

Circular law for non-central random matrices

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Abstract

Let $(X_{jk})_{j,k \geq 1}$ be an infinite array of i.i.d. complex random variables, with mean 0 and variance 1. Let $\lambda_{n,1}, \dots, \lambda_{n,n}$ be the eigenvalues of $(\frac{1}{\sqrt{n}}X_{jk})_{1 \leq j,k \leq n}$. The strong circular law theorem states that with probability one, the empirical spectral distribution $\frac{1}{n}(\delta_{\lambda_{n,1}} + \dots + \delta_{\lambda_{n,n}})$ converges weakly as $n \rightarrow \infty$ to the uniform law over the unit disc $\{z \in \mathbb{C}; |z| \leq 1\}$. In this short note, we provide an elementary argument that allows to add a deterministic matrix M to $(X_{jk})_{1 \leq j,k \leq n}$ provided that $\text{Tr}(MM^*) = O(n^2)$ and $\text{rank}(M) = O(n^\alpha)$ with $\alpha < 1$. Conveniently, the argument is similar to the one used for the non-central version of Wigner's and Marchenko-Pastur theorems.

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1 Introduction

For any square $n \times n$ matrix \mathbf{A} with complex entries, let the complex eigenvalues $\lambda_1(\mathbf{A}), \dots, \lambda_n(\mathbf{A})$ of \mathbf{A} be labeled so that $|\lambda_1(\mathbf{A})| \geq \dots \geq |\lambda_n(\mathbf{A})|$. The *empirical spectral distribution* of \mathbf{A} is the discrete probability measure $\mu_{\mathbf{A}} := \frac{1}{n} \sum_{k=1}^n \delta_{\lambda_k(\mathbf{A})}$. We denote by $s_1(\mathbf{A}) \geq \dots \geq s_n(\mathbf{A})$ the *singular values* of \mathbf{A} , i.e. the eigenvalues of the positive semi-definite Hermitian matrix $\sqrt{\mathbf{A}\mathbf{A}^*}$ where \mathbf{A}^* is the conjugate-transpose of \mathbf{A} . The *operator norm* is $s_1(\mathbf{A}) = \max_{\|x\|_2=1} \|\mathbf{A}x\|_2$ and the square *Hilbert-Schmidt* norm is $\|\mathbf{A}\|^2 := s_1(\mathbf{A})^2 + \dots + s_n(\mathbf{A})^2 = \text{Tr}(\mathbf{A}\mathbf{A}^*) = \sum_{j,k=1}^n |\mathbf{A}_{j,k}|^2$. Weyl's inequality $|\lambda_1(\mathbf{A})|^2 + \dots + |\lambda_n(\mathbf{A})|^2 \leq s_1(\mathbf{A})^2 + \dots + s_n(\mathbf{A})^2$ ensures that the second moment of $\mu_{\mathbf{A}}$ is always bounded above by $\frac{1}{n}\|\mathbf{A}\|^2$. The following result was recently obtained by Tao and Vu [19, Corollary 1.15].

Theorem 1.1 (Circular law for central random matrices). *Let $(X_{jk})_{j,k \geq 1}$ be i.i.d. complex random variables. Let $(M_{jk})_{j,k \geq 1}$ be deterministic complex numbers. For every integer $n \geq 1$, set $\mathbf{X}_n = (X_{jk})_{1 \leq j,k \leq n}$ and $\mathbf{M}_n = (M_{jk})_{1 \leq j,k \leq n}$. If*

- $\mathbb{E}[|X_{1,1}|^2] = 1$ and $\mathbb{E}[X_{1,1}] = 0$
- $\|\mathbf{M}_n\|^2 = O(n^2)$ and $\text{rank}(\mathbf{M}_n) = O(n^\alpha)$ for some $\alpha < 1$

then with probability one, $\mu_{\frac{1}{\sqrt{n}}(\mathbf{X}_n + \mathbf{M}_n)}$ tends weakly as $n \rightarrow \infty$ to the uniform distribution on the unit disc $\{z \in \mathbb{C}; |z| \leq 1\}$ (known as the circular law).

The aim of this note is to provide an alternative and elementary argument which reduces theorem 1.1 to the central case where $\mathbf{M}_n \equiv 0$ for every n . Conveniently, the approach is close in spirit to the one used by Bai [3] for the derivation of Wigner's and Marchenko-Pastur theorems for non-central random matrices.

