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Laurent Demonet

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**MUTATIONS OF GROUP SPECIES WITH POTENTIALS  
AND THEIR REPRESENTATIONS.  
APPLICATIONS TO CLUSTER ALGEBRAS.**

LAURENT DEMONET

ABSTRACT. This article tries to generalize former works of Derksen, Weyman and Zelevinsky about skew-symmetric cluster algebras to the skew-symmetrizable case. We introduce the notion of group species with potentials and their decorated representations. In good cases, we can define mutations of these objects in such a way that these mutations mimic the mutations of seeds defined by Fomin and Zelevinsky for a skew-symmetrizable exchange matrix defined from the group species. These good cases are called non-degenerate. Thus, when an exchange matrix can be associated to a non-degenerate group species with potential, we give an interpretation of the  $F$ -polynomials and the  $\mathbf{g}$ -vectors of Fomin and Zelevinsky in terms of the mutation of group species with potentials and their decorated representations. Hence, we can deduce a proof of a serie of combinatorial conjectures of Fomin and Zelevinsky in these cases. Moreover, we give, for certain skew-symmetrizable matrices a proof of the existence of a non-degenerate group species with potential realizing this matrix. On the other hand, we prove that certain skew-symmetrizable matrices can not be realized in this way.

CONTENTS

1. Introduction	1
2. Group species and path algebras	3
3. Potential and their Jacobian ideals	5
4. Group species with potentials	6
5. Mutations of group species with potential	8
6. Exchange matrices	10
7. Existence of nondegenerate potentials	11
8. Decorated representations and their mutations	12
9. $F$ -polynomials and $\mathbf{g}$ -vectors of decorated representations	15
10. $\mathcal{E}$ -invariant	22
11. Applications to cluster algebras	24
12. An example and a counterexample	25
Acknowledgments	28
References	28

1. INTRODUCTION

The aim of this paper is to extend the results of [DWZ2] and [DWZ1] to the case of skew-symmetrizable exchange matrices. Unfortunately, the

techniques presented here do not work in any situation, but nevertheless in some important cases.

For this, we introduce *group species with potential* (GSP), which can be seen as quivers with potential with more than one idempotent at each vertex. Thus, we can also define a Jacobian ideal and a Jacobian algebra and study their representations. More precisely, we define the notion of a group species with potential with a *decorated representation* (GSPDR) and the notion of the mutation of a GSPDR at a vertex  $k$  (which is called the direction of the mutation). In *good cases*, we can mutate a GSPDR as many times as we want in any direction. In this case, the underlying GSP is called *non-degenerate*. Moreover, we can associate to certain GSPs, called locally free, a skew-symmetrizable matrix in such a way that the mutation we introduce projects, when it exists, to the mutation of matrix introduced by Fomin and Zelevinsky [FZ1]. Any skew-symmetrizable matrix can be reached in this way using a locally free GSP. The hard problem is to find which skew-symmetrizable matrix can be reached using a non-degenerate GSP. It is the case of matrices of the form  $DS$  where  $D$  is diagonal with positive integer coefficients and  $S$  is skew-symmetric with integer coefficients. It is also the case for the skew-symmetrizable matrices which occur in the situation of [Dem], in particular in all acyclic cases. Nevertheless, it is not always true, as shown by the counterexample at the end of section 12. The techniques presented in [DWZ2] work here almost in the same way. The only problem is that it is not always the case that for any 2-cycle, there exists a potential canceling it (this fact is very easy in the context of [DWZ2]).

We now explain the content of this article in more details. Let  $K$  be an algebraically closed field. Let  $I$  be a finite set and  $E = \bigoplus_{i \in I} K[\Gamma_i]$  where, for each  $i$ ,  $\Gamma_i$  is a finite group whose cardinal is not divisible by the characteristic of  $K$ . Let also  $A$  be an  $(E, E)$ -bimodule. This data is called a *group species* and its *complete path algebra* is

$$E\langle\langle A \rangle\rangle = \prod_{n \in \mathbb{N}} A^{\otimes n}.$$

A potential  $S$  on this group species can be seen as a (maybe infinite) linear combination of cyclic path, up to rotation. It permits to construct a two sided ideal  $J(S)$ , called the *Jacobian ideal* and a quotient algebra  $\mathcal{P}(A, S) = E\langle\langle A \rangle\rangle/J(S)$  called the *Jacobian algebra*. A *decorated representation* of the GSP is a pair consisting of a  $\mathcal{P}(A, S)$ -module  $X$  and an  $E$ -module  $V$ . In sections 5 and 8, we define the mutation of a GSP with a decorated representation (GSPDR). This mutation is well defined if the group species has no loop and is 2-acyclic (that is, for any  $i \in I$ ,  $E_i(A \oplus A \otimes_E A)E_i = 0$ , where  $E_i = K[\Gamma_i] \subset E$ ).

In what follows, we suppose that the  $\Gamma_i$  are commutative and that the GSP is *locally free*, that is, for any  $i, j \in I$ ,  $E_i A E_j$  is a free  $E_i$ -left module and a free  $E_j$ -right module. In section 6, we define the exchange matrix  $B$  of a the group species by

$$b_{ij} = \dim_{E_j} A_{ji} - \dim_{E_j} A_{ij}^*.$$

Thus, the mutation of GSPDRs descends to the mutation of matrices defined by Fomin and Zelevinsky [FZ1]. In section 7, we discuss a class of matrices,

namely those of the form  $DS$ , for which there is always a non-degenerate GSP. Moreover, we remark that there exists also non-degenerate GSP in all cases which are categorified in [Dem] (because the endomorphisms rings of cluster-tilting objects constructed in [Dem] are Jacobian algebras). Remark also that there is no chance, with definitions given here, to construct non-degenerate GSPs for any skew-symmetrizable matrix, as shown by the counterexample ending section 12.

Following the ideas of [DWZ1], we explain in section 9 how to reinterpret the  $F$ -polynomials and  $\mathbf{g}$ -vectors defined in [FZ2] in terms of GSPDRs and their mutations. We deduce in section 11 that, when a skew-symmetrizable matrix can be obtained from a non-degenerate GSP, then the following conjectures are true:

**Conjecture** ([FZ2, conjecture 5.4]). *For any  $\mathbf{i} \in I^n$  and  $k \in I$ ,  $F_{k;\mathbf{i}}^B$  has constant term 1.*

**Conjecture** ([FZ2, conjecture 5.5]). *For any  $\mathbf{i} \in I^n$  and  $k \in I$ ,  $F_{k;\mathbf{i}}^B$  has a maximum monomial for divisibility order with coefficient 1.*

**Conjecture** ([FZ2, conjecture 7.12]). *For any  $\mathbf{i} \in I^n$ ,  $k \in I$ , we denote by  $k\mathbf{i}$  the concatenation of  $(k)$  and  $\mathbf{i}$ . Let  $j \in I$  and  $(g_i)_{i \in I} = \mathbf{g}_{j;\mathbf{i}}^B$  and  $(g'_i)_{i \in I} = \mathbf{g}_{j;k\mathbf{i}}^{\mu_k(B)}$ . Then we have, for any  $i \in I$ ,*

$$g'_i = \begin{cases} -g_i & \text{if } i = k; \\ g_i + \max(0, b_{ik})g_k - b_{jk} \min(g_k, 0) & \text{if } i \neq k. \end{cases}$$

**Conjecture** ([FZ2, conjecture 6.13]). *For any  $\mathbf{i} \in I^n$ , the vectors  $\mathbf{g}_{i;\mathbf{i}}^B$  for  $i \in I$  are sign-coherent. In other terms, for  $i, i', j \in I$ , the  $j$ -th components of  $\mathbf{g}_{i;\mathbf{i}}^B$  and  $\mathbf{g}_{i';\mathbf{i}}^B$  have the same sign.*

