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Minisuperspace limit of the AdS_3 WZNW model

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ABSTRACT: We derive the three-point function of the AdS_3 WZNW model in the minisuperspace limit by Wick rotation from the H_3^+ model. The result is expressed in terms of Clebsch-Gordan coefficients of the Lie algebra $s\ell(2,\mathbb{R})$. We also introduce a covariant basis of functions on AdS_3, which can be interpreted as bulk-boundary propagators.

Contents

1. Introduction

The AdS_3 Wess–Zumino–Novikov–Witten model is interesting in particular due to its string theory applications. A conjecture for the spectrum of this model was proposed by Maldacena and Ooguri [1], but the full solution of the model is still missing. In the sense of the conformal bootstrap, a full solution means the computation of the three-point functions of primary fields on the sphere, and the proof of crossing symmetry of the four-point functions. (Equivalently, the computation of operator product expansions of primary fields, and the proof of their associativity.)

The conjectured spectrum of the AdS_3 WZNW model is fairly complicated, as it contains both discrete and continuous series of representations of the symmetry algebra, and their images under the so-called spectral flow automorphism. On the other hand, as a geometrical space, AdS_3 is related by Wick rotation to the Euclidean space H_3^+, and the AdS_3 WZNW model is often assumed to be related to the H_3^+ WZNW model. The spectrum of the latter model is much simpler, as it contains only a continuous series of representations, and the H_3^+ model has been fully solved [2, 3]. An additional difficulty of the AdS_3 WZNW model is that the group AdS_3, which is the universal cover of $SL(2,\mathbb{R})$, has no realization as a group of finite-dimensional matrices. It follows that writing a simple basis of functions on AdS_3 is more difficult than in the cases of $SL(2,\mathbb{R})$ or H_3^+. Similarly, it is in general more complicated to write functions on the Anti-de Sitter space AdS_d

than on its Euclidean version H_d^+. Some works like [4] which are purportedly about AdS_d actually deal with H_d^+, thereby avoiding this difficulty (and other ones).

The presence of discrete representations and the lack of an obvious basis of functions on AdS_3 are two difficulties of the AdS_3 WZNW model which also affect its minisuperspace limit (also known as the zero-mode approximation), where the model reduces to the study of functions on the AdS_3 space. It is therefore interesting to solve the model in this limit. We will do this by starting from the well-understood minisuperspace H_3^+ model [5] and using Wick rotation. The main object we wish to compute is the minisuperspace analog of the operator product expansion, namely the product of functions on AdS_3. (Equivalently, the minisuperspace three-point function.)

We will start with a study of certain bases of functions on AdS_3, $SL(2,\mathbb{R})$ and H_3^+ (Section 2). In particular we will construct functions on AdS_3 which transform covariantly under the symmetries, and which can be interpreted as bulk-boundary propagators. Then we will study the Clebsch-Gordan coefficients of the Lie algebra $s\ell(2,\mathbb{R})$ (Section 3). Due to the symmetries of the AdS_3 WZNW model in the minisuperspace limit, the products of functions on AdS_3 can be expressed in terms of these coefficients. We will check this after obtaining these products of functions by Wick rotation from H_3^+ (Section 4). In conclusion we will comment on the Wick rotation and on the problem of solving the AdS_3 WZNW model (Section 5).

2. H_3^+, $SL(2,\mathbb{R})$, AdS_3 and functions thereon

In this section we will review the geometries of the spaces H_3^+, $SL(2,\mathbb{R})$ and AdS_3, and introduce bases of functions on these spaces. The sense in which such functions form bases of certain functional spaces will be explained in section 2.4. While H_3^+ and $SL(2,\mathbb{R})$ can be viewed as spaces of two-dimensional matrices, AdS_3 cannot, and this will make the descriptions of functions on AdS_3 more complicated.

2.1 Geometry and symmetry groups

Let us start with the group $SL(2,\mathbb{R})$ of real, size two matrices of determinant one. This group is not simply connected, since the subgroup of the matrices

$$g_\tau \equiv \left(\begin{smallmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{smallmatrix}\right) \tag{2.1}$$

is a non-contractible loop. Therefore, there exists a universal covering group, sometimes called $\widetilde{SL}(2,\mathbb{R})$, which we will call AdS_3. If $SL(2,\mathbb{R})$ elements are parametrized using three real coordinates (ρ,θ,τ) as

$$g = \begin{pmatrix} \cosh\rho\cos\tau + \sinh\rho\cos\theta & \sinh\rho\sin\theta + \cosh\rho\sin\tau \\ \sinh\rho\sin\theta - \cosh\rho\sin\tau & \cosh\rho\cos\tau - \sinh\rho\cos\theta \end{pmatrix} , \tag{2.2}$$

where θ and τ are 2π-periodic, then AdS_3 is obtained by decompactifying τ. Elements of AdS_3 can alternatively be parametrized as doublets $G = (g, I)$ where $g \in SL(2,\mathbb{R})$ and I is the integer part of $\frac{\tau}{2\pi}$. Writing $\frac{\tau}{2\pi} = I + F$, the group multiplication of AdS_3 can be written as

$$(g, I)(g', I') = (gg', I + I' + F(g) + F(g') - F(gg')) . \tag{2.3}$$

The $U(1)$ subgroup of the matrices $g_\tau \in SL(2,\mathbb{R})$ decompactifies into an $\mathbb{R}$ subgroup of elements $G_\tau \in AdS_3$, which we parametrize as

$$G_\tau = \exp \tau \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \ . \tag{2.4}$$

The group structures of AdS_3 and $SL(2,\mathbb{R})$ lead to left and right actions by group multiplication, in the AdS_3 case $(G_L, G_R)\cdot G = G_L G G_R$, which will be symmetries of the models under consideration. More precisely, the geometrical symmetry group of $SL(2,\mathbb{R})$ is $\frac{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}{\mathbb{Z}_2}$, where we must divide by the center $\mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\}$ of $SL(2,\mathbb{R})$ as its left and right actions are identical. The geometrical symmetry group of AdS_3 is $\frac{AdS_3\times AdS_3}{\mathbb{Z}}$, where the center of AdS_3 is freely generated by $(-\mathrm{id}, 0) = (\rho = 0, \theta = 0, \tau = \pi)$ and therefore isomorphic to $\mathbb{Z}$. These geometrical symmetry groups are not simply connected; their first fundamental groups are $\mathbb{Z}^2$ in the case of $SL(2,\mathbb{R})$ and $\mathbb{Z}$ in the case of AdS_3. This is the origin of the spectral flow symmetries of the corresponding WZNW models. (See for instance [1].)

Then H_3^+ is the space of hermitian, size two matrices of determinant one, which can be parametrized using three real coordinates (ρ, θ, τ) as

$$h = \begin{pmatrix} e^\tau \cosh\rho & e^{i\theta}\sinh\rho \\ e^{-i\theta}\sinh\rho & e^{-\tau}\cosh\rho \end{pmatrix} \ . \tag{2.5}$$

We have chosen identical names (ρ, θ, τ) for the coordinates on H_3^+ and AdS_3, thereby defining a bijection between these two spaces. This bijection gives rise to a map $\Phi(\rho,\theta,\tau) \to \Phi(\rho,\theta,i\tau)$ from the analytic functions on H_3^+ to the analytic functions on AdS_3, which is called the Wick rotation. Our bijection however does not relate the matrix forms of H_3^+ and $SL(2,\mathbb{R})$ which we have given. Notice that H_3^+ is not a group, rather a group coset, namely $SL(2,\mathbb{C})/SU(2)$. The geometrical symmetry group of H_3^+ is $\frac{SL(2,\mathbb{C})}{\mathbb{Z}_2}$, whose elements k act on $h \in H_3^+$ by $k\cdot h = khk^\dagger$.

2.2 Functions: t-bases

In both cases $SL(2,\mathbb{R})$ and H_3^+, the existence of a matrix realization allows us to write functions which transform very simply under the symmetries. In the case of H_3^+, we can indeed introduce the following "x-basis" of functions $\Phi^j_x(h)$, parametrized by their spin $j \in \mathbb{C}$ and isospin $x \in \mathbb{C}$:

$$\Phi^j_x(h) \equiv \frac{2j+1}{\pi} \left|(\bar{x}, 1) h \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right)\right|^{2j} \ \Rightarrow\ \Phi^j_x(khk^\dagger) = |cx+d|^{4j} \Phi^j_{\frac{ax+b}{cx+d}}(h) \ , \tag{2.6}$$

where we denote $k^\dagger = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$. Similarly, we can introduce the following "t-basis" of functions $\Phi^{j,\eta}_{t_L,t_R}(g)$ on $SL(2,\mathbb{R})$, with $(t_L, t_R) \in \mathbb{R}^2$:

$$\begin{aligned} \Phi^{j,\eta}_{t_L,t_R}(g) &\equiv \frac{2j+1}{\pi} \left|(1, -t_L) g \left(\begin{smallmatrix} t_R \\ 1 \end{smallmatrix} \right)\right|^{2j} \mathrm{sign}^{2\eta}(1,-t_L) g \left(\begin{smallmatrix} t_R \\ 1 \end{smallmatrix} \right) \ \Rightarrow\ \Phi^{j,\eta}_{t_L,t_R}(g_L^{-1} g g_R) \\ &= |(c_R t_R + d_R)(c_L t_L + d_L)|^{2j} \,\mathrm{sign}^{2\eta}(c_R t_R + d_R)(c_L t_L + d_L)\ \Phi^{j,\eta}_{\frac{a_L t_L + b_L}{c_L t_L + d_L}, \frac{a_R t_R + b_R}{c_R t_R + d_R}}(g) \ , \end{aligned} \tag{2.7}$$

where we denote $g_L = \left(\begin{smallmatrix} a_L & b_L \\ c_L & d_L \end{smallmatrix} \right)$ and $g_R = \left(\begin{smallmatrix} a_R & b_R \\ c_R & d_R \end{smallmatrix} \right)$. The parity $\eta \in \{0, \frac{1}{2}\}$ is the same for both actions of $SL(2,\mathbb{R})$ on itself by multiplications from the left and from the right (see the factor

$\text{sign}^{2\eta}(c_R t_R + d_R)(c_L t_L + d_L))$, because the parity characterizes the action of the central subgroup $\mathbb{Z}_2$.

