Tree-width of hypergraphs and surface duality

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Abstract

In Graph Minors III, Robertson and Seymour write:"It seems that the tree-width of a planar graph and the tree-width of its geometric dual are approximately equal — indeed, we have convinced ourselves that they differ by at most one." They never gave a proof of this. In this paper, we prove that given a hypergraph H on a surface of Euler genus k, the tree-width of H^* is at most the maximum of $\text{tw}(H) + 1 + k$ and the maximum size of a hyperedge of H^* minus one.

Keywords: Tree-width, duality, surface.

1 Introduction

Tree-width is a graph parameter that was first defined by Halin [Hal76], and which has been rediscovered many times (see [RS84, AP89]). In [AP89], Arnborg and Proskurovski introduced a general framework to solve NP-complete problems efficiently when restricted to graphs of bounded tree-width. Courcelle [Cou90] extended this framework by showing that any problem expressible in a certain logic on graph can be solved efficiently for a class of graph of bounded tree-width. Tree-width thus seems to be a good "complexity measure" for graphs.

Given a graph G embedded in a surface, it is easy to obtain the dual embedding G^*: just put a vertex in each face and for every edge e separating the faces f and g, add a dual edge fg. One could thus expect that G and G^* have the same "complexity", and indeed in [RS84], Robertson and Seymour claimed that for a plane graph G, $\text{tw}(G)$ and $\text{tw}(G^*)$ differ by at most one.

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In an unpublished paper, Lapoire [Lap96] gave a more general statement about hypergraphs on orientable surfaces. Nevertheless, his proof was rather long and technical. Later, Bouchitté et al. and Mazoit [BMT03, Maz04] gave easier proofs for plane graphs. Here we give a proof of the following theorem:

Theorem 1 *For any 2-cell embedding H of a hypergraph on a surface Σ,*

$$\mathrm{tw}(H^*) \leq \max\{\mathrm{tw}(H) + 1 + k_\Sigma, \alpha_{H^*} - 1\}.$$

in which α_{H^*} is the maximum size of an edge of H^* and k_Σ is the Euler genus of Σ.

We will of course define formally the required notions but "hand waving", the proof is as follows. We first define *p-trees* which can be seen as a restricted class of tree-decomposition. We then prove that

i. there always exists a p-tree T such that $\mathrm{tw}(T) = \mathrm{tw}(H)$;

ii. for any p-tree T, $\mathrm{tw}(T^*) \leq \max\{\mathrm{tw}(T) + 1 + k_\Sigma, \alpha_{H^*} - 1\}$.

In section 2, we give the basic definitions needed to properly state the theorem. Section 3 is devoted to the first step of the proof and we prove the second one in section 4.

2 Tree-width and embedded hypergraphs

2.1 Hypergraphs and partitioning trees

If X is a graph or hypergraph, we denote by V_X its vertex set, by E_X its edge or hyperedge set, and α_X the maximum size of an edge of X. Let H be a fixed hypergraph. A *tree-decomposition* of H is a pair $\mathcal{T} = (T, (X_v)_{v \in V_T})$ with T a tree and $(X_v)_{v \in V_T}$ a family of subsets of vertices of H called *bags* with:

i. $\bigcup_{v \in V_T} X_v = V_H$;

ii. $\forall e \in E_H$, $\exists v \in V_T$ with $e \subseteq X_v$;

iii. $\forall x$, y, $z \in V_T$ with y on the path from x to z, $X_x \cap X_z \subseteq X_y$.

The *width* of $\mathcal{T}$ is $\mathrm{tw}(\mathcal{T}) = \max\{|X_t| - 1 \ ; \ t \in V_T\}$ and the *tree-width* $\mathrm{tw}(H)$ of H is the minimum width of one of its tree-decompositions.

In this paper, we only consider partitions that do not contain the empty set. The *border* of a partition μ of E_H is the set of vertices $\delta_H(\mu)$ which are incident with edges in at least two parts of μ, and the *border* of $A \subseteq E_H$ is $\delta_H(A) = \delta_H(\{A, E_H \setminus A\})$. A *partitioning tree* of H is a tree T whose leaves are labelled by edges of H in a bijective way. Removing an internal node v of T results in a partition of the leaves of T and thus in a *node-partition* λ_v of E_H. It is straightforward to check that labelling each internal node v of T with $\delta_H(\lambda_v)$ turns T into a tree-decomposition. The *tree-width* of a partitioning tree is its *tree-width*, seen as a tree-decomposition.

