



## ON FULKERSON CONJECTURE

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ABSTRACT. If  $G$  is a bridgeless cubic graph, Fulkerson conjectured that we can find 6 perfect matchings (a *Fulkerson covering*) with the property that every edge of  $G$  is contained in exactly two of them. A consequence of the Fulkerson conjecture would be that every bridgeless cubic graph has 3 perfect matchings with empty intersection (this problem is known as the Fan Raspaud Conjecture). A *FR-triple* is a set of 3 such perfect matchings. We show here how to derive a Fulkerson covering from two FR-triples.

Moreover, we give a simple proof that the Fulkerson conjecture holds true for some classes of well known snarks.

### 1. INTRODUCTION

The following conjecture is due to Fulkerson, and appears first in [4].

**Conjecture 1.1.** *If  $G$  is a bridgeless cubic graph, then there exist 6 perfect matchings  $M_1, \dots, M_6$  of  $G$  with the property that every edge of  $G$  is contained in exactly two of  $M_1, \dots, M_6$ .*

We shall say that  $\mathcal{F} = \{M_1 \dots M_6\}$ , in the above conjecture, is a *Fulkerson covering*. A consequence of the Fulkerson conjecture would be that every bridgeless cubic graph has 3 perfect matchings with empty intersection (take any 3 of the 6 perfect matchings given by the conjecture). The following weakening of this (also suggested by Berge) is still open.

**Conjecture 1.2.** *There exists a fixed integer  $k$  such that every bridgeless cubic graph has a list of  $k$  perfect matchings with empty intersection.*

For  $k = 3$  this conjecture is known as the Fan Raspaud Conjecture.

**Conjecture 1.3.** [2] *Every bridgeless cubic graph contains perfect matching  $M_1, M_2, M_3$  such that*

$$M_1 \cap M_2 \cap M_3 = \emptyset$$

Let  $G$  be a cubic graph with 3 perfect matchings  $M_1, M_2$  and  $M_3$  having an empty intersection. Since  $G$  satisfies the Fan Raspaud conjecture, when considering these perfect matchings, we shall say that  $\mathcal{T} = (M_1, M_2, M_3)$  is a *FR-triple*. We define  $T_i \subset E(G)$  ( $i = 0..2$ ) as the set of edges of  $G$  which are covered  $i$  times by  $\mathcal{T}$ . It will be convenient to use  $T'_i$  ( $i = 0..2$ ) for the FR-triple  $\mathcal{T}$ .

### 2. FR-TRIPLES AND FULKERSON COVERING

In this section, we are concerned with the relationship between FR-triples and Fulkerson coverings.

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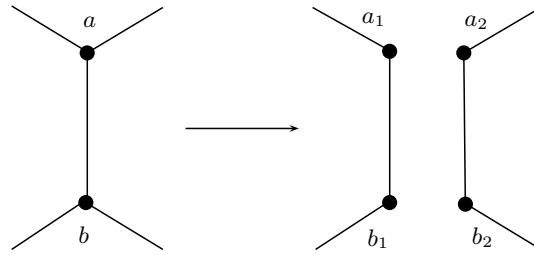


FIGURE 1. Splitting an edge

### 2.1. On FR-triples.

**Proposition 2.1.** *Let  $G$  be a bridgeless cubic graph with  $\mathcal{T}$  a FR-triple. Then  $T_0$  and  $T_2$  are disjoint matchings.*

**Proof** Let  $v$  be a vertex incident to an edge of  $T_0$ . Since  $v$  must be incident to each perfect matching of  $\mathcal{T}$  and since the three perfect matchings have an empty intersection, one of the remaining edges incident to  $v$  must be contained in 2 perfect matchings while the other is contained in exactly one perfect matching. The result follows.  $\square$

We introduce now concepts and definitions coming from [10]. Let  $ab$  be an edge of bridgeless cubic graph  $G$ . We shall say that we have *split* the edge  $ab$  when we have applied the operation depicted in Figure 1. The resulting graph is no longer cubic since we get 4 vertices with degree 2 instead of two vertices of degree 3. Let  $A_1$  and  $A_2$  be two disjoint matchings of  $G$  (we insist to say that these matchings are not, necessarily, perfect matchings). For  $i = 1, 2$ , let  $G_{A_i}$  be the graph obtained by splitting the edges of  $A_i$  and let  $\overline{G_{A_i}}$  be the graph homeomorphic to  $G_{A_i}$  when the degree 2 vertices are deleted. The connected component of  $\overline{G_{A_i}}$  are cubic graphs and *vertexless loop graphs* (graph with one edge and no vertex). We shall say that  $\overline{G_{A_i}}$  is 3-edge colourable whenever the cubic components are 3-edge colourable (any colour can be given to the vertexless loops).

The following Lemma can be obtained from the work of Hao and al. [10] when considering FR-triples.

**Lemma 2.2.** *Let  $G$  be a bridgeless cubic graph and let  $\mathcal{T}$  be a FR-triple. Then  $\overline{G_{T_2}}$  is 3-edge colourable.*

#### Proof

Assume that  $T = (M_1, M_2, M_3)$  is a FR-triple. Let  $ab$  be an edge of  $T_2$  then the two edges of  $T_1$  incident with  $ab$  must be in the same perfect matching of  $\mathcal{T}$ . Hence, these two edges are identified in some sens. If we colour the edges of  $T_1$  with 1, 2 or 3 when they are in  $M_1$ ,  $M_2$  or  $M_3$  respectively, we get a natural 3-edge colouring of  $\overline{G_{T_2}}$ .  $\square$

**Lemma 2.3.** *Let  $G$  be a bridgeless cubic graph containing two disjoint matchings  $A_1$  and  $A_2$  such that  $\overline{G_{A_1}}$  is 3-edge colourable and  $A_1 \cup A_2$  forms an union of disjoint cycles. Then  $G$  has a FR-triple  $\mathcal{T}$  where  $T_2 = A_1$  and  $T_0 = A_2$ .*

**Proof** Obviously,  $A_1 \cup A_2$  forms an union of disjoint even cycles in  $G$ . Let  $C = a_0 a_1 \dots a_{2p-1}$  be an even cycle of  $A_1 \cup A_2$  and assume that  $a_i a_{i+1} \in A_1$  when  $i \equiv 0(2)$ .

