metric).

The usual field theory considers the γ^μ as the four matrices *Dirac matrices*. They correspond to the first grade members of the space-time algebra.

3.6.3 Projectors and Weyl spinors

The complex 4×4 matrices matrices of this representation act on the space of Dirac spinors $S_{Dirac} \simeq \mathbb{C}^4$. We write a Dirac spinor as ψ_D. The $\gamma_{\mu\nu}$ are the generators of the Lorentz rotations.

From the chirality matrix, one construct the two *projectors*

$$P_{left} \equiv \frac{1}{2}\ (\mathbb{I} - \chi),\ P_{right} \equiv \frac{1}{2}\ (\mathbb{I} + \chi).$$

They project a Dirac spinor ψ_D onto its right and left components

$$\psi_R \equiv P_{right}\ \psi_D = \begin{pmatrix} 0 \\ \eta \end{pmatrix},\ \psi_L \equiv P_{left}\ \psi_D = \begin{pmatrix} \pi \\ 0 \end{pmatrix};\ \psi = \psi_R + \psi_L = \begin{pmatrix} \pi \\ \eta \end{pmatrix}.$$

Thus, S_{Dirac} decomposes as

$$S_{Dirac} = P_+\ S_{Dirac} \oplus P_-\ S_{Dirac},$$

where $P_\pm S_{Dirac}$ is the eigenspace of χ (in S_{Dirac}) with eigenvalue ± 1 of the chirality operator. Elements of $P_\pm S_{Dirac}$ are called right (resp. left) helicity Weyl spinors, and they identify with the definitions above. Each $P_\pm S_{Dirac}$ is an irreducible representation space for $\mathbb{C}\ell^{even}$ [4].

The Dirac and Weyl spinors also provide representations for the groups Pin and Spin.

3.7 Spinors in the the space-time-algebra

Spinors can be described as elements of the space-time-algebra itself. Namely the [Dirac] spinor space identifies with a [left or right] minimal ideal. of the space-timealgebra ([6] and references therein): the [spin] representation space lies inside the algebra operating on it.

Following [6], one may e.g. select the nilpotent element $f \stackrel{def}{=} \frac{1}{4}\ (e_0 - e_3)\ (e_1 - i\ e_2)$, and the vector space $C\ f$ appears as a minimal left ideal. This 4-dimensional vector space admits the basis $(1, \theta_1 \stackrel{def}{=} \frac{1}{2}\ (e_0 + e_3), \theta_2 \stackrel{def}{=} \frac{1}{2}\ (e_1 + i\ e_2), \theta_1\ \theta_2)$. It may be identified to the vector space S_D of Dirac spinors. Then, γ_5 acts as the projector and generates the splitting $S_D = S_{left} \oplus S_{right}$ such that

$$i\ \gamma_5\ \psi_L = \psi_L; \qquad i\ \gamma_5\ \psi_R = -\psi_R.$$

S_{left}, with the basis $(1, \theta_1\ \theta_2)$, and S_{right}, with the basis (θ_1, θ_2), may be seen as the Weyl spinor spaces in the space-time algebra.

Note that there are many different ways to embed the representation space into the space-time-algebra.

4 The Clifford bundle on a [pseudo-]Riemanian manifold

We have examined the space-time algebra, with its relations to spinors. Both were constructed over the Minkowski vector space, whose affine version is the the Minkowski spacetime. Physical fields require the construction of fiber bundles over space-time, i.e., a pseudo-Riemannian manifold whose tangent spaces are copies of the Minkowski vector space: the fibers are isomorphic to the Clifford algebra, to the spinor spaces, to the Spin groups ... The basis is space-time.

A space-time is considered as four-dimensional orientable pseudo-Riemannian manifold. A choice of time orientation (a polarization) allows to select those timelike or null vectors which are future directed.

4.1 Fiber bundles associated to a manifold

To a differential manifold $\mathcal{M}$ are associated *natural vector bundles* (i.e., on which the diffeomorphism group acts canonically), among which the tangent and the cotangent bundles:

$$\mathrm{T}\mathcal{M} \equiv \bigcup_{m \in \mathcal{M}} \mathrm{T}_m\mathcal{M}, \quad \mathrm{T}^*\mathcal{M} \equiv \bigcup_{m \in \mathcal{M}} \mathrm{T}^*_m\mathcal{M}.$$

Each $\mathrm{T}_m\mathcal{M}$ is a copy of the Minkowski vector space $\mathbb{M}$, each $\mathrm{T}^*_m\mathcal{M}$ is a copy of its dual $\mathbb{M}^*$.

One defines the $\begin{pmatrix} n \\ k \end{pmatrix}$ dimensional spaces of k-forms at m, $\bigwedge^k(\mathrm{T}^*_m\mathcal{M})$. Their union form the vector bundle of k-forms on $\mathcal{M}$, $\bigwedge^k(\mathrm{T}^*\mathcal{M})$. Its sections, the element of $\mathrm{Sect}(\bigwedge^k \mathrm{T}^*_x\mathcal{M})$, are the k-forms [fields] on $\mathcal{M}$.