Given a non-empty subset $A \subseteq E_H$, we define the *contracted hypergraph* $H_{/A}$ of H as the hypergraph with vertex set $\cup(E_H \setminus A)$ and with edge set $(E_H \setminus A) \cup \{e_A\}$ in which $e_A = \delta_H(A)$ is a new hyperedge. Let $\{A, B\}$ be a bipartition of E_H and $T_{/A}$ and $T_{/B}$ be respectively partitioning trees of $H_{/A}$ and $H_{/B}$. By removing from the disjoint union $T_{/A} \cup T_{/B}$ the leaves labelled e_A and e_B and adding a new edge between their respective neighbours, we obtain a partitioning tree T which is the *merge of $T_{/A}$ and $T_{/B}$*.

Lemma 1 *Let H be an hypergraph, let $\{A, B\}$ be a bipartition of E_H, and let $T_{/A}$ and $T_{/B}$ be partitioning trees of $H_{/A}$ and $H_{/B}$. Then the merge T of $T_{/A}$ and $T_{/B}$ is such that*

$$\mathrm{tw}(T) = \max\{\mathrm{tw}(T_{/A}), \mathrm{tw}(T_{/B})\}.$$

Proof. Let $C \subseteq E_H$ be disjoint from A. We claim that $\delta_H(C)$ and $\delta_{H_{/A}}(C)$ are equal.

Indeed, let $v \in \delta_{H_{/A}}(C)$. By definition, there exists $e \in E_{H_{/A}} \setminus C$ and $f \in C$ which contain v. If $e \neq e_A$, then $e \in E_H \setminus C$. Otherwise, $e = e_A = \delta_H(A)$ and there exists $e' \in A \subseteq E_H \setminus C$ which contains v. In both cases, $v \in \delta_H(C)$. Conversely, let $v \in \delta_H(C)$. By definition, there exists $e \in E_H \setminus C$ and $f \in C$ which contain v. If $e \notin A$, then $e \in E_{H_{/A}} \setminus C$. Otherwise $v \in \delta_H(A) = e_A$. In both cases $v \in \delta_{H_{/A}}(C)$.

Let u be an internal node of T. By symmetry, we can suppose that u belongs to $T_{/A}$. In $T_{/A}$, u corresponds to a partition $\lambda_{/A} = \{E_1 \cup \{e_A\}, E_2, \ldots, E_p\}$ of $E_{H_{/A}}$ and in $T_{/A}$, and to the partition $\lambda = \{E_1 \cup A, E_2, \ldots, E_p\}$ of E_H in T. The above claim implies that $\delta_H(\lambda) = \delta_{H_{/A}}(\lambda_{/A})$, and thus that $\mathrm{tw}(T) = \max\{\mathrm{tw}(T_{/A}), \mathrm{tw}(T_{/B})\}$. ■

Lemma 1 and the following folklore lemma are the key tools to our proof of Theorem 2 that there always exists a p-tree of optimal width.

Lemma 2 *Let H be an hypergraph. For any bipartition $\{A, B\}$ of E_H,*

$$\mathrm{tw}(H) \leq \max\{\mathrm{tw}(H_{/A}), \mathrm{tw}(H_{/B})\}.$$

If $\delta_H(\{A, B\})$ belongs to a bag of an optimal tree-decomposition, then

$$\mathrm{tw}(H) = \max\{\mathrm{tw}(H_{/A}), \mathrm{tw}(H_{/B})\}.$$

Proof. Let $\mathcal{T}_{/A} = (T_{/A}, (X_v)_{v \in V_{T_{/A}}})$ and $\mathcal{T}_{/B} = (T_{/B}, (Y_v)_{v \in V_{T_{/B}}})$ be respective optimal tree-decompositions of $H_{/A}$ and $H_{/B}$. Let $u \in V_{T_{/A}}$ and $v \in V_{T_{/B}}$ whose bags respectively contain e_A and e_B. By adding an edge uv to the disjoint union $T_{/A} \cup T_{/B}$, we obtain a tree-decomposition $\mathcal{T}$ of H such that $\mathrm{tw}(\mathcal{T}) = \max\{\mathrm{tw}(H_{/A}), \mathrm{tw}(H_{/B})\}$, which proves the first part of the lemma.