In the case of AdS_3, writing a similar "t-basis" of functions is more complicated. We define t-basis functions $\Phi^{j,\alpha}_{t_L,t_R}$ on AdS_3 by the assumption that they transform covariantly under the left and right actions of AdS_3 on itself, in a way which generalizes the transformation property of the t-basis functions $\Phi^{j,\eta}_{t_L,t_R}$ on $SL(2,\mathbb{R})$ eq. (2.7). (The AdS_3 parameter $\alpha \in [0,1)$ generalizes the $SL(2,\mathbb{R})$ parameter $\eta \in \{0, \frac{1}{2}\}$.) The appropriate generalization of the transformation property has been written in [6] (Section 4.1); it involves a function $N(G|t)$ on $AdS_3 \times \mathbb{R}$ such that $N(G'G|t) = N(G'|Gt) + N(G|t)$ and $\forall n \in \mathbb{Z},\ N((\text{id}, I)|t) = I$, where if $G = (g, I) = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), I)$ then $Gt \equiv gt = \frac{at+b}{ct+d}$. For instance, $N(G|t)$ can be taken as the number of times $G't$ crosses infinity as G' moves from $(\text{id}, 0)$ to G, in which case $N(G|t) \in \mathbb{Z}$, and $[G] \equiv N(G|t) - \frac{1}{2}\text{sign}(t + \frac{d}{c}) - \frac{1}{2}$ is a t-independent integer. Then, the axiom for $\Phi^{j,\alpha}_{t_L,t_R}$ is

$$\Phi^{j,\alpha}_{t_L,t_R}(G_L^{-1} G G_R) = |(c_R t_R + d_R)(c_L t_L + d_L)|^{2j}\, e^{2\pi i\alpha(N(G_L|t_L) - N(G_R|t_R))} \Phi^{j,\alpha}_{G_L t_L, G_R t_R}(G) \tag{2.8}$$

This axiom is obeyed by

$$\Phi^{j,\alpha}_{t_L,t_R}(G) = \frac{2j+1}{\pi} e^{2\pi i\alpha n(G|t_L,t_R)} \left|(1, -t_L) g \left(\begin{smallmatrix} t_R \\ 1 \end{smallmatrix}\right)\right|^{2j} , \tag{2.9}$$

provided the function $n(G|t_L, t_R)$ satisfies

$$n(G_L^{-1} G G_R | t_L, t_R) - n(G | G_L t_L, G_R t_R) = N(G_L|t_L) - N(G_R|t_R) . \tag{2.10}$$

This implies that the function $n(\text{id}|t_L, t_R)$ should satisfy

$$n(\text{id}|t_L, G t_R) - n(\text{id}|G^{-1} t_L, t_R) = N(G^{-1}|t_L) + N(G|t_R) , \tag{2.11}$$

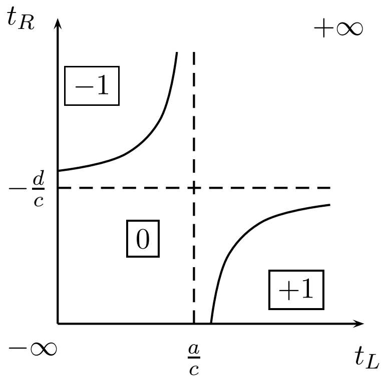
which, using $Gt = gt = \frac{at+b}{ct+d}$ and the properties of $N(G|t)$, amounts to

$$n(\text{id}|t_L, \tfrac{at_R+b}{ct_R+d}) - n(\text{id}|\tfrac{dt_L-b}{-ct_L+a}, t_R) = \tfrac{1}{2}\text{sign}(t_L - \tfrac{a}{c}) + \tfrac{1}{2}\text{sign}(t_R + \tfrac{d}{c}) . \tag{2.12}$$

A solution is found to be

$$n(\text{id}|t_L, t_R) = \tfrac{1}{2}\text{sign}(t_L - t_R) , \tag{2.13}$$

then $n(G|t_L, t_R) - \frac{1}{2}\text{sign}(t_L - t_R)$ is the number of times gt_L crosses t_R when g runs from id to G. Let us now study the behaviour of $n(G|t_L, t_R)$ as a function of t_L, t_R for a generic choice of G. Notice that $\tilde{n}(t_L, t_R) \equiv n(G|t_L, t_R) + [G] + \frac{1}{2} = \frac{1}{2}\left[\text{sign}\left((t_L - \frac{a}{c})(t_R + \frac{d}{c}) + \frac{1}{c^2}\right) - 1\right] \text{sign}(t_R + \frac{d}{c})$ takes values $0, \pm 1$, and jumps between these values occur on the hyperbola with equation $(1, -t_L) g \left(\begin{smallmatrix} t_R \\ 1 \end{smallmatrix}\right) = 0$. These values and these jumps are shown on the following plot:



(2.14)

We now propose that certain linear combinations of the functions $\Phi^{j,\alpha}_{t_L,t_R}$ can be interpreted as bulk-boundary propagators. These combinations are

$$\Phi^j_{(t_L,t_R,N)}(G) \equiv \int_0^1 d\alpha\, e^{-2i\pi\alpha N}\Phi^{j,\alpha}_{t_L,t_R}(G) = \delta_{N,n(G|t_L,t_R)}\left|(1,-t_L)g\left(\begin{smallmatrix} t_R \\ 1\end{smallmatrix}\right)\right|^{2j} . \quad (2.15)$$

We interpret $(t_L, t_R, N) \in \mathbb{R}\times\mathbb{R}\times\mathbb{Z}$ as coordinates on the boundary of AdS_3. The action of the symmetry group $AdS_3 \times AdS_3$ on the boundary is then given by

$$(G_L, G_R)\cdot(t_L,t_R,N) = (g_Lt_L, g_Rt_R, N - N(G_L|t_L) + N(G_R|t_R)) , \quad (2.16)$$

and the behaviour of $\Phi^{j,\alpha}_{t_L,t_R}$ under the action of $AdS_3\times AdS_3$ (2.8) implies the following behaviour of $\Phi^j_{(t_L,t_R,N)}$:

$$\Phi^j_{(t_L,t_R,N)}(G_L^{-1}GG_R) = |(c_Rt_R+d_R)(c_Lt_L+d_L)|^{2j}\,\Phi^j_{(G_L,G_R)\cdot(t_L,t_R,N)}(G) . \quad (2.17)$$

2.3 Functions: m-bases

The t-bases of functions behave simply under symmetry transformations, but t-bases in H_3^+ and AdS_3 are not related by the Wick rotation. This is because the matrix realizations (2.2) and (2.5) on which the t-bases are built are themselves not related by the Wick rotation. We will therefore introduce the more complicated “m-bases” of functions, which are better suited to the Wick rotation. In the case of H_3^+, the m-basis functions $\Phi^j_{m,\bar{m}}(h)$ are defined as

$$\Phi^j_{m,\bar{m}}(h) \equiv \int d^2x\, x^{-j-1+m}\bar{x}^{-j-1+\bar{m}}\Phi^j_x(h) \quad \text{with} \quad m-\bar{m}\in\mathbb{Z} . \quad (2.18)$$

The numbers $m, \bar{m}$ can be written in terms of an integer $n \in \mathbb{Z}$ and a momentum p, which is imaginary in the H_3^+ model:

$$m = \frac{1}{2}(n+p) \quad , \quad \bar{m} = \frac{1}{2}(-n+p) . \quad (2.19)$$

The explicit expression for $\Phi^j_{m,\bar{m}}(h)$ is found to be

$$\begin{aligned}\Phi^j_{m,\bar{m}}(h) = -4\frac{\Gamma(-j+\frac{|n|+p}{2})\Gamma(-j+\frac{|n|-p}{2})}{\Gamma(|n|+1)\Gamma(-2j-1)}e^{-p\tau+in\theta}\sinh^{|n|}\rho\,\cosh^p\rho \\ \times F(-j+\tfrac{|n|+p}{2}, j+1+\tfrac{|n|+p}{2}, |n|+1, -\sinh^2\rho) . \quad (2.20)\end{aligned}$$

Notice that this obeys the so-called reflection property

$$\begin{aligned}\Phi^j_{m,\bar{m}} = R^j_{m,\bar{m}}\Phi^{-j-1}_{m,\bar{m}} \quad , \quad R^j_{m,\bar{m}} &= \frac{\Gamma(2j+1)}{\Gamma(-2j-1)}\frac{\Gamma(-j+m)\Gamma(-j-\bar{m})}{\Gamma(j+1+m)\Gamma(j+1-\bar{m})} \quad (2.21) \\ &= \frac{\Gamma(2j+1)}{\Gamma(-2j-1)}\frac{\Gamma(-j+\frac{|n|+p}{2})\Gamma(-j+\frac{|n|-p}{2})}{\Gamma(j+1+\frac{|n|+p}{2})\Gamma(j+1+\frac{|n|-p}{2})} , \quad (2.22)\end{aligned}$$

where $R^j_{m,\bar{m}} = R^j_{\bar{m},m}$ due to $n = m-\bar{m}\in\mathbb{Z}$.

