Stochastic Utilities With a Given Optimal Portfolio : Approach by Stochastic Flows*†

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Abstract

The paper generalizes the construction by stochastic flows of consistent utilities processes introduced by M. Mrad and N. El Karoui in [17]. The market is incomplete and securities are modeled as locally bounded positive semimartingales. Making minimal assumptions and convex constraints on test-portfolios, we construct by composing two stochastic flows of homeomorphisms, all the consistent stochastic utilities whose the optimal wealth process is a given admissible portfolio, strictly increasing in initial capital. Proofs are essentially based on change of variables techniques.

1 Introduction.

The purpose of this paper is to generalize the construction by stochastic flows introduced in [17] in a Itô's framework where securities are modeled as continuous Itô's semimartingales. The concept of consistent stochastic utilities, also called "forward dynamic utilities", has been introduced by M. Musiela and T. Zariphopoulou in 2003 [21, 20, 22, 24, 26] ; since

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this notion appears in the literature in varied forms, in the work of T. Choulli, C. Stricker and L. Jia [1], V. Henderson and D. Hobson [4], F. Berrier, M. Tehranchi and Rogers [9], G. Zitkovic [36] and in the work of M. Mrad and N. El Karoui in [17]. Intuitively, a stochastic utility should represent, possibly changing over time, individual preferences of an agent. The agent's preferences are affected over time by the information available on the market represented by the filtration $(\mathcal{F}_t, t \geq 0)$ defined on the probability space $(\Omega, \mathbb{P}, \mathcal{F}_\infty)$. For this, the agent starts with today's specification of his utility, $u(x)$, and then builds the process $U(t,x)$ for $t > 0$ taking into account the information flow given by $(\mathcal{F}_t, t \geq 0)$. Consequently, its utility, denoted by $U(t,x)$ is a progressive process depending on time and wealth, t and x, which is as a function of x strictly increasing and concave. These utility processes will be called progressive utilities in that follows. In contrast to the classical literature, there is no pre-specified trading horizon at the end of which the utility datum is assigned.

Working on incomplete market, the progressive utility process of an investor may satisfy some additional properties on a convex class-test $\mathscr{X}$ of positive wealths. We study the optimality conditions in general way and we give an example of consistent utilities, based on simple utility function, to illustrate one of the main difficulties of our study. In paragraph 2.4, we show the stability of the notion of consistent utility by change of numeraire and, then, without loss of generality we consider the market martingale where admissible portfolios are local martingales.
Our main contribution is the new approach by stochastic flows of consistent dynamic utilities, proposed in Section 3. The idea is very simple and natural: suppose the optimal portfolio denoted by X^* is strictly increasing with respect to the initial capital and denoting by $\mathcal{X}$ the reverse flow, then using the duality identity $U_x(t, X_t^*(x)) = \mathcal{Y}(t,x)$ where $\mathcal{Y}/U_x(0,.)$ is the optimal process of the dual problem yields $U_x(t,x) = \mathcal{Y}(t, \mathcal{X}(t,x))$. Finally we get U by integration. We then, by stochastic flows techniques, construct all consistent utilities generating X^* as optimal portfolio.

The paper is organized as follows. In the next section, we introduce the market model and we define consistent stochastic utilities. Next we study the optimality conditions and we elaborate on the question of duality and numeraire change. In section 3, we present our new approach. We close the paper by a return (using numeraire change techniques) to initial market where wealths are not necessary local martingales.

2 Market Model and Consistent Stochastic Utilities.

The model of securities market consists of $d+1$ assets one riskless asset with price S^0 given by $dS^0_t = S^0_t r_t dt$ and d risky assets. The price of the d risky assets are modeled as a locally bounded positive semimartingales $S^i, \quad i = 1, \dots, d$ defined on a filtered probability space $(\Omega, \mathcal{F}_{t\geq 0}, \mathbb{P})$, with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, where $\mathcal{F}_0$ is assumed to be trivial, i.e., generated by the $\mathbb{P}$-null sets.
A (self−financing) portfolio is defined as a pair (x, ϕ) where the constant x is the initial value of the portfolio and $\phi = (\phi^i)_{1\leq i\leq d}$ is a predictable S-integrable process specifying the amount of each asset held in the portfolio. The value process $X = (X_t)_{t\geq 0}$ of such portfolio ϕ is given by

$$\frac{X_t}{S^0_t} = \frac{x}{S_0} + \int_0^t \frac{\phi_\alpha}{S^0_\alpha} d(\frac{S_\alpha}{S^0_\alpha}), \quad t \geq 0. \tag{1}$$

To facilitate the exposition, we restrict our study to the set $\mathbb{X}^+$ of wealth processes X which are positive almost surely. Define for any $X \in \mathbb{X}^+$ the portfolio proportion by

$$\pi_t = \frac{\phi_t}{X_t}, \quad \text{if } X_t \neq 0 \text{ and } \pi_s = 0 \quad \text{for } s \geq t \text{ if } X_t = 0. \tag{2}$$

a process which is $\{\mathcal{F}_t\}$-progressively measurable. The components of π_t represent the proportions of wealth X_t invested in the respective assets at time $t \geq 0$. The dynamics of wealth processes (1) becomes

$$\frac{dX_t}{X_t} = r_t dt + \pi_t.(\frac{dS_t}{S_t} - r_t \mathbf{1} dt), \quad t \geq 0 \tag{3}$$

where the d-dimensional vector such all components are equal to 1 is denoted by $\mathbf{1}$ and the inner scalar product is denoted by ".".

Definition 2.1. *A probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an equivalent local martingale measure if for any $X \in \mathbb{X}^+$, $\frac{X}{S^0}$ is a local martingale under $\mathbb{Q}$.*

The family of equivalent local martingale measures will be denoted by $\mathcal{M}$. We assume throughout this paper that

$$\mathcal{M} \neq \emptyset$$

This condition is intimately related to the absence of arbitrage opportunities on the security market. See [7], [5] for a precise statement and references.

2.1 Definition of $\mathscr{X}$-consistent Stochastic Utilities.

In this paragraph we recall the concept of consistent utilities which has been introduced by M. Musiela and T. Zariphopoulou [21, 20, 22, 24, 26] under the name "forward utilities", also called "forward performance processes". A *progressive utility* U is a positive adapted continuous random field $U(t,x)$, such that $t \geq 0$, $x > 0 \mapsto U(t,x)$ is an increasing concave function, (in short utility function).
Obviously, this very general definition of progressive utility has to be constrained to represent, possibly changing over time, the individual preferences of an investor in a given financial market. The idea is to calibrate these utilities with regard to some convex class (in particular vector space) of positive wealth processes, denoted by $\mathscr{X}$, on which utilities may have more properties.

Definition 2.2 ($\mathscr{X}$**-consistent Utility).** *A $\mathscr{X}$-consistent stochastic utility $U(t,x)$ process is a positive progressive utility, in particular $t \geq 0$, $x > 0 \mapsto U(t,x)$ is an increasing, strictly concave function such that $U_x(t,x) := \frac{\partial}{\partial x}U(t,x)$ exists and continuous with the following property:*

- **Consistency with the test-class** *For any admissible wealth process $X \in \mathscr{X}$, $\mathbb{E}(U(t,X_t)) < +\infty$ and*
$$\mathbb{E}(U(t,X_t)/\mathcal{F}_s) \leq U(s,X_s),\ \forall s \leq t \quad .a.s.$$
- **Existence of optimal wealth** *For any initial wealth $x > 0$, there exists an optimal wealth process $X^* \in \mathscr{X}$, such that $X^*_0 = x$), and $U(s,X^*_s) = \mathbb{E}(U(t,X^*_t)/\mathcal{F}_s)$ $\forall s \leq t$.*

In short for any admissible wealth $X \in \mathscr{X}$, $U(t,X_t)$ is a positive supermartingale and a martingale for the optimal-benchmark wealth X^*.