Suppose now that $\delta_H(\{A, B\})$ belongs to the bag of a vertex v of an optimal tree-decomposition $\mathcal{T} = (T, (Z_v)_{v \in V_T})$ of H. By removing $V \setminus (\cup B)$ from the bags of $\mathcal{T}$, we obtain a tree-decomposition $\mathcal{T}_{/A}$ of $H_{/A}$ such that $\mathrm{tw}(\mathcal{T}_{/A}) \leq \mathrm{tw}(H)$. Similarly, we obtain a tree-decomposition $\mathcal{T}_{/B}$ of $H_{/B}$ such that $\mathrm{tw}(\mathcal{T}_{/B}) \leq \mathrm{tw}(H)$. The second part of the lemma follows. ■

2.2 Embedded hypergraphs

A *surface* is a connected compact 2-manyfold without boundaries. Oriented surfaces can be obtained by adding "handles" to the sphere, and non-orientable surfaces, by adding "crosscaps" to the sphere. The *Euler genus* k_Σ of a surface Σ (or just *genus*) is twice the number of handles added if Σ is orientable and k_Σ is the number of crosscaps added otherwise. We denote by $\overline{X}$ the closure of a subset X of Σ. We say that two disjoint subsets X and Y of Σ are *incident* if $X \cap \overline{Y}$ or $Y \cap \overline{X}$ is non empty. Since we consider finite graphs and hypergraphs, we can assume that all the subsets of surfaces that we consider have enough regularity. This implies that connectivity and arc-connectivity coincide.

An *embedding of a graph G on a surface* Σ is a drawing G on Σ, i.e. each vertex is an element of Σ, each edge is an open curve between two vertices, and edges are pairwise disjoint. A bipartite embedded graph G with bipartition $V \cup V'$ can be seen as the incidence graph of an hypergraph. For each $v_e \in V'$, we merge v_e and its incident edges into an *edge* e and call v_e its *centre*. The edges of G which are incident to v_e are the *half edges of* e. Let E be the set of all *edges*. An *embedded hypergraph* on Σ is any such pair $H = (V, E)$. We denote by G_H the embedded incidence graph of an embedded hypergraph H, and we only consider embedded graph and hypergraphs up to homeomorphisms. Note that embedded graphs also are embedded hypergraphs, and since embedded graphs and embedded hypergraphs naturally have abstract counterparts, we apply graph theoretic notions to them without further notice. For example, we may consider an edge e as a subset of Σ or as a set of vertices. We also consider embedded hypergraphs on Σ as subsets of Σ.

A *face* of a hypergraph H is a component of $\Sigma \setminus H$. We denote by F_H the set of faces of H. An embedded hypergraph is *2-cell* if all its faces are homeomorphic to open discs. Let G be a 2-cell embedding of a graph in a surface Σ. Euler's formula links the number of vertices, edges and faces of G and the genus of the surface:

$$|V_G| - |E_G| + |F_G| = 2 - k_\Sigma.$$

We now let H be a 2-cell embedding of a hypergraph in a surface Σ. The *dual* of H is the hypergraph H^* such that:

i. Every vertex of H^* belongs to a face of H and every face of H contains exactly one vertex of H^*;

ii. For every edge e of H, there exists a dual edge e^* sharing its centre, and every edge of H^* corresponds to an edge of H.

iii. For every edge e of H with centre v_e, the half edges around e and the half edges around e^* alternate in their cyclic order.

Note that the construction does not need H to be 2-cell but if not, H^* is not unique and $(H^*)^*$ need not be H. If μ is a partition of E_H, then $\delta^*_H(\mu)$ if the set of faces of H which are incident with edges in at least two parts of μ.

Given a partitioning tree T of H, the *dual partitioning tree* is the partitioning tree T^* of H^* obtained by replacing in T each label e by the dual edge e^*.

3 P-trees

In this section, we define *p-trees* which are special partitioning trees, and we prove that, for any embedded hypergraph H, there always exists a p-tree whose tree-width is $\mathrm{tw}(H)$. Note that in this section, the hypergraphs are not required to be 2-cell but they must be connected and have at least one edge.