This note was motivated by the study of random Markov matrices, including the Dirichlet Markov Ensemble [8, 7], for which a circular law theorem is conjectured. The initial version of this note was written before the apparition of [19], and provided for the first time a non-central version of the circular law theorem. The initial version was based on potential theoretic tools. For convenience, the present version makes use instead of the replacement principle borrowed from [19].

Theorem 1.1 belongs to a sequence of works by many authors, including Mehta [13], Girko [10], Silverstein [12], Bai [2], Edelman [9] Śniady [17], Bai and Silverstein [4], Pan and Zhou [14], Götze and Tikhomirov [11], and Tao and Vu [18].

Remark 1.2 (Constant case). *Consider the case where the entries of \mathbf{M}_n are all equal to 1 in theorem 1.1. We have then $\text{rank}(\mathbf{M}_n) = 1$ and $s_1(\mathbf{M}_n) = n$. Suppose additionally that $X_{1,1}$ has finite fourth moment. Then, by Bai and Yin theorem [5], with probability one, $\lim_{n \rightarrow \infty} s_1(\frac{1}{\sqrt{n}}\mathbf{X}_n) = 2$, and thus $\frac{1}{\sqrt{n}}(\mathbf{X}_n + \mathbf{M}_n)$ is a random bounded perturbation of the rank one symmetric matrix $\frac{1}{\sqrt{n}}\mathbf{M}_n$ which has spectrum*

$$\lambda_n(\mathbf{M}_n) = \dots = \lambda_2(\mathbf{M}_n) = 0 \quad \text{and} \quad \lambda_1(\mathbf{M}_n) = \sqrt{n}.$$

From this observation, Silverstein [16] has shown, via perturbation techniques such as Bauer-Fike and Gerschgorin theorems, that with probability one,

$$\left| \lambda_2\left(\frac{1}{\sqrt{n}}(\mathbf{X}_n + \mathbf{M}_n)\right) \right| \leq 2 + o(1) \quad \text{and} \quad \left| \lambda_1\left(\frac{1}{\sqrt{n}}(\mathbf{X}_n + \mathbf{M}_n)\right) - \sqrt{n} \right| \leq 2 + o(1).$$

See also the work of Andrew [1]. Also, with probability one, as $n \rightarrow \infty$, the spectral radius $|\lambda_1(\frac{1}{\sqrt{n}}(\mathbf{X}_n + \mathbf{M}_n))|$ blows up while $\mu_{\frac{1}{\sqrt{n}}(\mathbf{X}_n + \mathbf{M}_n)}$ remains weakly localized.

2 Reduction to the central case

In order to show that theorem 1.1 reduces to the central case where $\mathbf{M}_n \equiv 0$ for every $n \geq 1$, it suffices to check the assumptions of the replacement principle of theorem 3.1 with $\mathbf{A}_n := \frac{1}{\sqrt{n}}\mathbf{X}_n$ and $\mathbf{B}_n := \frac{1}{\sqrt{n}}(\mathbf{X}_n + \mathbf{M}_n)$. By the strong law of large numbers and the assumption on $\|\mathbf{M}_n\|$, with probability one,

$$\frac{1}{n}\|\mathbf{A}_n\|^2 + \frac{1}{n}\|\mathbf{B}_n\|^2 = O(1).$$

Next, by theorem (3.4) and the first Borel-Cantelli lemma, for all $z \in \mathbb{C}$, with probability one, the random matrices $\mathbf{A}_n - z\mathbf{I}_n$ and $\mathbf{B}_n - z\mathbf{I}_n$ are invertible for large enough n . Let us define, for large enough n , the quantity

$$\Delta_{n,z} := \frac{1}{n} \log |\det(\mathbf{A}_n - z\mathbf{I}_n)| - \frac{1}{n} \log |\det(\mathbf{B}_n - z\mathbf{I}_n)|.$$