**Conjecture** ([FZ2, conjecture 7.10(2)]). *For any  $\mathbf{i} \in I^n$ , the vectors  $\mathbf{g}_{i;\mathbf{i}}^B$  for  $i \in I$  form a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^I$ .*

**Conjecture** ([FZ2, conjecture 7.10(1)]). *For any  $\mathbf{i}, \mathbf{i}' \in I^n$ , if we have*

$$\sum_{i \in I} a_i \mathbf{g}_{i;\mathbf{i}}^B = \sum_{i \in I} a'_i \mathbf{g}_{i;\mathbf{i}'}$$

*for some nonnegative integers  $(a_i)_{i \in I}$  and  $(a'_i)_{i \in I}$ , then there is a permutation  $\sigma \in \mathfrak{S}_I$  such that for every  $i \in I$ ,*

$$a_i = a'_{\sigma(i)} \quad \text{and} \quad a_i \neq 0 \Rightarrow \mathbf{g}_{i;\mathbf{i}}^B = \mathbf{g}_{\sigma(i);\mathbf{i}'}^B \quad \text{and} \quad a_i \neq 0 \Rightarrow F_{i;\mathbf{i}}^B = F_{\sigma(i);\mathbf{i}'}^B.$$

*In particular,  $F_{i;\mathbf{i}}^B$  is determined by  $\mathbf{g}_{i;\mathbf{i}}^B$ .*

Thus, as stated in [FZ2, remark 7.11], if  $B$  is a full rank skew-symmetrizable matrix which correspond to a non-degenerate GSP, then the cluster monomials of a cluster algebra with exchange matrix  $B$  are linearly independent.

## 2. GROUP SPECIES AND PATH ALGEBRAS

Let  $K$  be a field.

**Definition 2.1.** A *group species* is a triple  $(I, (\Gamma_i)_{i \in I}, (A_{ij})_{(i,j) \in I^2})$  where  $I$  is a finite set, for each  $i \in I$ ,  $\Gamma_i$  is a finite group and for each  $(i, j) \in I^2$ ,  $A_{ij}$  is a finite dimensional  $(K[\Gamma_i], K[\Gamma_j])$ -bimodule (the first acting on the left and the second on the right).

Fix now such a group species  $Q = (I, (\Gamma_i)_{i \in I}, (A_{ij})_{(i,j) \in I^2})$

**Definition 2.2.** A *representation* of  $Q$  is a pair  $((V_i)_{i \in I}, (x_{ij})_{(i,j) \in I^2})$  where for each  $i \in I$ ,  $V_i$  is a right finite dimensional  $K[\Gamma_i]$ -module and for each  $(i, j) \in I^2$ ,

$$x_{ij} \in \text{Hom}_{\Gamma_j}(V_i \otimes_{\Gamma_i} A_{ij}, V_j).$$

**Definition 2.3.** Let  $((V_i)_{i \in I}, (x_{ij})_{(i,j) \in I^2})$  and  $((V'_i)_{i \in I}, (x'_{ij})_{(i,j) \in I^2})$  be two representations of  $Q$ . A *morphism* from the first one to the second one is a family  $(f_i)_{i \in I} \in \prod_{i \in I} \text{Hom}_{\Gamma_i}(V_i, V'_i)$  such that for each  $(i, j) \in I^2$  the following diagram commute :

$$\begin{array}{ccc} V_i \otimes_{\Gamma_i} A_{ij} & \xrightarrow{x_{ij}} & V_j \\ f_i \otimes \text{Id}_{A_{ij}} \downarrow & & \downarrow f_j \\ V'_i \otimes_{\Gamma_i} A_{ij} & \xrightarrow{x'_{ij}} & V'_j \end{array}$$

*Remarks 2.4.*

- The previous definitions give rise to an abelian category.
- If for each  $i \in I$ ,  $\Gamma_i$  is the trivial group, we get back the classical definition of a quiver (up to the choice of a basis of each  $A_{ij}$ ) and of the category of representations of a quiver.
- If for each  $i \in I$ ,  $K[\Gamma_i]$  is replaced by a division algebra, we obtain the usual definition of a species (see for example [DR]).

**Definition 2.5.** For each  $i \in I$ , denote  $E_i = K[\Gamma_i]$ . Denote also  $E = \bigoplus_{i \in I} E_i$  and  $A = \bigoplus_{(i,j) \in I^2} A_{ij}$ . Thus, we put the natural  $(E, E)$ -bimodule structure on  $A$  and define the graded algebras

$$E\langle A \rangle = \bigoplus_{n \in \mathbb{N}} A^{\otimes n} \quad \text{and} \quad E\langle\langle A \rangle\rangle = \prod_{n \in \mathbb{N}} A^{\otimes n}$$

the first one being called the *path algebra* of the group species and the second one the *complete path algebra* of the group species (note that every tensor product is taken over  $E$ ).

*Remarks 2.6.*

- As usual for quiver, the category of representations of a group species is equivalent to the category of finite dimensional right modules over its path algebra. Moreover, the category of nilpotent representations of a group species is equivalent to the category of finite dimensional right modules over its complete path algebra.
- If one denotes

$$\mathfrak{m} = \prod_{n > 0} A^{\otimes n} \subset E\langle\langle A \rangle\rangle$$

which is clearly a two-sided ideal, then  $E\langle\langle A \rangle\rangle$  becomes a topological algebra for the  $\mathfrak{m}$ -adic topology and  $E\langle A \rangle$  is a dense subalgebra of it.

As in [DWZ2],  $\mathfrak{m}$  is the unique maximal two-sided ideal of  $E\langle\langle A \rangle\rangle$  not intersecting  $E$ . Moreover, if we have another group species with the same vertices whose arrows are encoded in the  $(E, E)$ -bimodule  $A'$ , then, again as in [DWZ2], the morphisms  $\varphi$  from  $E\langle\langle A \rangle\rangle$  to  $E\langle\langle A' \rangle\rangle$  such that  $\varphi|_E = \text{Id}_E$  (later called  $E$ -morphisms) are parameterized in an obvious way by a pair  $(\varphi^{(1)}, \varphi^{(2)})$  where  $\varphi^{(1)} : A \rightarrow A'$  and  $\varphi^{(2)} : A \rightarrow \mathfrak{m}^2$  are  $(E, E)$ -bimodule morphisms. Thus,  $\varphi$  is an isomorphism if and only if  $\varphi^{(1)}$  is an isomorphism. Introduce now the analogous of [DWZ2, definition 2.5]:

**Definition 2.7.** An  $E$ -automorphism  $\varphi$  of  $E\langle\langle A \rangle\rangle$  will be called a *change of arrows* if  $\varphi^{(2)} = 0$  and a *unitriangular automorphism* if  $\varphi^{(1)} = \text{Id}_A$ .

Finally, introduce the following useful notation:

**Notation 2.8.** For all  $i, j \in I$ ,

$$E\langle A \rangle_{ij} = E_i E\langle A \rangle E_j \quad \text{and} \quad E\langle\langle A \rangle\rangle_{ij} = E_i E\langle\langle A \rangle\rangle E_j$$

and for  $n \in \mathbb{N}$ ,

$$A_{ij}^{\otimes n} = A^{\otimes n} \cap E\langle A \rangle_{ij} = A^{\otimes n} \cap E\langle\langle A \rangle\rangle_{ij}$$

so that

$$E\langle A \rangle_{ij} = \bigoplus_{n \in \mathbb{N}} A_{ij}^{\otimes n} \quad \text{and} \quad E\langle\langle A \rangle\rangle_{ij} = \prod_{n \in \mathbb{N}} A_{ij}^{\otimes n}.$$

### 3. POTENTIAL AND THEIR JACOBIAN IDEALS

Following [DWZ2] define:

**Definition 3.1.** Define

$$E\langle\langle A \rangle\rangle_{\text{cyc}} = \frac{E\langle\langle A \rangle\rangle}{[E\langle\langle A \rangle\rangle, E\langle\langle A \rangle\rangle]}$$

whose elements are called *potentials* (here,  $[E\langle\langle A \rangle\rangle, E\langle\langle A \rangle\rangle]$  is the closure of the two-sided ideal generated by commutators). As  $[E\langle\langle A \rangle\rangle, E\langle\langle A \rangle\rangle]$  is generated by its homogeneous elements, we can decompose  $E\langle\langle A \rangle\rangle_{\text{cyc}} = \prod_{n \in \mathbb{N}} A_{\text{cyc}}^{\otimes n}$  where

$$A_{\text{cyc}}^{\otimes n} = \frac{A^{\otimes n}}{[E\langle\langle A \rangle\rangle, E\langle\langle A \rangle\rangle] \cap A^{\otimes n}}$$

and, if  $S \in E\langle\langle A \rangle\rangle_{\text{cyc}}$ , we write  $S^{(n)}$  its summand which lies in  $A_{\text{cyc}}^{\otimes n}$ .