Let H be an embedding of a hypergraph in a surface Σ. A *radial graph* of H is an embedding Π of a bipartite graph on Σ such that:

i. $\{V_H, V_\Pi \setminus V_H\}$ is a bipartition of Π, and $V_\Pi \setminus V_H$ contains exactly one vertex per face of H;

ii. each edge of H is contained in a face of Π and each face of Π contains exactly one edge of H.

First radial graphs do exist.

Lemma 3 *Every embedding H of a connected hypergraph with at least one edge on a surface Σ admits a radial graph.*

Proof. The set V_H being fixed, let us first choose one *face vertex* per face of H to get $V_\Pi \setminus V_H$.

Let $(D_e)_{e \in E_H}$ be pairwise disjoint open discs such that each D_e contains e. Such discs can be obtained by "thickening" each edge a little. We now continuously distort all the discs intersecting a given face so that they become incident with it corresponding "face vertex".

At this time, the union of the borders of the discs D_e correspond to the drawing of a bipartite graph that satisfies all the required condition except that some faces may be empty. If such a face exists, then we just remove one of its incident edge to merge it with an adjacent face, and thus decreasing the total number of empty faces. In the end, we obtain a radial graph Π of H. ■

Let Π be a radial graph of H. We say that an edge or a vertex of Π is *private to* a set F of faces of Π if all the faces it is incident to belong to F.

We now define several notions which all depend on Π. They thus are defined *with respect to a radial graph*. Let A be a set of edges of H. We denote by Π_A the open set that contains all the faces of Π corresponding edges in A, together with the edges and vertices of Π which are private to these faces. We say that A is Π-*connected* if Π_A is connected, and that a partition of E is Π-*connected* if its parts are Π-connected. Two edges e and f of H are Π-*adjacent* if $\Pi_{\{e,f\}}$ is Π-conencted. An edge e of H is *troublesome* is the partition $\{e, E \setminus \{e\}\}$ is not Π-connected.

If a vertex x of Π is private to a set of faces of Π, then so are all its incident edges. Thus if we denote by G^{Π} the graph whose vertices are the faces of Π, and in which two faces are adjacent if they are incident to a common edge, then Π-connected sets of edges of H exactly correspond to connected subgraphs of G^{Π}. Note that two edges e and f are Π-adjecent if and only if their corresponding faces in Π are adjacent in G_{Π}.

Let A be a Π-connected set of edges of H. Let us denote $\widetilde{A}$ and $\widetilde{H}$ the respective abstract counterparts of A and H. If we remove the part of H which is contained in Π_A and replace it by an edge e_A whose set of ends is $\delta_H(A)$ (which is possible because Π_A is connected), we obtain a hypergraph embedded on Σ whose abstract counterpart is $\widetilde{H}_{/\widetilde{A}}$. We thus denote this new embedding by $H_{/A}$. By removing from Π all the edges and vertices which are contained in Π_A, we obtains the *contracted radial graph* $\Pi_{/A}$ of $H_{/A}$. Any hypergraph $H_{/A}$ is implicitly equipped with his contracted radial graph. Note that, by construction of $\Pi_{/A}$, a partition $\{C \cup A, D\}$ is Π-connected if and only if the partition $\{C \cup \{e_A\}, D\}$ is $\Pi_{/A}$-connected.

Let H be an embedding of a connected hypergraph with at least one edge on a surface Σ, and let Π be a radial graph of H. A *p-tree* of H is a partitioning tree of H such that:

i. all node partitions are Π-connected;

ii. if v is an internal node of T whose degree is not three, then v is a neighbour of a leaf labelled by a troublesome edge e and $\delta_H(\lambda_v) \subseteq e$.

Lemma 4 *Let H be an embedding of a connected hypergraph with at least one edge on a surface Σ, let Π be a radial graph of H, and let $\{A, B\}$ be a Π-connected bipartition of E_H. Let $T_{/A}$ and $T_{/B}$ be respective p-tree of $H_{/A}$ and $H_{/B}$. The merged partitioning-tree T of $T_{/A}$ and $T_{/B}$ is a p-tree of H.*

Proof. Let v be an internal node of T. By symmetry, we can suppose that v is a node of $T_{/A}$.