We will use the functions on AdS_3 obtained from the above functions $\Phi^j_{m,\bar{m}}(h)$ by the Wick rotation $\tau \to i\tau$. In order for the resulting functions to be delta-function normalizable, we now need to assume the momentum p to be real, instead of imaginary in the H_3^+ case. We do not introduce a new notation for the resulting functions on AdS_3, but still call them $\Phi^j_{m,\bar{m}}(G)$ or $\Phi^j_{m,\bar{m}}$.

In contrast to t-basis functions, m-basis functions on AdS_3 do not transform simply under the action of the $AdS_3 \times AdS_3$ symmetry group. However, they do transform simply under the action of the $\mathbb{R} \times \mathbb{R}$ subgroup made of pairs (G_{τ_L}, G_{τ_R}), where G_τ was defined by eq. (2.4):

$$\Phi^{j,\alpha}_{m,\bar{m}}(G_{\tau_L} G G_{\tau_R}) = e^{-2i(m\tau_L + \bar{m}\tau_R)} \Phi^{j,\alpha}_{m,\bar{m}}(G) \ . \tag{2.23}$$

Notice that the identity $G_\pi G G_{-\pi} = G$ implies $m - \bar{m} \in \mathbb{Z}$. (In the particular case of $SL(2,\mathbb{R})$, we have the additional identity $g_{2\pi} = \mathrm{id}$, which implies $m, \bar{m} \in \frac{1}{2}\mathbb{Z}$.) Introducing

$$\alpha \in [0,1) \qquad \text{such that} \qquad m, \bar{m} \in \alpha + \mathbb{Z} \ , \tag{2.24}$$

this parameter α is identical to the parameter α of the t-basis functions $\Phi^{j,\alpha}_{t_L,t_R}$ (2.9). We will look for a relation of the type

$$\Phi^j_{m,\bar{m}} = c^{j,\alpha} \int_{\mathbb{R}} dt_L \ (1+t_L^2)^{-j-1} e^{i\pi m} \left(\tfrac{1-it_L}{1+it_L}\right)^m \int_{\mathbb{R}} dt_R \ (1+t_R^2)^{-j-1} \left(\tfrac{1+it_R}{1-it_R}\right)^{\bar{m}} \Phi^{j,\alpha}_{t_L,t_R} \tag{2.25}$$

where $c^{j,\alpha}$ is a normalization factor. We can check that the right-hand side of this relation obeys the transformation property (2.23), thanks to the behaviour eq. (2.8) of $\Phi^{j,\alpha}_{t_L,t_R}$. To see this it is useful to notice that the integrand in eq. (2.25) is continuous through $t_L = \infty$ and $t_R = \infty$, as can be deduced from the behaviour of the phase factor $e^{2i\pi\alpha n(G|t_L,t_R)}$ of $\Phi^{j,\alpha}_{t_L,t_R}$, which is depicted in the diagram (2.14). This makes it possible to perform translations of the variables φ_L, φ_R such that $t_{L,R} = \tan \frac{1}{2}\varphi_{L,R}$. The normalization factor $c^{j,\alpha}$ is easily computed in the limit $\rho \to \infty$, where the dependences of the integrand on t_L and t_R factorize. We find

$$c^{j,\alpha} = \frac{4^{2j} \sin \pi 2j}{\sin \pi(j-\alpha) \ \sin \pi(j+\alpha)} \ . \tag{2.26}$$

2.4 Completeness of the bases of functions

We have been considering functions on a space X with $X \in \{H_3^+, SL(2,\mathbb{R}), AdS_3\}$. Given X, let us consider the space of complex-valued square-integrable functions $L^2(X)$ with the scalar product $\langle f, g\rangle = \int_X d\mu \ \bar{f} g$, where the invariant measure can be written in all three cases as $d\mu = \sinh 2\rho \ d\rho \ d\theta \ d\tau$. Although our functions Φ do not necessarily belong to $L^2(X)$, they form orthogonal bases in the same sense as $\{e^{ipq} | p \in \mathbb{R}\}$ is an orthogonal basis of the space of functions on $\mathbb{R}$. Namely, there exist sets B_X of values of the parameters and $\{\Phi_b, b \in B_X\}$ of the corresponding functions such that any pair (f,g) of smooth, compactly supported functions on X obeys $\langle f, g\rangle = \sum_{b\in B_X} N(b) \langle f, \Phi_b\rangle \langle \Phi_b, g\rangle$, where $N(b)$ is a normalization factor, and the sum $\sum_{b\in B_X}$ becomes an integral whenever it involves continuous parameters.

More specifically, the x-bases of functions are

$$H_3^+ : \quad \{\Phi_x^j | j \in -\tfrac{1}{2} + i\mathbb{R}_+, x \in \mathbb{C}\} , \tag{2.27}$$

$$\begin{aligned} SL(2,\mathbb{R}) : \quad & \{\Phi^{j,\eta}_{t_L,t_R} | j \in -\tfrac{1}{2} + i\mathbb{R}_+, (t_L,t_R) \in \mathbb{R}^2, \eta \in \{0,\tfrac{1}{2}\}\} \\ & \cup \{\Phi^{j,\eta}_{t_L,t_R} | j \in -1 - \tfrac{1}{2}\mathbb{N}, (t_L,t_R) \in \mathbb{R}^2, \eta = j \mod 1\} , \end{aligned} \tag{2.28}$$

$$\begin{aligned} AdS_3 : \quad & \{\Phi^{j,\alpha}_{t_L,t_R} | j \in -\tfrac{1}{2} + i\mathbb{R}_+, (t_L,t_R) \in \mathbb{R}^2, \alpha \in [0,1)\} \\ & \cup \{\Phi^{j,\alpha}_{t_L,t_R} | j \in (-\tfrac{1}{2},\infty), (t_L,t_R) \in \mathbb{R}^2, \alpha = j \mod 1\} , \end{aligned} \tag{2.29}$$

and the corresponding m-bases are

$$H_3^+ : \quad \{\Phi^j_{m,\bar{m}} | j \in -\tfrac{1}{2} + i\mathbb{R}_+, m + \bar{m} \in i\mathbb{R}, m - \bar{m} \in \mathbb{Z}\} , \tag{2.30}$$

$$\begin{aligned} SL(2,\mathbb{R}) : \quad & \{\Phi^j_{m,\bar{m}} | j \in -\tfrac{1}{2} + i\mathbb{R}_+, m + \bar{m} \in \tfrac{1}{2}\mathbb{Z}, m - \bar{m} \in \mathbb{Z}\} \\ & \cup \{\Phi^j_{m,\bar{m}} | j \in -1 - \tfrac{1}{2}\mathbb{N}, m, \bar{m} \in \pm(j+1+\mathbb{N})\} , \end{aligned} \tag{2.31}$$

$$\begin{aligned} AdS_3 : \quad & \{\Phi^j_{m,\bar{m}} | j \in -\tfrac{1}{2} + i\mathbb{R}_+, m + \bar{m} \in \mathbb{R}, m - \bar{m} \in \mathbb{Z}\} \\ & \cup \{\Phi^j_{m,\bar{m}} | j \in (-\tfrac{1}{2},\infty), m, \bar{m} \in \pm(j+1+\mathbb{N})\} . \end{aligned} \tag{2.32}$$

The completeness of both the x- and m-bases of functions on H_3^+ was proved in [5]. In the case of AdS_3, the completeness of the m-basis follows from the results of [7], where a Plancherel formula for AdS_3 was proved. The completeness of the t-basis then follows from the integral relation (2.25). The case of $SL(2,\mathbb{R})$ can be deduced from the case of AdS_3 by noting that functions on $SL(2,\mathbb{R})$ correspond to τ-periodic functions on AdS_3 with period 2π.

Moreover, in each case the basis $\{\Phi_b, b \in B_X\}$ provides a spectral decomposition of the Laplacian on X, which is Hermitian with respect to the scalar product $\langle f, g\rangle$. A function of spin j is an eigenvector of the Laplacian for the eigenvalue $-j(j+1)$. In the cases $X \in \{SL(2,\mathbb{R}), AdS_3\}$ this follows from the transformation properties of such functions under the symmetries, and the fact that the Laplacian coincides with the Casimir differential operators associated with these symmetries. In the case of H_3^+ this can be deduced from the case of AdS_3 by Wick rotation.