Let  $M_1, M_2$  and  $M_3$  be the three matchings associated to a 3 edge-colouring of  $\overline{G_{A_1}}$ . Thanks to the construction of  $\overline{G_{A_1}}$  for some  $i \equiv 0[2]$ , the third edge incident to  $a_i$ , say  $e$ , and the third one incident to  $a_{i+1}$ , say  $e'$  lead to a unique edge of  $\overline{G_{A_1}}$ . Assume that this edge of  $\overline{G_{A_1}}$  is in  $M_1$ , then  $M_1$  can be extended naturally to a matching of  $G$  containing  $\{e, e'\}$ . Moreover we add  $a_i a_{i+1}$  to  $M_2$  and  $a_i a_{i+1}$  to  $M_3$ . When applying this process to all edges of  $A_1$  on all cycles of  $A_1 \cup A_2$  we extend the colours of  $\overline{G_{A_1}}$  into perfect matchings of  $G$ . Since every edge of  $G$  belongs to at most 2 matchings in  $\{M_1, M_2, M_3\}$  we have a FR-triple with  $\mathcal{T} = \{M_1, M_2, M_3\}$ . By construction, we have  $T_2 = A_1$  and  $T_0 = A_2$ , as claimed.  $\square$

**Proposition 2.4.** *Let  $G$  be a bridgeless cubic graph then  $G$  has a FR-triple if and only if  $G$  has two disjoint matchings  $A_1$  and  $A_2$  such that  $A_1 \cup A_2$  forms an union of disjoint cycles, moreover  $\overline{G_{A_1}}$  or  $\overline{G_{A_2}}$  is 3-edge colourable.*

**Proof** Assume that  $G$  has two disjoint matchings  $A_1$  and  $A_2$  such that, without loss of generality,  $\overline{G_{A_1}}$  is 3-edge colourable. From Lemma 2.3,  $G$  has a FR-triple  $\mathcal{T}$  where  $T_2 = A_1$  and  $T_0 = A_2$ .

Conversely, assume that  $\mathcal{T}$  is a FR-triple. From Lemma 2.2  $\overline{G_{T_2}}$  is 3-edge colourable. Let  $A_1 = T_0$  and  $A_2 = T_2$ . Then  $A_1$  and  $A_2$  are two disjoint matchings and  $\overline{G_{A_2}}$  is 3-edge colourable.  $\square$

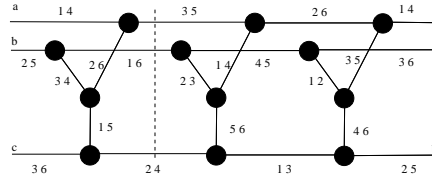
**2.2. On compatible FR-triples.** As pointed out in the introduction, any three perfect matchings in a Fulkerson covering lead to a FR-triple. Is it possible to get a Fulkerson covering when we know one or more FR-triples? In fact, we can characterize a Fulkerson covering in terms of FR-triples in the following way.

Let  $G$  be a bridgeless cubic graph with  $\mathcal{T} = (M_1, M_2, M_3)$  and  $\mathcal{T}' = (M'_1, M'_2, M'_3)$  two FR-triples. We shall say that  $\mathcal{T}$  and  $\mathcal{T}'$  are *compatible* whenever  $T_0 = T'_2$  and  $T_2 = T'_0$  (and hence  $T_1 = T'_1$ ).

**Theorem 2.5.** *Let  $G$  be a bridgeless cubic graph then  $G$  can be provided with a Fulkerson covering if and only if  $G$  has two compatible FR-triples.*

**Proof** Let  $\mathcal{F} = \{M_1 \dots M_6\}$  be a Fulkerson covering of  $G$  and let  $\mathcal{T} = (M_1, M_2, M_3)$  and  $\mathcal{T}' = (M_4, M_5, M_6)$ .  $\mathcal{T}$  and  $\mathcal{T}'$  are two FR-triples and we claim that they are compatible. Since each edge of  $G$  is covered exactly twice by  $\mathcal{F}$ ,  $T_1$  the set of edges covered only once by  $\mathcal{T}$  must be covered also only once by  $\mathcal{T}'$ ,  $T_0$  the set of edges not covered by  $\mathcal{T}$  must be covered exactly twice by  $\mathcal{T}'$  and  $T_2$  the set of edges covered exactly twice by  $\mathcal{T}$  is not covered by  $\mathcal{T}'$ . Which means that  $T_1 = T'_1$ ,  $T_0 = T'_2$  and  $T_2 = T'_0$ , that is  $\mathcal{T}$  and  $\mathcal{T}'$  are compatible.

Conversely, assume that  $\mathcal{T}$  and  $\mathcal{T}'$  are two FR-triples compatible. Then it is an easy task to check that each edge of  $G$  is contained in exactly 2 perfect matchings

FIGURE 2.  $J_3$ 

of the 6 perfect matchings involved in  $\mathcal{T}$  or  $\mathcal{T}'$ .  $\square$

**Proposition 2.6.** *Let  $G$  be a bridgeless cubic graph then  $G$  has two compatible FR-triples if and only if  $G$  has two disjoint matchings  $A_1$  and  $A_2$  such that  $A_1 \cup A_2$  forms an union of disjoint cycles and  $\overline{G_{A_1}}$  and  $\overline{G_{A_2}}$  are 3-edge colourable.*

**Proof** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be 2 compatible FR-triples. From Lemma 2.2 we know that  $\overline{G_{T_2}}$  and  $\overline{G_{T'_2}}$  are 3-edge colourable. Since  $T_0 = T'_2$  and  $T'_0 = T_2$  by the compatibility of  $\mathcal{T}$  and  $\mathcal{T}'$ , the result holds when we set  $A_1 = T_0$  and  $A_2 = T_2$ .

Conversely, assume that  $G$  has two disjoint matchings  $A_1$  and  $A_2$  such that  $\overline{G_{A_1}}$  and  $\overline{G_{A_2}}$  are 3-edge colourable. From Lemma 2.3,  $G$  has a FR-triple  $\mathcal{T}$  where  $T_2 = A_1$  and  $T_0 = A_2$  as well as a FR-triple  $\mathcal{T}'$  where  $T'_2 = A_2$  and  $T'_0 = A_1$ . These two FR-triples are obviously compatible.  $\square$

**Proposition 2.7.** [10] *Let  $G$  be a bridgeless cubic graph then  $G$  can be provided with a Fulkerson covering if and only if  $G$  has two disjoint matchings  $A_1$  and  $A_2$  such that  $A_1 \cup A_2$  forms an union of disjoint cycles and  $\overline{G_{A_1}}$  and  $\overline{G_{A_2}}$  are 3-edge colourable.*

**Proof** Obvious in view of Theorem 2.5 and Proposition 2.6.  $\square$

### 3. FULKERSON COVERING FOR SOME CLASSICAL SNARKS

A non 3-edge colourable, bridgeless, cyclically 4-edge-connected cubic graph is called a *snark*.

