FIXED POINTS AND LINES IN 2-METRIC SPACES

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Abstract. We consider bounded 2-metric spaces satisfying an additional axiom, and show that a contractive mapping has either a fixed point or a fixed line.

1. Introduction

Gähler introduced in the 1960's the notion of 2-metric space [7] [8] [9], and several authors have studied the question of fixed point theorems for mappings on such spaces. A 2-metric is a function $d(x, y, z)$ symmetric under permutations, satisfying the *tetrahedral inequality*

$$d(x, y, z) \leq d(x, y, a)+d(x, a, z)+d(a, y, z) \text{ for all } x, y, z, a \in X.$$

as well as conditions (Z) and (N) which will be recalled below. In the prototypical example, $d(x, y, z)$ is the area of the triangle spanned by x, y, z.

This notion has been considered by several authors (see [6]), who have notably generalized Banach's principle to obtain fixed point theorems, for example White [25], Iseki [11], Rhoades [23], Khan [12], Singh, Tiwari and Gupta [24], Naidu and Prasad [21], Naidu [22] and Zhang [14], Abd El-Monsef, Abu-Donia, Abd-Rabou [2], Ahmed [3] and others. The contractivity conditions used in these works are usually of the form

$$d(F(x), F(y), a) \leq \ldots$$

for any $a \in X$. We may think of this as meaning that $d(x, y, a)$ is a family of distance-like functions of x and y, indexed by $a \in X$. This interpretation intervenes in our transitivity condition (Trans) below. However, Hsiao has shown that these kinds of contractivity conditions don't have a wide range of applications, since they imply colinearity of the sequence of iterates starting with any point [10]. We thank B. Rhoades for pointing this out to us.

There have also been several different notions of a space together with a function of 3-variables. For example, Dhage [5] introduced the

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concept of D-metric space and proved the existence of a unique fixed point of a self-mapping satisfying a contractive condition. Dhage's definition uses the symmetry and tetrahedral axioms present in Gähler's definition, but includes the *coincidence* axiom that $d(x, y, z) = 0$ if and only if $x = y = z$.

A sequence $\{x_n\}$ in a D-metric space (X, d) is said by Dhage to be convergent to an element $x \in X$ (or d-convergent) [5] if given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_m, x_n, x) < \epsilon$ for all $m, n \geq N$. He calls a sequence $\{x_n\}$ in a D-metric space (X, d) Cauchy (or d-Cauchy) [5] if given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m, x_p) < \epsilon$ for all $n, m, p \geq N$.

These definitions, distinct from those used by Gähler et al, motivate the definition of the property $LIM(y, (x_i))$ in Definition 4.4 and studied in Theorem 4.8 below.

The question of fixed-point theorems on such spaces has proven to be somewhat delicate [18]. Mustafa and Sims introduced a notion of G-metric space [19] [20], in which the tetrahedral inequality is replaced by an inequality involving repetition of indices. In their point of view the function $d(x, y, z)$ is thought of as representing the perimeter of a triangle.

The question of fixed points for mappings on G-metric spaces has been considered by Abbas-Rhoades [1], Mustafa and co-authors [16], [17]. This is not an exhaustive description of the large literature on this subject.

In the present paper, we return to the notion of 2-metric space. The basic philosophy is that since a 2-metric measures area, a contraction should send the space towards a configuration of zero area, which is to say a line. Always assuming that d is globally bounded (B), we add an additional quadratic axiom (Trans) to the original definition of 2-metric. The axiom (Trans) will be shown to hold in the example $X = S^2$ where $d(x, y, z)$ is given by a determinant (Section 5), which has appeared in [15], as well as for the standard area 2-metric on $\mathbb{R}^n$. The abbreviation comes from the fact that (Trans) implies transitivity of the relation of colinearity, see Lemma 4.2. This axiom allows us to consider a notion of *fixed line* of a mapping F which is contractive in the sense that

$$d(F(x), F(y), F(z)) \leq kd(x, y, z)$$

for $k < 1$. With these hypotheses on d and under appropriate compactness assumptions we prove that such a mapping has either a fixed point or a fixed line.

The reader will remark that our contractivity condition is different from the kinds of conditions considered by many authors as recalled previously, in that the function F is applied to all three variables. Consequently, it turns out that there exist many mappings satisfying our contractivity condition, but not the triviality observed by Hsiao [10]. Some examples will be discussed in Section 7. In the example of S^2 with the norm of determinant 2-metric, one can take a neighborhood of the equator which contracts towards the equator, composed with a rotation. This will have the equator as fixed line, but no fixed point, and the successive iterates of a given point will not generally be colinear. Interesting examples of 2-metrics on manifolds are obtained from embedding in $\mathbb{R}^n$ and pulling back the standard area 2-metric. The properties in a local coordinate chart depend in some way on the curvature of the embedded submanifold. We consider a first case of patches on S^2 in the euclidean $\mathbb{R}^3$. These satisfy an estimate (Lemma 5.5) which allows to exhibit a "flabby" family of contractible mappings depending on functional parameters (Proposition 7.2). This shows that in a strong sense the objection of [10] doesn't apply.

The first section of the paper considers usual metric-like functions of two variables, pointing out that the classical triangle inequality may be weakened in various ways. A bounded 2-metric leads naturally to such a distance-like function $\varphi(x, y)$ but we also take the opportunity to sketch some directions for fixed point results in this general context, undoubtedly in the same direction as [4] but more elementary.

As a small motivation to readers more oriented towards abstract category theory, we would like to point out that a metric space (in the classical sense) may be considered as an enriched category: the ordered set $(\mathbf{R}_{\geq 0}, \leq)$ considered as a category has a monoidal structure $+$, and a metric space is just an $(\mathbf{R}_{\geq 0}, \leq, +)$-enriched category. We have learned this observation from Leinster and Willerton [13] although it was certainly known before. An interesting question is, what categorical structure corresponds to the notion of 2-metric?

2. Asymmetric triangle inequality

Suppose X is a set together with a function $\varphi(x, y)$ defined for $x, y \in X$ such that:
(R)—$\varphi(x, x) = 0$;
(S)—$\varphi(x, y) = \varphi(y, x)$;
(AT)—for a constant $C \geq 1$ saying

$$\varphi(x, y) \leq \varphi(x, z) + C\varphi(z, y).$$

In this case we say that (X, φ) satisfies the *asymmetric triangle inequality.*

It follows that $\varphi(x, y) \geq 0$ for all $x, y \in X$. Furthermore, if we introduce a relation $x \sim y$ when $\varphi(x, y) = 0$, then the three axioms imply that this is an equivalence relation, and furthermore when $x \sim x'$ and $y \sim y'$ then $\varphi(x, y) = \varphi(x', y')$. Thus, φ descends to a function on the quotient $X/\sim$ and on the quotient it has the property that $\varphi(\overline{x}, \overline{y}) = 0 \Leftrightarrow \overline{x} = \overline{y}$. In view of this discussion it is sometimes reasonable to add the *strict reflexivity* axiom
(SR)—if $\varphi(x, y) = 0$ then $x = y$.

B. Rhoades pointed out to us that the asymmetric triangle inequality implies the property $\varphi(x, y) \leq \gamma(\varphi(x, z) + \varphi(z, y))$ of a *quasidistance* used by Beppo [4] and it seems likely that the following discussion could be a consequence of her fixed point result for (φ, i, j, k)-mappings, although that deduction doesn't seem immediate.

It is easy to see for (X, φ) satisfying the asymmetric triangle inequality, that the notion of limit for the distance function φ makes sense, similarly the notion of Cauchy sequence for φ makes sense, and we can say that (X, φ) is *complete* if every Cauchy sequence has a limit. If a sequence has a limit then it is Cauchy. The function φ is continuous, i.e., it transforms limits into limits. If furthermore the strictness axiom (SR) satisfied, then limit is unique.