3. Representations and Clebsch-Gordan coefficients

3.1 Representations of $s\ell(2,\mathbb{R})$

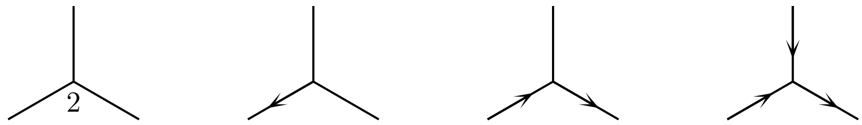
The minisuperspace limit of the spectrum of the AdS_3 WZNW model is the space of delta-function normalizable functions on AdS_3. It is subject to the action of the geometrical symmetry group $\frac{AdS_3 \times AdS_3}{\mathbb{Z}}$, and therefore of its Lie algebra $s\ell(2,\mathbb{R}) \times s\ell(2,\mathbb{R})$. Three types of unitary representations of $s\ell(2,\mathbb{R})$ appear in the minisuperspace spectrum: continuous representations, and two series of discrete representations. Continuous representations $C^{j,\alpha}$ are parametrized by a spin j and a number $\alpha \in [0,1)$ such that $m \in \alpha + \mathbb{Z}$. Discrete representations $D^{j,\pm}$ are parametrized by a spin $j \in (-\frac{1}{2},\infty)$, and their states obey $m \in \pm(j+1+\mathbb{N})$. All these representations of $s\ell(2,\mathbb{R})$ extend to representations of the group AdS_3. However, only representations with $m \in \frac{1}{2}\mathbb{Z}$, namely $C^{j,\alpha}$ with $\alpha \in \frac{1}{2}\mathbb{Z}$ and $D^{j,\pm}$ with $j \in \frac{1}{2}\mathbb{N}$, extend to representations of the group $SL(2,\mathbb{R})$.

More precisely, given the $s\ell(2,\mathbb{R})$ algebra with generators $J^3, J^\pm$ and relations $[J^3, J^\pm] = \pm J^\pm$, $[J^+, J^-] = -2J^3$, the spin j is defined by $(J^3)^2 - \frac{1}{2}(J^+J^- + J^-J^+) = j(j+1)$, and the states $|m\rangle$ are such that

$$J^3|m\rangle = m|m\rangle \quad , \quad J^+|m\rangle = (m+j+1)|m+1\rangle \quad , \quad J^-|m\rangle = (m-j-1)|m-1\rangle \ . \tag{3.1}$$

These conventions are incompatible with the unit normalization of the states (which would mean $\langle m|m'\rangle = \delta_{m,m'}$), however they will turn out to agree with the behaviour of our functions $\Phi^j_{m,\bar{m}}$ eq. (2.20).

The tensor product laws for $s\ell(2,\mathbb{R})$ representations are well-known. They are equivalent to knowing the three-point invariants, which we schematically depict here in the cases when they do not vanish:



$$C\otimes C\otimes C \quad , \quad D^-\otimes C\otimes C \quad , \quad D^-\otimes D^+\otimes C \quad , \quad D^+\otimes D^+\otimes D^- \tag{3.2}$$

For instance, the first diagram means that any continuous representation $C^{j,\alpha}$ appears twice in the tensor product $C^{j_1,\alpha_1}\otimes C^{j_2,\alpha_2}$ of two continuous representations. (The m-conservation rule $\alpha = \alpha_1+\alpha_2 \mod 1$ is implicitly assumed.) The fourth diagram means that $D^{j,+}\subset D^{j_1,+}\otimes D^{j_2,+}$. (The rule $j\in j_1+j_2+1+\mathbb{N}$ is implicitly assumed.) The fourth diagram also means that $D^{j,-}$ may appear once in $D^{j_1,-}\otimes D^{j_2,+}$. (This happens if $j\in j_2-j_1-1-\mathbb{N}$.) We omit the diagrams obtained by reverting the arrows in the second and fourth diagrams, namely $D^+\otimes C\otimes C$ and $D^-\otimes D^-\otimes D^+$.

3.2 Clebsch-Gordan coefficients: m basis

We will rederive the tensor product rules by studying the Clebsch-Gordan coefficients. These coefficients are the three-point invariants, viewed as functions $C(j_1,j_2,j_3|m_1,m_2,m_3)$ subject to the equations

$$\sum_{i=1}^{3}(m_i+j_i+1)C(m_i+1) = \sum_{i=1}^{3}(m_i-j_i-1)C(m_i-1) = \sum_{i=1}^{3} m_i C = 0 \ . \tag{3.3}$$

It is of course possible to prove a priori that these equations are obeyed by the three-point function $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i}\right\rangle \equiv \int_{AdS_3} dG \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i}(G)$. To do this, we would introduce a realization of the Lie algebra $s\ell(2,\mathbb{R})$ as first-order differential operators D^a wrt ρ,θ,τ, such that $D^+\Phi^j_{m,\bar{m}}(G) = (m+j+1)\Phi^j_{m,\bar{m}}(G)$, $D^-\Phi^j_{m,\bar{m}}(G) = (m-j-1)\Phi^j_{m,\bar{m}}(G)$ and $D^3\Phi^j_{m,\bar{m}}(G) = m\Phi^j_{m,\bar{m}}(G)$. Then eq. (3.3) would follow from the identity $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i}(G)\right\rangle = \left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i}(G_L G)\right\rangle$. We however abstain from doing this, as we will later explicitly compute the three-point function $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i}\right\rangle$ and write it in terms of solutions of the equation (3.3).

Given three irreducible representations of $s\ell(2,\mathbb{R})$, there exist zero, one or two linearly independent solutions of the equation (3.3). In the case of three continuous representations, the momenta (m_1,m_2) belong to a two-dimensional lattice of the type $\prod_{i=1}^2(\alpha_i+\mathbb{Z})$ (with of course $m_3 = -m_1-m_2$), and the coefficients $m_i\pm(j_i+1)$ never vanish. In this situation, a solution of eq. (3.3) is determined once the values of C at two neighbouring points of the lattice are given.

In the case when at least one representation is discrete, say $m_1 \in j_1 + 1 + \mathbb{N}$, the lattice becomes semi-infinite in one direction, and a solution is determined once the value of C at one point is given.

Let us introduce the function

$$G\begin{pmatrix} a & b & c \\ & e & f \end{pmatrix} \equiv \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)}\ {}_3F_2\left(\begin{matrix} a & b & c \\ & e & f \end{matrix}\middle| 1\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{\Gamma(e+n)\Gamma(f+n)} , \qquad (3.4)$$

where the sum converges provided $a+b+c-e-f<0$, and the poles of G are the same as those of $\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(e+f-a-b-c)$. Knowing the identity

$$(a-e+1)G\begin{pmatrix} a & b & c \\ & e & f \end{pmatrix} + (b-f)G\begin{pmatrix} a+1 & b & c \\ & e & f+1 \end{pmatrix} + (c-1)G\begin{pmatrix} a & b & c-1 \\ & e-1 & f \end{pmatrix} = 0 , \qquad (3.5)$$

we can use this function for writing a solutions of eq. (3.3):

$$C = \delta(m_1+m_2+m_3)\, G\begin{pmatrix} & -j_2+m_2 & -j_3-m_3 & -j_{23}^1 \\ 1+j_1-j_3+m_2 & 1+j_1-j_2-m_3 & \end{pmatrix} \equiv \delta(m_1+m_2+m_3) g^{23} \qquad (3.6)$$

which is well-defined provided $2+j_{123}>0$, where we use the notations $j_{123}=j_1+j_2+j_3$ and $j_{23}^1 \equiv j_2+j_3-j_1$. Of course five other solutions of the type g^{ab} with $a \neq b \in \{1,2,3\}$ can be obtained by permutations of indices. These solutions are not linearly independent, as can be shown with the help of the identity

$$\begin{aligned} s(b)s(c-a)G\begin{pmatrix} a & b & c \\ & e & f \end{pmatrix} &= s(e-a)s(f-a)G\begin{pmatrix} a & a-e+1 & a-f+1 \\ & a-b+1 & a-c+1 \end{pmatrix} \\ &\quad - s(c-e)s(c-f)G\begin{pmatrix} c & c-e+1 & c-f+1 \\ & c-a+1 & c-b+1 \end{pmatrix} , \end{aligned} \qquad (3.7)$$

where $s(x) \equiv \sin \pi x$, and we will also use $c(x) \equiv \cos \pi x$. Thus we obtain

$$\begin{pmatrix} g^{21} \\ g^{12} \end{pmatrix} = M_{13} \begin{pmatrix} g^{32} \\ g^{23} \end{pmatrix} \quad , \quad M_{13} = \frac{1}{s(j_{12}^3)} \begin{pmatrix} \frac{s(j_2+m_2)s(j_3-m_3)}{s(j_1+m_1)} & \frac{s(j_3+m_3)s(j_3-j_1+m_2)}{s(j_1+m_1)} \\ \frac{s(j_3-m_3)s(j_3-j_1-m_2)}{s(j_1-m_1)} & \frac{s(j_2-m_2)s(j_3+m_3)}{s(j_1-m_1)} \end{pmatrix} . \qquad (3.8)$$

Together with the other identities obtained by permuting the indices, this shows that at most two of the solutions g^{ab} are linearly independent.