For an odd  $k \geq 3$ , let  $J_k$  be the cubic graph on  $4k$  vertices  $x_0, x_1, \dots, x_{k-1}$ ,  $y_0, y_1, \dots, y_{k-1}$ ,  $z_0, z_1, \dots, z_{k-1}$ ,  $t_0, t_1, \dots, t_{k-1}$  such that  $x_0 x_1 \dots x_{k-1}$  is an induced cycle of length  $k$ ,  $y_0 y_1 \dots y_{k-1} z_0 z_1 \dots z_{k-1}$  is an induced cycle of length  $2k$  and for  $i = 0 \dots k-1$  the vertex  $t_i$  is adjacent to  $x_i$ ,  $y_i$  and  $z_i$ . The set  $\{t_i, x_i, y_i, z_i\}$  induces the claw  $C_i$ . In Figure 2 we have a representation of  $J_3$ , the half edges (to the left and to the right in the figure) with same labels are identified. For  $k \geq 5$  those graphs were introduced by Isaacs in [6] under the name of flower snarks in order to provide an infinite family of snarks.

Proposition 2.7 is essentially used in [10] in order to show that the so called flower snarks and Goldberg snarks can be provided with a Fulkerson covering. We shall see, in this section, that this result can be directly obtained.

**Theorem 3.1.** *For any odd  $k \geq 3$ ,  $J_k$  can be provided with a Fulkerson covering.*

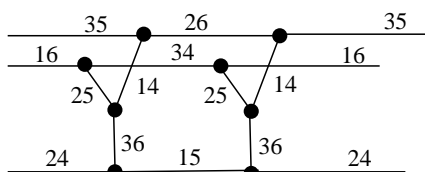
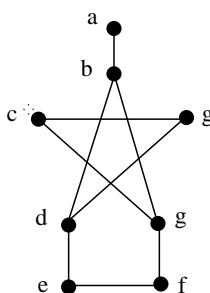


FIGURE 3. A block for the flower snark

FIGURE 4.  $H$ 

**Proof** For  $k = 3$  the Fulkerson covering is given in Figure 2. We obtain a Fulkerson covering of  $J_k$  by inserting a suitable number of subgraphs isomorphic to the subgraph depicted in Figure 3 when we cut  $J_3$  along the dashed line of Figure 2. The labels of the edges of the two sets of three semi-edges (left and right) are identical which insures that the process can be repeated as long as necessary. These labels lead to the perfect matchings of the Fulkerson covering.  $\square$

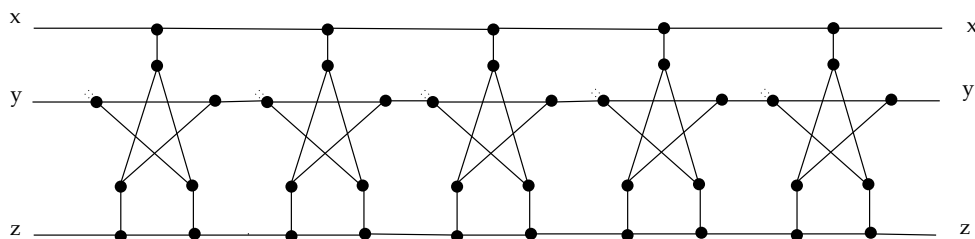
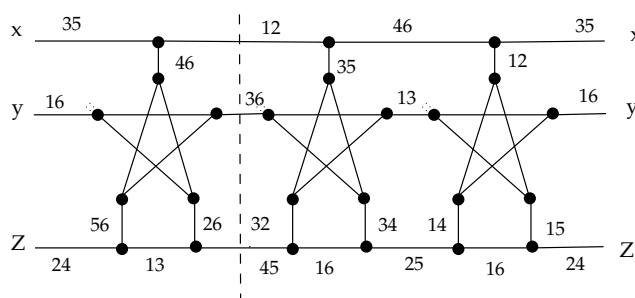
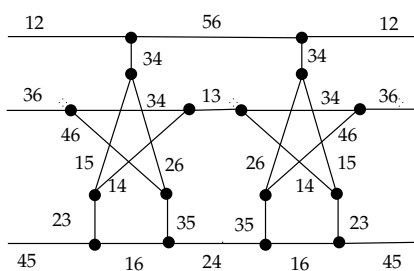
Let  $H$  be the graph depicted in Figure 4

Let  $G_k$  ( $k$  odd) be a cubic graph obtained from  $k$  copies of  $H$  ( $H_0 \dots H_{k-1}$  where the name of vertices are indexed by  $i$ ) by adding edges  $a_i a_{i+1}$ ,  $c_i c_{i+1}$ ,  $e_i e_{i+1}$ ,  $f_i f_{i+1}$  and  $h_i h_{i+1}$  (subscripts are taken modulo  $k$ ).

If  $k = 5$ , then  $G_k$  is known as the Goldberg snark (see [5]). Accordingly, we refer to all graphs  $G_k$  as Goldberg graphs. The graph  $G_5$  is shown in Figure 5. The half edges (to the left and to the right in the figure) with same labels are identified.

**Theorem 3.2.** For any odd  $k \geq 5$ ,  $G_k$  can be provided with a Fulkerson covering.

**Proof** We give first a Fulkerson covering of  $G_3$  in Figure 6(a). The reader will complete easily the matchings along the 5-cycles by remarking that these cycles are incident to 5 edges with a common label from 1 to 6 and to exactly one edge of each remaining label. We obtain a Fulkerson covering of  $G_k$  with odd  $k \geq 5$  by inserting a suitable number of subgraphs isomorphic to the subgraph depicted in Figure 6(b) when we cut  $G_3$  along the dashed line. The labels of the edges of the two sets of three semi-edges (left and right) are identical which insures that

FIGURE 5. Goldberg snark  $G_5$ (a) A Fulkerson covering for  $G_3$ 

(b) A block for the Goldberg snark

FIGURE 6. Fulkerson covering for the Goldberg Snarks

the process can be repeated as long as necessary. These labels lead to the perfect matchings of the Fulkerson covering.  $\square$

#### 4. A TECHNICAL TOOL

Let  $M$  be a perfect matching, a set  $A \subseteq E(G)$  is an  $M$ -balanced matching when we can find a perfect matching  $M'$  such that  $A = M \cap M'$ . Assume that  $\mathcal{M} = \{A, B, C, D\}$  are 4 pairwise disjoint  $M$ -balanced matchings, we shall say that  $\mathcal{M}$  is an  $F$ -family for  $M$  whenever the three following conditions are fulfilled:

- i Every odd cycle of  $G \setminus M$  has exactly one vertex incident with one edge of each matching in  $\mathcal{M}$ .
- ii Every even cycle of  $G \setminus M$  incident with some matching in  $\mathcal{M}$  contains 4 vertices such that two of them are incident to one matching in  $\mathcal{M}$  while the other are incident to another matching in  $\mathcal{M}$  or the 4 vertices are incident to the same matching in  $\mathcal{M}$ .
- iii The subgraph induced by 4 vertices so determined in the previous items has a matching.