A *point of accumulation* of a sequence $(x_i)_{i\in\mathbb{N}}$ is a limit of a subsequence, that is to say a point y such that there exists a subsequence $(x_{i(j)})_{j\in\mathbb{N}}$ with $i(j)$ increasing, such that $y = \lim_{j\to\infty} x_{i(j)}$. A set X provided with a distance function satisfying the asymmetric triangle inequality (i.e. (R), (S) and (AT)), is *compact* if every sequence has a point of accumulation. In other words, every sequence admits a convergent subsequence. This notion should perhaps be called "sequentially compact" but it is the only compactness notion which will be used in what follows.

Lemma 2.1. *Suppose (X, φ) satisfies the asymmetric triangle inequality with the constant C. Suppose $F : X \to X$ is a map such that $\varphi(Fx, Fy) \leq k\varphi(x, y)$ with $k < (1/C)$. Then for any $x \in X$ the sequence $\{F^i(x)\}$ is Cauchy. If (X, φ) is complete and strictly reflexive then its limit is the unique fixed point of F.*

Proof. Let x_0 be an arbitrary point in X and $\{x_n\}$ the sequence defined by $x_{n+1} = F(x_n) = F^n(x_0)$ for all positive integer n. We have

$$\varphi(x_{n+1}, x_n) = \varphi(Fx_n, Fx_{n-1}) \leq k\varphi(x_n, x_{n-1}).$$

By induction, we obtain

$$\varphi(x_{n+1}, x_n) \leq k^n \varphi(x_0, x_1)$$

Using the *asymmetric triangle inequality* several times we get for all positive integers n, m such that $m > n$

$$\varphi(x_n, x_m) \leq \varphi(x_n, x_{n+1}) + C\varphi(x_{n+1}, x_{n+2}) + C^2\varphi(x_{n+2}, x_{n+3}) + ...$$
$$... + C^{m-n-1}\varphi(x_{m-1}, x_m).$$

Then

$$\varphi(x_n, x_m) \leq \ k^n\varphi(x_0, x_1) + Ck^{n+1}\varphi(x_0, x_1) + C^2k^{n+2}\varphi(x_0, x_1) +$$
$$... + C^{m-n-1}k^{m-1}\varphi(x_0, x_1).$$

Therefore

$$\varphi(x_n, x_m) \leq \ (1 + Ck + C^2k^2 + ... + C^{m-n-1}k^{m-n-1})k^n\varphi(x_0, x_1)$$

and so

$$\varphi(x_n, x_m) < \frac{k^n}{1 - Ck}\varphi(x_0, x_1)$$

Hence, the sequence $\{x_n\}$ is Cauchy. Since (X, φ) is complete, it converges to some $x \in X$. Now, we show that z is a fixed point of F. Suppose not. Then

$$\varphi(Fz, Fx_n) \leq k\varphi(z, x_{n-1})$$

As n tends to infinity we get $z = Fz$ using (SR). The uniqueness of z follows easily. □

Corollary 2.2. *Suppose (X, φ) satisfies the asymmetric triangle inequality, is strictly reflexive and complete. If $F : X \to X$ is a map such that $\varphi(Fx, Fy) \leq k\varphi(x, y)$ with $k < 1$, then F has a unique fixed point.*

Proof. Since $k < 1$ there exists $a_0 \geq 1$ such that $k^a < (1/C)$ for any $a \geq a_0$. Then the previous lemma applies to F^a whenever $a \geq a = 0$, and F^a has a unique fixed point z_a. Choose $b \geq a_0$ and let z_b be the unique fixed point of F^b. Then

$$F^{ab}(z_b) = (F^b)^a(z_b) = z_b,$$

but also

$$F^{ab}(z_a) = (F^a)^b(z_a) = z_a.$$

Thus z_a and z_b are both fixed points of F^{ab}; as $ab \geq a_0$ its fixed point is unique so $z_a = z_b$. Apply this with $b = a + 1$, so

$$F(z_a) = F(F^a(z_a)) = F^b(z_a) = F^b(z_b) = z_b = z_a.$$

Thus z_a is a fixed point of F. If z is another fixed point of F then it is also a fixed point of F^a so $z = z_a$; this proves uniqueness. $\square$

2.1. **Triangle inequality with cost.** If $d(x,y,z)$ is a function of three variables, the "triangle inequality with cost" is

$$\varphi(x,y) \leq \varphi(x,z) + \varphi(z,y) + d(x,y,z). \tag{2.1}$$

This enters into Lemma 3.2 below.

We mention in passing a "triangle inequality with multiplicative cost": suppose given a function $\varphi(x,y)$ plus a function of 3 variables $\psi(x,y,z)$ such that

$$\varphi(x,y) \leq (\varphi(x,z) + \varphi(z,y))e^{\psi(x,y,z)}. \tag{2.2}$$

Assume also that φ is invariant under transposition, with $\varphi(x,y) = 0 \Leftrightarrow x = y$ and that ψ is bounded above and below. We can define limits and Cauchy sequences, hence completeness and the function φ is continuous. The following fixed point statement is not used elsewhere but seems interesting on its own.

Proposition 2.3. *Suppose given φ, ψ satisfying the triangle inequality with multiplicative cost* (2.2) *as above. If F is a map such that*

$$\varphi(F(x),F(y)) \leq k\varphi(x,y) \text{ and}$$
$$\psi(F(x),F(y),F(z)) \leq k\psi(x,y,z)$$

whenever both sides are positive, then we get a Cauchy sequence $F^k(x)$. If (X,φ) is complete then the limit of this Cauchy sequence is the unique fixed point of F.

Proof. Let x_0 be an arbitrary point in X and $\{x_n\}$ the sequence defined by $x_{n+1} = F(x_n) = F^n(x_0)$ for all positive integer n. We have

$$\begin{aligned}
&\varphi(x_n,x_m) \leq (\varphi(x_n,x_{n+1}) + \varphi(x_{n+1},x_m))e^{\psi(x_n,x_m,x_{n+1})}\\
&\leq k^n\varphi(x_0,x_1)e^{\psi(x_n,x_m,x_{n+1})} + \varphi(x_{n+1},x_m)e^{\psi(x_n,x_m,x_{n+1})}\\
&\leq k^n\varphi(x_0,x_1)e^{\psi(x_n,x_m,x_{n+1})} +\\
&(\varphi(x_{n+1},x_{n+2}) + \varphi(x_{n+2},x_m))e^{\psi(x_{n+1},x_m,x_{n+2})+\psi(x_n,x_m,x_{n+1})}\\
&\leq k^n\varphi(x_0,x_1)e^{\psi(x_n,x_m,x_{n+1})} +\\
&k^{n+1}\varphi(x_0,x_1)e^{\psi(x_{n+1},x_m,x_{n+2})+\psi(x_n,x_m,x_{n+1})} +\\
&\varphi(x_{n+2},x_m)e^{\psi(x_{n+1},x_m,x_{n+2})+\psi(x_n,x_m,x_{n+1})}\\
&\leq k^n\varphi(x_0,x_1)e^{\psi(x_n,x_m,x_{n+1})} +\\
&k^{n+1}\varphi(x_0,x_1)e^{\psi(x_{n+1},x_m,x_{n+2})+\psi(x_n,x_m,x_{n+1})} +\\
&k^{n+2}\varphi(x_0,x_1)e^{\psi(x_{n+2},x_m,x_{n+3})+\psi(x_{n+1},x_m,x_{n+2})+\psi(x_n,x_m,x_{n+1})} +\\
&\varphi(x_{n+3},x_m)e^{\psi(x_{n+2},x_m,x_{n+3})+\psi(x_{n+1},x_m,x_{n+2})+\psi(x_n,x_m,x_{n+1})}
\end{aligned}$$