**Definition 3.2.** Define the continuous linear map

$$\partial : (E\langle\langle A \rangle\rangle)^* \otimes_k E\langle\langle A \rangle\rangle \rightarrow E\langle\langle A \rangle\rangle$$

in the following way. First remark that  $(E\langle\langle A \rangle\rangle)^* \simeq \bigoplus_{n \in \mathbb{N}} (A^{\otimes n})^*$ . Then, if  $\xi \in (A^{\otimes n})^*$  and  $a_1, a_2, \dots, a_\ell \in A$  define  $\partial_\xi(a_1 a_2 \dots a_\ell) = 0$  if  $\ell < n$  and

$$\partial_\xi(a_1 a_2 \dots a_\ell) = \sum_{j=1}^{\ell} \sum_{g, h \in \mathcal{B}} \xi(g^{-1} a_j a_{j+1} \dots a_{j+n-1} h) h^{-1} a_{j+n} a_{j+n+1} \dots a_{j-1} g$$

if  $\ell \geq n$  where all indices are taken modulo  $\ell$  and  $\mathcal{B} = \bigcup_{i \in I} \Gamma_i \subset E$ . It is easy to see that  $\partial$  is well defined and moreover that it vanishes on commutators. Thus, we can descend  $\partial$  to a continuous linear map

$$\partial : (E\langle\langle A \rangle\rangle)^* \otimes_k E\langle\langle A \rangle\rangle_{\text{cyc}} \rightarrow E\langle\langle A \rangle\rangle.$$

*Remark 3.3.* With the natural structure of  $(E, E)$ -bimodule on  $(E\langle\langle A \rangle\rangle)^*$ , one gets, for any  $S \in E\langle\langle A \rangle\rangle_{\text{cyc}}$ , that  $\xi \mapsto \partial_\xi S$  is a morphism of  $(E, E)$ -bimodules.

**Definition 3.4.** For a potential  $S \in E\langle\langle A \rangle\rangle_{\text{cyc}}$ , define the *Jacobian ideal*  $J(S)$  to be the closure of the two-sided ideal of  $E\langle\langle A \rangle\rangle$  generated by the  $\partial_\xi(S)$  for  $\xi \in A^*$ . The quotient  $E\langle\langle A \rangle\rangle/J(S)$  is called the *Jacobian algebra* and is denoted by  $\mathcal{P}(A, S)$  (we do not keep trace of  $(I, (\Gamma_i))$  in the notation because it will be fixed).

Note that every  $E$ -morphism  $\varphi : E\langle\langle A \rangle\rangle \rightarrow E\langle\langle A' \rangle\rangle$  descends to  $\varphi : E\langle\langle A \rangle\rangle_{\text{cyc}} \rightarrow E\langle\langle A' \rangle\rangle_{\text{cyc}}$ .

It is now easy to adapt the proof of [DWZ2, proposition 3.7]:

**Proposition 3.5.** *Let  $S \in E\langle\langle A \rangle\rangle_{\text{cyc}}$ . Every  $E$ -isomorphism  $\varphi : E\langle\langle A \rangle\rangle \rightarrow E\langle\langle A' \rangle\rangle$  maps  $J(S)$  to  $J(\varphi(S))$  and therefore induces an isomorphism*

$$\mathcal{P}(A, S) \rightarrow \mathcal{P}(A', \varphi(S)).$$

#### 4. GROUP SPECIES WITH POTENTIALS

For the rest of this article, the data  $(I, (\Gamma_i))$  and so  $E$  will be fixed. Following the ideas of [DWZ2], define:

**Definition 4.1.** As before,  $A$  is an  $(E, E)$ -bimodule and we take  $S \in E\langle\langle A \rangle\rangle_{\text{cyc}}$ . We say that  $(A, S)$  is a group species with potential (GSP for short) if the species has no loop (for all  $i \in I$ ,  $E_i A E_i = \{0\}$ ) and  $S \in \prod_{n>1} A_{\text{cyc}}^{\otimes n}$ .

**Definition 4.2.** Let  $(A, S)$  and  $(A', S')$  be two GSPs. One says that an  $E$ -isomorphism  $\varphi : E\langle\langle A \rangle\rangle \rightarrow E\langle\langle A' \rangle\rangle$  is a right-equivalence if  $\varphi(S) = S'$ .

Note that this definition induces an equivalence relation. Moreover, a right equivalence  $(A, S) \simeq (A', S')$  induces isomorphisms of  $(E, E)$ -bimodules  $A \simeq A'$ ,  $J(S) \simeq J(S')$  and  $\mathcal{P}(A, S) \simeq \mathcal{P}(A', S')$  as said before.

**Notation 4.3.** If  $(A, S)$  and  $(A', S')$  are two GSPs, define  $(A, S) \oplus (A', S') = (A \oplus A', S + S')$  so that  $\mathcal{P}((A, S) \oplus (A', S'))$  is the completion of  $\mathcal{P}(A, S) \oplus \mathcal{P}(A', S')$  for the product topology.

**Definition 4.4.** We say that a GSP  $(A, S)$  is *trivial* if  $S \in A_{\text{cyc}}^{\otimes 2}$  and  $\{\partial_\xi(S) \mid \xi \in A^*\} = A$ , or, equivalently, if  $\mathcal{P}(A, S) = E$ .

The following easy proposition is an adaptation of [DWZ2, proposition 4.4]:

**Proposition 4.5.** *A GSP  $(A, S)$  is trivial if and only if there exist an  $(E, E)$ -bimodule  $B$  and an  $(E, E)$ -bimodules isomorphism  $\varphi : A \rightarrow B \oplus B^*$  such that*

$$\varphi(S) = \sum_{b \in \mathcal{B}} b \otimes b^*$$

where  $\varphi$  is naturally extended to an isomorphism  $E\langle\langle A \rangle\rangle_{\text{cyc}} \rightarrow E\langle\langle B \oplus B^* \rangle\rangle_{\text{cyc}}$  and the right member does not depend of the choice of a basis  $\mathcal{B}$  of  $B$ .

One gets also this proposition, similar to [DWZ2, proposition 4.5]:

**Proposition 4.6.** *If  $(A, S)$  is a GSP and  $(B, T)$  is a trivial GSP, then the canonical embedding  $E\langle\langle A \rangle\rangle \hookrightarrow E\langle\langle A \oplus B \rangle\rangle$  induces an isomorphism  $\mathcal{P}(A, S) \simeq \mathcal{P}(A \oplus B, S + T)$ .*

For a GSP  $(A, S)$ , we define the *trivial* and *reduced* part of  $A$  as the  $(E, E)$ -bimodules

$$A_{\text{triv}} = \{\partial_\xi S^{(2)} \mid \xi \in A^*\} \quad \text{and} \quad A_{\text{red}} = A/A_{\text{triv}}.$$

Moreover, we say that  $(A, S)$  is reduced if  $S^{(2)} = 0$ , or, equivalently, if  $A_{\text{triv}} = \{0\}$ .