It will be convenient to denote the set of edges described in the third item by  $N$ .

**Theorem 4.1.** *Let  $G$  be a bridgeless cubic graph together with a perfect matching  $M$  and an  $F$ -family for  $M$ . Then  $G$  can be provided with a Fulkerson covering.*

**Proof** Since  $A, B, C$  and  $D$  are  $M$ -balanced matchings, we can find 4 perfect matchings  $M_A, M_B, M_C$  and  $M_D$  such that

$$M \cap M_A = A \quad M \cap M_B = B \quad M \cap M_C = C \quad M \cap M_D = D$$

Let  $M' = M \setminus \{A, B, C, D\} \cup N$ , we will prove that  $\mathcal{F} = \{M, M_A, M_B, M_C, M_D, M'\}$  is a Fulkerson covering of  $G$ .

**Claim 4.1.1.**  *$M'$  is a perfect matching*

**Proof** The vertices of  $G$  which are not incident with some edge in  $M \setminus \{A, B, C, D\}$  are precisely those which are end vertices of edges in  $M_A \cup M_B \cup M_C \cup M_D$ . From the definition of an  $F$ -family, the 4 vertices defined on each cycle of  $\{C_i | i = 1 \dots k\}$  incident to edges of  $\mathcal{M}$  form a matching with two edges, which insures that  $M'$  is a perfect matching. □

Let  $\mathcal{C} = \{\Gamma_i | i = 1 \dots k\}$  be the set of cycles of  $G \setminus M$  and let  $X$  and  $Y$  be two distinct members of  $\mathcal{M}$ .

**Claim 4.1.2.** *Let  $\Gamma \in \mathcal{C}$  be an odd cycle. Assume that  $X$  and  $Y$  have ends  $x$  and  $y$  on  $\Gamma$ . Then  $xy$  is the only edge of  $\Gamma$  not covered by  $M_X \cup M_Y$*

**Proof** Since  $M_X$  ( $M_Y$  respectively) is a perfect matching, the edges of  $M_X$  ( $M_Y$  respectively) contained in  $\Gamma$  saturate every vertex of  $\Gamma$  with the exception of  $x$  ( $y$  respectively). The result follows. □

**Claim 4.1.3.** *Let  $\Gamma \in \mathcal{C}$  be an even cycle. Assume that  $X$  and  $Y$  have ends  $x_1, x_2$  and  $y_1, y_2$  on  $\Gamma$  with  $x_1y_1 \in N$  and  $x_2y_2 \in N$ . Then  $x_1y_1$  and  $x_2y_2$  are the only edges of  $\Gamma$  not covered by  $M_X \cup M_Y$*

**Proof** The perfect matching  $M_X$  must saturate every vertex of  $\Gamma$  with the exception of  $x_1$  and  $x_2$ . The same holds with  $M_Y$  and  $y_1$  and  $y_2$ . Since  $x_1y_1$  and  $x_2y_2$  are edges of  $\Gamma$ , these two edges are not covered by  $M_X \cup M_Y$  and we can easily check that the other edges are covered. □

**Claim 4.1.4.** *Let  $\Gamma \in \mathcal{C}$  be an even cycle. Assume that  $X$  and  $Y$  have ends  $x_1, x_2$  and  $y_1, y_2$  on  $\Gamma$  with  $x_1x_2 \in N$  and  $y_1y_2 \in N$ . Then either  $x_1x_2$  and  $y_1y_2$  are the only edges of  $\Gamma$  not covered by  $M_X \cup M_Y$  or  $M_X \cup M_Y$  induces a perfect matching*

on  $\Gamma$  such that every edge in that perfect matching is covered by  $M_X$  and  $M_Y$  with the exception of  $x_1x_2$  which belongs to  $M_Y$  and  $y_1y_2$  which belongs to  $M_X$ .

**Proof** The perfect matching  $M_X$  must saturate every vertex of  $\Gamma$  with the exception of the two consecutive vertices  $x_1$  and  $x_2$ . The same holds with  $M_Y$  and  $y_1$  and  $y_2$ .

Let us recall here that, since  $X$  ( $Y$  respectively) is a balanced matching, the paths determined by  $x_1$  and  $x_2$  on  $\Gamma$  have odd lengths (the paths determined by  $y_1$  and  $y_2$  respectively). Two cases may occur.

**case 1:** *The two paths obtained on  $\Gamma$  by deleting the edges  $x_1x_2$  and  $y_1y_2$  have odd lengths* We can check that  $M_X \cup M_Y$  determines a perfect matching on  $\Gamma$  such that every edge in that perfect matching is covered by  $M_X$  and  $M_Y$  with the exception of  $x_1x_2$  which belongs to  $M_Y$  and  $y_1y_2$  which belongs to  $M_X$

**case 2:** *The two paths obtained on  $\Gamma$  by deleting  $x_1x_2$  and  $y_1y_2$  have even lengths* We can check that  $M_X \cup M_Y$  covers every edge of  $\Gamma$  with the exception of  $x_1x_2$  and  $y_1y_2$ .

□

**Claim 4.1.5.** *Let  $\Gamma \in \mathcal{C}$  be an even cycle. Assume that  $X$  have ends  $x_1, x_2, x_3$  and  $x_4$  on  $\Gamma$  with  $x_1x_2 \in N$  and  $x_3x_4 \in N$ . Then we can choose a perfect matching  $M_Y$  in such a way that  $x_1x_2$  and  $x_3x_4$  are the only edges of  $\Gamma$  not covered by  $M_X \cup M_Y$ .*

**Proof** Since  $M_X$  is a perfect matching, the edges of  $M_X$  contained in  $\Gamma$  saturate every vertex of  $\Gamma$  with the exception of  $x_1, x_2, x_3$  and  $x_4$ . Since  $Y$  is not incident to  $\Gamma$  the perfect matching  $M_Y$  can be chosen in two ways (taking one of the two perfect matchings contained in this cycle). We can see easily that we can choose  $M_Y$  in such a way that every edge distinct from  $x_1x_2$  and  $x_3x_4$  is covered by  $M_X$  or  $M_Y$ . □

Since  $\{A, B, C, D, M' \cap M\}$  is a partition of  $M$ , each edge of  $M$  is covered twice by some perfect matchings of  $\mathcal{F}$ .