$\leq k^n\varphi(x_0, x_1)e^{\psi(x_n,x_m,x_{n+1})}+$
$k^{n+1}\varphi(x_0, x_1)e^{\psi(x_{n+1},x_m,x_{n+2})+\psi(x_n,x_m,x_{n+1})}+$
$k^{n+2}\varphi(x_0, x_1)e^{\psi(x_{n+2},x_m,x_{n+3})+\psi(x_{n+1},x_m,x_{n+2})+\psi(x_n,x_m,x_{n+1})} + ...+$
$k^{m-1}\varphi(x_0, x_1)e^{\psi(x_n,x_m,x_{n+1})+\psi(x_{n+1},x_m,x_{n+2})+...+\psi(x_{m-2},x_m,x_{m-1})}$
$\leq k^n\varphi(x_0, x_1)(e^{k^n\psi(x_0,x_p,x_1)} + ke^{k^n\psi(x_0,x_p,x_1)+k^{n+1}\psi(x_0,x_p,x_1)}+$
$k^2e^{k^n\psi(x_0,x_p,x_1)+k^{n+1}\psi(x_0,x_p,x_1)+k^{n+2}\psi(x_0,x_p,x_1)}+$
$\ldots+$
$k^{m-n-1}e^{k^n\psi(x_0,x_p,x_1)+k^{n+1}\psi(x_0,x_p,x_1)+...+k^{m-1}\psi(x_0,x_p,x_1)})$
$\leq k^n\varphi(x_0, x_1)(e^{Mk^n} + ke^{M(k^n+k^{n+1})} + k^2e^{M(k^n+k^{n+1}+k^{n+2})}+$
$\ldots+$
$k^{m-n-1}e^{M(k^n+k^{n+1}+k^{n+2}+...+k^{m-1})})$

since ψ is bounded. Hence, the sequence $\{x_n\}$ is Cauchy. Since (X, φ) is complete, it converges to some $x \in X$. The rest of the proof follows as in Lemma 2.1. □

3. Bounded 2-metric spaces

Gähler defined the notion of 2-metric space to be a set X with function $d : X^3 \to \mathbb{R}$ denoted $(x, y, z) \mapsto d(x, y, z)$ satisfying the following axioms [7] [8] [9]:
(Sym)—that $d(x, y, z)$ is invariant under permutations of the variables x, y, z.
(Tetr)—for all a, b, c, x we have

$$d(a, b, c) \leq d(a, b, x) + d(b, c, x) + d(a, c, x).$$

(Z)—for all a, b we have $d(a, b, b) = 0$.
(N)—for all a, b there exists c such that $d(a, b, c) \neq 0$.

One can think of $d(x, y, z)$ as measuring how far are x, y, z from being "aligned" or "colinear".

The 2-metric spaces (X, d) have been the subject of much study, see [1] and [2] for example. The prototypical example of a 2-metric space is obtained by setting $d(x, y, z)$ equal to the area of the triangle spanned by x, y, z.

Assume that the 2-metric is *bounded*, and by rescaling the bound can be supposed equal to 1:
(B)—the function is bounded by $d(x, y, z) \leq 1$ for all $x, y, z \in X$.

Define the *associated distance* by

$$\varphi(x, y) := \sup_{z\in X} d(x, y, z).$$

Lemma 3.1. *We have $d(x, y, z) \geq 0$ and hence $\varphi(x, y) \geq 0$. Also $\varphi(x, x) = 0$ and $\varphi(x, y) = \varphi(y, x)$.*

Proof. Applying the axiom (Tetr) with $b = c$, we get

$$d(a, b, b) \leq d(a, b, x) + d(b, a, x) + d(a, b, x).$$

By the axiom (Z) and the symmetry of d we obtain $d(a, b, x) \geq 0$ and so $d(x, y, z) \geq 0$. Then, $\varphi(x, y) \geq 0$. Symmetry of φ follows from invariance of d under permutations (Sym). □

Lemma 3.2. *We have the triangle inequality with cost* (2.1)

$$\varphi(x, y) \leq \varphi(x, z) + \varphi(z, y) + d(x, y, z).$$

Therefore

$$\varphi(x, y) \leq \varphi(x, z) + \varphi(z, y) + \min(\varphi(x, z), \varphi(z, y))$$

and hence the asymmetric triangle inequality (AT)

$$\varphi(x, y) \leq \varphi(x, z) + 2\varphi(z, y).$$

Proof. We have

$$d(x, y, z_0) \leq d(x, y, z) + d(y, z_0, z) + d(x, z_0, z)$$

$$\leq \varphi(x, z) + \varphi(z, y) + d(x, y, z).$$

For the next statement, note that by definition

$$d(x, y, z) \leq \min(\varphi(x, z), \varphi(z, y)),$$

and for the last statement, $\min(\varphi(x, z), \varphi(z, y)) \leq \varphi(z, y)$. □

In particular the distance φ satisfies the axioms (R), (S) and (AT) of Section 2. This allows us to speak of limits, Cauchy sequences, points of accumulation, completeness and compactness, see also [4]. For clarity it will usually be specified that these notions concern the function φ. Axiom (N) for d is equivalent to strict reflexivity (SR) for φ; if this is not assumed from the start, it can be fixed as follows.

3.1. **Nondegeneracy.** It is possible to start without supposing the nondegeneracy axiom (N), define an equivalence relation, and obtain a 2-metric on the quotient satisfying (N). For the next lemma and its corollary, we assume that d satisfies all of (Sym), (Tetr), (Z), (B), but not necessarily (N).

Lemma 3.3. *If a, b, x, y are any points then*

$$|d(a, b, x) - d(a, b, y)| \leq 2\varphi(x, y).$$

Proof. By condition (Tetr),

$$d(a,b,y) \leq d(a,b,x)+d(b,y,x)+d(a,y,x) \leq d(a,b,x)+2\varphi(x,y).$$

The same in the other direction gives the required estimate. □

Corollary 3.4. *If x, y are two points with $\varphi(x,y) = 0$ then for any a, b we have $d(a,b,x) = d(a,b,y)$. Therefore, if $\sim$ is the equivalence relation considered in the second paragraph of Section 2, the function d descends to a function $(X/\sim)^3 \to \mathbb{R}$ satisfying the same properties but in addition its associated distance function is strictly reflexive and d satisfies (N).*

Proof. For the first statement, apply the previous lemma. This invariance applies in each of the three arguments since d is invariant under permutations, which in turn yields the descent of d to a function on $(X/\sim)^3$. The associated distance function is the descent of φ which is strictly reflexive. □

In view of this lemma, we shall henceforth assume that φ satisfies (SR) or equivalently d satisfies (N) too. In particular the limit of a sequence is unique if it exists.

3.2. **Surjective mappings.** If F is surjective, then a boundedness condition for d implies the same for φ. Since we are assuming that d is globally bounded (condition (B)), a surjective mapping cannot be strictly contractive:

Lemma 3.5. *Suppose $F : X \to X$ is a map such that*

$$d(F(x), F(y), F(z)) \leq kd(x,y,z)$$

for some constant $k > 0$. If F is surjective then $\varphi(F(x), F(y)) \leq k\varphi(x,y)$. The global boundedness condition implies that $k \geq 1$ in this case.