The class $\mathscr{X}$ is a class of test portfolios which only allows to define the stochastic utility. Once his utility defined, an investor can then turn to a portfolio optimization problem on the general financial market to establish his optimal strategy or to calculate indifference prices. They are two interpretation of $\mathscr{X}$. $\mathscr{X}$ is chosen because it is rather rich with high liquidity, so that the investor is able to specify his preferences. Second, the investor have no interest to invest in this class and for this reason he use it only to define his utility.

The important novel feature of our definition of consistent dynamic utilities and this is where our notion differs from that in the work of Musiela and Zariphopoulou [21, 20, 22, 24, 26], Barrier et al. [9] and Zitkovic [36] is that: First, the wealth process X is not required to be discounted; this variation opens the door to a more general analysis as the question of numeraire change. Second, the notion of class-test, that has not been introduced in the previous literature gives more sense to the notion of progressive "forward" utility, as explained above. Note also that in the literature, consistent stochastic utilities are, in general, defined on the general set $\mathbb{X}^+$. But one might wonder what remains to optimize after having built the utility.

Admissible Wealth Processes As the initial conditions of wealth processes has a particularly important role in this work, more precisions and definitions are necessary for the rest of the paper.

Definition 2.3. • *For $s \geq 0$, a $\mathcal{F}_s$-measurable random variable η is said to be s-attainable if there exists $X \in \mathscr{X}$ such that $X_s = \eta$ a.s.*
• *A wealth process X is said to be an admissible test portfolio if $X \in \mathscr{X}$. Furthermore, a wealth process $X(s,\eta)$ starting at time s from η is said to be an admissible test portfolio, and we write $X(s,\eta) \in \mathscr{X}(s,\eta)$, if there exists $X \in \mathscr{X}$ such that $X_s = \eta$, and $X_t = X_t(s,\eta)$ for $t \geq \eta$.*

To continue the investigations, a more precise assumption on the structure of the class $\mathscr{X}$ is also needed.

Assumption 2.1 (Convexity)**.** *The class $\mathscr{X}$ is closed and convex in the sense that is for $s \geq 0$, $\varepsilon \in [0,1]$ and for any η, η' s-admissible wealths*

$$\varepsilon \mathscr{X}(s,\eta) + (1-\varepsilon)\mathscr{X}(s,\eta') \subset \mathscr{X}(s, \varepsilon\eta + (1-\varepsilon)\eta').$$

2.2 Optimality Conditions.

In the remainder of this section we shall focus on optimality conditions. Assuming the optimal portfolio X^* to be increasing with respect to x, we shall prove the existence of a $\mathscr{X}$-consistent dynamic utility. Then, in Section 3.4.2 we extend this result to a fully characterization of all $\mathscr{X}$-consistent dynamic utilities that generate X^* as optimal wealth.

Theorem 2.1. *Let U be an $\mathscr{X}$-consistent stochastic utility with optimal wealth process X^*. Suppose that $U_x = \frac{\partial}{\partial x}U$ exists and it is continuous, then*
(OC) *For any s-attainable wealths $\eta, \eta' > 0$ and for any $X(s,\eta) \in \mathscr{X}(s,\eta')$ the process $\big((X_t(s,\eta') - X^*_t(s,\eta))U_x(t, X^*_t(s,\eta)),\ t \geq s\big)$ is a supermartingale.*
Furthermore, if the convex set $\mathscr{X}$ is homogeneous that is for any $\lambda > 0$ $\lambda\mathscr{X} \subset \mathscr{X}$, then

(i) *The process $\big(X^*_t(s,\eta)U_x(t, X^*_t(s,\eta))\big)_{t\geq s}$ is martingale.*

(ii) *For any s-attainable wealths $\eta, \eta' > 0$, and any test-wealth $X \in \mathscr{X}(s,\eta')$, the process $\big(X_t(s,\eta')U_x(t, X^*_t(s,\eta))\big)_{t\geq s}$ is a supermartingale.*

Before proceeding to the proof of this result, it is interesting to note that this optimality conditions established in a general way are quite different from those of [17]. Indeed, in the last paper the process $U_x(t, X_t^*)$ is a state density process, in turn for any admissible wealth X, $X_t U_x(t, X_t^*)$ is a local martingale and a martingale if $X = X^*$. This is due essentially to the structure of the class $\mathscr{X}$ which is, only, assumed to be convex in the present paper and $\mathscr{X} = \mathbb{X}^+$ (set of all positive wealth processes) in [17].

Proof. First remark, by convexity of $\mathscr{X}$, that for any admissible wealth-test $X(s,\eta') \in \mathscr{X}(s,\eta')$ and any $\varepsilon \in [0,1]$, the process $\varepsilon\big(X(s,\eta') - X^*(s,\eta)\big) + X^*(s,\eta')$ is an admissible process starting from $\varepsilon(\eta' - \eta) + \eta$ at time $t = s$. Consequently, by consistency property with the class $\mathscr{X}$ and by martingale property of $U(., X^*_.(s,\eta))$, it follows for $t \geq \alpha \geq s \geq 0$

$$\begin{aligned}
&\mathbb{E}\big(U\big(t, \varepsilon(X_t(s,\eta') - X_t^*(s,\eta)) + X_t^*(s,\eta)\big) - U(t, X_t^*(s,\eta))/\mathcal{F}_\alpha\big) \\
&\leq U\big(\alpha, \varepsilon(X_\alpha(s,\eta') - X_\alpha^*(s,\eta)) + X_\alpha^*(s,\eta)\big) - U\big(\alpha, X_\alpha^*(s,\eta)\big).
\end{aligned} \tag{4}$$

Dividing by $\varepsilon > 0$ and denoting, for any t, η and η', $f(\varepsilon)$ the functional defined by

$$f(t,\varepsilon) := \frac{1}{\varepsilon}\Big[U\big(t, \varepsilon\big(X_t(s,\eta') - X_t^*(s,\eta)\big) + X_t^*(s,\eta')\big) - U\big(t, X_t^*(s,\eta')\big)\Big].$$

From the derivability assumption of U for any t, $f(t,\varepsilon)$ goes to $f(t,0)$ when $\varepsilon \mapsto 0$. By this, the right hand side of last inequality converge almost surely to $f(t,0) = (X_t(s,\eta') - X_t^*(s,\eta))U_x(t, X_t^*(s,\eta))$. To conclude, it remains to justify the passage to the limit under the expectation sign. For this end, remark that by concavity and the increasing properties of $U(t,.)$, $\varepsilon \mapsto f(t,\varepsilon)$ is a decreasing function with the same sign as $\big(X_t(s,\eta') - X_t^*(s,\eta)\big)$. Then, on the set $X_t(s,\eta') - X_t^*(s,\eta) \geq 0$, $f(t,\varepsilon)$ is positive and increase to $f(t,0)$. Letting $\varepsilon \searrow 0$, the conditional monotone convergence theorem implies

$$\mathbb{E}\big(f(t,\varepsilon)\mathbf{1}_{X_t(s,\eta') \geq X_t^*(s,\eta)}/\mathcal{F}_\alpha\big) \longrightarrow \mathbb{E}\big(f(0)\mathbf{1}_{X_t(s,\eta') \geq X_t^*(s,\eta)}/\mathcal{F}_\alpha\big)$$

On the other hand, on the set $X_t(s,\eta') - X_t^*(s,\eta) \leq 0$, $-f(t,\varepsilon)$ is positive and increase to $-f(t,0)$. The dominated convergence theorem applied, yields for $t \geq \alpha \geq s \geq 0$

$$\mathbb{E}\big(f(t,\varepsilon)\mathbf{1}_{X_t(s,\eta') \leq X_t^*(s,\eta)}/\mathcal{F}_\alpha\big) \longrightarrow \mathbb{E}\big(f(0)\mathbf{1}_{X_t(s,\eta') \leq X_t^*(s,\eta)}/\mathcal{F}_\alpha\big)$$

This justifies the passage to the limit on the inequality (4). Hence, it follows that

$$\mathbb{E}\Big(\big(X_t(s,\eta') - X_t^*(s,\eta)\big)U_x(t, X_t^*(s,\eta))/\mathcal{F}_\alpha\Big) \leq \big(X_\alpha(s,\eta') - X_\alpha^*(s,\eta)\big)U_x(\alpha, X_\alpha^*(s,\eta)). \tag{5}$$

Which proves **(OC)**.