Let  $\Gamma \in \mathcal{C}$  be an odd cycle, each edge of  $\Gamma$  distinct from the two edges of  $N$  (Claim 4.1.2) is covered twice by some perfect matchings of  $\mathcal{F}$ . The two edges of  $N$  are covered by exactly one perfect matching belonging to  $\{M_A, M_B, M_C, M_D\}$  and by the perfect matching  $M'$ . Hence every edge of  $\Gamma$  is covered twice by  $\mathcal{F}$ .

Let  $\Gamma \in \mathcal{C}$  be an even cycle. Assume first that 4 vertices of  $\Gamma$  are ends of some edges in  $A$  while no other set of  $\mathcal{M}$  is incident with  $\Gamma$ . From Claim 4.1.5 we can choose  $M_B$  in such a way that every edge distinct from the two edges of  $N$  is covered by  $M_A$  or  $M_B$ . We can then choose  $M_C$  in such a way that one of the two edges of  $N$  belongs to  $M_C$ . Finally, we can choose  $M_D$  in order to cover the other edge of  $N$ . Each edge of  $\Gamma$  distinct from the two edges of  $N$  (Claim 4.1.5) is covered twice by some perfect matchings of  $\mathcal{F}$ . The two edges of  $N$  are covered by exactly one perfect matching belonging to  $\{M_A, M_B, M_C, M_D\}$  and by the perfect matching  $M'$ . Hence every edge of  $\Gamma$  is covered twice by  $\mathcal{F}$ .

Assume now that 2 vertices of  $\Gamma$  are ends of some edges in  $A$  (say  $a_1$  and  $a_2$ ) and 2 other vertices are ends of some edges in  $B$  (say  $b_1$  and  $b_2$ ).

**case 1:**  $a_1b_1 \in N$  and  $a_2b_2 \in N$ . We can choose  $M_C$  and  $M_D$  in order to cover every edge of  $\Gamma$ . From Claim 4.1.3 every edge of  $\Gamma$  is covered by  $M_A \cup M_B$  with the exception of  $a_1b_1$  and  $a_2b_2$ . Hence every edge of  $\Gamma$  is covered twice by

$M_A \cup M_B \cup M_C \cup M_D$  while  $a_1b_1$  and  $a_2b_2$  are covered twice by  $M_C \cup M_D \cup M'$ . Hence every edge of  $\Gamma$  is covered twice by  $\mathcal{F}$ .

**case 2:**  $a_1a_2 \in N$  and  $b_1b_2 \in N$ . Assume that  $a_1a_2$  and  $b_1b_2$  are the only edges of  $\Gamma$  not covered by  $M_A \cup M_B$  (Claim 4.1.4). Then we can choose  $M_C$  and  $M_D$  in such a way that every edge of  $\Gamma$  is covered by  $M_C \cup M_D$ . In that case every edge of  $\Gamma$  is covered twice by  $M_A \cup M_B \cup M_C \cup M_D$  with the exception of  $a_1a_2$  and  $b_1b_2$  which are covered twice by  $M_C \cup M_D \cup M'$ .

Assume now that  $M_A \cup M_B$  induces a perfect matching on  $\Gamma$  where  $a_1a_2 \in M_B$  and  $b_1b_2 \in M_A$  while the other edges of this perfect matchings are in  $M_A \cap M_B$  (Claim 4.1.4). Then we can choose  $M_C$  and  $M_D$  such that every edge of  $\Gamma$  not contained in  $M_A \cup M_B$  is covered twice by  $M_C \cup M_D$  ( $M_C \cup M_D$  induces a perfect matching on  $\Gamma$ ). hence every edge of  $\Gamma$  is covered twice by  $M_C \cup M_D$  or by  $M_A \cup M_B$  with the exception of  $a_1a_2$  which is covered twice by  $M_B \cup M'$  and  $b_1b_2$  which is covered twice by  $M_A \cup M'$ .

Finally, assume that  $\Gamma$  has no vertex as end of some edge in  $\mathcal{M}$ . Then we can choose easily  $M_A, M_B, M_C$  and  $M_D$  such that every edge of  $\Gamma$  is covered twice by  $M_A \cup M_B \cup M_C \cup M_D$ .

Hence  $\mathcal{F}$  is a Fulkerson covering of  $G$ .  $\square$

**Remark 4.2.** Observe that the matchings of the Fulkerson covering described in the above proof are all distinct.

**4.1. Dot products which preserve an F-family.** In [6] Isaacs defined the *dot product* operation in order to describe infinites families of non trivial snarks.

Let  $G_1, G_2$  be two bridgeless cubic graphs and  $e_1 = u_1v_1, e_2 = u_2v_2 \in E(G_1)$  and  $e_3 = x_1x_2 \in E(G_2)$  with  $N_{G_2}(x_1) = \{y_1, y_2, x_2\}$  and  $N_{G_2}(x_2) = \{z_1, z_2, x_1\}$ .

The *dot product* of  $G_1$  and  $G_2$ , denoted by  $G_1 \cdot G_2$  is the bridgeless cubic graph  $G$  defined as (see Figure 7) :

$$G = [G_1 \setminus \{e_1, e_2\}] \cup [G_2 \setminus \{x_1, x_2\}] \cup \{u_1y_1, v_1y_2, u_2z_1, v_2z_2\}$$

It is well known that the dot product of two snarks remains to be a snark. It must be pointed out that in general the dot product operation does not permit to extend a Fulkerson covering, in other words, whenever  $G_1$  and  $G_2$  are snarks together with a Fulkerson covering, we do not know how to get a Fulkerson covering for  $G_1 \cdot G_2$ .

However, in some cases, the dot product operation can preserve, in some sense, an  $F$ -family, leading thus to a Fulkerson covering of the resulting graph.

**Proposition 4.3.** Let  $M_1$  be a perfect matching of a snark  $G_1$  such that  $G_1 \setminus M_1$  contains only two (odd) cycles, namely  $C$  and  $C'$ . Let  $ab$  be an edge of  $C$  and  $a'b'$  be an edge of  $C'$ .