If we set $\mu_{n,z} := \mu_{\sqrt{(\mathbf{A}_n - z\mathbf{I}_n)(\mathbf{A}_n - z\mathbf{I}_n)^*}}$ and $\nu_{n,z} := \mu_{\sqrt{(\mathbf{B}_n - z\mathbf{I}_n)(\mathbf{B}_n - z\mathbf{I}_n)^*}}$ then

$$\Delta_{n,z} = \int_0^\infty \log(t) d(\mu_{n,z} - \nu_{n,z})(t).$$

By the strong law of large numbers and the assumption on $\|\mathbf{M}_n\|$, for all $z \in \mathbb{C}$, with probability one, there exists $a > 0$ such that

$$\max(s_1(\mathbf{A}_n - z\mathbf{I}_n), s_1(\mathbf{B}_n - z\mathbf{I}_n)) \leq n^a$$

for large enough n . On the other hand, by theorem (3.4) and the first Borel-Cantelli lemma, for all $z \in \mathbb{C}$, with probability one, there exists $b > 0$ such that

$$\min(s_n(\mathbf{A}_n - z\mathbf{I}_n), s_n(\mathbf{B}_n - z\mathbf{I}_n)) \geq n^{-b}$$

for large enough n . Therefore, with $\alpha_n := n^{-b}$ and $\beta_n := n^a$, and large enough n ,

$$\Delta_{n,z} = \int_{\alpha_n}^{\beta_n} \log(t) d(\mu_{n,z} - \nu_{n,z})(t).$$

Let $F_{n,z}$ and $G_{n,z}$ be the cumulative distribution functions of the real probability measures $\mu_{n,z}$ and $\nu_{n,z}$. By lemma 3.3 and the assumption on $\text{rank}(\mathbf{M}_n)$, for almost all $z \in \mathbb{C}$, with probability one, there exists $\varepsilon > 0$ such that

$$\|F_{n,z} - G_{n,z}\|_\infty = O(n^{-\varepsilon}).$$

Therefore, by lemma 3.2, we obtain, for almost all $z \in \mathbb{C}$, with probability one,

$$|\Delta_{n,z}| \leq (\log(\beta_n) - \log(\alpha_n)) \|F_{n,z} - G_{n,z}\|_\infty = o(1).$$

3 Tools

This section gathers some tools used in our proof of theorem 1.1. By Green's theorem, for any complex polynomial P and smooth compactly supported $f : \mathbb{C} \rightarrow \mathbb{R}$,

$$\int_{\mathbb{C}} f d\mu = \frac{1}{2\pi} \int_{\mathbb{C}} \Delta f \log |P| dx dy$$

where $\mu := \delta_{\lambda_1} + \dots + \delta_{\lambda_n}$ is the counting measure of the roots $\lambda_1, \dots, \lambda_n$ of P in \mathbb{C} . Used for characteristic polynomials of random matrices, this identity provides, via dominated convergence arguments, the following theorem, see [19, Theorem 2.1].

Theorem 3.1 (Replacement principle). *Let $(\mathbf{A}_n)_{n \geq 1}$ and $(\mathbf{B}_n)_{n \geq 1}$ be two sequences of complex random matrices where $\mathbf{A}_n, \mathbf{B}_n$ are $n \times n$, without any assumptions. If*

- with probability one $\frac{1}{n} \|\mathbf{A}_n\|^2 + \frac{1}{n} \|\mathbf{B}_n\|^2 = O(1)$
- for almost all $z \in \mathbb{C}$, with probability one, the random matrices $\mathbf{A}_n - z\mathbf{I}_n$ and $\mathbf{B}_n - z\mathbf{I}_n$ are invertible for large enough n

- for almost all $z \in \mathbb{C}$, with probability one,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n} \log |\det(\mathbf{A}_n - z\mathbf{I}_n)| - \frac{1}{n} \log |\det(\mathbf{B}_n - z\mathbf{I}_n)| \right) = 0$$

then with probability one, $\mu_{\mathbf{A}_n} - \mu_{\mathbf{B}_n}$ tends weakly to zero as $n \rightarrow \infty$.

The following lemma is a special case of the integration by parts formula for the Lebesgue-Stieltjes integral (with atoms). We give a short proof for convenience.