Proof. Suppose $x, y \in X$. For any $z \in X$, choose a preimage $w \in X$ such that $F(w) = z$ by surjectivity of F. Then

$$d(F(x), F(y), z) = d(F(x), F(y), F(w)) \leq kd(x,y,w) \leq k\varphi(x,y).$$

It follows that

$$\varphi(F(x), F(y)) = \sup_{z \in X} d(F(x), F(y), z) \leq k\varphi(x,y).$$

Suppose now that $k < 1$. Let B be the supremum of $\varphi(x,y)$ for $x, y \in X$. Then $0 < B < 1$ by conditions (N) and (B). Therefore there exist x, y such that $kB < \varphi(x,y)$, but this contradicts the existence of u and v such that $F(u) = x$ and $F(v) = y$. This shows that $k \geq 1$. □

4. Colinearity

Consider a bounded 2-metric space (X, d) , that is to say satisfying axioms (Sym), (Tetr), (Z), (N) and (B), and require the following additional *transitivity axiom*:
(Trans)—for all a, b, c, x, y we have

$$d(a, b, x)d(c, x, y) \le d(a, x, y) + d(b, x, y).$$

In Section 5 below we will see that the standard area function, as well as a form of geodesic area function on $\mathbb{RP}^2$, satisfy this additional axiom. The terminology "transitivity" comes from the fact that this condition implies a transitivity property of the relation of colinearity, see Lemma 4.2 below.

The term $d(c, x, y)$ may be replaced by its sup over c which is $\varphi(x, y)$. If we think of $d(a, b, x)$ as being a family of distance-like functions of a and b, indexed by $x \in X$, (Trans) can be rewritten

$$d(a, b, x) \le (d(a, y, x) + d(y, b, x))\varphi(x, y)^{-1}$$

for $y \ne x$. This formulation may be related to the notion of "triangle inequality with multiplicative cost" (2.2) discussed in Section 2.1.

Definition 4.1. *Say that* (x, y, z) *are* colinear *if* $d(x, y, z) = 0$*. A* line *is a maximal subset* $Y \subset X$ *consisting of colinear points, that is to say satisfying*

$$\forall x, y, z \in Y, \quad d(x, y, z) = 0. \tag{4.1}$$

The colinearity condition is symmetric under permutations by (Sym).

Lemma 4.2. *Using all of the above axioms including (N) and assumption (Trans), colinearity satisfies the following transitivity property: if* x, y, z *are colinear,* y, z, w *are colinear, and* $y \ne z$*, then* x, y, w *and* x, z, w *are colinear.*

Proof. By (N), $\varphi(y, z) \ne 0$, then use the above version of (Trans) rewritten after some permutations as

$$d(x, y, w) \le (d(x, y, z) + d(y, z, w))\varphi(y, z)^{-1}.$$

This shows that x, y, w are colinear. Symmetrically, the same for x, z, w. □

A line is nonempty, by maximality since $d(y, y, y) = 0$ by (Z).

Lemma 4.3. *If* $x \ne y$ *are two points then there is a unique line* Y *containing* x *and* y*, and* Y *is the set of points* a *colinear with* x *and* y*, i.e. such that* $d(a, x, y) = 0$*.*

Proof. The set $\{x, y\}$ satisfies Condition (4.1), so there is at least one maximal such set Y containing x and y. Choose one such Y. If $a \in Y$ then automatically $d(a, x, y) = 0$.

Suppose $d(a, x, y) = 0$. By Lemma 4.2, $d(a, x, u) = 0$ for any $u \in Y$.

Now suppose $u, v \in Y$. If $u = x$ then the preceding shows that $d(a, u, v) = 0$. If $u \neq x$ then, since $d(x, u, v) = 0$ and $d(a, x, u) = 0$, Lemma 4.2 implies that $d(a, u, v) = 0$. This shows that a is colinear with any two points of Y. In particular, $Y \cup \{a\}$ also satisfies Condition (4.1) so by maximality, $a \in Y$. This shows that Y is the set of points a such that $d(a, x, y) = 0$, which characterizes it uniquely. □

The notions of colinearity and lines come from the geometric examples of 2-metrics which will be discussed in Section 5 below. It should be pointed out that there can be interesting examples of 2-metric spaces which don't satisfy the transitivity condition of Lemma 4.2 and which therefore don't satisfy Axiom (Trans). The remainder of our discussion doesn't apply to such examples.

We assume Axiom (Trans) from now on. It allows us to look at the question of fixed subsets of a contractive mapping F when F is not surjective. In addition to the possibility of having a fixed point, there will also be the possibility of having a fixed line. We see in examples below that this can happen.

Definition 4.4. *Consider a sequence of points $x_i \in X$. The property $LIM(y, (x_i))$ is defined to mean:*

$$\forall \epsilon > 0 \ \exists a_\epsilon, \ \forall i, j \geq a_\epsilon, \ d(y, x_i, x_j) < \epsilon.$$

Suppose $LIM(y, (x_i))$ and $LIM(y', (x_i))$. We would like to show that $d(y, y', x_i) \to 0$. However, this is not necessarily true: if (x_i) is Cauchy then the properties LIM are automatic (see Proposition 4.9 below). So, we need to include the hypothesis that our sequence is not Cauchy, in the following statements.

Lemma 4.5. *Suppose (x_i) is not Cauchy. If both $LIM(y, (x_i))$ and $LIM(y', (x_i))$ hold, then $d(y, y', x_i) \to 0$ as $i \to \infty$.*

Proof. The sequence (x_i) is supposed not to be Cauchy for φ, so there exists $\epsilon_0 > 0$ such that for any $m \geq 0$ there are $i, j \geq m$ with $\varphi(x_i, x_j) \geq \epsilon_0$. Therefore, in view of the definition of φ, for any m there exist $i(m), j(m) \geq m$ and a point $z(m) \in X$ such that $d(x_{i(m)}, x_{j(m)}, z(m)) \geq \epsilon_0/2$.

We now use condition (Trans) with $x = x_{i(m)}$ and $y = x_{j(m)}$ and $c = z(m)$, for $a = y$ and $b = y'$. This says

$$d(y, y', x_{i(m)})\epsilon_0/2 \leq d(y, x_{i(m)}, x_{j(m)}) + d(y', x_{i(m)}, x_{j(m)}).$$

If $LIM(y,(x_i))$, $LIM(y',(x_i))$, then for any ϵ we can assume m is big enough so that

$$d(y, x_{i(m)}, x_{j(m)}) \leq \epsilon\epsilon_0/4$$

and

$$d(y', x_{i(m)}, x_{j(m)}) \leq \epsilon\epsilon_0/4.$$

Putting these together gives $d(y, y', x_{i(m)}) \leq \epsilon$.

Choose m so that for all $j, k \geq m$ we have $d(y, x_j, x_k) \leq \epsilon$ and the same for y'. Then we have by (Tetr), for any $j \geq m$

$$d(y', y, x_j) \leq d(x_{i(m)}, y, x_j) + d(y', x_{i(m)}, x_j) + d(y', y, x_{i(m)}) \leq 3\epsilon.$$

Changing ϵ by a factor of three, we obtain the following statment: for any $\epsilon > 0$ there exists m such that for all $i \geq m$ we have $d(y', y, x_i) \leq \epsilon$. This is the required convergence. □

Corollary 4.6. *If the sequence (x_i) is not Cauchy for the distance φ, then the following property holds:*
—if $LIM(y,(x_i))$, $LIM(y',(x_i))$, and $LIM(y'',(x_i))$ then (y, y', y'') are colinear.

Proof. We use the fact that

$$d(y, y', y'') \leq d(y, y', x_i) + d(y', y'', x_i) + d(y, y'', x_i).$$

By Lemma 4.5, all three terms on the right approach 0 as $i \to \infty$. This proves that $d(y, y', y'') = 0$. □

Lemma 4.7. *Suppose that (x_i) is not Cauchy for the distance φ. If $LIM(y,(x_i))$, $LIM(y',(x_i))$, $\varphi(y, y') > 0$ and y'' is a point such that (y, y', y'') are colinear, then also $LIM(y'',(x_i))$.*

Proof. By Lemma 4.5, for any $\epsilon > 0$ there exists m such that for all $i \geq m$ we have $d(y', y, x_i) \leq \epsilon$.