Due to our convention $j \in (-\frac{1}{2}, \infty)$ for discrete representations, we have $\Re j \geq -\frac{1}{2}$ for all representations of interest. This ensures that the sum in eq. (3.4) converges, so that g^{ab} is well-defined provided the summand is finite, which occurs unless $\Gamma(-j_a+m_a)\Gamma(-j_b-m_b)\Gamma(-j_{ab}^c)$ has a pole.

Case $C \otimes C \otimes C$. In this case, two given solutions say g^{23}, g^{32} are linearly independent, and they provide a basis of the two-dimensional space of invariants.

Case $D^- \otimes C \otimes C$**.** We assume for example $m_2 = -j_2 - 1 - \ell$ with $\ell \in \mathbb{N}$. Some relations of the type of eq. (3.8) simplify, and we find

$$g^{21} = (-1)^\ell \frac{s(j_3 + m_3)}{s(j_1 + m_1)} g^{23} = -\frac{s(j_{13}^2)}{s(2j_2)} g^{31} = -(-1)^\ell \frac{s(j_3 + m_3)}{s(j_1 + m_1)} \frac{s(j_{13}^2)}{s(2j_2)} g^{13} \,. \tag{3.9}$$

Since the space of invariants is one-dimensional, the two remaining functions g^{12} and g^{32} must also be proportional to the other four. However, this proportionality relation is not very simple, as can be seen in the case of the highest-weight state $\ell = 0$ when g^{12} and g^{32} fail to become expressible as products of Γ-functions, in contrast to the other four solutions.

Case $D^+ \otimes C \otimes C$**.** The situation is completely analogous to the previous case. We assume for example $m_2 = j_2 + 1 + \ell$ with $\ell \in \mathbb{N}$ and find

$$g^{12} = (-1)^\ell \frac{s(j_3 - m_3)}{s(j_1 - m_1)} g^{32} = -\frac{s(j_{13}^2)}{s(2j_2)} g^{13} = -(-1)^\ell \frac{s(j_3 - m_3)}{s(j_1 - m_1)} \frac{s(j_{13}^2)}{s(2j_2)} g^{31} \,. \tag{3.10}$$

Case $D^- \otimes D^- \otimes C$**.** We expect no invariants to exist in this case. Let us check this, assuming for example $m_2 \in -j_2 - 1 - \ell_2$ and $m_3 \in -j_3 - 1 - \ell_3$ with $\ell_2, \ell_3 \in \mathbb{N}$. Equation (3.9) implies two incompatible relations between g^{21} and g^{31}, which must therefore both vanish. An apparent paradox comes from the non-vanishing of $g^{12}, g^{32}, g^{23}, g^{13}$. However, these functions do not provide solutions to eq. (3.3), because they become infinite at $\ell_2 = -1$ or $\ell_3 = -1$. For instance, if $C(m_1, -j_2, m_3) = \infty$, then $(m_1 + j_1 + 1)C(m_1 + 1, -j_2 - 1, m_3) + (-j_2 + j_2)C(m_1, -j_2, m_3) + (m_3 + j_3 + 1)C(m_1, -j_2 - 1, m_3 + 1) = 0$ may have an unwanted nonvanishing second term. Therefore, the analysis of g^{ab} agrees with the representation-theoretic expectations that no invariant exists.

Case $D^+ \otimes D^- \otimes C$**.** We assume for example $m_2 = -j_2 - 1 - \ell_2$ and $m_3 = j_3 + 1 + \ell_3$ with $\ell_2, \ell_3 \in \mathbb{N}$. We find the relations

$$g^{31} = g^{12} = -\frac{s(2j_3)}{s(j_{12}^3)} g^{13} = -\frac{s(2j_2)}{s(j_{13}^2)} g^{21} = \frac{s(2j_2)s(2j_3)}{s(j_{13}^2)s(j_{12}^3)} g^{23} \,. \tag{3.11}$$

The functions g^{13}, g^{21}, g^{23} stay finite for any values $\ell_2, \ell_3 \in \mathbb{Z}$, and therefore provide three proportional invariants. The functions g^{31} and g^{12} become infinite if $\ell_3 < 0$ and $\ell_2 < 0$ respectively, so that it is not a priori clear that they provide invariants. That they actually do is guaranteed by the above relations.

Case $D^- \otimes D^- \otimes D^+$**.** We assume for example $m_1 = j_1 + 1 + \ell_1$, $m_2 \in -j_2 - 1 - \ell_2$ and $m_3 \in -j_3 - 1 - \ell_3$, with $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$. Noticing $s(2j_2)s(2j_3) = s(j_{13}^2)s(j_{12}^3)$, we find the relations

$$g^{12} = g^{23} = -\frac{s(2j_3)}{s(j_{12}^3)} g^{13} = -\frac{s(2j_3)}{s(j_{12}^3)} g^{32} \,. \tag{3.12}$$

These four proportional functions provide the invariant in this case. In particular, g^{12} and g^{13} no longer become infinite at $\ell_2 = -1$ or $\ell_3 = -1$ respectively, as happened in the case $D^- \otimes D^- \otimes C$. The remaining two functions g^{21}, g^{31} vanish, as they already did in the case $D^- \otimes D^- \otimes C$. Notice that the selection rule $j_1 \in j_2 + j_3 + 1 + \mathbb{N}$ manifests itself as g^{23} becoming infinite if $j_1 \in j_2 + j_3 - \mathbb{N}$, due to a series of poles which correspond to those of $\Gamma(-j_{23}^1)$.

3.3 Clebsch-Gordan coefficients: t-basis

Our m-basis invariants g^{ab} are not symmetric under permutations of the indices, but the equation (3.3) which they solve is. In the t-basis, we will now show that there exist natural permutation-symmetric invariants, and we will relate them to combinations of the g^{ab} invariants. A similar analysis was already performed for the Clebsch-Gordan coefficients of $SO(2,1)=\frac{SL(2,\mathbb{R})}{\mathbb{Z}_2}$, in the articles [8, 9]. The representations of $SO(2,1)$ correspond to representations of $s\ell(2,\mathbb{R})$ such that $m\in\mathbb{Z}$, or in other words $\alpha=0$. We will perform the generalization to arbitrary values of α.

In the t-basis, AdS_3 invariants should be solutions of

$$C(\{t_i\})=C(\{\tfrac{at_i+b}{ct_i+d}\})\prod_{i=1}^{3}|ct_i+d|^{2j_i}e^{-2i\pi\alpha_i N(G|t_i)}\ , \tag{3.13}$$

for any AdS_3 element G whose projection onto $SL(2,\mathbb{R})$ is $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. Solutions exist provided $\alpha_1+\alpha_2+\alpha_3\in\mathbb{Z}$, and we will assume $\alpha_1+\alpha_2+\alpha_3=0$. The solutions are [10]

$$C(\{t_i\})=|t_{12}|^{j_{12}^3}|t_{23}|^{j_{23}^1}|t_{31}|^{j_{31}^2}\ e^{i\pi(\alpha_{12}\text{sign}t_{12}+\alpha_{23}\text{sign}t_{23}+\alpha_{31}\text{sign}t_{31})}\ , \tag{3.14}$$

where $\alpha_{12},\alpha_{23},\alpha_{31}$ should obey the equation $\alpha_{12}-\alpha_{23}=\alpha_2\mod\mathbb{Z}$ and the two other equations obtained by even permutations thereof. The solutions to such equations are

$$\alpha_{ab}=\tfrac{1}{3}(\alpha_b-\alpha_a)+\alpha_0\ , \tag{3.15}$$

where α_0 is an arbitrary constant. (In $SL(2,\mathbb{R})$ we have $\alpha_a=\eta_a\in\frac{1}{2}\mathbb{Z}$ and it is more convenient to adopt the convention $\alpha_{ab}=\eta_a+\eta_b+\alpha_0$. Equivalently, we can use the above formula provided we assume $\alpha_a\in\frac{3}{2}\mathbb{Z}$.) Let us now apply the change of basis (2.25) to $C(\{t_i\})$. After the change of integration variables $t_i=\tan\frac{\varphi_i}{2}$ the m-basis version of $C(\{t_i\})$ is

$$\begin{aligned}C(\{m_i\})=\prod_{i=1}^{3}\left[\int_{-\pi}^{\pi}d\varphi_i\ e^{im_i\varphi_i}\right]|\sin\tfrac{1}{2}\varphi_{12}|^{j_{12}^3}|\sin\tfrac{1}{2}\varphi_{23}|^{j_{23}^1}|\sin\tfrac{1}{2}\varphi_{31}|^{j_{13}^2}\\ \times e^{i\pi\left(\alpha_{12}\text{sign}\sin\frac{1}{2}\varphi_{12}+\alpha_{23}\text{sign}\sin\frac{1}{2}\varphi_{23}+\alpha_{31}\text{sign}\sin\frac{1}{2}\varphi_{31}\right)}\ ,\end{aligned} \tag{3.16}$$

where the integrand has the same values at $\varphi_i=\pi$ and $\varphi_i=-\pi$. This integral can be explicitly evaluated by generalizing the computations of [8], and in particular using the formula, valid for $\varphi\in(-\pi,\pi)$:

$$\left|\sin\tfrac{\varphi}{2}\right|^{-2a}e^{i\pi\alpha\text{sign}\sin\frac{1}{2}\varphi}=s(a+\alpha)e^{-i\pi\alpha}2^{2a}\Gamma(1-2a)\sum_{n=-\infty}^{\infty}e^{i(n+\alpha)\varphi}\frac{\Gamma(a+n+\alpha)}{\Gamma(1-a+n+\alpha)} \tag{3.17}$$