At each point, one may define the two Clifford algebras $\mathrm{C}\ell(\mathrm{T}_m\mathcal{M})$ and $\mathrm{C}\ell(\mathrm{T}^*_m\mathcal{M})$. Their unions define

- the *Clifford bundle of multivector fields* on $\mathcal{M}$:

  $$\mathrm{C}\ell(\mathrm{T}\mathcal{M}) \equiv \bigcup_m \mathrm{C}\ell(\mathrm{T}_m\mathcal{M}).$$

  Each fibre $\mathrm{C}\ell(\mathrm{T}_m\mathcal{M})$ is a copy of the space-time algebra $\mathrm{C}\ell(1,3)$.

- the *Clifford bundle of [differential] multiforms* on $\mathcal{M}$:

  $$\mathrm{C}\ell(\mathrm{T}^*\mathcal{M}) \equiv \bigcup_m \mathrm{C}\ell(\mathrm{T}^*_m\mathcal{M}).$$

  Each fibre $\mathrm{C}\ell(\mathrm{T}^*_m\mathcal{M})$ is also a copy of $\mathrm{C}\ell(1,3)$. $\mathrm{C}\ell(\mathrm{T}^*M)$ is similar to the Cartan bundle $\bigwedge \mathrm{T}^*\mathcal{M} \stackrel{def}{=} \cup_m \bigwedge \mathrm{T}^*_m\mathcal{M}$, the difference lying in the possibility of addition of forms of different degrees in $\mathrm{C}\ell(\mathrm{T}^*\mathcal{M})$, not in $\bigwedge \mathrm{T}^*\mathcal{M}$. The latter can be seen as embedded in $\mathrm{C}\ell(\mathrm{T}^*\mathcal{M})$. Here, $\cup_m \bigwedge \mathrm{T}^*_m\mathcal{M} = \bigcup_{k=0}^n \bigwedge^k \mathrm{T}^*_x M$, where $\mathrm{Sect}(\bigwedge^k \mathrm{T}^*_x M)$ is the $\begin{pmatrix} n \\ k \end{pmatrix}$ - dimensional space of k-forms.

There is a complete isomorphism between $\mathrm{C}\ell(\mathrm{T}^*M)$ and $\mathrm{C}\ell(\mathrm{T}M)$, which may be explicited by an extension of the canonical metric (musical) isomorphism.

We recall that a scalar product in a vector space is naturally extended to the Clifford algebra (see 1.2.4). Here, the metric of $\mathcal{M}$ is extended to the tensor bundles, and thus to the [sections of] the Clifford bundles, i.e., to multiforms and multivectors.

A metric compatible connection acts on the tensor bundle. It is extended to define a covariant derivative acting on Clifford fields (i.e., section of the Clifford bundles).

The Hodge duality, that we defined for arbitrary polyvectors (1.2.4) extends naturally to the Clifford bundles.

4.2 Spin structure and spin bundle

We recall the principal fiber bundles defined on a [pseudo-]Riemanian manifold:

- the *frame bundle* $\mathrm{Fr}(\mathcal{M})$ (also written $\mathrm{Fr} \mapsto \mathcal{M}$) is a GL-principal bundle on $\mathcal{M}$, with structure group the general linear group GL. A section is a moving frame of $\mathcal{M}$: a choice of a vector basis for the tangent space, at each point of the manifold.

- the *special orthogonal frame bundle* (or *tetrad bundle*) $\mathrm{Fr}^{\mathrm{SO}} \mapsto \mathcal{M}$ has structure group SO. A section is an oriented ON frame of $\mathcal{M}$, or oriented tetrad. Orientability requires the vanishing of the first Stiefel–Whitney class.

- the *time-oriented special orthogonal frame bundle* (or time-oriented *tetrad bundle*) $\mathrm{Fr}^{\mathrm{SO}\uparrow} \mapsto \mathcal{M}$ has structure group $\mathrm{SO}^{\uparrow}$. A section is an oriented and time-oriented ON frame of $\mathcal{M}$.

These bundles are well defined in a [pseudo-]Riemannian manifold. In a differential manifold, one may consider $\mathrm{Fr}^{\mathrm{SO}}$ as the result of a process of *fiber bundle reduction* (see, e.g., [10]) from $\mathrm{Fr}^{\mathrm{GL}}$, equivalent to a choice of metric.

A *Spin-structure* will be defined as a Spin-principal fiber bundle $\mathrm{Fr}^{\mathrm{Spin}} \mapsto \mathcal{M}$ called the *Spin bundle* (also written $\mathrm{Spin}(\mathcal{M})$). A section is called a *spin frame*. The Spin bundle is an extension of $\mathrm{Fr}^{\mathrm{SO}} \mapsto \mathcal{M}$ by the group Z^2.