Let u, v denote some x_i or x_j. By hypothesis $\varphi(y, y') > 0$ so there is a point z such that $d(y, y', z) = \epsilon_1 > 0$. Then condition (Trans) applied with $a = y'', b = u, x = y', c = z, y = y$ gives

$$d(y'', u, y')d(z, y', y) \leq d(y'', y', y) + d(u, y', y).$$

Hence

$$d(y'', u, y') \leq d(u, y', y)/\epsilon_1.$$

We can do the same for v, and also interchanging y and y', to get

$$d(y'', v, y') \leq d(v, y', y)/\epsilon_1,$$
$$d(y'', u, y) \leq d(u, y, y')/\epsilon_1,$$
$$d(y'', v, y) \leq d(v, y, y')/\epsilon_1.$$

We have

$$d(y'', u, v) \leq d(y, u, v) + d(y'', y, v) + d(y'', u, y)$$
$$\leq d(y, u, v) + (d(u, y, y') + d(v, y, y'))/\epsilon_1.$$

For $u = x_i$ and $v = x_j$ with $i, j \geq m$ as previously we get

$$d(y'', x_i, x_j) \leq d(y, x_i, x_j) + (d(x_i, y, y') + d(x_j, y, y'))/\epsilon_1.$$

By choosing m big enough this can be made arbitrarily small, thus giving the condition $LIM(y'', (x_i))$. □

Theorem 4.8. *Suppose that (X, d) is a bounded 2-metric space satisfying axiom (Trans) as above. Suppose (x_i) is a sequence. Then there are the following possibilities (not necessarily exclusive):*
—there is no point y with $LIM(y, (x_i))$;
—there is exactly one point y with $LIM(y, (x_i))$;
—the sequence (x_i) is Cauchy for the distance φ; or
—the subset $Y \subset X$ of points y such that $LIM(y, (x_i))$, is a line.

Proof. Consider the subset $Y \subset X$ of points y such that $LIM(y, (x_i))$ holds. We may assume that there are two distinct points $y_1 \neq y_2$ in Y, for otherwise one of the first two possibilities would hold. Suppose that (x_i) is not Cauchy for φ; in particular, Lemmas 4.5, 4.7 and Corollary 4.6 apply.

If y, y', y'' are any three points in Y, then by Corollary 4.6, they are colinear. Thus Y is a subset satisfying Condition (4.1) in the definition of a line; to show that it is a line, we have to show that it is a maximal such subset.

Suppose $Y \subset Y_1$ and Y_1 also satisfies (4.1). Since $y_1 \neq y_2$, and we are assuming that φ satisfies strict reflexivity (SR), we have $\varphi(y_1, y_2) \neq 0$. By Lemma 3.1, $\varphi(y_1, y_2) > 0$. If $y \in Y_1$ then by (4.1), $d(y, y_1, y_2) = 0$. By Lemma 4.7, y must also satisfy $LIM(y, (x_i))$, thus $y \in Y$. This shows that $Y_1 \subset Y$, giving maximality of Y. Thus, Y is a line. □

The following proposition shows that the case when (x_i) Cauchy has to be included in the statement of the theorem.

Proposition 4.9. *If (x_i), (y_j) and (z_k) are Cauchy sequences, then the sequence $d(x_i, y_j, z_k)$ is Cauchy in the sense that for any $\epsilon > 0$ there exists M such that for $i, j, k, p, q, r \geq M$ then $|d(x_i, y_j, z_k) - d(x_p, y_q, z_r)| < \epsilon$. In particular d is continuous. If (x_i) is Cauchy then $LIM(y, (x_i))$ holds for any point $y \in X$.*

Proof. For given ϵ, by the Cauchy condition there is M such that for $i, j, k, p, q, r \geq M$ we have $\varphi(x_i, x_p) < \epsilon/6$, $\varphi(y_j, y_q) < \epsilon/6$, and

$\varphi(z_k, z_r) < \epsilon/6$. Then by Lemma 3.3

$$|d(x_i, y_j, z_k) - d(x_i, y_j, z_r)| \leq \epsilon/3,$$

$$|d(x_i, y_j, z_r) - d(x_i, y_q, z_r)| \leq \epsilon/3,$$

and

$$|d(x_i, q, z_r) - d(x_p, y_q, z_r)| \leq \epsilon/3.$$

These give the Cauchy property

$$|d(x_i, y_j, z_k) - d(x_p, y_q, z_r)| \leq \epsilon.$$

This shows in particular that d is continuous. Suppose (x_i) is Cauchy and y is any point. Then the sequence $d(y, x_i, x_j)$ is Cauchy in the above sense in the two variables i, j, which gives exactly the condition $LIM(y, (x_i))$. □

We say that a sequence (x_i) is *tri-Cauchy* if

$$\forall \epsilon > 0, \ \ \exists m_\epsilon, \ \ i, j, k \geq m_\epsilon \Rightarrow d(x_i, x_j, x_k) < \epsilon.$$

Lemma 4.10. *Suppose (x_i) is a tri-Cauchy sequence, and $y \in X$ is an accumulation point of the sequence with respect to the distance φ. Then $LIM(y, (x_i))$.*

Proof. The condition that y is an accumulation point means that there exists a subsequence $(x_{u(k)})$ such that $(x_{u(k)}) \to y$ with respect to the distance φ. We have by (Tetr)

$$\begin{aligned} d(y, x_i, x_j) &\leq d(x_{u(k)}, x_i, x_j) + d(y, x_{u(k)}, x_j) + d(y, x_i, x_{u(k)}) \\ &\leq d(x_{u(k)}, x_i, x_j) + 2\varphi(y, x_{u(k)}), \end{aligned}$$

and both terms on the right become small, for i, j big in the original sequence and k big in the subsequence. Hence $d(y, x_i, x_j) \to 0$ as $i, j \gg 0$, which is exactly the condition $LIM(y, (x_i))$. □

We say that (X, d) is *tri-complete* if, for any tri-Cauchy sequence, the set Y of points satisfying $LIM(y, (x_i))$ is nonempty. By Theorem 4.8, Y is either a single point, a line, or (in case (x_i) is Cauchy) all of X.

Lemma 4.11. *Suppose (X, φ) is compact. Then it is tri-complete, and for any tri-Cauchy sequence (x_i) we have one of the following two possibilities:*
—(x_i) has a limit; or
—the subset Y of points y with $LIM(y, (x_i))$ is a line.

Proof. Suppose (x_i) is a tri-Cauchy sequence. By compactness there is at least one point of accumulation, so the set Y of points y with $LIM(y,(x_i))$ is nonempty by Lemma 4.10. This rules out the first possibility of Theorem 4.8.

Suppose Y consists of a single point y. We claim then that $x_i \to y$. Suppose not: then there is a subsequence which doesn't contain y in its closure, but since X is compact after going to a further subsequence we may assume that the subsequence has a limit point $y' \neq y$. But again by Lemma 4.10, we would have $LIM(y',(x_i))$, a contradiction. So in this case, the sequence (x_i) is Cauchy for φ and has y as its limit; thus we are also in the situation of the third possibility. Note however that, since (x_i) is Cauchy, the set of points Y consists of all of X by Proposition 4.9, so the second possibility doesn't occur unless X is a singleton.

From Theorem 4.8 the remaining cases are that (x_i) is Cauchy, in which case it has a limit by compactness; or that Y is a line. □

5. Some examples

The classic example of a 2-metric is the function on $\mathbb{R}^2$ defined by setting $d(x,y,z)$ to be the area of the triangle spanned by x,y,z. Before discussing this example we look at a related example on $\mathbb{RP}^2$ which is also very canonical.