Thus, we find

$$C(\{m_i\})=-\delta(\textstyle\sum m_i)2^{-j_{123}}\frac{1}{\pi^2}\Gamma(1+j_{12}^3)\Gamma(1+j_{23}^1)\Gamma(1+j_{31}^2)\ e^{-i\pi(\alpha_{12}+\alpha_{23}+\alpha_{31})}\ g^{\alpha_0}\ , \tag{3.18}$$

where we introduce the invariants

$$\begin{aligned}g^{\alpha_0}=\pi^2 s(\tfrac{1}{2}j_{12}^3-\alpha_{12})s(\tfrac{1}{2}j_{23}^1-\alpha_{23})s(\tfrac{1}{2}j_{31}^2-\alpha_{31})\\ \times\sum_{n\in\mathbb{Z}}\frac{\Gamma(-\frac{1}{2}j_{12}^3+\alpha_{12}+n)}{\Gamma(1+\frac{1}{2}j_{12}^3+\alpha_{12}+n)}\frac{\Gamma(-\frac{1}{2}j_{13}^2+m_1+\alpha_{12}+n)}{\Gamma(1+\frac{1}{2}j_{13}^2+m_1+\alpha_{12}+n)}\frac{\Gamma(-\frac{1}{2}j_{23}^1-m_2+\alpha_{12}+n)}{\Gamma(1+\frac{1}{2}j_{23}^1-m_2+\alpha_{12}+n)}\ ,\end{aligned} \tag{3.19}$$

which can be expressed in terms of the invariants g^{ab} (3.6) as

$$g^{\alpha_0} = s(\tfrac{1}{2}j_{13}^2 - \alpha_{31})s(j_3 - \alpha_3)s(j_1 + \alpha_1)g^{31} + s(\tfrac{1}{2}j_{13}^2 + \alpha_{31})s(j_1 - \alpha_1)s(j_3 + \alpha_3)g^{13} \quad (3.20)$$

In this formula, g^{α_0} depends on the paramter α_0 only through α_{31} eq. (3.15). There are two particularly interesting special values of α_0, namely 0 and $\frac{1}{2}$. In the case $\alpha_0 = 0$ then $C(\{t_i\})$ and g^0 are invariant under permutations. In the case $\alpha_0 = \frac{1}{2}$ then $C(\{t_i\})$ and $g^{\frac{1}{2}}$ are odd under permutations i.e. invariant up to a sign. When the three involved representations are continuous, g^0 and $g^{\frac{1}{2}}$ can serve as a basis of the two-dimensional space of invariants. Notice that in the case of $SL(2,\mathbb{R})$ only these two values of α_0 are possible.

4. Products of functions

In the conformal bootstrap approach to the AdS_3 WZNW model, all correlation functions can in principle be constructed from the knowledge of three objects: the spectrum, the two-point correlation functions on a sphere, and the operator product expansions – or equivalently the three-point correlation functions on a sphere. We will now determine these objects in the minisuperspace limit. We first recall their definitions. Given n functions $\Phi^i(G)$ on AdS_3, the corresponding correlation function is $\left\langle \prod_{i=1}^n \Phi^i \right\rangle \equiv \int dG\ \prod_{i=1}^n \Phi^i(G)$ where dG is the invariant measure. If $\{\Phi^i\}_{i\in S}$ form an orthogonal basis of the spectrum (that is $\left\langle \Phi^i\Phi^j \right\rangle = 0$ if $i \neq j$), the product of functions is schematically $\Phi^1\Phi^2 = \sum_{i\in S} \frac{\left\langle \Phi^1\Phi^2\Phi^i \right\rangle}{\left\langle \Phi^i\Phi^i \right\rangle}\Phi^i$. This product is obviously associative and commutative.

The functions $\Phi^j_{m,\bar{m}}(G)$ on AdS_3 are related to corresponding functions $\Phi^j_{m,\bar{m}}(h)$ on H_3^+ by a Wick rotation, and therefore their correlation functions can be deduced from H_3^+ correlation functions by that Wick rotation. This will involve some subtleties, because the discrete representations which appear in the minisuperspace spectrum on AdS_3 are absent in H_3^+. But let us first review the products of functions on H_3^+.

4.1 Products of functions on H_3^+

The minisuperspace spectrum of H_3^+ is generated by the functions

$$\{\Phi^j_{n,p}(h) | j \in -\tfrac{1}{2} + i\mathbb{R},\ n \in \mathbb{Z},\ p \in i\mathbb{R}\}\ . \quad (4.1)$$

The correlation functions $\left\langle \prod_{i=1}^n \Phi^{j_i}_{n_i,p_i} \right\rangle \equiv \int dh\ \prod_{i=1}^n \Phi^{j_i}_{n_i,p_i}(h)$ are obtained by integrating products of such functions with respect to the invariant measure $dh = \sinh 2\rho\ d\rho\ d\theta\ d\tau$. The two-point functions can be computed from the expression (2.20) of $\Phi^j_{n,p}(h)$:

$$\left\langle \Phi^j_{n,p}\Phi^{j'}_{n',p'} \right\rangle = 128\pi^2\delta_{n+n',0}\ \delta(p+p')\left[\delta(j+j'+1) + R^j_{n,p}\delta(j-j')\right]\ , \quad (4.2)$$

where the reflection coefficient $R^j_{n,p}$ was defined in eq. (2.21). A similar direct computation of the three-point function seems complicated. Instead, we will make use of the known x-basis three-point function [5]

$$\left\langle \prod_{i=1}^3 \Phi^{j_i}_{x_i} \right\rangle = C(j_1,j_2,j_3)|x_{12}|^{2j_{12}^3}|x_{23}|^{2j_{23}^1}|x_{31}|^{2j_{31}^2}\ , \quad (4.3)$$

$$C(j_1,j_2,j_3) \equiv \pi^{-3}\Gamma(-j_{123}-1)\frac{\Gamma(-j_{12}^3)\Gamma(-j_{23}^1)\Gamma(-j_{31}^2)}{\Gamma(-2j_1-1)\Gamma(-2j_2-1)\Gamma(-2j_3-1)}\ . \quad (4.4)$$

The transformation to the m-basis (2.18) can be performed thanks to an integral formula of Fukuda and Hosomichi [11]. The result can be written in terms of the Clebsch-Gordan coefficients g^{ab} (3.6):

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle = C(j_1,j_2,j_3)\, \delta^{(2)}(\textstyle\sum m_i)\, K^2 s(j^2_{13})^2 s(j^1_{23})^2$$
$$\times \left[g^{23}\bar{g}^{31} + \frac{s(2j_2)}{s(j^2_{13})} g^{23}\bar{g}^{32} + \frac{s(2j_1)}{s(j^1_{23})} g^{13}\bar{g}^{31} + g^{13}\bar{g}^{32} \right] , \quad (4.5)$$

where $\bar{g}^{ab}$ denotes g^{ab} with m_i replaced by $\bar{m}_i$, and we introduced the factor

$$K \equiv \frac{1}{\pi^2} \Gamma(1+j^3_{12})\Gamma(1+j^2_{13})\Gamma(1+j^1_{23}) \, . \quad (4.6)$$

It can be checked that the two- and three-point functions have the behaviour under reflection which is expected from the behaviour of $\Phi^j_{m,\bar{m}}$ (2.21). Now, using the three-point function, products of functions on H_3^+ can be written as

$$\Phi^{j_1}_{m_1,\bar{m}_1} \Phi^{j_2}_{m_2,\bar{m}_2} = \frac{1}{256\pi^2} \int_{-\frac{1}{2}+i\mathbb{R}} dj_3 \, \left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \frac{1}{R^{j_3}_{m_3,\bar{m}_3}} \Phi^{j_3}_{-m_3,-\bar{m}_3} \, , \quad (4.7)$$

where we use the notation $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle = \delta^{(2)}(m_1+m_2+m_3) \left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle'$.