All these G-principal fiber bundles have associated vector bundles with an action of the principal group G. The fibers are copies of a representation vector space of G. This is the tangent bundle $\mathrm{T}\mathcal{M}$ for the three first. For $\mathrm{Spin}(\mathcal{M})$, they are called the *spinor bundles* (Weyl spinor bundles, with a fibre of complex dimension 2; Dirac spinor bundles, with a fibre of complex dimension 4). The fiber is a representation space for the group Spin, i.e., a spinor space (see, e.g., [11]).

Spin structure

The Spin group is a the double universal covering of the group SO. We recall the double covering group homomorphisms

$$H : \mathrm{Spin} \mapsto \mathrm{SO},$$

$$H : \mathrm{Spin}^{\uparrow} \mapsto \mathrm{SO}^{\uparrow}.$$

Given a [pseudo-]Riemannian manifold $\mathcal{M}$, the special-orthogonal bundle $\mathrm{Fr}^{\mathrm{SO}}(\mathcal{M})$ is a SO-principal fiber bundle. A section is an oriented ON frame (an oriented tetrad).

Even when $\mathrm{Fr}^{\mathrm{SO}}(\mathcal{M})$ does exist, they may be some *topological obstruction* to the existence of a spin structure. This requires the vanishing of the second stiefel-Whitney class. Also, the existence of a spin structure is equivalent to the requirement that $\mathrm{Fr}^{\mathrm{SO}} \mapsto \mathcal{M}$ is a trivial bundle. Since this is a principal bundle, this means that it admits global sections (see, e.g., [8]), which are global SO-tetrads. This implies that its universal covering $\mathrm{Spin}(\mathcal{M})$ also admits global sections, which are the *spin frames*.

The transition functions of the bundle f_{ij} of $\mathrm{Fr}^{\mathrm{SO}} \mapsto \mathcal{M}$ take their values in SO. A spin structure $\mathrm{Spin}(\mathcal{M})$, when it exists, is defined by its transition functions $\tilde{f}_{ij}$, with values in Spin, and such that $H(\widetilde{f_{ij}}) = f_{ij}$. Note that $\mathcal{M}$ admits in general many spin structures, depending on the choice of the $\widetilde{f_{ij}}$.

The 2-1 homomorphism

$$\widetilde{H} : \ \mathrm{Spin}(\mathcal{M}) \mapsto \mathrm{Fr}^{\mathrm{SO}}(\mathcal{M})$$

maps a fiber onto a fiber, so that $\widetilde{H}(u\ \lambda) = \widetilde{H}(u)\ H(\lambda)$, with $u \in \mathrm{Spin}(\mathcal{M})$ (the fiber bundle) and $\lambda \in \mathrm{Spin}$ (the group). Considering the 4 connected components of SO, the possible combinations give the possibility to 8 different spin structures corresponding to the choices of signs for P^2, T^2 and $(PT)^2$.

Proper spin structure

The bundle $\mathrm{Fr}^{\mathrm{SO}}(\mathcal{M})$ has for sections the oriented tetrads (= SO-frames). The bundle $\mathrm{Fr}^{\mathrm{SO}\uparrow}(\mathcal{M})$ of *time oriented* and oriented tetrads is obtained after the [fiber bundle] reduction of the Lorentz group SO to the *proper* Lorentz group $\mathrm{SO}^{\uparrow}$. Then a *proper spin structure* is defined as a principal bundle $\mathrm{Spin}^{\uparrow}(\mathcal{M})$ over $\mathcal{M}$, with structure group $\mathrm{Spin}^{\uparrow}$.

Spinor fields

The group Spin acts on its representations which are the spinor [vector]spaces. A vector bundle over space-time, whose fibre is such a representation space for the group Spin is a *spinor bundle*; a section is called a *spinor field*. This is a spinor-valued function (0-form) on space-time. In particular, the Dirac and the (left and right) Weyl spinor bundles have for sections the corresponding Dirac or Weyl *spinor fields*. Representations of the group Spin act on them.

Spin connections

A *linear connection* on a differential manifold $\mathcal{M}$ identifies with a principal connection on the principal fiber bundle $\mathrm{Fr} \mapsto \mathcal{M}$, with the linear group GL as principal group. A metric structure allows (is equivalent to) a fiber bundle bundle reduction from $\mathrm{Fr} \mapsto \mathcal{M}$ to the orthogonal frame bundle $\mathrm{Fr}^{\mathrm{SO}} \mapsto \mathcal{M}$. The linear connection (with values in the Lie algebra $\underline{\mathfrak{g}\ell}$) is reduced to a *Lorentz connection*, with values in the Lie algebra $\underline{\mathfrak{so}}$. This is a principal connection on the principal fiber bundle $\mathrm{Fr}^{\mathrm{SO}} \mapsto \mathcal{M}$. When a spin structure exists, the latter defines a connection of the principal fiber bundle $\mathrm{Fr}^{\mathrm{Spin}} \mapsto \mathcal{M}$, which is called a *spin connection*.

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