We first prove that the node partition $\lambda = \{E_1 \cup A, E_2, \ldots, E_p\}$ of v in T is Π-connected. In $T_{/A}$, the node partition of v is $\lambda_{/A} = \{E_1 \cup \{e_A\}, E_2, \ldots, E_p\}$. Since $\lambda_{/A}$ is $\Pi_{/A}$-connected, $(\Pi_{/A})_{E_1 \cup \{e_A\}}$ and all $(\Pi_{/A})_{E_i}$ $(2 \leq i \leq p)$ are connected. But, by definition of Π_A, $\Pi_{E_1 \cup A} = (\Pi_{/A})_{E_1 \cup \{e_A\}}$ and $\Pi_{E_i} = (\Pi_{/A})_{E_i}$ for $2 \leq i \leq p$. This implies that λ is Π-connected.

Suppose for now that the degree of v is not three in T. Since this is also the case in $T_{/A}$, v is the neighbour of a leaf u which is labelled by a troublesome edge e of $H_{/A}$ and $\delta_{H_{/A}}(\lambda_{/A}) \subseteq e$. Since $\{A, B\}$ is Π-connected, then e_A is not a troublesome edge in $H_{/A}$. The node v is thus a neighbour of a leaf u labelled by the troublesome edge e in H, and $\delta_H(\lambda_v) \subseteq e$. This finishes the proof. ■

We can now prove that

Theorem 2 *Let H be an embedding of a connected hypergraph with at least one edge on a surface Σ, and let Π be a radial graph for H. There exists a p-tree T such that* $\mathrm{tw}(T) = \mathrm{tw}(H)$.

Proof. We proceed by induction on $|V_H| + |E_H|$. We say that a hypergraph H is *smaller* than another hypergraph H' if $|V_H| + |E_H| < |V_{H'}| + |E_{H'}|$. Let $\mathcal{T}$ be a tree-decomposition of H of optimal width. By eventually merging them, we can suppose that $\mathcal{T}$ has no two neighbouring bags, one of which contains the other.

Suppose that we find a Π-connected bipartition $\{A, B\}$ of E_H such that $\delta_H(\{A, B\})$ is contained in a bag of $\mathcal{T}$, and such that $H_{/A}$ and $H_{/B}$ are smaller than H. Such a partition is *good*. By induction, let $T_{/A}$ and $T_{/B}$ be p-trees of $H_{/A}$ and $H_{/B}$ of optimal width, and let T be the merge of $T_{/A}$ and $T_{/B}$. By Lemma 4, T is a p-tree. By Lemma 1, its tree-width is $\max\{\mathrm{tw}(T_{/A}), \mathrm{tw}(T_{/B})\} = \max\{\mathrm{tw}(H_{/A}), \mathrm{tw}(H_{/B})\}$. Since, $\delta_H\{A, B\}$ is contained in a bag of $\mathcal{T}$, Lemma 2 implies that $\mathrm{tw}(H) = \max\{\mathrm{tw}(H_{/A}), \mathrm{tw}(H_{/B})\}$, and thus, $\mathrm{tw}(T) = \mathrm{tw}(H)$. We thus only have to find good partitions.

Suppose that H contains a troublesome edge e. If e contains and separates all the other edges of H, then the partitioning star with one internal node is a p-tree of optimal width. Otherwise, there exists a set of edges A such that Π_A is connected component of $\Pi_{E_H \setminus \{e\}}$, and either A contains at least two edges or the set of vertices of $H_{/A}$ is a strict subset of V_H. Since $\delta_H(A) \subseteq e$, e is contained in least one bag of $\mathcal{T}$, and both $H_{/A}$ and $H_{/E(H) \setminus A}$ are smaller than H, then $\{A, E_H \setminus A\}$ is good. So let us thus suppose that H contains no troublesome edge.

Now two cases arise:

- $\mathcal{T}$ contains at least two nodes.

In any tree-decomposition of H with no bag being a subset of a neighbouring one, the intersection of two neighbouring bags is a separator of H. There thus exists a separator S which is contained in a bag of $\mathcal{T}$.