Let  $M_2$  be a perfect matching of a snark  $G_2$  where  $\{A, B, C, D\}$  is an  $F$ -family for  $M_2$ . Let  $xy$  be an edge of  $M_2 \setminus \{A \cup B \cup C \cup D\}$ , with  $x$  and  $y$  vertices of two distinct odd cycles of  $G_2 \setminus M_2$ .

Then  $\{A, B, C, D\}$  is an  $F$ -family for the perfect matching  $M$  of  $G = G_1 \cdot G_2$  with  $M = M_1 \cup M_2 \setminus \{xy\}$ .

**Proof** Obvious by the definition of the  $F$ -family and the construction of the graph resulting of the dot product.  $\square$

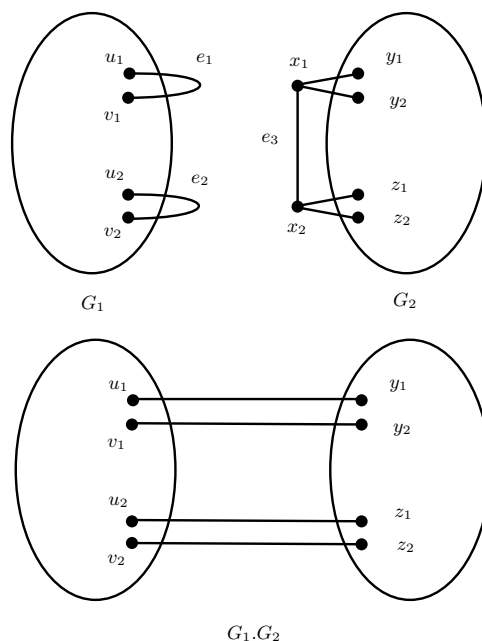


FIGURE 7. The dot product operation

**Proposition 4.4.** *Let  $M_1$  be a perfect matching of a snark  $G_1$  where  $\{A, B, C, D\}$  is an  $F$ -family for  $M_1$ . Let  $xy$  and  $zt$  be two edges of  $E(G_1) \setminus M_1$  not contained in  $N$ .*

*Let  $M_2$  be a perfect matching of a snark  $G_2$  such that  $G_2 \setminus M_2$  contains only two (odd) cycles, namely  $C$  and  $C'$ . Let  $xy \in M_2$ , with  $x \in V(C)$  and  $y \in V(C')$ .*

*Then  $\{A, B, C, D\}$  is an  $F$ -family for the perfect matching  $M$  of  $G = G_1 \cdot G_2$  with  $M = M_1 \cup M_2 \setminus \{xy\}$ .*

**Proof** Obvious by the definition of the  $F$ -family and the construction of the graph resulting of the dot product.  $\square$

We remark that the graphs obtained via Propositions 4.3 and 4.4 can be provided with a Fulkerson covering by Theorem 4.1.

The dot product operations described in Propositions 4.3 and 4.4 will be said to *preserve* the  $F$ -family.

## 5. APPLICATIONS

**5.1. Fulkerson coverings, more examples.** Figures 8 and 9(a) show that the Petersen Graph as well as the flower snark  $J_5$  have oddness 2 and have an  $F$ -family (the dashed edges denote the related perfect matching).

Moreover, as shown in Figure 9(b) the  $F$ -family of  $J_5$  can be extended by induction to all the  $J_k$ 's ( $k$  odd).

Thus, following the above Propositions we can define a sequence  $(G_n)_{n \in \mathbb{N}}$  of cubic graphs as follows :

- Let  $G_0$  be the Petersen graph or the flower snark  $J_k$  ( $k > 3$ ,  $k$  odd).

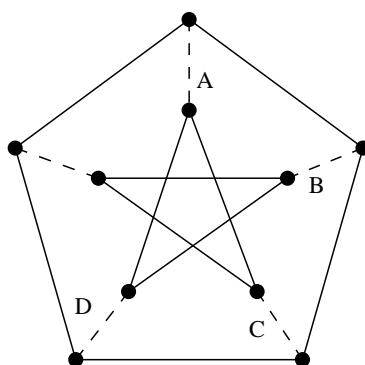
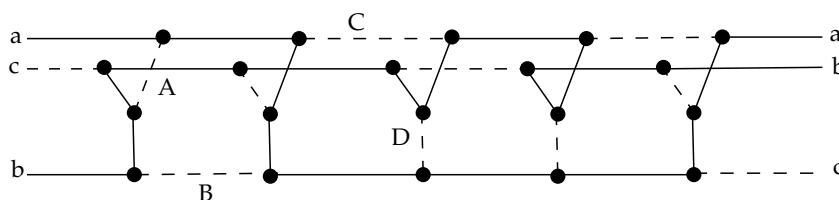
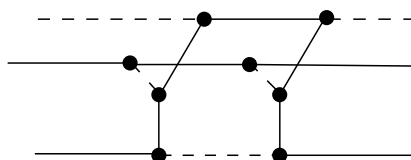


FIGURE 8. An F-family  $\{A, B, C, D\}$  for the Petersen graph.



(a) An F-family  $\{A, B, C, D\}$  for the flower snark  $J_5$ .



(b) Two more Claws.

FIGURE 9. An F-family  $\{A, B, C, D\}$  for the flower snark  $J_k$ .

- For  $n \in \mathbb{N}^*$ ,  $G_n = G_{n-1}.G$  where  $G$  is either the Petersen graph or the flower snark  $J_k$  ( $k > 3$ ,  $k$  odd) and the dot product operation preserves the F-family.

As a matter of fact this sequence of iterated dot products of the Petersen graph and/or the flower snark  $J_k$  forms a family of exponentially many snarks including the Szekeres Snark (see Figure 10) as well as the two types of generalized Blanuša snarks proposed by Watkins in [9] (see Figure 11).

The family obtained when reducing the possible values of  $k$  to  $k = 5$  has already been defined by Skupień in [8], in order to provide a family of hypohamiltonian snarks in using the so-called *Flip-flop construction* introduced by Chvátal in [1].

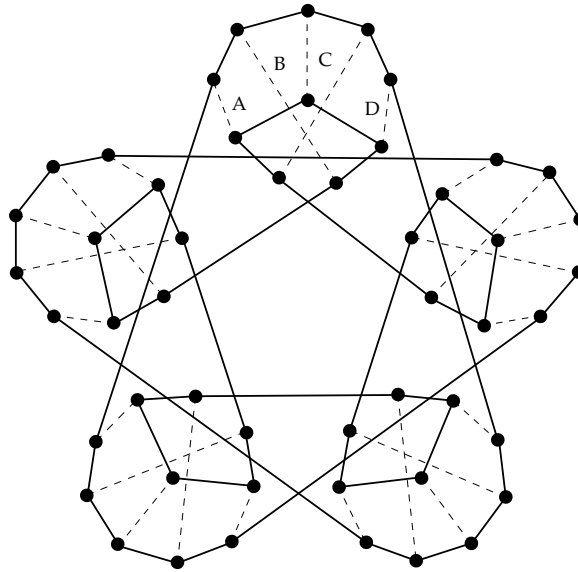


FIGURE 10. An F-family  $\{A, B, C, D\}$  for the Szekeres Snark.