Let , now, focus on the the case where the convex set $\mathscr{X}$ is homogeneous. In this case stability property of $\mathscr{X}$ by a positive multiplication implies that for any $\varepsilon > -1$, the wealth $(1+\varepsilon)X^*(s,\eta)$ still admissible and hence, by the same argument as above, we deduce for $-1 < \varepsilon < 0$ respectively $\varepsilon > 0$, the following inequalities

$$\frac{1}{\varepsilon}\mathbb{E}\Big(U\big(t,(1+\varepsilon)X_t^*(s,\eta)\big) - U\big(t,X_t^*(s,\eta)\big)/\mathcal{F}_\alpha\Big) \geq \frac{1}{\varepsilon}\Big(U\big(\alpha,(1+\varepsilon)X_\alpha^*(s,\eta)\big) - U\big(\alpha,X_\alpha^*(s,\eta)\big)\Big)$$
$$\frac{1}{\varepsilon}\mathbb{E}\Big(U\big(t,(1+\varepsilon)X_t^*(s,\eta)\big) - U\big(t,X_t^*(s,\eta)\big)/\mathcal{F}_\alpha\Big) \leq \frac{1}{\varepsilon}\Big(U\big(\alpha,(1+\varepsilon)X_\alpha^*(s,\eta)\big) - U\big(\alpha,X_\alpha^*(s,\eta)\big)\Big).$$

Passing to the limit $\varepsilon \to 0$, yields

$$\mathbb{E}\big(X_t^*(s,\eta)U_x(t,X_t^*(s,\eta))/\mathcal{F}_\alpha\big) = X_\alpha^*(s,\eta)U_x(\alpha,X_\alpha^*(s,\eta)),\ \forall t \geq \alpha \geq s \geq 0$$

and (i) hold. Reconciling (i) and **(OC)** yields (ii). □

2.3 Duality.

The use of convex duality in utility maximization and optimal stochastic control in general has proven extremely fruitful. As it is established in [17] analysis of utility random fields is no exception, the process $U_x(t,X_t^*),\ t \geq 0$ is a State price density process which is optimal to some dual problem. The idea here is to adopt a similar approach by duality in order to prove the dual optimality of $U_x(t,X_t^*),\ t \geq 0$. This will support the intuition and allows us a constructive intuition on different difficulties encountered in the study of consistent progressive utilities.

We start with a straightforward translation of the well-known Fenchel-Legendre conjugacy to the random-field case. For a utility random field U we define the dual random field $\tilde{U} : [0,+\infty[\times[0,+\infty[\times\Omega$, by

$$\tilde{U}(t,y) \stackrel{def}{=} \max_{x\in\mathbb{Q}_+^*}\Big(U(t,x) - xy\Big),\ fort \geq 0, \quad y \geq 0 \tag{6}$$

By a simple derivation with respect to x, the maximum is achieved at $x_t^* = (U_x)^{-1}(t,y) = -\tilde{U}_y(t,y)$. In turn

$$\tilde{U}(t,y) = U(t,I(t,y)) - y(U_x)^{-1}(t,y) \tag{7}$$

For the remainder of this section, assume that the set $\mathscr{X}$ to be homogeneous. Let $\mathscr{Y}$ denote the set of all positive processes Y such that YX is a supermartingale for any $X \in \mathscr{X}$. Moreover, define the class

$$\mathscr{Y}(s,y) := \{Y(s,y) \geq 0 : (Y_t(s,y)X_t,\ t \geq s) \text{ supermartingale, } \forall X \in \mathscr{X} \text{ and } Y_s(s,y) = y\}$$

and note that this sets contain the process $U_x(t,X^*)$.

As in the definition of stochastic utilities, we introduce in the following definition the s attainability of a dual random variable κ.

Definition 2.4. *For fixed $s \geq 0$, a random variable κ is s-attainable if there exists an s-attainable wealth η such that $\kappa = U_x(s,\eta)$ a.s.*

The goal of this section is now, the proof of the following theorem :

Theorem 2.2 (Duality)**.** *Assume the set $\mathscr{X}$ to be homogeneous. Then the convex conjugate $\tilde{U}$ of an $\mathscr{X}$-Consistent utility U, given by (6), satisfies*

(i) for any $t \geq 0$, $y \mapsto \tilde{U}(t,y)$ is convex decreasing function.

(ii) for any $Y(s,\kappa) \in \mathscr{Y}(s,\kappa)$, we have

$$\tilde{U}(s,\kappa) \leq \mathbb{E}(\tilde{U}(t, Y_t(s,\kappa))/\mathcal{F}_s), \;\; t \geq s, \;\; \kappa > 0. \tag{8}$$

(iii) Suppose that κ is s-attainable that is, there exists an s-attainable wealth η such that $\kappa = U_x(s,\eta)$ a.s. Then there exists a unique optimal process $Y^*_t(s,\kappa)$ s.t.

$$\tilde{U}(s,\kappa) = \mathbb{E}(\tilde{U}(t, Y^*_t(s,\kappa))/\mathcal{F}_s) = \inf_{Y(s,\kappa)\in\mathscr{Y}(s,\kappa)} \mathbb{E}(\tilde{U}(t, Y_t(s,y))/\mathcal{F}_s), \;\; t \geq s \tag{9}$$

Furthermore, $Y^*_t(s, U_x(s,\eta)) = U_x(t, X^*_t(s,\eta))$ where we recall that $X^*_t(s,\eta)$ denote the optimal wealth process at time t associated to U, starting from the s-attainable capital η at $t = s$.

The reader should note the difference between assertions (ii) and (iii) of this theorem. Indeed, (ii) assert that inequality (8) is satisfied for any $\kappa > 0$ while assertion (iii) says that equality in (8) hold if κ is s-admissible. This is, essentially , due to the fact that sets $\mathscr{X}(s,.)$ and $\mathscr{Y}(s,.)$ are not in perfect duality because $(U_x)^{-1}(.,\mathscr{Y}(s,.)) \nsubseteq \mathscr{X}(s,.)$, in general. In other terms, existence of solutions is intimately related to the inverse image of U_x, i.e. $(U_x)^{-1}(.,\mathscr{Y}(s,.))$. For more details see [18] for the classical case of optimization problem. For example, if $U(0,.)$ satisfies Inada conditions that is $\lim_{x\mapsto 0} U_x(0,x) = +\infty$, $\lim_{x\mapsto +\infty} U_x(0,x) = 0$ (or such that asymptotic elasticity, introduced in [18], is less than 1) then any $y > 0$ is 0-admissible which implies that for any $y > 0$ the dual problem (9) at $s = 0$ (replacing s by 0) admits a unique solution.
Let us also point out that the homogeneity property of the set $\mathscr{X}$ plays a crucial role in Theorem 2.2 in particular to establish inequality (8). Within this property, by optimality condition **(OC)**, we easily sees that $\mathscr{Y}$ is not the adequate set in which belongs $U_x(t, X^*_t)$. Furthermore, within homogeneity property the dual problem is ill posed because there is no clear characterization of the dual set of process Y satisfying **(OC)**.