5.1. **A projective area function.** Let $X := S^2 := \{(x_1,x_2,x_3) \in \mathbb{R}^3,\ \ x_1^2+x_2^2+x_3^2 = 1\}$. Define the function $d(x,y,z)$ by taking the absolute value of the determinant of the matrix containing x,y,z as column vectors:

$$d\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) := \left|\det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}\right|.$$

This has appeared in Example 2.2 of [15].

Proposition 5.1. *This function satisfies axioms (Sym), (Tetr), (Z), (B), (Trans).*

Proof. Invariance under permutations (Sym) comes from the corresponding fact for determinants. Condition (Z) comes from vanishing of a determinant with two columns which are the same. Condition (B) comes from the fact that the determinant of a matrix whose columns have norm 1, is in $[-1,1]$. We have to verify (Tetr) and (Trans).

For condition (Tetr), suppose given vectors $x,y,z \in S^2$ as above, and suppose that $d(x,y,z) > 0$ i.e. they are linearly independent. Suppose given another vector $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ too. Then the determinants $d(a, y, z)$, $d(x, a, y)$ and $d(x, y, a)$ are the absolute values of the numerators appearing in Cramer's rule. This means that if we write

$$a = \alpha x + \beta y + \gamma z$$

then

$$\frac{d(a, y, z)}{d(x, y, z)} = |\alpha|,$$

$$\frac{d(x, a, z)}{d(x, y, z)} = |\beta|,$$

and

$$\frac{d(x, y, a)}{d(x, y, z)} = |\gamma|.$$

Now by the triangle inequality in $\mathbb{R}^3$ we have

$$1 = \|a\| \leq |\alpha| + |\beta| + |\gamma|$$

which gives exactly the relation (Tetr).

To prove (Trans), notice that it is invariant under orthogonal transformations of $\mathbb{R}^3$ so we may assume that

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}, \quad u^2 + v^2 = 1.$$

In this case, $\sup_{c \in S^2} d(c, x, y) = |v|$, so we are reduced to considering

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Now $d(a, x, y) = |a_3 v|$ and $d(b, x, y) = |b_3 v|$, so we have to show that

$$d(a, b, x)|v| \leq |a_3 v| + |b_3 v|.$$

But $d(a, b, x) = |a_2 b_3 - a_3 b_2|$ so this inequality is true (since $|a_2| \leq 1$ and $|b_2| \leq 1$). This completes the proof of (Trans). □

Corollary 5.2. *If $X \subset S^2$ then with the same function $d(x, y, z)$, it still satisfies the axioms.*

□

The "lines" for the function d defined above, are the great circles or geodesics on S^2.

Remark 5.3. *The distance function φ is not strictly reflexive in this example, indeed the associated equivalence relation identifies antipodal points. The quotient $S^2/\sim$ is the real projective plane. The corresponding function on $\mathbb{RP}^2$ is a bounded 2-metric satisfying (Trans).*

5.2. **The Euclidean area function.** We next go back and consider the standard area function on Euclidean space, which we denote by α.

Proposition 5.4. *Let $U \subset \mathbb{R}^n$ be a ball of diameter ≤ 1. Let $\alpha(x, y, z)$ be the area of the triangle spanned by $x, y, z \in U$. Then α satisfies axioms (Sym), (Tetr), (Z), (N), (B), (Trans).*

Proof. Conditions (Sym), (Tetr), (Z), (N) are classical. Condition (B) comes from the bound on the size of the ball. It remains to show Condition (Trans). By invariance under orthogonal transformations, we may assume that $x = (0, \ldots, 0)$ and $y = (y_1, 0, \ldots, 0)$ with $y_1 > 0$. Again by the bound on the size of the ball, for any c we have $\alpha(x, y, c) \leq y_1$. Thus, we need to show

$$\alpha(x, a, b)y_1 \leq \alpha(x, y, a) + \alpha(x, y, b).$$

Write $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. Then

$$\alpha(x, y, a) = y_1\|\tilde{a}\|$$

where $\tilde{a} = (0, a_2, \ldots, a_n)$ is the projection of a on the plane orthogonal to (xy). Similarly $d(x, y, b) = y_1\|\tilde{b}\|$. To complete the proof it suffices to show that $\alpha(x, a, b) \leq \|\tilde{a}\| + \|\tilde{b}\|$.

Write $a = a_1e_1 + \tilde{a}$ and $b = b_1e_1 + \tilde{b}$ where e_1 is the first basis vector. Then

$$\begin{aligned}\alpha(x, a, b) &= \frac{1}{2}\|a \wedge b\| \\ &= \frac{1}{2}\|a_1e_1 \wedge \tilde{b} - b_1e_1 \wedge \tilde{a} + \tilde{a} \wedge \tilde{b}\| \\ &\leq \|\tilde{a}\| + \|\tilde{b}\|\end{aligned}$$

since $a_1 \leq 1$, $b_1 \leq 1$, $\|\tilde{a}\| \leq 1$, and $\|\tilde{b}\| \leq 1$. □

5.3. **Euclidean submanifolds.** If $U \subset \mathbb{R}^n$ is any subset contained in a ball of diameter 1 but not in a single line, then the function $d(x, y, z)$ on U^3 induced by the standard Euclidean area function on $\mathbb{R}^n$ is again a bounded isometric 2-metric. This is interesting in case U is a patch inside a submanifold. The induced 2-metric will have different properties depending on the curvature of the submanifold. We consider $U \subset S^2 \subset \mathbb{R}^3$ as an example.

Choose a patch U on S^2, contained in the neighborhood of the south pole of radius $r < 1/4$. Let $p : U \stackrel{\cong}{\to} V \subset \mathbb{R}^2$ be the vertical projection. Let $h(x,y,z) := \alpha(p^{-1}(x), p^{-1}(y), p^{-1}(z))$ be the 2-metric on V induced from the standard Euclidean area function α of $S^2 \subset \mathbb{R}^3$ via the projection p. This satisfies upper and lower bounds which mirror the convexity of the piece of surface U:

Lemma 5.5. *Keep the above notations for $h(x,y,z)$. Let α_2 denote the standard area function on $V \subset \mathbb{R}^2$. Define a function*

$$\rho(x,y,z) := \|x-y\| \cdot \|x-z\| \cdot \|y-z\|.$$

Then there is a constant $C > 1$ such that

$$(5.1) \qquad \begin{array}{rcl} \frac{1}{C}\left(\alpha_2(x,y,z) + \rho(x,y,z)\right) & \leq & h(x,y,z) \\ h(x,y,z) & \leq & C\left(\alpha_2(x,y,z) + \rho(x,y,z)\right) \end{array}$$

for $x,y,z \in V$.

Proof. If any two points coincide the estimate holds, so we may assume the three points x,y,z are distinct, lying on a unique plane P. In particular, they lie on a circle $S^2 \cap P$ of radius ≤ 1. A simple calculation on the circle gives a bound of the form $h(x,y,z) \geq (1/C)\rho(x,y,z)$. The vertical projection from P to $\mathbb{R}^2$ has Jacobian determinant ≤ 1 so $h(x,y,z) \geq \alpha_2(x,y,z)$. Together these give the lower bound.

For the upper bound, consider the unit normal vector to P and let u_3 be the vertical component. If $|u_3| \geq 1/4$ then the plane is somewhat horizontal and $h(x,y,z) \leq C\alpha_2(x,y,z)$. If $|u_3| \leq 1/4$ then the plane is almost vertical, and in view of the fact that the patch is near the south pole, the intersection $P \cap S^2$ is a circle of radius $\geq 1/4$. In this case $h(x,y,z) \leq C\rho(x,y,z)$. From these two cases we get the required upper bound. □

6. Fixed points of a map

Suppose that (X, φ) is *compact* meaning that every sequence has a convergent subsequence. Suppose $F : X \to X$ is a d-decreasing map i.e. one with $d(F(x), F(y), F(z)) \leq kd(x,y,z)$ for $0 < k < 1$.