Let C be a connected component of $H \setminus S$, and let E^C be the sets of edges which are incident with vertices in C. The set E^C is thus Π-connected. Let Π_{E_1}, ..., Π_{E_p} be the connected components of $\Pi_{E \setminus E^C}$. Since $S' := \delta_H(E^C) \subseteq S$ is a separator, then there exists a connected component D of $H \setminus S'$ which is not C. The set of edges which are incident with D is Π-connected, and is thus a subset of, say, E_1. Since the sets $\overline{\Pi_{E_i}}$ $(2 \leq i \leq p)$ are incident with Π_{E^C}, the partition $\mu := \{E_1, E \setminus E_1\}$ is Π-connected. Since both $H_{/E_1}$ and $H_{/E_H \setminus E_1}$ contain fewer vertices than H, and since $\delta_H(\mu) \subseteq S' \subseteq S$ is contained in a bag of $\mathcal{T}$, then μ is good.

- $\mathcal{T}$ is the trivial one node decomposition of H.

  If H contains at most three edges, then all partitioning trees are p-trees, so we can suppose that H contains at least four edges. Since no edge of H is troublesome, G^Π contains at least two vertices of degree two. And since G^Π is connected, it contains at least two disjoint edges which can be extended in a non trivial bipartition of G^Π in two connected sets. This partition corresponds to a Π-connected partition $\mu := \{A, B\}$ of E_H. Since $H_{/A}$ and $H_{/B}$ contain fewer edges than H, then μ is good.

■

4 P-trees and duality

In this section, we prove that the tree-width of a p-tree and that of its dual cannot differ too much. In fact, we prove something stronger: if T is a p-tree, then for every node v of T, the corresponding bags X_v in T and X_v^* in T^* have roughly the same size. To do so, if v is an internal node of T, then we construct a 2-cell embedded hypergraph H' with vertex set X_v and with face set X_v^*. We then apply Euler's formula on the incidence graph of H' to obtain our bound. It is easy to define an embedded hypergraph with vertex set X_v. Indeed, if the node partition of v is $\{A, B, C\}$, then $((H_{/A})_{/B})_{/C}$ will do. The problem is that it is not 2-cell, and its face set need not be X_v^*. But if we are a little more careful when contracting, then we can obtain such an embedding.

Lemma 5 *Let H be a 2-cell embedding on a surface Σ of a hypergraph with at least three edges, let Π be a radial graph of H, and let $\mu = \{A, B, C\}$ be a Π-connected partition of E_H.*

There exists a 2-cell embedded hypergraph H' on a surface Σ', and a partition $\mu' = \{A', B', C'\}$ of $E_{H'}$ such that

i. $V_{H'} = \delta_H(\mu)$, $F_{H'} = \delta^*_H(\mu) = \delta^*_{H'}(\mu')$;

ii. $k_{\Sigma'} \leq k_\Sigma + 3 - |E_{H'}|$.

Proof. Since Π_A, Π_B and Π_C are disjoint, we can work independently in each one of them. We thus start with Π_A.

First we claim that we can suppose that $H \cap \Pi_A$ is a connected subset of Π_a. Since Π_A is Π-connected, it corresponds to a connected subgraph of G^Π. If all the pairs $(e, f) \in A^2$ of Π-connected edges are such that $(e \cup f) \cap \Pi_A$ are connected, then we are done. So let $(e, f) \in A^2$ Π-connected edges, and $(e \cup f) \cap \Pi_A$ is not connected. This can happen if e and f are incident with a vertex v in $\delta_H(A)$. In this case, let D be a small disc around v. We add a new vertex v_e in $e \cap D$, a new vertex v_f in $f \cap D$, and an edge $v_e v_f$ in $D \cap \Pi_A$. When doing so, we split a face of H which may have belong to $\delta^*_H(A)$ but the new triangle face $v v_e v_f$ is only incident with edges in Π_A, and the other face belongs to $\delta^*_H(A)$ if and only if the original face did. The claim follows.

Since $H \cap \Pi_A$ is connected, we can contract a spanning tree of $G_H \cap \Pi_A$ so that it contains a single vertex linked to $\delta_H(A)$ together with some loops. Then we remove all the loops as follows:

- If e can be contracted, then we do so.
- If e is incident with two faces at least one of which is not in $\delta^*_H(A)$, then we remove e.
- If e is incident with only one face F, just removing e will create a crosscap, but then it is possible to remove F and replace it by a new face which is a disc. In this case, we obtain a new surface with a lower genus.
- If e is incident to two faces which belong to $\delta^*_H(A)$, then we cut the surface along e. This creates two holes that we can fill with discs that

we contract. While doing so, we have created a copy of v_A but we also have replaced Σ by a surface of lower genus.