Again, the proof of [DWZ2, theorem 4.6] is easy to adapt:

**Theorem 4.7.** *For any GSP  $(A, S)$ , there exist  $S_{\text{triv}} \in E\langle\langle A_{\text{triv}} \rangle\rangle$  and  $S_{\text{red}} \in E\langle\langle A_{\text{red}} \rangle\rangle$  such that  $(A, S)$  is right equivalent to  $(A_{\text{triv}}, S_{\text{triv}}) \oplus (A_{\text{red}}, S_{\text{red}})$ .*

*Moreover, the right equivalence classes of  $(A_{\text{triv}}, S_{\text{triv}})$  and  $(A_{\text{red}}, S_{\text{red}})$  are uniquely determined by the right equivalence class of  $(A, S)$ .*

**Definition 4.8.** A group species  $(I, (\Gamma_i), A)$  is called *2-acyclic* if, for any  $i \in I$ ,  $E_i A^{\otimes 2} E_i = \{0\}$ .

We will see now how to find, as in [DWZ2], algebraic conditions guaranteeing the 2-acyclicity of the reduced part of a group species. Let  $K[E\langle\langle A \rangle\rangle_{\text{cyc}}]$  be the ring of polynomial functions on  $E\langle\langle A \rangle\rangle_{\text{cyc}}$  vanishing on all but a finite number of the  $A_{\text{cyc}}^{\otimes n}$ .

For each  $S \in E\langle\langle A \rangle\rangle_{\text{cyc}}$  and  $i, j \in I$ , define the bilinear form  $\alpha_{S, ij}$  by:

$$\begin{aligned} A_{ij}^* \times A_{ji}^* &\rightarrow K \\ (f, g) &\mapsto \sum_{\substack{\gamma \in \Gamma_i \\ \gamma' \in \Gamma_j}} \left[ (\gamma' f \gamma^{-1} \otimes \gamma g \gamma'^{-1}) (S^{(2)}) + (\gamma g \gamma'^{-1} \gamma' f \gamma^{-1}) (S^{(2)}) \right]. \end{aligned}$$

First, an easy lemma:

**Lemma 4.9.** *Let  $i, j \in I$ . The followings are equivalent:*

- (i) *there exists  $S \in E\langle\langle A \rangle\rangle_{\text{cyc}}$  such that  $\alpha_{S, ij}$  is of maximal rank;*
- (ii) *either  $A_{ij}^*$  is a subbimodule of  $A_{ji}$  or  $A_{ji}^*$  is a subbimodule of  $A_{ij}$ .*

*Proof.* We clearly have  $\alpha_{S, ij} = \alpha_{S, ji}$  for any  $S$  and therefore, one can suppose without loss of generality that  $\dim_K A_{ij} \leq \dim_K A_{ji}$ . Suppose that  $\alpha_{S, ij}$  is of maximal rank. In any basis, the matrix of  $\alpha_{S, ij}$  is the matrix of  $A_{ij}^* \rightarrow A_{ji} : \xi \mapsto \partial_\xi(S^{(2)})$  and therefore,  $A_{ij}^*$  is a subbimodule of  $A_{ji}$ .

Reciprocally, suppose that  $A_{ij}^*$  is a subbimodule of  $A_{ji}$ . Thus, if  $\mathcal{B}$  is a basis of  $A_{ij}$ , define

$$S = \sum_{a \in \mathcal{B}} a \otimes a^*$$

where  $a^* \in A_{ij}^*$  is identified with its image in  $A_{ji}$ . Then, it is clear that  $\alpha_{S, ij}$  is of maximal rank.  $\square$

Again, it is easy to generalize [DWZ2, proposition 4.15]:

**Proposition 4.10.** *The reduced part of a GSP  $(A, S)$  is 2-acyclic if and only if, for any  $i, j \in I$ ,  $\alpha_{S, ij}$  is of maximal rank. This condition is open. Moreover, if, for any  $i, j \in I$ , either  $A_{ij}^*$  is a subbimodule of  $A_{ji}$ , either*

$A_{ji}^*$  is a subbimodule of  $A_{ij}$ , then there is a non empty Zariski open subset  $U$  of  $E\langle\langle A \rangle\rangle_{\text{cyc}}$ , a 2-acyclic  $(E, E)$ -bimodule  $A'$  and a regular map  $H : U \rightarrow E\langle\langle A' \rangle\rangle_{\text{cyc}}$  such that for any  $S \in U$ ,  $(A_{\text{red}}, S_{\text{red}})$  is right equivalent to  $(A', H(S))$ .

*Proof.* The arguments are the same than in [DWZ2]. For each  $i, j \in I^2$ , choose  $\overline{A}_{ij}^* \subset A_{ij}^*$  such that  $\overline{A}_{ij}^* = A_{ij}^*$  or  $\overline{A}_{ij}^* \simeq A_{ji}$ . Let  $U$  to be the non-empty open subset of  $E\langle\langle A \rangle\rangle_{\text{cyc}}$  containing the  $S$  such that for all  $i, j \in I$ ,  $\alpha_{S, ij}|_{\overline{A}_{ij}^* \times \overline{A}_{ji}^*}$  is non-degenerate (it corresponds to the non-vanishing of a fixed maximal minor of  $\alpha_{S, ij}$ ). Define  $A'$  to be the intersection of the kernels of the elements of the  $\overline{A}_{ij}^*$ . Then the construction of  $H$  follows the proof of [DWZ2, theorem 4.6].  $\square$

## 5. MUTATIONS OF GROUP SPECIES WITH POTENTIAL

Let  $(A, S)$  and  $k \in I$  be a vertex such that  $E_k A^{\otimes 2} E_k = \{0\}$  (we say that  $(A, S)$  is 2-acyclic at  $k$ ). We suppose also that for any  $i \in I$ , the characteristic of  $K$  does not divide  $\#\Gamma_i$ . As in [DWZ2, §5], one defines  $\tilde{\mu}_k(A, S) = (\tilde{A}, \tilde{S})$  where, if  $i, j \in I$ ,

$$\tilde{A}_{ij} = \begin{cases} A_{ji}^* & \text{if } k \in \{i, j\}; \\ A_{ij} \oplus A_{ik} \otimes_{E_k} A_{kj} & \text{otherwise.} \end{cases}$$

In other terms,

$$\tilde{A} = \overline{E}_k A \overline{E}_k \oplus A E_k A \oplus (E_k A)^* \oplus (A E_k)^*$$

where  $\overline{E}_k = \bigoplus_{i \neq k} E_i$ . Let now  $[-] : \overline{E}_k E\langle\langle A \rangle\rangle \overline{E}_k \rightarrow E\langle\langle \tilde{A} \rangle\rangle$  be the morphism of  $k$ -algebras generated by  $[a] = a$  if  $a \in \overline{E}_k A \overline{E}_k$  and  $[ab] = ab \in A E_k A$  if  $a \in A E_k$  and  $b \in E_k A$  which is well defined because  $(A, S)$  has no loop. Again, because  $(A, S)$  has no loop, every potential  $S \in E\langle\langle A \rangle\rangle_{\text{cyc}}$  has a representative in  $\overline{E}_k E\langle\langle A \rangle\rangle \overline{E}_k$  and it is easy to see that  $[-]$  descends to a map

$$[-] : E\langle\langle A \rangle\rangle_{\text{cyc}} \rightarrow E\langle\langle \tilde{A} \rangle\rangle_{\text{cyc}}.$$

Moreover, as for any  $i \in I$  the characteristic of  $K$  does not divide  $\#\Gamma_i$ , we have a canonical sequence of isomorphisms

$$\begin{aligned} \text{Hom}_E(A E_k A, A E_k A) &\simeq (A E_k A)^* \otimes_E A E_k A \simeq (A E_k \otimes_E E_k A)^* \otimes_E A E_k A \\ &\simeq (E_k A)^* \otimes_E (A E_k)^* \otimes_E A E_k A \subset E\langle\langle \tilde{A} \rangle\rangle \end{aligned}$$

and we define  $\Delta_k(A)$  to be the image of  $\text{Id}_{A E_k A}$  through this isomorphism. Thus, define

$$\tilde{S} = [S] + \Delta_k(A).$$

The proof of [DWZ2, proposition 5.1] can be easily generalized:

