$A-$infinity $GL(N)-$equivariant matrix integrals

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$S = \sum_{g\geq 0,i} \hbar^{2g-1+i} S_{g,i}$ where

$$S_{g,i} \in Symm^i(C_\lambda[1-d]),\ \ C_\lambda = (\oplus_{j=0}^{\infty}((V[1]^{\otimes j})^\vee)^{\mathbb{Z}/j\mathbb{Z}})$$

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- In the case of the algebra $e \cdot e = e$, the answer is *the matrix Airy integral* $\int \exp(\frac{1}{6}Tr(Y^3) - \frac{1}{2}Tr(\Lambda Y^2))dY$

The A-infinity equivariant matrix integrals ([B2],2006)

- The asymptotic expansion as $\Lambda \rightarrow \infty$ ⇒sum over stable ribbon graphs⇒cohomology classes in $H^*(\overline{\mathcal{M}}^K_{g,n})$ (in $H^*(\overline{\mathcal{M}}^K_{g,n}, \mathcal{L})$ for odd d)

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- My A_∞ equivariant matrix integrals give an integration framework in the noncommutative (derived algebraic) geometry, particularly adobted to the equation $\{m_{A_\infty}, m_{A_\infty}\} = 0$

$$\int \exp \widehat{S}(X, \Lambda)\widehat{\varphi} dX$$

$$\varphi \in Ker(\hbar\Delta + \{S, \cdot\}), \quad \varphi_{g,i} \in Symm^i(C_\lambda[1-d])$$

CY complex projective variety (g=0 calculations)

- M- projective manifold/$\mathbb{C}$, $c_1(T_M) = 0$

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- for γ^W, normalized via a filtration W opposite to F^{Hodge}([B5]):

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$$C_{kij}(t) = \partial^3(\text{genus} = 0 \ \ \text{GW-potential of} \ \ M^{mirror})$$

Noncommutative Hodge structures ([B5])

- The class $[\exp(\gamma^W)\varpi] = \Omega(t, \hbar)$ is obtained as intersection

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- Over moduli space of complex structures

$$\mathcal{L}(t) = \sum_r (F^{Hodge})^r \hbar^{-r}[[\hbar]]$$

Noncommutative Hodge structures ([B5]), cont'd

- $\mathcal{L}(t)$ corresponds via HKR and formality isomorphisms for $C^*(A,A) + k[\xi, \frac{\partial}{\partial \xi}]-$module $C_*(A)$ to

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where

$$rk_{C[[\hbar]]} HC^-(A) = rk_{C((\hbar))} HP$$

assumed, i.e. the degeneration of nc Hodge -to-De Rham spectral sequence, proven (Kaledin) for $A-$smooth and compact, Z_+-graded, then $HC^- \subset HP$,

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- Real structure on HP in the case of arbitrary $A_\infty-$algebra?

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- Define the noncommutative BV differential on F via

$$\Delta(x_{\rho_1}\dots x_{\rho_r})_\lambda(x_{\tau_1}\dots x_{\tau_t})_\lambda =$$

$$= \sum_{p,q}(-1)^{\varepsilon} l_{\rho_p\tau_q}(x_{\rho_1}\dots x_{\rho_{p-1}}x_{\tau_{q+1}}\dots x_{\tau_{q-1}}x_{\rho_{p+1}}\dots x_{\rho_r})_\lambda +$$

$$\sum_{p\pm1\neq q}(-1)^{\widetilde{\varepsilon}} l_{\rho_p\rho_q}(x_{\rho_1}\dots x_{\rho_{p-1}}x_{\rho_{q+1}}\dots x_{\rho_r})_\lambda(x_{\rho_{p+1}}\dots x_{\rho_{q-1}})_\lambda(x_{\tau_1}\dots x_{\tau_t})_\lambda$$

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- $\Delta_1 + \hbar\Delta_2$ →non-commutative Batalin–Vilkovisky geometry, stable ribbon graphs, compactified moduli spaces $H_*(\overline{\mathcal{M}}^K_{g,n})$([B1])

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- $Ker \Delta_1 + \Delta_2 = \mathrm{Im}\,\Delta_1 + \Delta_2$

Solutions to nc BV equation

- Conjecture ([B1]). Counting of holomorphic curves $(\Sigma, \partial\Sigma, p_i) \to (M, \coprod L_i, \oplus H_*(L_i \bigcap L_j))$, with $\mathbb{Z}/2\mathbb{Z}$-graded local systems, gives solution to the nc-BV equations.

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- Theorem ([B6]). Summation over ribbon graphs→solution to the nc Batalin-Vilkovisky equation from dg-associative algebras (summation over trees→A-infinity algebra structure)

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- Theorem ([B2],[B3]) This data →Cohomology classes in $H^*(\overline{\mathcal{M}}^K_{g,n})$
- Example $q(N)$, $q(N) = \{[X, \pi] = 0 | X \in gl(N|N)\}$,where $\pi-$odd involution, $q(N)$ has *odd trace otr*, $I = [\Xi, \cdot]$, Ξ- odd element $\Xi = \left(0 \quad | diag(\lambda_1, \ldots, \lambda_n)\right)$, ($I^2 \neq 0$ (!))

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- Theorem ([B2],[B3]) This is the generating function for products of classes $c_1(T_i)$.
- Similarly, with even scalar product and an *odd derivation, in particular for* $gl(N|N)$ and $I = [\Xi, \cdot]$, $\Xi \in gl(N|N)_{odd}$.

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