Pick a point $x_0 \in X$ and define the *sequence of iterates* inductively by $x_{i+1} := F(x_i)$.

Corollary 6.1. *Suppose (X, φ) is compact and F is a d-decreasing map. Pick a point x_0 and define the sequence of iterates (x_i) with $x_{i+1} = F(x_i)$. This sequence is tri-Cauchy; hence either it is Cauchy with a unique limit point $y \in X$, or else the subset $Y \subset X$ of points y with $LIM(y, (x_i))$ is a line.*

Proof. Note first that the sequence (x_i) is tri-Cauchy. If $m \leq i, j, k$ then

$$x_i = F^m(x_{i-m}), \quad x_j = F^m(x_{j-m}), \quad x_k = F^m(x_{k-m}).$$

Hence using the global bound (B),

$$d(x_i, x_j, x_k) \leq k^m d(x_{i-m}, x_{j-m}, x_{k-m}) \leq k^m.$$

As $0 < k < 1$, for any ϵ there exists m such that $k^m < \epsilon$; this gives the tri-Cauchy property of the sequence of iterates.

Then by Lemma 4.11, either $(x_i) \to y$ or else the set Y of points y with $LIM(y, (x_i))$ is a line. □

Theorem 6.2. *Suppose X is nonempty and d is a bounded transitive 2-metric (i.e. one satisfying (B) and (Trans)), such that (X, φ) is compact. Suppose F is a d-decreasing map for a constant $0 < k < 1$. Then, either F has a fixed point, or there is a line Y fixed in the sense that $F(Y) \subset Y$ and Y is the only line containing $F(Y)$.*

Proof. Pick $x_0 \in X$ and define the sequence of iterates (x_i) as above. By the previous corollary, either $x_i \to y$ or else the set Y of points y with $LIM(y, (x_i))$ is a line.

Suppose we are in the second possibility but not the first; thus (x_i) has more than one accumulation point. Suppose y is a point with property $LIM(y, (x_i))$, which means that $d(y, x_i, x_j) < \epsilon$ for $i, j \geq m_\epsilon$. Hence if $i, j \geq m_\epsilon + 1$ then

$$d(F(y), x_i, x_j) = d(F(y), F(x_{i-1}), F(x_{j-1})) < k\epsilon.$$

This shows the property $LIM(F(y), (x_i))$. Thus, $F(Y) \subset Y$.

Suppose Y_1 is a line with $F(Y) \subset Y_1$. If $F(Y)$ contains at least two distinct points then there is at most one line containing $F(Y)$ by Lemma 4.3 and we obtain the second conclusion of the theorem. Suppose $F(Y) = \{y_0\}$ consists of a single point. Then, $y_0 \in Y$ so $F(y_0) \in F(Y)$, which shows that $F(y_0) = y_0$; in this case F admits a fixed point.

We may now assume that we are in the first case of the first paragraph: $x_i \to y$. If $F(y) = y$ we have a fixed point, so assume[1] $F(y) \neq y$. Let Y be the unique line containing y and $F(y)$ by Lemma 4.3 which also says Y is the set of points colinear with y and $F(y)$.

We claim that $F(X) \subset Y$. If $z \in X$ then $LIM(z, (x_i))$ by the last part of Proposition 4.9. For a given ϵ there is m_ϵ such that $i, j \geq m_\epsilon \Rightarrow$

[1]We need to consider this case: as F is not assumed to be surjective, it is not necessarily continuous for φ so the convergence of the sequence of iterates towards y doesn't directly imply that y is a fixed point.

$d(x_i, x_j, z) < \epsilon$. However, for i fixed we have $x_j \to y$ by hypothesis, and the continuity of d (Proposition 4.9) implies that $d(x_i, y, z) < \epsilon$. Apply F, giving

$$\forall i \geq m_\epsilon, \quad d(x_{i+1}, F(y), F(z)) < k\epsilon.$$

Again using continuity of d, we have

$$d(x_{i+1}, F(y), F(z)) \to d(y, F(y), F(z))$$

and the above then implies that $d(y, F(y), F(z)) < k\epsilon$ for any $\epsilon > 0$. Hence $d(y, F(y), F(z)) = 0$ which means $F(z) \in Y$. This proves that $F(X) \subset Y$ *a fortiori* $F(Y) \subset Y$. If $F(F(y))$ is distinct from $F(y)$ then Y is the unique line containing $F(Y)$, otherwise $F(y)$ is a fixed point of F. This completes the proof of the theorem. □

Corollary 6.3. *Suppose $X \subset S^2$ is a closed subset. Define the function $d(x, y, z)$ by a determinant as in Proposition 5.1. If $F : X \to X$ is a function such that $d(F(x), F(y), F(z)) \leq kd(x, y, z)$ for $0 < k < 1$ then either it has a fixed pair of antipodal points, or a fixed great circle.*

Proof. Recall that we should really be working with the image of X in the real projective space $\mathbb{RP}^2 = S^2/\sim$ (Remark 5.3). On this quotient, the previous theorem applies. Note that by a "fixed great circle" we mean a subset $Y \subset X$ which is the intersection of X with a great circle, and such that Y is the intersection of X with the unique great circle containing $F(Y)$. □

7. Examples of contractive mappings

Here are some basic examples which show that the phenomenon pointed out by Hsiao [10] doesn't affect our notion of contractive mapping. He showed that for a number of different contraction conditions for a mapping F on a 2-metric space, the iterates x_i defined by $x_{i+1} := F(x_i)$ starting with any x_0, are all colinear i.e. $d(x_i, x_j, x_k) = 0$ for all i, j, k.

7.1. Consider first the standard 2-metric given by the area of the spanned triangle in $\mathbb{R}^n$, see Proposition 5.4. Suppose U is a convex region containing 0, contained in a ball of diameter ≤ 1. Choose $0 < k < 1$ and put $G(x_1, \ldots, x_n) := (kx_1, \ldots, kx_n)$. Then clearly

$$d(Gx, Gy, Gz) \leq k^2 d(x, y, z).$$

Even though G itself satisfies the property of colinearity of iterates, suppose $M \in O(n)$ is a linear orthogonal transformation of $\mathbb{R}^n$ preserving

U. Then $d(Mx, My, Mz) = d(x, y, z)$, so if we put $F(x) := M(G(x))$ then

$$d(Fx, Fy, Fz) \le k^2 d(x, y, z).$$

Hence F is a contractive mapping. If M is some kind of rotation for example, the iterates $F^i(x_0)$ will not all be colinear, so Hsiao's conclusion doesn't hold in this case. Of course, in this example there is a unique fixed point $0 \in U$ so the phenomenon of fixed lines hasn't shown up yet.

7.2. Look now at the determinant 2-metric on $\mathbb{RP}^2$ of Proposition 5.1, viewed for convenience as a function on S^2. Fix $0 < e < 1$ and consider a subset $U \subset S^2$ defined by the condition

$$U = \{(x_1, x_2, x_3) \subset S^2, \text{ s.t. } \|(x_1, x_2, 0\| \ge e\}.$$

This is a tubular neighborhood of the equator $x_3 = 0$. Choose $0 < k < 1$ and put

$$G(x_1, x_2, x_3) := \frac{(x_1, x_2, kx_3)}{\|(x_1, x_2, kx_3)\|} = \frac{\tilde{x}}{\|\tilde{x}\|},$$

where $\tilde{x} := (x_1, x_2, kx_3)$. The mapping G preserves U, in fact it is a contraction towards the equator. Note that $\|\tilde{x}\| \ge \|(x_1, x_2, 0)\| \ge e$, so

$$d(Gx, Gy, Gz) = \frac{kd(x, y, z)}{\|\tilde{x}\| \, \|\tilde{y}\| \, \|\tilde{z}\|} \le ke^{-3} d(x, y, z).$$

If $k < e^3$ then this mapping is contractive. Every point of the equator is a fixed point, and indeed G is interesting from the point of view of studying uniqueness, since it has the equator (containing all of the fixed points), as well as all of the longitudes (each containing a single fixed point), as fixed lines.