The invariance of the three-point function (4.5) under permutations of the indices is not manifest, but can be checked using linear relations between the g^{ab} such as eq. (3.8). It seems that a reasonably simple, manifestly permutation-symmetric expression exists only in the case $m_i, \bar{m}_i \in \eta_i + \mathbb{Z}$ with $\eta_i \in \{0, \frac{1}{2}\}$, which corresponds to functions on $SL(2,\mathbb{R})$. In this case, we can use the invariants $g^0, g^{\frac{1}{2}}$ (3.20), and we find

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle = -\tfrac{1}{2\pi^5} C(j_1,j_2,j_3)\, \delta_{\sum m_i,0} \delta_{\sum \bar{m}_i,0} \, \frac{\gamma(1+j^3_{12})\gamma(1+j^1_{23})\gamma(1+j^2_{31})}{\prod_{i=1}^3 s(j_i+\eta_i)}$$
$$\times \sum_{\epsilon\in\{0,\frac{1}{2}\}} \frac{s(\frac{1}{2}j_{123}+\epsilon)}{s(\frac{1}{2}j^3_{12}+\eta_3+\epsilon) s(\frac{1}{2}j^1_{23}+\eta_1+\epsilon) s(\frac{1}{2}j^2_{31}+\eta_2+\epsilon)} \, g^\epsilon \bar{g}^\epsilon \, , \quad (4.8)$$

where we use $\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$. From this, we can reconstruct the t-basis three-point function

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i,\eta_i}_{t_i^L,t_i^R} \right\rangle = \frac{2^{2j_{123}}}{2\pi^4} C(j_1,j_2,j_3) \prod_{i=1}^{3} c(j_i-\eta_i) \times$$
$$\left|t^L_{12}t^R_{12}\right|^{j^3_{12}} \left|t^L_{23}t^R_{23}\right|^{j^1_{23}} \left|t^L_{31}t^R_{31}\right|^{j^2_{31}} e^{i\pi\left[\eta_3(\mathrm{sign} t^L_{12}+\mathrm{sign} t^R_{12})+\eta_1(\mathrm{sign} t^L_{23}+\mathrm{sign} t^R_{23})+\eta_2(\mathrm{sign} t^L_{31}+\mathrm{sign} t^R_{31})\right]} \times$$
$$\sum_{\epsilon\in\{0,\frac{1}{2}\}} s(\tfrac{1}{2}j_{123}-\epsilon) c(\tfrac{1}{2}j^3_{12}+\eta_3+\epsilon) c(\tfrac{1}{2}j^1_{23}+\eta_1+\epsilon) c(\tfrac{1}{2}j^2_{31}+\eta_2+\epsilon) \, \left(\mathrm{sign} t^L_{12}t^R_{12}t^L_{23}t^R_{23}t^L_{31}t^R_{31}\right)^{2\epsilon} \, .$$
$$(4.9)$$

Comparing this formula to the H_3^+ three-point function in the x-basis eq. (4.3), we obtain a confirmation of the lack of a simple relation between the x-basis in H_3^+ and the t-basis in $SL(2,\mathbb{R})$ or AdS_3.

In the more general case of functions on AdS_3, the three-point function can still be expressed in terms of the invariants g^0 and $g^{\frac{1}{2}}$, but the formula is more complicated than eq. (4.8) and in particular the "mixed" terms $g^0\bar{g}^{\frac{1}{2}}$ and $g^{\frac{1}{2}}\bar{g}^0$ are present. Their absence in the case of $SL(2,\mathbb{R})$ can be attributed to the exterior automorphism ω of $SL(2,\mathbb{R})$, namely $\omega(g) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) g \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, which is such that $\Phi^{j,\eta}_{t_L,t_R}(\omega(g)) = (-1)^{2\eta}\Phi^{j,\eta}_{-t_L,-t_R}(g)$. In the case of AdS_3 this action still exists and can be expressed as $\omega(\rho,\theta,\tau) = (\rho,-\theta,-\tau)$. But it does not act simply on the function $\Phi^{j,\alpha}_{t_L,t_R}(G)$.[1]

4.2 Products of functions on AdS_3

Functions $\Phi^j_{m,\bar{m}}$ on AdS_3 are obtained from the corresponding functions on H_3^+ by performing the Wick rotation $\tau \to i\tau$ and continuing $p = m + \bar{m}$ from $i\mathbb{R}$ to $\mathbb{R}$. If we do not modify the value of the spin $j \in -\frac{1}{2} + i\mathbb{R}$, this yields functions transforming in the continuous representations of $AdS_3 \times AdS_3$, namely $\Phi^j_{m,\bar{m}} \in C^{j,\alpha} \otimes C^{j,\alpha}$ where $m,\bar{m} \in \alpha + \mathbb{Z}$. We may in addition obtain functions transforming in the discrete representation by continuing j to real values such that $m,\bar{m} \in \pm j \pm \mathbb{Z}$. More precisely, functions $\Phi^j_{m,\bar{m}} \in D^{j,\pm} \otimes D^{j,\pm}$ correspond to

$$j \in (-\tfrac{1}{2},\infty) \quad \text{and} \quad m,\bar{m} \in \pm(j+1+\mathbb{N}) \Leftrightarrow -j-1-\tfrac{1}{2}|n| \pm \tfrac{1}{2}p \in \mathbb{N} \tag{4.10}$$

$$\text{or} \quad j \in (-\infty,-\tfrac{1}{2}) \quad \text{and} \quad m,\bar{m} \in \pm(-j+\mathbb{N}) \Leftrightarrow j-\tfrac{1}{2}|n| \pm \tfrac{1}{2}p \in \mathbb{N}\,. \tag{4.11}$$

These two possibilities are related by the reflection $j \to -j-1$ and they are equivalent. We will only consider the first possibility, because our invariants g^{ab} (3.6) are well-defined for $\Re j \geq -\frac{1}{2}$. We will see that the set of these discrete and continuous functions is closed under products, consistently with the fact that they generate the space of functions on AdS_3 as we saw in section 2.4.

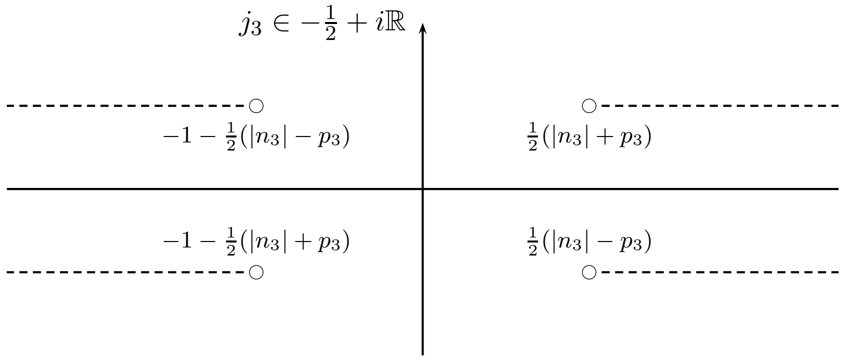
We now derive the products of functions on AdS_3 by continuing the products of functions on H_3^+ (4.7) to the relevant values of spins j and momenta p. We will examine various cases, according to the nature – discrete or continuous – of the fields $\Phi^{j_1}_{m_1,\bar{m}_1}$ and $\Phi^{j_2}_{m_2,\bar{m}_2}$. For example, the case when $j_1 \in -\frac{1}{2}+i\mathbb{R}$ and $m_2,\bar{m}_2 \in j_2+1+\mathbb{Z}$ will be denoted $C \times D^+$. We will check that the terms which appear in a given product are those which are allowed by the well-known tensor product laws for $s\ell(2,\mathbb{R})$ representations (3.2).

Case $C \times C$. We should continue p_1,p_2,p_3 from imaginary to real values in eq. (4.7). This is problematic only when the integrand, viewed as a function of j_3, has poles which cross the integration line. Such p_i-dependent poles of the integrand may come from either of its three factors. The poles coming from the second factor $\frac{1}{R^{j_3}_{m_3,\bar{m}_3}}$ are easily seen from eq. (2.21), and the poles from the other factors are obtained from these by the reflection[2] $j_3 \to -j_3-1$. All these poles fall

[1]Let us give the behaviour of certain objects of section 2.2: $N(\omega(G)|t) = -N(G|-t)$, $[\omega(G)] = -[G]-1$ and $n(\omega(G)|t_L,t_R) = -n(G|-t_L,-t_R)$.

[2]It is also possible to study the m-dependent poles of $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle$ directly from the formula (4.5). For example, $\bar{g}^{32}$ has poles at $j_2+\bar{m}_2 \in \mathbb{N}$. But the coefficient of $\bar{g}^{32}$ is a combination of g^{13} and g^{23} of the type $s(2j_2)g^{23} + s(j_{13}^2)g^{13} = \frac{s(j_3-m_3)}{s(j_1-m_1)}s(j_2+m_2)g^{32}$, where we used $j_2+m_2 \in \mathbb{Z}$ (which follows from $j_2+\bar{m}_2 \in \mathbb{N}$) and eq. (3.8).

on the four dashed half-lines in the following diagram, which depicts the j_3 complex plane. The two half-lines on the left correspond to the poles of $\frac{1}{R^{j_3}_{m_3,\bar{m}_3}}$.