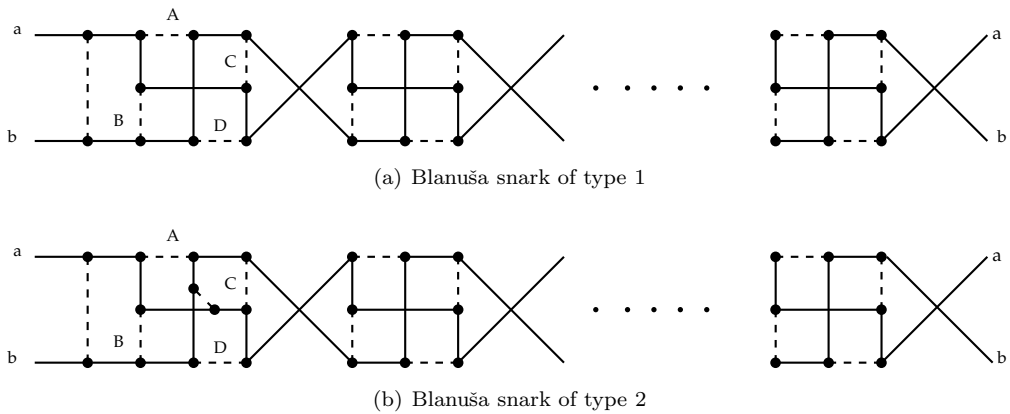


FIGURE 11. An F-family  $\{A, B, C, D\}$  for the Generalized Blanuša snarks

As far as we know there is no Fulkerson family for the Golberg snark.

**5.2. Graphs with a 2-factor of  $C_5$ 's.** Let  $G$  be a bridgeless cubic graph having a 2-factor where each cycle is isomorphic to a chordless  $C_5$ . We denote by  $G^*$  the multigraph obtained from  $G$  by shrinking each  $C_5$  of this 2-factor in a single vertex. The graph  $G^*$  is 5-regular and we can easily check that it is bridgeless.

**Theorem 5.1.** *Let  $G$  be a bridgeless cubic graph having a 2-factor of chordless  $C_5$ . Assume that  $G^*$  has chromatic index 5. Then  $G$  can be provided with a Fulkerson covering.*

**Proof** Let  $M$  be the perfect matching complementary of the 2-factor of  $C_5$ . Let  $\{A, B, C, D, E\}$  be a 5-edge colouring of  $G^*$ . Each colour corresponds to a matching of  $G$  (let us denote these matchings by  $A, B, C, D$  and  $E$ ). Then it is an easy task to see that  $\mathcal{M} = \{A, B, C, D\}$  is an F-family for  $M$  and the result follows from Theorem 4.1.  $\square$

**Theorem 5.2.** *Let  $G$  be a bridgeless cubic graph having a 2-factor of chordless  $C_5$ . Assume that  $G^*$  is bipartite. Then  $G$  can be provided with a Fulkerson covering.*

**Proof** It is well known, in that case, the chromatic index of  $G^*$  is 5. the result follows from Theorem 5.1.  $\square$

Remark that, when considering the Petersen graph  $P$ , the graph associated  $P^*$  is reduced to two vertices and is thus bipartite.

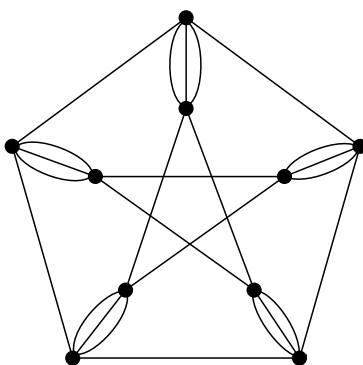
We can construct cubic graphs with chromatic index 4 which are cyclically 4-edge connected (*snarks* in the literature) and having a 2-factor of  $C_5$ 's. Indeed, let  $G$  be cyclically 4-edge connected snark of size  $n$  and  $M$  be a perfect matching of  $G$ ,  $M = \{x_i y_i | i = 1 \dots \frac{n}{2}\}$ . Let  $G_1 \dots G_{\frac{n}{2}}$  be  $\frac{n}{2}$  cyclically 4-edge connected snarks (each of them having a 2-factor of  $C_5$ ). For each  $G_i$  ( $i = 1 \dots \frac{n}{2}$ ) we consider two edges  $e_i^1$  and  $e_i^2$  of the perfect matching which is the complement of the 2-factor.

We construct then a new cyclically 4-edge connected snark  $H$  by applying the dot-product operation on  $\{e_i^1, e_i^2\}$  and the edge  $x_i y_i$  ( $i = 1 \dots \frac{n}{2}$ ). We remark that the vertices of  $G$  vanish in the operation and the resulting graph  $H$  has a 2 factor of  $C_5$ , which is the union of the 2-factors of  $C_5$  of the  $G_i$ . Unfortunately, when considering the graph  $H^*$ , derived from  $H$ , we cannot insure, in general, that  $H^*$  is 5-edge colourable in order to apply Theorem 5.1 and obtain hence a Fulkerson covering of  $H$ .

An interesting case is obtained when, in the above construction of  $H$ , each graph  $G_i$  is isomorphic to the Petersen graph. Indeed, the 2-factor of  $C_5$ 's obtained then is such that we can find a partition of the vertex set of  $H$  in sets of 2  $C_5$  joined by 3 edges. These sets lead to pairs of vertices of  $H^*$  joined by three parallel edges. We can thus see  $H^*$  as a cubic graph where a perfect matching is taken 3 times. Let us denote by  $\tilde{H}$  this cubic graph (by the way  $\tilde{H}$  is 3-connected). It is an easy task to see that, when  $\tilde{H}$  is 3-edge colourable,  $H^*$  is 5-edge colourable and hence, Theorem 5.1 can be applied.