Proof. Assertion (i) is a simple consequence of the definition of the convex conjugate. Let prove (ii) and (iii). From definition of Fenchel Transform, it is immediate that for any $Y(y) \in \mathscr{Y}(y)$,

$$\tilde{U}(t, Y_t(y)) \geq U(t, X_t^*(x)) - Y_t(y)X_t^*(x).$$

Using definition of $\mathscr{Y}(s,\kappa)$ and the martingale property of $(U(t, X_t^*))_{t\geq 0}$, one easily, sees that, for any s-attainable wealth η

$$\mathbb{E}(\tilde{U}(t, Y_t(s,\kappa))/\mathcal{F}_s) \geq \mathbb{E}(U(t, X_t^*(s,\eta))/\mathcal{F}_s) - \mathbb{E}(Y_t(\kappa)X_t^*(s,\eta)/\mathcal{F}_s) \geq U(s,\eta) - \kappa\eta.$$

In turn, by the homogeneity property of $\mathscr{X}$, we get for any $\lambda > 0$

$$\mathbb{E}(\tilde{U}(t, Y_t(s,\kappa))/\mathcal{F}_s) \geq U(s, \lambda\eta) - \lambda\kappa\eta.$$

let λ such that $\lambda\eta = (U_x)^{-1}(s,\kappa)$ then

$$\mathbb{E}(\tilde{U}(t, Y_t(s,\kappa))/\mathcal{F}_s) \geq U(s, (U_x)^{-1}(s,\kappa)) - \kappa(U_x)^{-1}(s,\kappa) = \tilde{U}(s,\kappa).$$

Which proves (ii). Now we turn to the case where κ is s-admissible. By assumption, there exists an s-admissible wealth η such that $U_x(s,\eta) = \kappa$. Denote by $\big(X_t^*(s,\eta)\big)_{t\geq s}$ the associated optimal process and by $\big(Y_t^*(s,\kappa)\big)_{t\geq s}$ the process defined by

$$Y_t^*(s,\kappa) = U_x(t, X_t^*(s, (U_x)^{-1}(s,\kappa))) > 0.$$

and observe that

$$(U_x)^{-1}(t, Y_t^*(s,\kappa)) = X_t^*(s, (U_x)^{-1}(s,\kappa))$$

which implies

$$\tilde{U}(t, Y_t^*(s,\kappa)) = U(t, X_t^*(s, (U_x)^{-1}(s,\kappa))) - Y_t^*(s,\kappa)X_t^*(s, (U_x)^{-1}(s,\kappa))).$$

Since U is an $\mathscr{X}$ consistent stochastic utility, from the martingale property of processes $\big(X_t^*(s,\eta)U_x(t, X_t^*(s,\eta))\big)_{t\geq s}$ and $\big(U(t, X_t^*(s,\eta))\big)_{t\geq s}$ and by definition of $\big(Y_t^*(s,\kappa)\big)_{t\geq s}$ we get that $\big(\tilde{U}(t, Y_t^*(s,\kappa))\big)_{t\geq s}$ is martingale. In order to conclude, let us observe that by optimality conditions (Theorem 2.1) we have, also, that for any s-admissible η' and any wealth-test $X \in \mathscr{X}(s,\eta')$, the process $\big(X_t(s,\eta')Y_t^*(s,\kappa)\big)_{t\geq s}$ is a positive supermartingale and hence $Y^*(s,\kappa) \in \mathscr{Y}(s,\kappa)$. Finally using (ii),

$$\begin{aligned}
\inf_{Y(s,\kappa)\in\mathscr{Y}(s,\kappa)} \mathbb{E}(\tilde{U}(t, Y_t(s,\kappa))/\mathcal{F}_s) \geq \tilde{U}(s,\kappa) &= \mathbb{E}(\tilde{U}(t, Y_t^*(s,\kappa))/\mathcal{F}_s) \\
&\geq \inf_{Y(s,\kappa)\in\mathscr{Y}(s,\kappa)} \mathbb{E}(\tilde{U}(t, Y_t(s,\kappa))/\mathcal{F}_s)
\end{aligned}$$

Which achieves the proof. □

2.4 Stability by numeraire change.

We saw in the previous sections, how optimality conditions, in non-homogeneous case, which satisfy the $\mathscr{X}$-consistent utilities are not intuitive. Because it is more convenient and more simpler to work with local martingales then semimartingales, the idea of this paragraph is to simplify the market model, which allow us to simplify the approach and to develop a constructive intuition about this study. The class $\mathscr{X}$ is only assumed convex, the goal of this paragraph is then to prove the following result.

Theorem 2.3 (Stability by numeraire change)**.**
Let $U(t,x)$ be a stochastic field and let Y be a positive semimartingale, and denote by $\mathscr{X}^Y$ the class of processes defined by $\mathscr{X}^Y = \{\frac{X}{Y},\ X \in \mathscr{X}\}$, then the process V defined by

$$V(t,x) = U(t, xY_t) \tag{10}$$

is an $\mathscr{X}^Y$-consistent stochastic utility if and only if U is an $\mathscr{X}$-consistent stochastic utility.

Roughly speaking, the theorem says, that the notion of $\mathscr{X}$-consistent stochastic utility is preserved by numeraire change. In particular, as the set of equivalent martingale measure is a nonempty, for any equivalent martingale measure M, this theorem shows that studying $\mathscr{X}$-consistent stochastic utilities is equivalent to study the $\mathscr{X}^M$-consistent utilities. The advantage, here, is that the new wealth processes in $\mathscr{X}^M$ are positive local martingales (in particular a supermartingales). From this point, in the sequel, we will deep the study of our utilities in the new market $\mathscr{X}^M, M \in \mathcal{M}(\mathscr{X})$.

Proof. To show this result it is enough to verify assertions of definition 2.2.

- Concavity : for $t \geq 0$, $x \mapsto V(t,x)$ is increasing concave function, by definition .
- Consistency with the test-class $\mathscr{X}^Y$: For any test-process $\tilde{X} \in \mathscr{X}^Y$, $\mathbb{E}\big(V(t,\tilde{X}_t) = U(t,X_t)\big) < +\infty$ a.s. and

$$\mathbb{E}(V(t,\tilde{X}_t)/\mathcal{F}_s) = \mathbb{E}(U(t,X_t)/\mathcal{F}_s) \leq U(s,X_s) \overset{def}{=} V(s,\tilde{X}_s)$$

- Existence of optimal: Let $\tilde{\eta}$ be an s-admissible wealth. As U is an $\mathscr{X}$-consistent utility and $\eta = Y_s\tilde{\eta}$ is s-admissible wealth in the initial market, there exists an optimal wealth process $X^*(s,\eta) \in \mathscr{X}(s,\eta)$,

$$U(s,\eta) = \mathbb{E}(U(t,X^*_t(s,\eta))/\mathcal{F}_s) = \operatorname{ess\,sup}_{X\in\mathscr{X}(s,\eta)} \mathbb{E}(U(t,X_t(s,\eta))/\mathcal{F}_s),\ \forall s \leq t.$$