**Proposition 5.1.** *If  $(A', S')$  is another GSP such that  $E_k A' = A' E_k = \{0\}$ , then*

$$\tilde{\mu}_k(A \oplus A', S + S') = \mu_k(A, S) \oplus (A', S').$$

Now, the proof of [DWZ2, theorem 5.2] is easy to generalize:

**Theorem 5.2.** *The right-equivalence class of the GSP  $\tilde{\mu}_k(A, S)$  is fully determined by the right-equivalence class of  $(A, S)$ .*

**Definition 5.3.** Using theorem 5.2 together with theorem 4.7, the right-equivalence class of the reduced part of  $\tilde{\mu}_k(A, S)$  is fully determined by the right-equivalence class of  $(A, S)$ . Thus we can define the map  $\mu_k$  from the set of right-equivalence classes which are 2-acyclic at  $k$  to itself. It is called the *mutation at vertex  $k$* .

Again, the proof of [DWZ2, theorem 5.7] is easy to generalize:

**Theorem 5.4.**  $\mu_k$  is an involution.

Let us also remark that [DWZ2, proposition 6.1], [DWZ2, proposition 6.4] and [DWZ2, corollary 6.6] can be generalized:

**Proposition 5.5.** The algebras  $\overline{E}_k \mathcal{P}(A, S) \overline{E}_k$  and  $\overline{E}_k \mathcal{P}(\tilde{\mu}_k(A, S)) \overline{E}_k$  are isomorphic.

**Proposition 5.6.** The Jacobian algebra  $\mathcal{P}(A, S)$  is finite-dimensional if and only if  $\mathcal{P}(\tilde{\mu}_k(A, S))$  is.

**Corollary 5.7.** The Jacobian algebras  $\overline{E}_k \mathcal{P}(A, S) \overline{E}_k$  and  $\overline{E}_k \mathcal{P}(\mu_k(A, S)) \overline{E}_k$  are isomorphic and  $\mathcal{P}(A, S)$  is finite-dimensional if and only if  $\mathcal{P}(\mu_k(A, S))$  is.

As stated in [DWZ2, remark 6.8], the following definition makes sense:

**Definition 5.8.** We define the *deformation space* of  $(A, S)$  to be

$$\text{Def}(A, S) = \frac{\mathcal{P}(A, S)}{E + [\mathcal{P}(A, S), \mathcal{P}(A, S)]}$$

where  $[\mathcal{P}(A, S), \mathcal{P}(A, S)]$  is the closure of the two-sided ideal of  $\mathcal{P}(A, S)$  generated by the commutators.

Thus, let us introduce the following extension of [DWZ2, proposition 6.9]:

**Proposition 5.9.** We have an isomorphism:

$$\text{Def}(A, S) \simeq \text{Def}(\tilde{\mu}_k(A, S)).$$

*Proof.* It is enough to prove that

$$\frac{\overline{E}_k \mathcal{P}(A, S) \overline{E}_k}{\overline{E}_k + [\overline{E}_k \mathcal{P}(A, S) \overline{E}_k, \overline{E}_k \mathcal{P}(A, S) \overline{E}_k]} \hookrightarrow \text{Def}(A, S)$$

is in fact an isomorphism (which is true because  $A$  has no loop) and to use proposition 5.5.

As in [DWZ2],

**Definition 5.10.** The GSP  $(A, S)$  is called *rigid* if  $\text{Def}(A, S) = \{0\}$ .

**Corollary 5.11.** The GSP  $(A, S)$  is rigid if and only if  $\mu_k(A, S)$  is.

## 6. EXCHANGE MATRICES

We suppose now that  $A$  has neither loop nor 2-cycle (that is  $A_{\text{cyc}}^{\otimes 1} = A_{\text{cyc}}^{\otimes 2} = \{0\}$ ). We suppose also that for any  $(i, j) \in I^2$ ,  $A_{ij}$  is a free left  $E_i$ -module and a free right  $E_j$ -module (we will call it a *locally free GSP*). Define the matrix  $B = B(A) = B(A, S)$  to be the matrix with rows and columns indexed by  $I$  and coefficients

$$b_{ij} = \dim_{E_j} A_{ji} - \dim_{E_j} A_{ij}^*$$

(by default, dimension are taken relatively to the left module structure). This matrix is clearly skew-symmetrizable since

$$\#\Gamma_j \times b_{ij} = \dim_K A_{ji} - \dim_K A_{ij}^*.$$

**Definition 6.1.** The matrix  $B$  is called the *exchange matrix* of  $A$ .

The following proposition justifies this generalization of [DWZ2]:

**Proposition 6.2.** *Every skew-symmetrizable matrix  $B$  can be reached in this way from a GSP.*

*Proof.* Let  $B$  be a skew-symmetrizable matrix and  $D = (d_i)_{i \in I}$  be a diagonal matrix with positive integer coefficients such that  $BD$  is skew-symmetric. Let  $\Gamma_i = \mathbb{Z}/d_i\mathbb{Z}$  and for  $(i, j) \in I^2$  such that  $b_{ij} > 0$ ,

$$A_{ji} = K[\mathbb{Z}/(d_j b_{ij})\mathbb{Z}] = K[\mathbb{Z}/(-d_i b_{ji})\mathbb{Z}]$$

which is a left and right free  $(\Gamma_j, \Gamma_i)$ -bimodule. It is clear that  $A = \bigoplus_{i, j \in I} A_{ij}$  has exchange matrix  $B$ .  $\square$

**Proposition 6.3.** *Let  $k \in I$ .*

- (i) *The GSP  $\tilde{\mu}_k(A, S)$  is locally free.*
- (ii) *If  $\mu_k(A, S)$  is 2-acyclic then it is locally free.*
- (iii) *If  $\mu_k(A, S)$  is 2-acyclic then*

$$\mu_k(B(A, S)) = B(\mu_k(A, S))$$

where the  $\mu_k$  on the left hand is the one defined in [FZ1]. Namely:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + \frac{b_{ik}|b_{kj}| + |b_{ik}|b_{kj}}{2} & \text{otherwise} \end{cases}$$

if  $B' = \mu_k(B)$ .

*Proof.* (i) First of all, it is clear that for  $i \in I$ ,  $E_i^* \simeq E_i$  as  $(E_i, E_i)$ -bimodules (as  $E_i$  is finite dimensional). Thus, for any  $i$ ,  $A_{ik}^*$  and  $A_{ki}^*$  are left and right free modules. Moreover, as a right module,

$$A_{ik} \otimes_{E_k} A_{kj} \simeq A_{kj}^{\dim_{E_k}(A_{ik}^*)}$$

and, as a left module,

$$A_{ik} \otimes_{E_k} A_{kj} \simeq A_{ik}^{\dim_{E_k}(A_{kj})}$$

which ends the proof that  $\tilde{\mu}_k(A, S)$  is locally free.

(ii) If one denotes  $(\tilde{A}, \tilde{S}) = \tilde{\mu}_k(A, S)$ , one has

$$\tilde{A} = \tilde{A}_{\text{red}} \oplus \tilde{A}_{\text{triv}}$$

As  $\tilde{A}_{\text{red}}$  is 2-acyclic, for any  $i, j \in I$ ,  $\tilde{A}_{\text{red},ij} = 0$  or  $\tilde{A}_{\text{red},ji} = 0$ . Suppose that  $\tilde{A}_{\text{red},ij} = 0$ . Hence  $\tilde{A}_{\text{triv},ji} \simeq \tilde{A}_{\text{triv},ij}^* \simeq \tilde{A}_{ij}^*$  is left and right free (thanks to the previous point). Moreover,  $\tilde{A}_{ji} = \tilde{A}_{\text{red},ji} \oplus \tilde{A}_{\text{triv},ji}$  and, as the categories of left  $E_j$ -modules and right  $E_i$ -modules are Krull-Schmidt,  $\tilde{A}_{\text{red},ji}$  is left and right free.