In the end, we have removed all the faces which were only incident with edges in A, and kept all the others. We have replaced A by, say, p hyperedges and done some surgery on Σ to keep the embedding 2-cell, and the surgery resulted in a decrease of genus of at least $p-1$. The lemma follows. ∎

We are now ready to prove:

Theorem 3 *Let H be a 2-cell embedding of a hypergraph with at least one edge on a surface Σ and let Π be a radial graph of H.*

For any p-tree T of H,

$$\mathrm{tw}(T^*) \leq \max\{\mathrm{tw}(T) + 1 + k_\Sigma, \alpha_{H^*} - 1\}.$$

Proof. Let v be a vertex of T, let X_v be its bag in T and let X_v^* be its bag in T^*. If v is a leaf labelled by an edge e, then $X_v^* = e^*$ and $|X_v^*| - 1 \leq \max\{\mathrm{tw}(T) + 1 + k_\Sigma, \alpha_{H^*} - 1\}$. If v is the neighbour of a leaf labelled by a troublesome edge e, then $X_v \subseteq e$ which implies $X_v^* \subseteq e^*$ and $|X_v^*| - 1 \leq \max\{\mathrm{tw}(T) + 1 + k_\Sigma, \alpha_{H^*} - 1\}$.

We can thus suppose that v is an internal node of T whose node partition is $\lambda_v = \{A, B, C\}$. We then have $X_v = \delta_H(\lambda_v)$, and $X_v^* = \delta_H^*(\lambda_v)$. Let H' be given by Lemma 5 with $\mu = \lambda_v$. Let $G_{H'}$ be the incidence graph of H'. Since $G_{H'}$ is bipartite, and since they are incident with edges in at least two parts among A, B and C, its faces are incident with at least 4 edges. If F_{2k} denotes the set of $2k$-gones faces of $G_{H'}$, we have $2|E_{G_{H'}}| = 4|F_4| + 6|F_6| + \cdots \geq 4|F_{G_{H'}}|$, and thus since the face set of $G_{H'}$ is exactly $\delta_H^*(\lambda_v) = X_v^*$,

$$|E_{G_{H'}}| \geq 2|X_v^*|. \tag{1}$$

Since $G_{H'}$ has $|X_v^*|$ faces, $|X_v| + |E_{H'}|$ vertices, and since $k_{\Sigma'} \leq k_\Sigma + 3 - |E_{H'}|$, by replacing these in Euler's formula, we obtain:

$$|X_v| + |E_{H'}| - |E_{G_{H'}}| + |X_v^*| \geq 2 - k_\Sigma - 3 + |E_{H'}|. \tag{2}$$

Adding (1) and (2), we get

$$|X_v| + 1 + k_\Sigma \geq |X_v^*|$$

which proves that $|X_v^*| - 1 \leq \max\{\mathrm{tw}(T) + 1 + k_\Sigma, \alpha_{H^*} - 1\}$, and thus $\mathrm{tw}(T^*) \leq \max\{\mathrm{tw}(T) + 1 + k_\Sigma, \alpha_{H^*} - 1\}$. ∎

The following theorem is a direct corollary of Theorem 2 and Theorem 3.

Theorem 4 *For any 2-cell embedding of a hypergraph H on a surface Σ,*

$$\mathrm{tw}(H^*) \leq \max\{\mathrm{tw}(H) + 1 + k_\Sigma, \alpha_{H^*} - 1\}.$$

Note that we dropped the condition that H contains at least one edge but the results is quite obvious for the one node 2-cell edge-less hypergraph.

5 Conclusion and open questions

In this paper, we show that tree-width is quite a robust parameter considering surface duality. Indeed, our proof says more than just "the difference between the tree-width of H and that of H^* is not too big". Our proof says that there always exists a decomposition which is optimal for H and very good for H^*. This leads to a natural question: For any embedding G of a graph (or hypergraph) on a surface, does there always exists a p-tree T such that $\mathrm{tw}(T) = \mathrm{tw}(G)$ and $\mathrm{tw}(T^*) = \mathrm{tw}(G^*)$? To our knowledge, the question is open, even for planar graphs. Another natural question is: Is the bound tight?

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