Let us consider by example the graph  $H$  obtained with 5 copies of the Petersen graph following the above construction (let us remark that the graph  $G$  involved in our construction must be isomorphic also to the Petersen graph). This graph is a snark on 50 vertices. Since  $\tilde{H}$  is a bridgeless cubic graph, the only case for which we cannot say whether  $H$  has a Fulkerson covering when  $\tilde{H}$  is isomorphic to the Petersen graph and, hence  $H^*$  is isomorphic to the *unslicable* graph  $\mathcal{P}(3)$  described by Rizzi [7] (see Figure 12). As a matter of fact we do not know if it is possible to construct a graph  $H$  as described above such that  $H^*$  is isomorphic to the graph  $\mathcal{P}(3)$ .

By the way, we do not know example of cyclically 5-edge connected snarks (excepted the Petersen graph) with a 2-factor of induced cycles of length 5. We have proposed in [3] the following problem.

FIGURE 12.  $H^*$  isomorphic to  $\mathcal{P}(3)$ 

**Problem 5.3.** *Is there any 5-edge connected snark distinct from the Petersen graph with a 2-factor of  $C_5$ 's ?*

#### 6. ON PROPER FULKERSON COVERING

As noticed in the introduction, when a cubic graph is 3-edge colourable, we can find a Fulkerson covering by using a 3-edge colouring and considering each colour twice.

**Proposition 6.1.** *Let  $G$  be a bridgeless cubic graph with chromatic index 4. Assume that  $G$  has a Fulkerson covering  $\mathcal{F} = \{M_1, M_2, M_3, M_4, M_5, M_6\}$  of its edge set. Then the 6 perfect matchings are distinct.*

**Proof** Assume, without loss of generality that  $M_1 = M_2$ . Since each edge is contained in exactly 2 perfect matchings of  $\mathcal{F}$ , we must have  $M_3 \cap M_1 = \emptyset$ . Hence  $G$  is 3-edge colourable, a contradiction.  $\square$

Let us say that a Fulkerson covering is *proper* whenever the 6 perfect matchings involved in this covering are distinct. An interesting question is thus to determine which cubic bridgeless graph have a proper Fulkerson covering.

A 3-edge colourable graph is said to be *bi-hamiltonian* whenever in any 3-edge colouring, there are at least two colours whose removing leads to an hamiltonian 2-factor.

**Proposition 6.2.** *Let  $G$  be a bridgeless 3-edge colourable cubic graph which is not bi-hamiltonian. Then  $G$  has a proper Fulkerson covering.*

**Proof** Let  $\Phi : E(G) \rightarrow \{\alpha, \beta, \gamma\}$  be a 3-edge colouring of  $G$ . When  $x$  and  $y$  are colours in  $\{\alpha, \beta, \gamma\}$ ,  $\Phi(x, y)$  denotes the set of disjoint even cycles induced by the two colours  $x$  and  $y$ .

Since the graph  $G$  is not bi-hamiltonian we may assume that the 2-factors  $\Phi(\alpha, \beta)$  and  $\Phi(\beta, \gamma)$  are not hamiltonian cycles. Let  $C$  be a cycle in  $\Phi(\alpha, \beta)$ , we get a new 3-edge colouring  $\Phi'$  by exchanging the two colours  $\alpha$  and  $\beta$  along  $C$ . We get hence a partition of  $E(G)$  into 3 perfect matching  $\alpha', \beta'$  and  $\gamma$ . In the same way,

when considering a cycle  $D$  in  $\Phi(\beta, \gamma)$ , we get a new 3-edge colouring  $\Phi''$  of  $G$  by exchanging  $\beta$  and  $\gamma$  along  $D$ . Let  $\alpha, \beta''$  and  $\gamma''$  be the 3 perfect matchings so obtained.

Since we have two distinct 3-edge colourings of  $G$ ,  $\Phi'$  and  $\Phi''$ , the set of 6 perfect matchings so involved  $\{\alpha, \alpha', \beta', \beta'', \gamma, \gamma''\}$  is a Fulkerson covering. It remains to show that this set is actually a proper Fulkerson covering.

The exchange operated in order to get  $\Phi'$  involve some edges in  $\alpha$  and some edges in  $\beta$  (those which are on  $C_1$ ) while the other edges keep their colour. In the same way, the exchange operated in order to get  $\Phi''$  involve some edges in  $\beta$  and some edges in  $\gamma$  (those which are on  $D_1$ ) while the other edges keep their colour.

The 3 perfect matchings of  $\Phi'$  ( $\alpha', \beta'$  and  $\gamma$ ) are pairwise disjoint as well as those of  $\Phi''$  ( $\alpha, \beta''$  and  $\gamma''$ ). We have  $\alpha \neq \alpha'$  since  $\alpha'$  contains some edges of  $\beta$ . We have  $\alpha \cap \beta'' = \emptyset$  and  $\alpha \cap \gamma'' = \emptyset$  since we have exchanged  $\beta$  and  $\gamma$  in order to obtain  $\beta''$  and  $\gamma''$ . We have  $\beta' \neq \beta''$  since  $\beta'$  contains some edges of  $\alpha$  while  $\beta''$  contains some edges of  $\gamma$ . We have  $\beta' \neq \gamma''$  since  $\beta'$  contains some edges of  $\alpha$  and  $\gamma''$  contains only edges in  $\beta$  or in  $\gamma$ . We have  $\gamma \neq \gamma''$  since  $\gamma''$  contains some edges of  $\beta$ .

Hence  $\{\alpha, \alpha', \beta', \beta'', \gamma, \gamma''\}$  is a proper Fulkerson covering. □

The *theta graph* (2 vertices joined by 3 edges),  $K_4$ ,  $K_{3,3}$  are examples of small bridgeless cubic graph without proper Fulkerson covering. The infinite family of bridgeless cubic bi-hamiltonian graphs obtained by doubling the edges of a perfect matching of an even cycle has no proper Fulkerson covering. On the other hand, we can provide a bi-hamiltonian graph together with a proper Fulkerson covering. Consider for example the graph  $G$  on 10 vertices which have a 2 factor of  $C_5$ 's, namely  $abcde$  and  $12345$  with the additional edges edges  $a2$ ,  $b4$ ,  $c3$ ,  $d5$  and  $e1$ , it is not difficult to check that this graph is bi-hamiltonian. Moreover since the following four balanced matchings  $\{a2\}$ ,  $\{b4\}$ ,  $\{c3\}$  and  $\{d5\}$  form an F-family for the perfect matching  $\{a2, b4, c3, d5, e1\}$ , due to Theorem 4.1 and Remark 4.2, the graph  $G$  has a proper Fulkerson covering.

A challenging problem is thus to characterize those bridgeless cubic graphs having a proper Fulkerson covering.

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