(iii) It is enough to remark that

$$\dim_{E_i} A_{ik} \otimes_{E_k} A_{kj} = \dim_{E_i} A_{ik}^{\dim_{E_k} A_{kj}} = \dim_{E_i}(A_{ik}) \dim_{E_k}(A_{kj})$$

and that

$$\dim_{E_i}(A_{jk} \otimes_{E_k} A_{ki})^* = \dim_{E_i}(A_{ki}^*)^{\dim_{E_k} A_{jk}^*} = \dim_{E_i}(A_{ki}^*) \dim_{E_k}(A_{jk}^*)$$

and to use the definition and the duality  $A_{\text{triv},ij} \simeq A_{\text{triv},ji}^*$ .  $\square$

**Definition 6.4.** The group species is said to be *globally free* if, for any  $i, j \in I$ ,  $A_{ij}$  is a free  $(E_i, E_j)$ -bimodule (that is a free  $E_i \otimes_K E_j^{\text{op}}$ -module).

*Remark 6.5.* The class of globally free group species is stable under mutation.

**Proposition 6.6.** *If a matrix is of the form  $DB$ , where  $D$  is diagonal with positive integer coefficients and  $B$  is skew-symmetric, then the group species constructed in proposition 6.2 is globally free.*

## 7. EXISTANCE OF NONDEGENERATE POTENTIALS

If  $(I, (\Gamma_i), A)$  is a group species without loop nor 2-cycle, a potential  $S \in E\langle\langle A \rangle\rangle_{\text{cyc}}$  will be said to be *non-degenerate* if every sequence of mutation going from  $(A, S)$  yields to a 2-acyclic GSP.

We cite the following adapted result, whose proof is the same than the proof of [DWZ2, corollary 7.4]:

**Theorem 7.1.** *If the group species is globally free then there is a countable number of non-constant polynomials in  $K[E\langle\langle A \rangle\rangle_{\text{cyc}}]$  such that the non-vanishing of these polynomials on  $S \in E\langle\langle A \rangle\rangle_{\text{cyc}}$  implies that  $S$  is non-degenerate. In particular if  $K$  is uncountable, there exist non-degenerate potentials.*

*Proof.* The only thing to change is that, if the group species is globally free, then for each  $i, j \in I$ , either  $A_{ij}^*$  is a subbimodule of  $A_{ji}$ , or  $A_{ji}^*$  is a subbimodule of  $A_{ij}$  and, therefore, proposition 4.10 can be applied.  $\square$

*Remark 7.2.* It is also easy to prove that for any skew-symmetrizable matrix  $B$  coming from the categories with an action of a group  $\Gamma$  considered in [Dem], there is a non-degenerate GSP realizing it. More precisely, the endomorphism ring of a  $\Gamma$ -stable cluster-tilting object in the stable category of a category constructed in [Dem] can be realized by a non-degenerate GSP (it is the case because  $\Gamma$ -2-cycles do not appear after mutations). In particular, the only potential for an acyclic group species is non-degenerate.

Another proposition linking rigid and non-degenerate potentials can be adapted from [DWZ2, proposition 8.1 and corollary 8.2]:

**Proposition 7.3.** *Every rigid globally free GSP  $(A, S)$  is 2-acyclic and, in this case,  $S$  is non-degenerate.*

As in [DWZ2, §8], there exist group species without rigid potentials. The techniques of [DWZ2, §8] work also in the context of this article.

## 8. DECORATED REPRESENTATIONS AND THEIR MUTATIONS

The aim of this section is to adapt the results of [DWZ2, §10]. We suppose here that for any  $i \in I$ , the characteristic of  $K$  does not divide the cardinal of  $\Gamma_i$ .

Following [DWZ2, definition 10.1],

**Definition 8.1.** A *decorated representation* of a GSP  $(A, S)$  is a pair  $(X, V)$  where  $X$  is a  $\mathcal{P}(A, S)$ -module and  $V$  is a  $E$ -module.

In the following, we will look at pairs consisting of a GSP  $(A, S)$  and a decorated representation of it. We will denote this type of objects by  $(A, S, X, V)$  and call them *group species with potential and decorated representation* (GSPDR).

Following [DWZ2, definition 10.2],

**Definition 8.2.** A right-equivalence between two GSPDRs  $(A, S, X, V)$  and  $(A', S', X', V')$  is a triple  $(\varphi, \psi, \eta)$  such that:

- $\varphi : E\langle\langle A \rangle\rangle \rightarrow E\langle\langle A' \rangle\rangle$  is a right-equivalence from  $(A, S)$  to  $(A', S')$  (see definition 4.2);
- $\psi : X \rightarrow X'$  is a linear isomorphism such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{u_X} & X \\ \psi \downarrow & & \downarrow \psi \\ X' & \xrightarrow{\varphi(u)_{X'}} & X' \end{array}$$

for any  $u \in E\langle\langle A \rangle\rangle$ ;

- $\eta : V \rightarrow V'$  is an isomorphism.

Using proposition 4.6, for each GSPDR  $(A, S, X, V)$ , the decorated representation  $(X, V)$  can be seen as a representation of  $(A_{\text{red}}, S_{\text{red}})$ . Thus, we can call  $(A_{\text{red}}, S_{\text{red}}, X, V)$  the *reduced part* of  $(A, S, X, V)$ . As in [DWZ2, proposition 10.5], the right-equivalence class of the reduced part of a GSPDR is fully determined by the right-equivalence class of this GSPDR.

Now, we can define the mutation of a GSPDR  $(A, S, X, V)$ . Let  $k \in I$ . Our aim is to define a GSPDR  $\mu_k(A, S, X, V) = (A', S', X', V')$  such that  $(A', S') = \mu_k(A, S)$ . Denote:

$$X_{\text{in}} = X \otimes_E A E_k \quad \text{and} \quad X_{\text{out}} = X \otimes_E A^* E_k.$$

Thus, we can define two right  $E_k$ -module morphisms. One,  $\alpha$ , from  $X_{\text{in}}$  to  $X_k = X E_k$  which is the application  $(x \otimes a) \mapsto xa$  and one from  $X_k$  to  $X_{\text{out}}$  which is defined by

$$\beta(x) = \sum_{b \in \mathcal{B}} x b \otimes b^*$$

which does not depend on the basis  $\mathcal{B}$  of  $E_k A$ . Observe also that we have a canonical sequence of isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{E_k}(X_{\mathrm{out}}, X_{\mathrm{in}}) &\simeq \mathrm{Hom}_E(X \otimes_E A^* E_k \otimes_{E_k} E_k A^*, X) \\ &\simeq \mathrm{Hom}_E(X \otimes_E (AE_k A)^*, X) \end{aligned}$$

It is not hard to see that  $[x \otimes \xi \mapsto x(\partial_\xi S)] \in \mathrm{Hom}_E(X \otimes_E (AE_k A)^*, X)$ . Let  $\gamma$  be the corresponding element of  $\mathrm{Hom}_{E_k}(X_{\mathrm{out}}, X_{\mathrm{in}})$ .