Taking $\tilde{X}^*(s,\tilde{\eta}) = X^*(s,x)/Y$ yields, by definition of V and that of $\mathscr{X}^Y$ we get

$$\begin{aligned} V(s,\tilde{\eta}) &= U(s,\eta) = \mathbb{E}(U(t,X^*_t(s,\eta))/\mathcal{F}_s) = \sup_{X\in\mathscr{X}(s,\eta)} \mathbb{E}(U(t,X_t(s,\eta))/\mathcal{F}_s) \\ &= \mathbb{E}(V(t,\tilde{X}^*_t(s,\tilde{\eta}))/\mathcal{F}_s) = \sup_{\tilde{X}\in\mathscr{X}^Y(s,\tilde{\eta})} \mathbb{E}(V(t,\tilde{X}_t(s,\tilde{\eta}))/\mathcal{F}_s),\ \forall s \leq t. \end{aligned}$$

The proof is complete. □

3 New approach by stochastic flows.

In this section, where $\mathscr{X}$ is only assumed to be convex class, we generalize the construction of consistent progressive utilities proposed in [17] where the market securities are modeled as a continuous semimartingale in a brownien market and where $\mathscr{X}$ is the set of all positives wealth processes. We remind the reader that the results of the following sections are stated in the martingale market and that similar results can be obtained for the initial market by using results of Theorem 2.3. To simplify, we keep same notations as in Section 2: X for wealth's processes and $\mathscr{X}$ for the class-test. The main contribution of this section is the explicit construction of progressive dynamic utilities by techniques of stochastic flows composition.

3.1 Optimality Conditions: martingale market.

We remind in this paragraph some results and notations, established in the previous section, which will play crucial role in the sequel. Let U be an $\mathscr{X}$-consistent stochastic utility, optimality conditions in the martingale market imply that the derivative U_x taken over the optimal portfolio X^*, i.e. $(U_x(t,X^*_t(x)))$ plays the role of dual process in our study (Theorem 2.1). In the case of homogeneous constraint $(U_x(t,X^*_t(x)))$ is a positive local martingale. Furthermore, the local martingale normalized $U_x(t,X^*_t(x))/U_x(0,x)$ is the optimal dual process of the dual optimization problem (9). For simplicity we denote it by $(\mathcal{Y}(t,x))_{t\geq 0,x>0}$. Notation $\mathcal{Y}$ serves for recalling that this process plays the role of dual process and not a wealth process. We remind, also, the conditions which have to satisfy necessarily optimum processes $X^*(x)$ and $\mathcal{Y}(.,x)$ as we established them in Theorem 2.1 of paragraph 2.2. For it, we begin by defining a set of properties to which we shall often refer afterwards. Except opposite mention, we assume that the reference market is "the martingale market".

Definition 3.1. *Let* $(X^*_t(x);\ t\geq 0)$ *and* $(\mathcal{Y}(t,x);\ t\geq 0)$ *be two given stochastic fields. Conditions* $(\mathcal{O}^*)$ *are :*

(O1) *For all* $x>0$, $(X^*_t(x); t\geq 0)$ *is a local martingale and an admissible wealth process.*

(OC) *For any initial positive wealth's* $x,\ x'$ *and any admissible wealth process* $X\in\mathscr{X}(x')$, $(X_t(x')-X^*_t(x))\mathcal{Y}(t,x)); t\geq 0)$ *is submartingale.*

Recall that conditions $(\mathcal{O}^*)$ are necessary conditions satisfied by the optimal portfolio $(X_t^*(x);\ t \geq 0)$ and the random variable $(U_x(t, X_t^*(x));\ t \geq 0)$.

3.2 Main Idea.

Because we know several properties of U_x, the derivative of an $\mathscr{X}$-consistent utility, along the optimal trajectory, i.e, $(U_x(t, X_t^*(x))))$ given in Theorem 2.1, the question is the following one: can we obtain more information about the process $U_x(t,x)$, itself, from these properties?
Although this can appear too much to ask, because we try to characterize the derivative of a stochastic utility from its behavior on a very particular trajectory, but the answer to this question is positive and simple. Suppose that *the optimal wealth process X^* is strictly increasing with respect to its initial condition x*. In turn the process $\mathcal{Y}(t,x)$ defined by

$$\mathcal{Y}(t,x) \stackrel{def}{=} U_x(t, X_t^*(x)) \tag{11}$$

is strictly decreasing with respect to x because U is strictly concave. Denoting by $\mathcal{X}(t,z)$ the inverse flow of $X^*(t,z)$, one, easily, sees that last identity becomes,

$$U_x(t,z) = \mathcal{Y}(t, \mathcal{X}(t,z)),\ \forall t \geq 0, z > 0.$$

Integrating yields

$$U(t,x) = \int_0^x \mathcal{Y}(t, \mathcal{X}(t,z))dz,\ \forall t \geq 0, z > 0.$$

This identity is the key of the construction which we propose, in the sequel, to characterize $\mathscr{X}$-consistent stochastic utilities.

Note that monotony assumption of the optimal wealth process is very natural.. For example, in the results of Example **??**, the optimal wealth is strictly monotonous and even twice differentiable with respect to the initial capital x, under certain additional hypotheses. This is still true within the framework of decreasing (in the time) consistent "forward" utilities, studied by M. Musiela et al [28] and Berrier et al. [9]. We can also find these properties of the optimal wealth in the classic framework of portfolio optimization in the case of power, logarithmic, exponential utilities and in the multitude of examples proposed by Huyên Pham in [10] and by Ioannis Karatzas and Steven Shreve in [16]. To conclude, let us notice that, by absence of arbitrage opportunities on the security market, the optimal wealth can be only increasing with regard to the initial wealth, because otherwise by investing less money we could obtain the same gain. Mathematically, technical problems can appear, what leads to put this property as a hypothesis.

Assumption 3.1. *Suppose the process* $(X_t^*(x); t \geq 0)$ *satisfying*

$$\forall t \geq 0, \quad x \mapsto X_t^*(x) \quad \text{continuous and strictly increasing, s.t.} \quad X_t^*(0) = 0 \quad X_t^*(\infty) = \infty.$$

Remark 3.1. *As a direct consequence of this hypothesis, we note that as the process* $\mathcal{Y}(t,x)$ *plays the role of* $U_x(t, X_t^*(x))$ *(equation (11)),* $\mathcal{Y}$ *should satisfy also,*

$$\forall t \geq 0, x \mapsto \mathcal{Y}(t,x), \text{ positive strictly decreasing, and s.t. Inada conditions hold if}$$
$$\mathcal{Y}(t,0) = +\infty, \quad \mathcal{Y}(t,\infty) = 0.$$

3.3 Optimal wealth process as a stochastic flow.

Hypothesis 3.1 of monotony of the wealth process $X_t^*(x)$ brings us naturally to consider it as the value, leaving from x at $t = 0$, of a stochastic flow $(X_t^*(s,x))_{s \leq t}$, which we define below. We can then consider the wealth as leaving from condition x at $t = 0$ or leaving from condition z at date s.

Proposition 3.1. *Let* $(X_t^* x))$ *be a strictly monotonous flow with respect to* x *with values in* $[0, \infty)$*. Its inverse* $\mathcal{X}(t,z) = (X_t^*(.))^{-1}(z)$ *is also a strictly monotonous stochastic flow, defined on* $[0,\infty)$*. We prolong the flow* X^* *and its inverse* $\mathcal{X}$ *in the intermediate dates* $(s < t)$ *in the following way*

$$\begin{aligned} X_t^*(s,x) &= X_t^*(\mathcal{X}(s,x)) \\ \mathcal{X}_s(t,z) &= (X_t^*(s,.))^{-1}(z) = X_s^*(\mathcal{X}(t,z)). \end{aligned} \tag{12}$$

In particular, we have the following properties

(i) *Equality* $X_t^*(s,x) = X_t^*(\alpha, X_\alpha^*(s,x))$ *hold true for all* $0 \leq \alpha \leq s \leq t$ *a.s..*
Identity $\mathcal{X}_s(t,z) = \mathcal{X}_s(\alpha, \mathcal{X}_\alpha(t,z))$ *hold true for all* $0 \leq s \leq \alpha \leq t$ *a.s..*

(ii) *Moreover,* $X_t^*(t,x) = x, \mathcal{X}_t(t,z) = z$*, and*
$\mathcal{X}_s(t, X_t^*(s,x)) = x, \quad X_t^*(s, \mathcal{X}_s(t,x)) = x$*, for all* $0 \leq s \leq t$*.*

For more details, we invite the reader to see H.Kunita [19] for the general theory of stochastic flows.