Lemma 3.2 (Integration by parts). *If $a_1, \dots, a_n, b_1, \dots, b_n \in [\alpha, \beta] \subset \mathbb{R}$, and F_μ and F_ν are the cumulative distribution functions of $\mu = \frac{1}{n}(\delta_{a_1} + \dots + \delta_{a_n})$ and $\nu = \frac{1}{n}(\delta_{b_1} + \dots + \delta_{b_n})$ respectively, then for any smooth $f : [\alpha, \beta] \rightarrow \mathbb{R}$,*

$$\int_\alpha^\beta f(x) d\mu(x) - \int_\alpha^\beta f(x) d\nu(x) = \int_\alpha^\beta f'(x)(F_\mu(x) - F_\nu(x)) dx.$$

In particular, when f is non decreasing,

$$\left| \int_\alpha^\beta f(x) d\mu(x) - \int_\alpha^\beta f(x) d\nu(x) \right| \leq (f(\beta) - f(\alpha)) \|F_\mu - F_\nu\|_\infty.$$

Proof. One can assume by continuity that $a_1, \dots, a_n, b_1, \dots, b_n$ are all different. We reorder $a_1, \dots, a_n, b_1, \dots, b_n$ into $c_1 \leq \dots \leq c_{2n}$. For every $1 \leq k \leq 2n$, set $\varepsilon_k = +1$ if $c_k \in \{a_1, \dots, a_n\}$ and $\varepsilon_k = -1$ if $c_k \in \{b_1, \dots, b_n\}$. We have

$$\int_\alpha^\beta f(x) d\mu(x) - \int_\alpha^\beta f(x) d\nu(x) = \frac{1}{n} \sum_{k=1}^n (f(a_i) - f(b_i)) = \frac{1}{n} \sum_{k=1}^{2n} \varepsilon_k f(c_k).$$

By an Abel transform, we get by denoting $S_k = \varepsilon_1 + \dots + \varepsilon_k$,

$$\sum_{k=1}^{2n} \varepsilon_k f(c_k) = - \sum_{k=1}^{2n-1} S_k (f(c_{k+1}) - f(c_k)) + S_{2n} f(c_{2n}).$$

Since $F_\mu - F_\nu$ is constant and equal to S_k on $[c_k, c_{k+1}[$,

$$S_k (f(c_{k+1}) - f(c_k)) = \int_{c_k}^{c_{k+1}} f'(x) (F_\mu(x) - F_\nu(x)) dx.$$

It remains to notice that $S_{2n} = F_\mu(c_{2n}) - F_\nu(c_{2n}) = 0$. \square

The following lemma is a direct consequence of interlacing inequalities for singular values obtained by Thompson [20] in 1976. It was also obtained by Bai [3] and generalized by Benaych-Georges and Rao [6]. It is worthwhile to mention that it gives neither an upper bound for $s_1(\mathbf{B}), \dots, s_k(\mathbf{B})$ nor a lower bound for $s_{n-k+1}(\mathbf{B}), \dots, s_n(\mathbf{B})$ where $k := \text{rank}(\mathbf{A} - \mathbf{B})$, even in the case $k = 1$.

Lemma 3.3 (Rank inequality). *Let \mathbf{A} and \mathbf{B} be two $n \times m$ complex matrices. Let $F_{\sqrt{\mathbf{A}\mathbf{A}^*}}, F_{\sqrt{\mathbf{B}\mathbf{B}^*}}$ be the cumulative distribution functions of $\mu_{\sqrt{\mathbf{A}\mathbf{A}^*}}$ and $\mu_{\sqrt{\mathbf{B}\mathbf{B}^*}}$. Then*

$$\|F_{\sqrt{\mathbf{A}\mathbf{A}^*}} - F_{\sqrt{\mathbf{B}\mathbf{B}^*}}\|_\infty \leq \frac{1}{n} \text{rank}(\mathbf{A} - \mathbf{B}).$$

The following theorem is due to Tao and Vu [18, Theorem 2.1], and is inspired from the work of Rudelson and Vershynin [15].

Theorem 3.4 (Polynomial bounds for smallest singular values). *Let L be a probability distribution on \mathbb{C} with finite and non-zero variance. For every constants $A > 0$ and $C_1 > 0$, there exists constants $B > 0$ and $C_2 > 0$ such that for every $n \times n$ random matrix \mathbf{X} with i.i.d. entries of law L and every $n \times n$ deterministic matrix \mathbf{C} with $s_1(\mathbf{C}) \leq n^{C_1}$, we have*

$$\mathbb{P}(s_n(\mathbf{X} + \mathbf{C}) \leq n^{-B}) \leq C_2 n^{-A}.$$

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