So we get, as in [DWZ2] a commutative diagram of right  $E_k$ -modules:

$$\begin{array}{ccc} & X_k & \\ \alpha \nearrow & & \searrow \beta \\ X_{\mathrm{in}} & \xleftarrow{\gamma} & X_{\mathrm{out}} \end{array}$$

with  $\alpha\gamma = \gamma\beta = 0$  [DWZ2, lemma 10.6]. For  $i \in I$ , define:

$$X'_i = \begin{cases} X_i & \text{if } i \neq k \\ \frac{\ker \gamma}{\mathrm{im} \beta} \oplus \mathrm{im} \gamma \oplus \frac{\ker \alpha}{\mathrm{im} \gamma} \oplus V_i & \text{if } i = k \end{cases}$$

and

$$V'_i = \begin{cases} V_i & \text{if } i \neq k \\ \frac{\ker \beta}{\ker \beta \cap \mathrm{im} \alpha} & \text{if } i = k \end{cases}$$

To get the structure of an  $\mathcal{P}(A', S')$ -module on  $X'$ , we must define the way  $\tilde{A}$  acts on it where  $(\tilde{A}, \tilde{S}) = \tilde{\mu}_k(A, S)$  (as  $\mathcal{P}(A', S') \simeq \mathcal{P}(\tilde{A}, \tilde{S})$ ). Recall from §5, that

$$\tilde{A} = \overline{E}_k A \overline{E}_k \oplus A E_k A \oplus (E_k A)^* \oplus (A E_k)^*.$$

First of all,  $\overline{E}_k A \overline{E}_k \oplus A E_k A \subset \overline{E}_k E \langle\langle A \rangle\rangle \overline{E}_k$  and for the vertices outside  $k$ ,  $X'_k = X_k$ . Therefore, we can take the same action for this part of  $\tilde{A}$ . For the rest, we have  $\tilde{A} E_k = A^* E_k$  and  $\tilde{A}^* E_k = A E_k$  and therefore, we have to define:

$$\alpha' : X'_{\mathrm{in}} = X' \otimes_E \tilde{A} E_k = X \otimes_E A^* E_k = X_{\mathrm{out}} \rightarrow X'_k$$

and

$$\beta' : X'_k \rightarrow X'_{\mathrm{out}} = X' \otimes_E \tilde{A}^* E_k = X \otimes_E A E_k = X_{\mathrm{in}}$$

As in [DWZ2], we have to choose a *splitting data*:

- let  $\rho : X_{\mathrm{out}} \twoheadrightarrow \ker \gamma$  be a splitting of  $\ker \gamma \hookrightarrow X_{\mathrm{out}}$  in the category mod  $E_k$  (it is possible, as the characteristic of  $K$  does not divide the cardinal of  $\Gamma_k$ );
- let  $\sigma : \ker \alpha / \mathrm{im} \gamma \hookrightarrow \ker \alpha$  a splitting of  $\ker \alpha \twoheadrightarrow \ker \alpha / \mathrm{im} \gamma$  in mod  $E_k$ .

Now, using the direct sum decomposition

$$X'_k = \frac{\ker \gamma}{\mathrm{im} \beta} \oplus \mathrm{im} \gamma \oplus \frac{\ker \alpha}{\mathrm{im} \gamma} \oplus V_i,$$

define

$$\alpha' = \begin{pmatrix} -\pi\rho \\ -\gamma \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta' = (0 \quad \iota \quad \iota\sigma \quad 0)$$

where  $\pi$  designs the canonical projection and  $\iota$  the canonical injections.

Again, [DWZ2, proposition 10.7] can be adapted:

**Proposition 8.3.** *The above definition gives rise to a decorated representation of  $(\tilde{A}, \tilde{S})$  and, therefore, through the isomorphism  $\mathcal{P}(\tilde{A}, \tilde{S}) \simeq \mathcal{P}(A', S')$ , to a decorated representation of  $(A', S')$ .*

**Notation 8.4.** We denote

$$\tilde{\mu}_k(A, S, X, V) = (\tilde{A}, \tilde{S}, X', V') \quad \text{and} \quad \mu_k(A, S, X, V) = (A', S', X', V').$$

We can adapt [DWZ2, proposition 10.9]:

**Proposition 8.5.** *The isomorphism class of the GSPDR  $\tilde{\mu}_k(A, S, X, V)$  does not depend on the choice of the splitting data.*

and [DWZ2, proposition 10.10 and corollary 10.12]:

**Proposition 8.6.** *The right-equivalence classes of the GSPDRs*

$$\tilde{\mu}_k(A, S, X, V) \quad \text{and} \quad \mu_k(A, S, X, V)$$

*depend only on the right-equivalence class of  $(A, S, X, V)$ .*

Now an important theorem whose proof is the same as the one of [DWZ2, theorem 10.13]:

**Theorem 8.7.** *On the right-equivalence classes of GSPDRs which are 2-acyclic at  $k$ ,  $\mu_k$  is an involution.*

It is easy to define the notion of a direct sum of two decorated representations of a GSP and, therefore, the notion of an indecomposable decorated representation of a GSP. Thus, as  $\mu_k$  clearly commutes with this type of direct sums,  $\mu_k$  acts on GSPs with indecomposable decorated representations. We call a GSPDR  $(A, S, X, V)$  positive if  $V = \{0\}$  and negative if  $X = \{0\}$ . Moreover, it is called *simple* at  $i \in I$  if  $X \oplus V$  is an indecomposable  $E_i$ -module. Then we adapt [DWZ2, proposition 10.15]:

**Proposition 8.8.** *An indecomposable GSPDR is either positive, or negative simple. The mutation  $\mu_k$  exchange a positive simple at  $k$  with the corresponding negative simple at  $k$ . Moreover, it is the only case where a mutation interchanges positive and negative indecomposable GSPDRs.*

As in [DWZ1, §6], denote, for  $k \in I$  and  $X, X' \in \text{mod } \mathcal{P}(A, S)$ ,

$$\text{Hom}_{\mathcal{P}(A, S)}^{[k]}(X, X') = \left\{ f \in \text{Hom}_{\mathcal{P}(A, S)}(X, X') \mid f|_{X\bar{E}_k} = 0 \right\}.$$

Cite now easy to adapt [DWZ1, proposition 6.1]:

**Proposition 8.9.** *The mutation  $\mu_k$  induces an isomorphism*

$$\frac{\text{Hom}_{\mathcal{P}(A, S)}(X, X')}{\text{Hom}_{\mathcal{P}(A, S)}^{[k]}(X, X')} \simeq \frac{\text{Hom}_{\mathcal{P}(\mu_k(A, S))}(\mu_k(X), \mu_k(X'))}{\text{Hom}_{\mathcal{P}(\mu_k(A, S))}^{[k]}(\mu_k(X), \mu_k(X'))}.$$

*Remark 8.10.* As claimed in [DWZ1, §6], the isomorphism of proposition 8.9 can be seen as a functorial isomorphism by introducing adapted quotient categories.

### 9. $F$ -POLYNOMIALS AND $\mathbf{g}$ -VECTORS OF DECORATED REPRESENTATIONS

The aim of this section is to define the notions of the  $F$ -polynomial and the  $\mathbf{g}$ -vector of a GSPDR and to give a link with the usual notion (see [FZ2]). It is an extension of [DWZ1]. As before,  $(I, (\Gamma_i))$  and therefore  $E$  are fixed. We suppose also that the characteristic of  $K$  does not divide any of the cardinals of the groups  $\Gamma_i$ . We suppose moreover that  $K$  is algebraically closed and that all the  $\Gamma_i$  are commutative (as seen in section 6, this case is sufficient to realize skew-symmetrizable exchange matrices).

**Notation 9.1.** For any  $i \in I$ , denote  $\text{irr}_i = \text{irr}(\Gamma_i)$  the set of isomorphism classes of irreducible representations of  $\Gamma_i$ . One defines  $\text{irr} = \bigcup_{i \in I} \{i\} \times \text{irr}_i$  and for  $i \in I$ ,  $C_i = K_0(\Gamma_i) \simeq \mathbb{Z}^{\text{irr}_i}$ . We also denote  $C = K_0(E) = \bigoplus_{i \in I} C_i \simeq \mathbb{Z}^{\text{irr}}$ . If  $V \in \text{mod } E$  (resp.  $V \in \text{mod } E_i$ ),  $[V]$  is its class in  $C$  (resp. in  $C_i$ ). If  $\mathbf{e} \in C$  (resp.  $\mathbf{e} \in C_i$ ) and  $(j, \rho) \in \text{irr}$  (resp.  $\rho \in \text{irr}_i$ ) then  $\mathbf{e}_{j, \rho}$  (resp.  $\mathbf{e}_\rho$ ) is the coefficient of  $(j, \rho)$  (resp.  $\rho$ ) in  $\mathbf{e}$ .