3.4 Construction of $\mathscr{X}$-consistent utilities for a given optimum portfolio: martingale market.

3.4.1 Existence of $\mathscr{X}$-consistent utilities for a given optimum portfolio.

As we announced it in the introduction of this section, our objective, under strictly monotonous hypothesis of wealth X^*, is to construct $\mathscr{X}$-consistent utility of given optimum wealth X^* on the financial martingale market. The previous study shows that if optimum wealth is martingale, (and not only a local martingale), the process $\mathcal{Y}$ s.t. $\mathcal{Y}(t,x)/\mathcal{Y}(0,x)=1$ is admissible in the sense that the pair $(X^*,\mathcal{Y}(0,x))$ satisfy conditions $\mathcal{O}^*$ of Definition 3.1. Let us choose initial condition of $\mathcal{Y}(0,x)=u_x(x)$ as the derivative of an utility function, satisfying Inada conditions . The main idea (equation (12)) suggests a very simple form of an $\mathscr{X}$-consistent utility $u(t,x)$ of given monotonous optimum wealth . If $\mathcal{X}(t,z)$ denote the inverse of $X^*_t(x)$, the concave increasing process $U(t,x)$ such that $U_x(t,x)=u_x(\mathcal{X}(t,x))$ is a good candidate to be an $\mathscr{X}$-consistent utility. Another remarkable property of this stochastic process is that $U_x(t,X^*_t(x))=u_x(x)$, what is in another way to express that optimal dual process $\mathcal{Y}(t,x)/\mathcal{Y}(0,x)$ is constant. This is the main idea of the following result.

Theorem 3.2. *Let $X^*_t(x)$ an admissible portfolio assumed to be* **martingale** *and strictly increasing with respect to the initial wealth. Denote by $\mathcal{X}(t,z)$ its inverse flow. Then for all utility function u such that $u_x(\mathcal{X}(t,z))$ is locally integrable near $z=0$, the stochastic process U defined by*

$$U(t,x)=\int_0^x u_x(\mathcal{X}(t,z))dz,\quad U(t,0)=0 \tag{13}$$

is an $\mathscr{X}$-consistent utility in the martingale market. The associated optimal wealth process is X^* and the optimal dual process is constant equal to 1. Further, the convex conjugate of U denoted by $\tilde U$, is given by

$$\tilde U(t,y)=\int_y^{+\infty} X^*(t,-\tilde u_y(z))dz, \tag{14}$$

The proof of Theorem 3.2 will be broken into several steps.

Lemma 3.3. *For any s-attainable wealth η and any test-process $(X_t(s,\eta);s\le t)\in\mathscr{X}(s,\eta)$, we have*

$$\mathbb{E}\big(U(t,X_t(s,\eta)/\mathcal{F}_s\big)\le\mathbb{E}\big(U(t,X^*_t(s,\eta))/\mathcal{F}_s\big) \tag{15}$$

Proof. By concavity of the process $U(t,x)$, we have

$$U\big(t,X_t(s,\eta)\big)-U\big(t,X^*_t(s,\eta)\big)\le\big(X_t(s,\eta)-X^*_t(s,\eta)\big)U_x\big(t,X^*_t(s,\eta)\big).$$

From Definition (22) of U, we get that $U_x\big(t,X^*_t(s,\eta)\big)=u_x(\mathcal{X}(t,X^*_t(s,\eta)))$. On the other hand, using proposition 3.1, we have $X^*_t(s,\eta)=X^*_t(\mathcal{X}(s,\eta))$ and hence, by definition of $\mathcal{X}$, we obtain $U_x\big(t,X^*_t(s,\eta)\big)=u_x(\mathcal{X}(s,\eta))=U_x(s,\eta)$. The inequality bellow becomes

$$U\big(t,X_t(s,\eta)\big)-U\big(t,X^*_t(s,\eta)\big)\le\big(X_t(s,\eta)-X^*_t(s,\eta)\big)U_x\big(t,\eta\big). \tag{16}$$

We have also that $(X^*_t(s,\eta),\ t\geq 0)$ is a martingale by assumption and $(X_t(s,\eta),\ t\geq 0)$ is a supermartingale because $X_t(s,\eta)$ is an admissible wealth-test in the martingale market. Those properties, together with (16), imply

$$\mathbb{E}\Big(U\big(t,X_t(s,\eta)\big)-U\big(t,X^*_t(s,\eta)\big)/\mathcal{F}_s\Big)\leq \mathbb{E}\big((X_t(s,\eta)-X^*_t(s,\eta))/\mathcal{F}_s\big)U_x\big(s,\eta\big)\leq 0.$$

This will prove the validity of (15).

□

Lemma 3.4. *For all $s\geq 0$ and for any s-admissible wealth test η,*

$$U(t,X^*_t(s,\eta))=U_x(s,\eta)X^*_t(s,\eta)-\int_0^{\mathcal{X}(s,\eta)}X^*_t(z)du_x(z)$$

and it is a martingale.

Proof. By definition,

$$U(t,X^*_t(s,x))=\int_0^{X^*_t(s,x)}u_x(\mathcal{X}(t,z))dz$$

Consider the increasing change of variable $z'=\mathcal{X}(t,z)$ or equivalently $z=X^*_t(z')$. Using identity $\mathcal{X}(t,X^*_t(s,x))=\mathcal{X}(s,z)$ it follows

$$U(t,X^*_t(s,x))=\int_0^{\mathcal{X}(s,x)}u_x(z)\,d_z\,X^*_t(z)$$

Integration by parts with integrability assumptions imply

$$U(t,X^*_t(s,x))=u_x(\mathcal{X}(s,x))X^*_t(s,x)-\int_0^{\mathcal{X}(s,x)}X^*_t(z)du_x(z).$$

Replacing x by η and using the fact that $u_x(\mathcal{X}(s,x))=U_x(s,x)$ yields the desired identity

$$U(t,X^*_t(s,\eta))=U_x(s,\eta)X^*_t(s,\eta)-\int_0^{\mathcal{X}(s,\eta)}X^*_t(z)du_x(z).$$

While $(X^*_t(s,\eta),\ t\geq s)$ is a martingale and $U_x(s,\eta)$ is $\mathcal{F}_s$-mesurable $U_x(s,\eta)X^*_t(s,\eta),\ t\geq s$ is a martingale. Using the Fubini-Tonelli theorem, the integral on $u_x(z)$ of $X^*_t(z)$ is a martingale. Consequently, as a sum of two martingales, the sequence of random variables $(U(t,X^*_t(s,x)),\ t\geq s)$ is a martingale.

□

We are now able to proof Theorem 3.2.