(4.12)

This diagram assumes $p_3 \in i\mathbb{R}$. When p_3 moves to real values, let us consider the poles from the left which may cross the integration line and end up on the right with $j_3 \in (-\frac{1}{2}, \infty)$. Such poles belong to the (possibly empty) sets $j_3 \in -1 - \frac{1}{2}|n_3| \pm \frac{1}{2}p_3 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$. Therefore, according to eq. (4.10), they correspond to functions $\Phi^{j_3}_{m_3,\bar{m}_3}$ in the $D^{j_3,\pm}$ representations. We deduce the formula for the products of two "continuous" functions on AdS_3:

$$\Phi^{j_1}_{m_1,\bar{m}_1}\Phi^{j_2}_{m_2,\bar{m}_2} = \frac{1}{256\pi^2}\int_{-\frac{1}{2}+i\mathbb{R}} dj_3 \left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \frac{1}{R^{j_3}_{m_3,\bar{m}_3}} \Phi^{j_3}_{-m_3,-\bar{m}_3}$$
$$+ \frac{2}{256\pi^2} \sum_{j_3 \in -1-\frac{1}{2}|n_3| \pm \frac{1}{2}p_3 - \mathbb{N}\cap(-\frac{1}{2},\infty)} \left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' 2\pi i \mathrm{Res} \frac{1}{R^{j_3}_{m_3,\bar{m}_3}} \Phi^{j_3}_{-m_3,-\bar{m}_3} \ , \quad (4.13)$$

where the factor 2 in the discrete term is due to the contribution of the poles with $j_3 \in (-\infty, -\frac{1}{2})$. Notice that the general expression (4.5) for the three-point function $\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle'$ simplifies in the case $j_3 \in -1 - \frac{1}{2}|n_3| \pm \frac{1}{2}p_3 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$ due to formulas of the type of eqs. (3.9) and (3.10). Examples of simplified expressions are:

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \underset{D^{j_3,+}}{=} -C(j_1,j_2,j_3)K^2 s(j^3_{12})s(j^1_{23})s(j^2_{31})s(j_{123})\, g^{21}\bar{g}^{13} \ , \quad (4.14)$$
$$g^{21}\bar{g}^{13} = \bar{g}^{21}g^{13} = g^{12}\bar{g}^{23} = \bar{g}^{12}g^{23} \ ,$$
$$\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle' \underset{D^{j_3,-}}{=} -C(j_1,j_2,j_3)K^2 s(j^3_{12})s(j^1_{23})s(j^2_{31})s(j_{123})\, g^{12}\bar{g}^{31} \ , \quad (4.15)$$
$$g^{12}\bar{g}^{31} = \bar{g}^{12}g^{31} = g^{21}\bar{g}^{32} = \bar{g}^{21}g^{32} \ .$$

Case $D^+ \times C$. After moving p_i to real values as in the previous case $C \times C$, we should move j_1 to $-1 - \frac{1}{2}|n_1| + \frac{1}{2}p_1 - \mathbb{N} \cap (-\frac{1}{2}, \infty)$. Let us show that no further poles cross the integration line in eq.

This vanishes, unless g^{32} itself has a pole. This shows that $\left\langle \prod_{i=1}^{3} \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle$ has simple poles when both $m_2, \bar{m}_2$ belong to $-j_2 + \mathbb{N}$, but not when only $\bar{m}_2$ does.

(4.13) during this operation. We are looking for possible j_1-dependent poles in $\left\langle\prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar m_i}\right\rangle'$, viewed as a function of j_3. We use formulas of the type of eq. (3.10) to obtain

$$\left\langle\prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar m_i}\right\rangle' \underset{D^{j_1,+}}{=} C(j_1,j_2,j_3)K^2 s(j^3_{12})s(j^1_{23})s(j^2_{31})s(j_{123})\frac{s(j^1_{23})}{s(2j_1)}g^{23}\bar g^{32} \ . \tag{4.16}$$

Potential poles come from factors $\Gamma(1+j^1_{23})$ in K (4.6) and $\Gamma(-1-j_{123})\Gamma(-j^3_{12})\Gamma(-j^2_{31})$ in $C(j_1,j_2,j_3)$ (4.4), but they are all cancelled by appropriate $\sin$ factors. On the other hand, the poles from the factors $\Gamma(1+j^3_{12})\Gamma(1+j^2_{31})$ in K, from the factor $\Gamma(-j^1_{23})$ in $C(j_1,j_2,j_3)$, and the poles of $\Gamma(-j^1_{23})$ which come from g^{23} and $\bar g^{32}$, cannot be reached because $\Re j_1 \geq -\frac12$.

This shows that the formula (4.13) still holds for products of functions in $D^+\times C$. Of course, simplified expressions for $\left\langle\prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar m_i}\right\rangle'$ can be used for both continuous and discrete values of j_3. We can moreover check that terms corresponding to $D^{j_3,+}$ actually vanish. This is due to

$$\left\langle\prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar m_i}\right\rangle' \underset{D^{j_1,+},D^{j_3,+}}{=} 0 \ , \tag{4.17}$$

which follows from eq. (4.14) if we notice that $g^{21}=g^{23}=0$ in this case due to eq. (3.10). This equation holds for generic values of j_2, in particular the values $j_2\in -\frac12+i\mathbb{R}$ which correspond to C^{j_2}.

Case $D^+\times D^+$. The formula (4.13) for the product of functions still holds, but the continuous term $\int_{-\frac12+i\mathbb{R}} dj_3\cdots$ vanishes due to eq. (4.17). Terms corresponding to $D^{j_3,+}$ representations also vanish by the same argument, but the equation $\left\langle\prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar m_i}\right\rangle' \underset{D^{j_1,+},D^{j_2,+}}{=} 0$ may fail if a $D^{j_3,-}$ representation is present, due to poles from the factor K^2 in eq. (4.5). To analyze this matter it is convenient to start with the identity

$$\left\langle\prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar m_i}\right\rangle' \underset{D^{j_2,+},D^{j_3,-}}{=} -C(j_1,j_2,j_3)K^2\frac{s(j^3_{12})^2 s(j^2_{31})^2 s(j^1_{23})s(j_{123})}{s(2j_2)s(2j_3)}g^{13}\bar g^{13} \ . \tag{4.18}$$

We then send j_1 to values corresponding to discrete representations $D^{j_1,+}$. Due to momentum conservation we must have $j_1\in j_2-j_3+\mathbb{Z}$. If $j_1\in j_2-j_3-1-\mathbb{N}$ then a double pole from K^2 cancels the double zero from $s(j^2_{13})^2$ and the result is finite. If $j_1\in j_2-j_3+\mathbb{N}$ then the simple pole from $C(j_1,j_2,j_3)$ does not cancel the double zero, and the result vanishes. The formula (4.13) therefore reduces to

$$\Phi^{j_1}_{m_1,\bar m_1}\Phi^{j_2}_{m_2,\bar m_2} = \frac{2}{256\pi^2}\sum_{j_3\in j_1+j_2+1+\mathbb{N}}\left\langle\prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar m_i}\right\rangle' 2\pi i \mathrm{Res}\frac{1}{R^{j_3}_{m_3,\bar m_3}}\Phi^{j_3}_{-m_3,-\bar m_3} \ . \tag{4.19}$$

Case $D^+\times D^-$. The formula (4.13) for the product of functions still holds, and the analysis of eq. (4.18) in the previous case determines which terms may vanish. Nonvanishing $D^{j_3,+}$ terms occur for $j_3\in j_2-j_1-1-\mathbb{N}\cap(-\frac12,\infty)$ and nonvanishing $D^{j_3,-}$ terms occur for $j_3\in j_1-j_2-1-\mathbb{N}\cap(-\frac12,\infty)$. Depending on the values of j_1,j_2 we can have either $D^{j_3,+}$ terms, or $D^{j_3,-}$ terms, or no discrete terms at all in the case $|j_1-j_2|\leq\frac12$.

5. Conclusion

At the level of symmetry algebras, the Wick rotation from H_3^+ to AdS_3 amounts to a map from $s\ell(2,\mathbb{C})$ to $s\ell(2,\mathbb{R})\times s\ell(2,\mathbb{R})$, which can be viewed as two different real forms of the same algebra $s\ell(2,\mathbb{C})^{\mathbb{C}} = s\ell(2,\mathbb{C}) \times s\ell(2,\mathbb{C})$. In particular, the Wick rotation maps the continuous representation C^j of $s\ell(2,\mathbb{C})$ to the representation $\int_0^1 d\alpha\, C^{j,\alpha}\otimes C^{j,\alpha}$ of $s\ell(2,\mathbb{R})\times s\ell(2,\mathbb{R})$. The fact that such an irreducible representation is mapped to a reducible one implies that the symmetry constraints are weaker in AdS_3 than in H_3^+. Namely, the H_3^+ three-point function should be $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle = C(j_1,j_2,j_3)H(j_i,m_i,\bar{m}_i)$ where H is determined by $s\ell(2,\mathbb{C})$ symmetry; while the AdS_3 three-point function can in principle be $\left\langle \prod_{i=1}^3 \Phi^{j_i}_{m_i,\bar{m}_i} \right\rangle = C'(j_1,j_2,j_3|\alpha_1,\alpha_2,\alpha_3)H'(j_i,m_i,\bar{m}_i)$ with $m_i,\bar{m}_i \in \alpha_i+\mathbb{Z}$, where the $s\ell(2,\mathbb{R})\times s\ell(2,\mathbb{R})$ symmetry determines H' but not the α_i-dependence.

For the full H_3^+ and AdS_3 WZNW models (and not just their minisuperspace limits), the assumption that these models are related by Wick rotation therefore determines part of the AdS_3 structure constants (analogs of C') in terms of the H_3^+ conformal blocks (analogs of H). This assumption is therefore rather nontrivial and it should be carefully justified. The best justification may come a posteriori, if an ansatz for the AdS_3 three-point function derived by Wick rotation can be shown to obey crossing symmetry. Such questions did not arise in the minisuperspace limit, as the bases of functions $\Phi^j_{m,\bar{m}}$ on H_3^+ and AdS_3 are related by Wick rotation by definition, and crossing symmetry amounts to the associativity of the product of functions on these spaces. But proving crossing symmetry – or equivalently the associativity of the operator product expansion – certainly is the most important and difficult task in solving the AdS_3 WZNW model.

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