To get an example where there are no fixed points but a fixed line, let $M \in O(3)$ be any orthogonal linear transformation of $\mathbb{R}^3$, which preserves S^2. Suppose that M preserves U; this will be the case for example if M is a rotation preserving the equator. Then M preserves the 2-metric d, so $F(x) := M(G(x))$ is a contractive mapping $U \to U$. If M is a nontrivial rotation, then it F has no fixed points, but the equator is a fixed line. It is easy to see that the iterates $F^i(x_0)$ are not colinear in general. Thus Hsiao's remark [10] doesn't apply to our notion of contractive mapping.

7.3. Come back to the standard area 2-metric on $\mathbb{R}^n$. It should be pointed out that our contractivity condition automatically implies that F preserves the relation of colinearity, hence it preserves lines. So, for example, if U is a compact region in $\mathbb{R}^n$ of diameter ≤ 1, and if U has dense and connected interior, then any contractive mapping $F : U \rightarrow U$ in our sense must be the restriction of a projective transformation of $\mathbb{RP}^n$, that is given by an element of $PGL(n+1, \mathbb{R})$. In this sense the Euclidean 2-metric itself has a strong rigidity property.

On the other hand, we can easily consider examples which don't contain any lines with more than two points. This is the case if U is contained in the boundary of a strictly convex region. Furthermore, that reduces the possibility of a fixed line in Theorem 6.2 to the case of two points interchanged, whence:

Corollary 7.1. *Suppose U is a compact subset of the boundary of a strictly convex region of diameter ≤ 1 in $\mathbb{R}^n$, and let d be the restriction of the standard area 2-metric (Proposition 5.4). If $F : U \rightarrow U$ is any mapping such that $d(F(x), F(y), F(z)) \leq kd(x, y, z)$ for a constant $0 < k < 1$, then either F has a fixed point, or else it interchanges two points in a fixed pair.*

Proof. In the boundary of a strictly convex region, the "lines" are automatically subsets consisting of only two points. □

7.4. For (U, d) as in the previous corollary, for example a region on the sphere S^2 with the restriction of the area 2-metric from $\mathbb{R}^3$ as in Lemma 5.5, there exist many contractive mappings F. It requires a little bit of calculation to show this.

Suppose $V' \subset V \subset \mathbb{R}^2$ are open sets. For a C^2 mapping $F : V' \rightarrow V$, let $J(F, x)$ denote the Jacobian matrix of F at a point x, viewed as an actual 2×2 matrix using the standard frame for the tangent bundle of $\mathbb{R}^2$. Let $dJ(F)$ denote the derivative of this matrix, i.e. the Hessian matrix of F.

Proposition 7.2. *Suppose d is a 2-metric on V satisfying a convexity bound of the form* (5.1)

$$\frac{1}{C}\left(\alpha_2(x, y, z) + \rho(x, y, z)\right) \leq d(x, y, z) \leq C\left(\alpha_2(x, y, z) + \rho(x, y, z)\right)$$

for a constant $C > 1$. Such d exists by Lemma 5.5. Given $C_A > 0$, there is a constant $C' > 0$ such that the following holds.

Suppose A is a fixed 2×2 matrix with $\det(A) \neq 0$ and $\|A\| \leq C_A$. Then there is a constant $c' > 0$ depending on A, such that if $F : V' \rightarrow V$ is a C^2 mapping with $\|J(F, x) - A\| \leq c'$ and $\|dJ(F)\| \leq c'$ over V', *then*

$$d(F(x), F(y), F(z)) \leq C'|\det(A)|d(x, y, z)$$

for any $x, y, z \in V'$.

Proof. In what follows the constants C_i and C' will depend only on C and the bound C_A for $\|A\|$, assuming c' to be chosen small enough depending on A. By choosing c' small enough we may assume that F is locally a diffeomorphism, as $J(F, x)$ is invertible, being close to the invertible matrix A. By Rolle's theorem there is a point y' on the segment joining x to y, and a positive real number u, such that

$$F(y) - F(x) = uJ(F, y')(y - x).$$

Furthermore $u < C_1$, if c' is small enough. We thank N. Mestrano for this argument.

Similarly there is a point z' on the segment joining x to z, and a positive real number $v < C_1$ such that

$$F(z) - F(x) = vJ(F, z')(z - x).$$

Put $S := J(F, y') - J(F, x)$ and $T := J(F, z') - J(F, x)$. Now the bound $\|dJ(F)\| \leq c'$ implies that

$$\|S\| \leq 4c'\|y - x\|, \quad \|T\| \leq 4c'\|z - x\|.$$

We have

$$\begin{aligned}
&\alpha_2(F(x), F(y), F(z)) = |(F(y) - F(x)) \wedge (F(z) - F(x))| \\
&= |(uJ(F, y')(y - x)) \wedge (vJ(F, z')(z - x))| \\
&= uv\,|(J(F, x) + S)(y - x) \wedge (J(F, x) + T)(z - x)| \\
&\leq uv|\det J(F, x)|\alpha_2(x, y, z) \\
&\quad + uv[(\|S\| + \|T\|)\|J(F, x)\| + \|S\|\|T\|] \cdot \|y - x\| \cdot \|z - x\|.
\end{aligned}$$

Note that

$$\|y-x\|^2\|z-x\|+\|y-x\|\|z-x\|^2+\|y-x\|^2\|z-x\|^2 \leq C_2\rho(x, y, z),$$

and $\|J(F, x)\| \leq C_3$. By chosing c' small enough depending on A, we may assume that $|\det J(F, x)| \leq C_4|\det A|$. So again possibly by reducing c', we get

$$\alpha_2(F(x), F(y), F(z)) \leq C_4|\det(A)|\alpha_2(x, y, z)+C_1C_2C_3\rho(x, y, z).$$

It is also clear that $\rho(F(x), F(y), F(z)) \leq C_5\rho(x, y, z)$. Using the convexity estimate in the hypothesis of the proposition, we get

$$d(F(x), F(y), F(z)) \leq C'|\det(A)|d(x, y, z)$$

for a constant C' which depends on C and C_A but not on A itself. $\square$

Suppose V is a disc centered at the origin in $\mathbb{R}^2$ with a 2-metric d satisfying the convexity estimate of the form (5.1). For example V could be the projection of a patch in the Euclidean S^2 as in Lemma 5.5. Fix a bound $C_A = 2$ for example. We get a constant C' from the previous proposition. Choose then a matrix A such that $C'|\det(A)| < 1$ and $A \cdot V \subset V$. Then there is c' given by the previous proposition. There exist plenty of C^2 mappings $F : V \to V$ with $\|J(F,x) - A\| \leq c'$ and $\|dJ(F)\| \leq c'$. By the conclusion of the proposition, these are contractive.

This example shows in a strong sense that there are no colinearity constraints such as [10] for mappings which are contractive in our sense.

As a 2-metric moves from a convex one towards the flat Euclidean one (for example for patches of the same size but on spheres with bigger and bigger radii) we have some kind of a deformation from a nonrigid situation to a rigid one. This provides a model for investigating this general phenomenon in geometry.

The question of understanding the geometry of contractive mappings, or even mappings F with $d(F(x), F(y), F(z)) \leq K d(x, y, z)$ for any constant K, seems to be an interesting geometric problem. Following the extensive literature in this subject, more complicated situations involving several compatible mappings and additional terms in the contractivity condition may also be envisioned.

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