Proof. (Theorem 3.2) Since u is an utility function [1] and $\mathcal{X}$ is strictly increasing, $U(t,.)$ is a strictly concave and increasing function. To conclude, we have to check that the above Lemmas imply assertions ii) and iii) of Definition 2.2.
Let $(X_t(s,\eta); s \leq t) \in \mathscr{X}(s,\eta)$ be an admissible wealth process, we have, using Lemmas 3.3 and 3.4,

$$\mathbb{E}\big(U(t, X_t(s,\eta)/\mathcal{F}_s\big) \leq \mathbb{E}\big(U(t, X_t^*(s,\eta))/\mathcal{F}_s\big) = U(s,\eta)$$

Which proves the consistency with the class-test $\mathscr{X}$. Existence and uniqueness of optimal is a simple consequence of X^*-admissibility and strict concavity of U, so that we may deduce that U is an $\mathscr{X}$-consistent stochastic utility with X^* as optimal portfolio. Note that (23) holds from definition of U and that optimal dual process is given by $U_x(t, X_t^*(s,\eta))/U_x(s,\eta)$ which is equal to one by construction. □

Remark 3.2. *Let us note that, if the process $X_t(s,x)$ defined by*

$$X_t(s,x) = X_t(\mathcal{X}(s,x)) \tag{17}$$

are admissible wealth-test, then we can replace η in the previous two Lemmas, simply, by x. There is no modifications to be brought in proofs. In other words, if we can start at any time s from any $x \in \mathbb{R}_+$ then, replacing $X_t(s,\eta)$ by $X_t(s,x)$ and $X_t^*(s,\eta)$ by $X^*(s,x)$, Lemmas 3.3 and 3.4 still valid.

3.4.2 Construction of all $\mathscr{X}$-consistent utilities for a given optimum portfolio.

We showed in Theorem 3.2 that for every increasing wealth process X^*, such X^* is a martingale, we can construct a consistent utilities of optimal wealth X^*. The feature of these consistent utilities, defined by (22), is that the optimal dual process is fixed to 1. In order to characterize all consistent utilities with given optimal portfolio X^*, we consider more general class of processes $\mathcal{Y}$ such that optimality conditions $\mathcal{O}^*$ are satisfied for the pair $(X^*, \mathcal{Y})$. As we saw it, the intuition is to characterize utilities U such that $U_x(t,x) = \mathcal{Y} o \mathcal{X}(t,x)$, where $\mathcal{X}(t,x)$ is the inverse flow of X^*. The monotony condition of X^* draw away that the stochastic flow $\mathcal{Y}$ must be decreasing to guarantee that $U_x(t,x)$ is decreasing. To resume, in the sequel we, only, consider pairs $(X^*, \mathcal{Y})$ of processes satisfying

Assumption 3.2. (A1) *The process $(X_t^*(x); x \geq 0, t \geq 0)$ is strictly increasing from 0 to $+\infty$ while $(\mathcal{Y}(t,x); x \geq 0, t \geq 0)$, according to remark 3.1, is strictly decreasing*

[1] u is a strictly concave and increasing function

from $+\infty$ to 0 such that $\mathcal{Y}(t,x)$ is locally integrable near $x=0$. $\mathcal{Y}(0,x)$ is a strictly decreasing functional, denoted by $u_x(x)$.

(A2) *The pair $(X^*,\mathcal{Y})$ satisfy $\mathcal{O}^*$.*

It is important to notice that martingale property (assertion (ii) of Theorem 2.1) of the process $\big(X^*(x)_t\mathcal{Y}(t,x); t\geq 0\big)$ hold true with X^* replaced by his derivative (if it exists) D_xX^x with respect to x . To justify passage on the limit and furthermore, in order to establish the main result of this section, we suppose that assertions of the following domination assumption hold true.

Assumption 3.3. H1 local) *For all x, there exists an integrable positif adapted process, $U_t(x)>0$ such that, if we denote by $\mathbf{B}(x,\alpha)$ the ball of radius $\alpha>0$ centered at x,*

$$\forall y,y'\in\mathbf{B}(x,\alpha), |X^*_t(y)-X^*_t(y')|<|y-y'|\,U_t(x),\ \ t\geq 0 \tag{18}$$

H2 global) *$U_t(x)$ is increasing with respect to x and $U^I_t(x)=\int_0^x\mathcal{Y}(t,z)U_t(z)dz$ is integrable for all $t\geq 0$.*

Let us point out that this hypothesis is introduced only to justify result of the following proposition. Summing up, under this assumption

Proposition 3.5. *Let assumption 3.3 hold. If the derivative with respect to x of the increasing process $X^*_t(x)$ denoted by $D_x\,X^*_t(x)$ exists in any point x, then $\mathcal{Y}(t,x)D_x\,X^*_t(x)$ is a martingale. Otherwise, without derivability assumption, the process*

$$\int_0^x\mathcal{Y}(t,z)d_zX^*_t(z), \tag{19}$$

is also a martingale.

We show in the proof of Theorem 3.6, that quantity

$$\int_0^{\mathcal{X}(s,\eta)}\mathcal{Y}(t,z)d_zX^*_t(z).$$

correspond to $U(t,X^*_t(s,\eta))$ where U is a process which we define afterwards. Particularly, this proposition is other than a generalization of Lemma 3.4 where we replace deterministic quantity u_x with process $\mathcal{Y}$.

Proof. **a)** First, suppose $X^*_t(x)$ is differentiable with respect to x. For $0<\epsilon<\alpha$, the process $\mathcal{Y}(t,x)\big(X^*_t(x+\epsilon)-X^*_t(x)\big)$ is a positive supermartingale (assertion (**(OC)** of Theorem 2.1). By assumption 3.3 the right derivative with respect to ϵ, $\mathcal{Y}(t,x)D^+_x\,X^*_t(x)$ is a

positive supermartingale.
Furthermore, the case when the set $\mathscr{X}$ of admissible wealth is homogeneous,

$$\mathcal{Y}(t,x)\big(X_t^*(x) - X_t^*(x-\epsilon)\big)$$

is a positive local martingale. Assumption 3.3 implies at once that this local martingale is a martingale and that this martingale property is preserved in passing to the limit when ϵ goes to 0. The result follows by differentiability of $X_t^*(x)$.
In the general case, $\mathcal{Y}(t,x)\big(X_t^*(x) - X_t^*(x-\epsilon)\big)$ is a positive submartingale. Once again, the hypothesis 3.3 is used to show that we can again pass to the limit and deduct that $\mathcal{Y}(t,x)D_x^- X_t^*(x)$ is a positive submartingale. From derivability of X^*, $D_x^- X_t^*(x) = D_x^+ X_t^*(x) = D_x X_t^*(x)$ and then the process $\mathcal{Y}(t,x)D_x X_t^*(x)$ is, consequently, a sub and supermartingale and therefore martingale.

b) In the general case, without differentiability assumption on X^*, we use Darboux sum to study the properties of $S(x) = \int_0^x \mathcal{Y}(t,z)d_z X_t^*(z)$. We partition the interval $[0,x]$ into N subintervals $]z_n, z_{n+1}]$ where the mesh approaches zero. To approach the integral (19) by below respectively by above we consider respectively the following sequences

$$\begin{aligned} S_N(t,x) &= \sum_{n=0}^{n=N-1} \mathcal{Y}(t,z_n)\big(X_t^*(z_{n+1}) - X_t^*(z_n)\big) \\ S'_N(t,x) &= \sum_{n=0}^{n=N-1} \mathcal{Y}(t,z_{n+1})\big(X_t^*(z_{n+1}) - X_t^*(z_n)\big). \end{aligned}$$

By the same arguments as above, the sequence $S_N(t,x)$ is a positive supermartingale, while the sequence $S'_N(t,x)$ is a positive sub-martingale, and a positive local martingale if $\mathscr{X}$ is homogeneous. In all cases, by hypothesis 3.3, the positive processes $S_N(t,x)$ and $S'_N(t,x)$ are bounded above by

$$\bar{S}_N(t,x) := \sum_{n=0}^{n=N-1} \mathcal{Y}(t,z_{n+1})U_t(z_{n+1})$$

Moreover, under assertion H2 global) of hypothesis 3.3 , $\bar{S}_N(t,x)$ is bounded above by $U_t^I(x) = \int_0^x \mathcal{Y}(t,z)U_t(z)dz$. As the properties of sub and supermartingale are preserved in passing to the limit it follows that $\int_0^x \mathcal{Y}(t,z)d_z X_t^*(x)$ is a martingale. □

We have now all elements to characterize consistent utilities of given optimal wealth.

Theorem 3.6. *Let $(X^*, \mathcal{Y})$ a pair of strictly positif processes satisfying assumptions 3.2 and 3.3, such that $\mathcal{Y}(t,x)$ is locally integrable near $x=0$ and $\mathcal{Y}(t,0)=+\infty, \quad \mathcal{Y}(t,+\infty)=0$. Let $\mathcal{X}$ the inverse flow of X^*, then the concave increasing process U defined by*

$$U(t,x) = \int_0^x \mathcal{Y}(t, \mathcal{X}(t,z))dz \tag{20}$$

is an $\mathscr{X}$-consistent stochastic utility on the martingale market with u as the initial function, X^* as the optimal wealth process. The optimal dual process is $\mathcal{Y}(t,x)/\mathcal{Y}(0,x)$ and the convex conjugate is given by

$$\tilde{U}(t,y) = \int_y^{+\infty} X_t^*((\mathcal{Y})^{-1}(t,z))dz. \tag{21}$$

In Theorem 3.2, for a given initial utility, we construct an $\mathscr{X}$-consistent utility of given optimal portfolio (martingale). The extension which we give here which, up-technical points, characterizes all the $\mathscr{X}$-consistent utilities equivalent to the previous one (in the sense that they gives the same optimal portfolio process). This characterization expresses only how we have to diffuse the function $u_x(x)$ to stay within the framework of the $\mathscr{X}$-consistent utilities. The answer is intuitive because it expresses that it is enough to keep a monotonous flow of martingale $\mathcal{Y}$ which do not influence the reference market.

Proof. The proof is made in two step, as in the construction of the previous section. The consistency with the universe of investment is based on two essential properties:
– On one hand on the fact that $(U(t, X_t^*(s,\eta)),\ t \geq s)$ is a martingale.
– On the other hand, the consistency with the class-test $\mathscr{X}(s,\eta)$.
There is no modifications to be brought to the proof of the theorem 3.2, the consistency with the universe of investment is automatic if we show $U(t, X_t^*(x))$ is a martingale. To be made, we proceed as in the previous example by writing that $U(t, X_t^*(s,\eta)) = \int_0^{X_t^*(s,\eta)} \mathcal{Y} o \mathcal{X}(t,z')dz'$. Let us make the change of variable $\mathcal{X}(t,z') = z$, consequently, because $\mathcal{X}(t, X_t^*(s,\eta)) = \mathcal{X}(s,\eta)$, we get

$$U(t, X_t^*(s,\eta)) = \int_0^{\mathcal{X}(s,\eta)} \mathcal{Y}(t,z) d_z(X_t^*(z))$$

Finally, by proposition 3.5, $\left(\int_0^{\mathcal{X}(s,\eta)} \mathcal{Y}(t,z)d_z(X_t^*(z)),\ t \geq s \right)$ is a martingale and, hence, $\left(U(t, X_t^*(s,\eta)),\ t \geq s \right)$ is a martingale and the proof is complete. □

4 Return to Initial Market.

As our purpose at the beginning of this paper is to study the consistent utilities in the universe of investment describes in the paragraph 2, we give in this section the equivalents of the results of the previous paragraphs established in the framework of martingale market.

Theorem 4.1. *Let M be an equivalent local martingale and let $X^*_t(x)$ be an admissible strictly increasing with respect to the initial wealth. Denote by $\mathcal{X}(t,z)$ its inverse flow. Assume that X^*M is a* **martingale**. *Then for all utility function u such that $u_x(\mathcal{X}(t,z))$ is locally integrable near $z=0$, the stochastic process U defined by*

$$U(t,x) = M_t \int_0^x u_x(\mathcal{X}(t,z))dz, \quad U(t,0)=0 \tag{22}$$

is an $\mathscr{X}$-consistent utility in the initial market. The associated optimal wealth process is X^* and the optimal dual process is M. Further, the convex conjugate of U denoted by $\tilde{U}$, is given by

$$\tilde{U}(t,y) = \int_y^{+\infty} X^*(t, -\tilde{u}_y(\frac{z}{M_t}))dz, \tag{23}$$

The proof of this result proceeds exactly as the proof of Theorem 3.2 and it is based on Lemma 3.3 (established in this context using the fact that $(M_{s,t}X^*_t(s,\eta))_{t\geq s}$ is a martingale) and the equivalent of Lemma 3.4 which is

Lemma 4.2. *Denote by $(M_{s,t})_{t\geq s}$ the process $(M_t/M_s)_{t\geq s}$, for all $s\geq 0$ and for any s-admissible wealth test η,*

$$U(t, X^*_t(s,\eta)) = U_x(s,\eta)M_{s,t}X^*_t(s,\eta) - M_{s,t}\int_0^{\mathcal{X}(s,\eta)} X^*_t(z)du_x(z)$$

and it is a martingale.

Also the identity in this result is obtained by the same change of variable as in Lemma 3.4 and the martingale property is achieved using the fact that $(M_{s,t}X^*_t(s,\eta))_{t\geq s}$ is a martingale)

Now, let us turn to the equivalent of Theorem 3.6. For that purpose, it is important to to remind some notations and results of the previous section. We recall essentially the optimality conditions in this market.

Let $(X^*,\mathcal{Y})$ be a pair of processes satisfying

Assumption 4.1.

(O1') *The process $(X^*_t(x); x\geq 0, t\geq 0)$ is admissible wealth process strictly increasing from 0 to $+\infty$.*

(O2') *While $(\mathcal{Y}(t,x); x\geq 0, t\geq 0)$ is strictly decreasing from $+\infty$ to 0 such that $\mathcal{Y}(t,x)$ is a locally integrable near $x=0$.*

(OC) *For any initial positif wealth's* x, x' *and any admissible wealth process* $X \in \mathscr{X}(x')$, $(X_t(x') - X_t^*(x))\mathcal{Y}(t,x)); t \geq 0)$ *is a submartingale.*

Remark 4.1. *Note that this hypothesis is the same as assumptions 3.2 in the martingale market. In fact they are valid for the initial and the martingal market.*

In analogy to Theorem 3.6, we establish the following main result in the market described in 2.

Theorem 4.3 (Return to Initial Market)**.** *Let* $(X^*, \mathcal{Y})$ *be a pair of strictly positif processes satisfying assumptions 4.1 and 3.3. Let* $\mathcal{X}$ *the inverse flow of* X^* *then the concave increasing process* U *defined by*

$$U(t,x) = \int_0^x \mathcal{Y}(t, \mathcal{X}(t,z))dz \tag{24}$$

is an $\mathscr{X}$*-consistent stochastic utility on the martingale market with u as the initial function,* X^* *as the optimal wealth process. The optimal dual process is* $\mathcal{Y}(t,x)/\mathcal{Y}(0,x)$ *and the convex conjugate is given by*

$$\tilde{U}(t,y) = \int_y^{+\infty} X_t^*((\mathcal{Y})^{-1}(t,z))dz. \tag{25}$$

Proof. There is no modifications to be brought in the proof of Theorem 3.6 because the fact that the test-wealths $X \in \mathscr{X}$ are local martingales is not so important and only optimality conditions **(OC)** and monotony properties of X^* and $\mathcal{Y}$ are needed, and they are the same as in the market martingale as we saw above. ☐

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