If  $(Y_j)_{j \in \text{irr}}$  (resp.  $(Y_j)_{j \in \text{irr}_i}$ ) is a family of indeterminates or of elements of a ring, and  $\mathbf{e} \in C$  (resp.  $\mathbf{e} \in C_i$ ), one denotes

$$Y^{\mathbf{e}} = \prod_{\substack{j \in \text{irr} \\ (\text{resp. } j \in \text{irr}_i)}} Y_j^{\mathbf{e}_j}.$$

If  $(A, S)$  is a GSP,  $X$  a representation of it,  $[X]$  is its class, seen as an  $E$ -module, in  $C$ . If  $\mathbf{e} \in C$  then  $\text{Gr}_{\mathbf{e}}(X)$  is the Grassmanian of the  $\mathcal{P}(A, S)$ -submodules  $X'$  of  $X$  such that  $[X'] = \mathbf{e}$ .

Let  $(A, S, X, V)$  be a GSPDR, we recall the diagram of section 8, by changing a little the notation:

$$\begin{array}{ccc} & X(k) & \\ \alpha_k \nearrow & & \searrow \beta_k \\ X_{\text{in}}(k) & \xleftarrow{\gamma_k} & X_{\text{out}}(k) \end{array}$$

**Definition 9.2.** One defines the  $F$ -polynomial  $F_X$  of  $X$  to be a polynomial in  $\mathbb{Z}[(Y_i)_{i \in \text{irr}}]$  defined by:

$$F_X(Y) = \sum_{\mathbf{e} \in C} \chi(\text{Gr}_{\mathbf{e}}(X)) Y^{\mathbf{e}}$$

where  $\chi$  is the Euler characteristic. One define also the  $\mathbf{g}$ -vector  $\mathbf{g}_{X, V} = (g_k)_{k \in I} \in C = \bigoplus_{k \in I} C_k$  by

$$g_k = [\ker \gamma_k] - [X(k)] + [V(k)].$$

With the same indexing, define  $\mathbf{h}_{X, V} = (h_k)_{k \in I}$  by

$$h_k = -[\ker \beta_k].$$

**Notation 9.3.** If  $(Y)$  is a family of indeterminates, we denote by  $\mathbb{Q}_+(Y)$  the free commutative semifield generated by its elements. If  $(y)$  is a family of elements of a commutative semifield, we denote by  $\mathbb{Q}_+(y)$  the subsemifield generated by its elements.

Then, it is easy to adapt [DWZ1, proposition 3.1], [DWZ1, proposition 3.2] and [DWZ1, proposition 3.3]:

**Proposition 9.4.** *The polynomial  $F_X(Y)$  has constant term 1 and maximum term (for divisibility of monomials)  $Y^{[X]}$ .*

**Proposition 9.5.** *If  $X'$  is another  $\mathcal{P}(A, S)$ -module then  $F_{X \oplus X'} = F_X F_{X'}$ .*

**Proposition 9.6.** *If  $F_X \in \mathbb{Q}_+(Y)$ , then  $F_X$  can be evaluated in the semifield  $\text{Trop}(Y')$  where  $(Y')_{i \in \text{irr}}$  is a family of indeterminates. Then  $\mathbf{h}_X$  and  $F_X$  are related by the following formula:*

$$Y^{\mathbf{h}_X} = F_X|_{\text{Trop}(Y')} \left( Y'_{i,\rho}{}^{-1} Y'^{[\rho \otimes_{E_i} E_i A^*]} \right)_{(i,\rho) \in \text{irr}}.$$

*Proof.* We follow the proof of [DWZ1]. Remark that for any  $\mathbf{e} \in C$ ,

$$(Y^{\mathbf{e}})|_{\text{Trop}(Y')} \left( Y'_{i,\rho}{}^{-1} Y'^{[\rho \otimes_{E_i} E_i A^*]} \right)_{(i,\rho) \in \text{irr}} = Y'^{-\mathbf{e} + [\mathbf{e} \otimes_E A^*]}.$$

For  $i \in I$ , the exponent of  $Y'_i = (Y_{i,\rho})_{\rho \in \text{irr}_i}$  can be rewritten as

$$-\mathbf{e}_i + [\mathbf{e} \otimes_E A^* E_i]$$

which can be interpreted as

$$-[X'(i)] + [X'_{\text{out}}(i)]$$

for any submodule  $X'$  of  $X$  such that  $[X'] = \mathbf{e}$ . Thus, the end of the proof is the same as in [DWZ1].  $\square$

Recall the definition of a  $Y$ -seed:

**Definition 9.7** ([DWZ1, §2]). A  $Y$ -seed is a pair  $(y, B)$  where  $y$  is a family of elements of a semifield indexed by  $I$  and  $B$  is a skew-symmetrizable matrix. For  $k \in I$ , we define  $\mu_k(y, B) = (y', \mu_k(B))$  where, for  $i \in I$ ,

$$y'_i = \begin{cases} y_i^{-1} & \text{if } i = k \\ y_i y_k^{\max(0, b_{ki})} (1 + y_k)^{-b_{ki}} & \text{if } i \neq k. \end{cases}$$

Now, define the notion of an extended  $Y$ -seed:

**Definition 9.8.** A *extended  $Y$ -seed* is a pair  $(y, (A, S))$  where  $y$  is a family of elements of a semifield indexed by  $\text{irr}$  and  $(A, S)$  is a non-degenerate GSP. For  $k \in I$ , we define  $\mu_k(y, (A, S)) = (y', \mu_k(A, S))$  where, for  $(i, \rho) \in \text{irr}$ ,

$$y'_{i,\rho} = \begin{cases} y_{i,\rho}^{-1} & \text{if } i = k \\ y_{i,\rho} y_k^{[\rho \otimes_{E_i} A_{ik}]} (1 + y_k)^{[\rho \otimes_{E_i} A_{ki}^*] - [\rho \otimes_{E_i} A_{ik}]} & \text{if } i \neq k. \end{cases}$$

*Remark 9.9.* The mutation of extended  $Y$ -seeds is involutive.

**Definition 9.10.** A  $Y$ -seed or an extended  $Y$ -seed will be called *free* if its variables  $y$  are algebraically independent.

*Remark 9.11.* The notion of freeness for a  $Y$ -seed (or an extended  $Y$ -seed) is stable under mutations. The semifield  $\mathbb{Z}_+(y)$  and the algebra  $\mathbb{Z}[y]$  are also stable under mutation, as the mutation is involutive.

**Definition 9.12.** Let  $(y, (A, S))$  be a free extended  $Y$ -seed and  $(z, B(A))$  be a  $Y$ -seed (for the same  $A$ ). The following morphism of algebra is called the *specialization map*:

$$\begin{aligned} \Phi_{y \rightarrow z} : \mathbb{Z}_+(y) &\rightarrow \mathbb{Z}_+(z) \\ y_{i,\rho} &\mapsto z_i. \end{aligned}$$

The analogous for  $\mathbb{Z}[y]$  and  $\mathbb{Z}[z]$  is also denoted by  $\Phi$ .

**Proposition 9.13.** *Let  $(y, (A, S))$  be a free extended  $Y$ -seed such that  $(A, S)$  is a locally free GSP, and  $(z, B(A))$  be a  $Y$ -seed. Let  $k \in I$ . Denote  $y' = \mu_k(y)$ , and  $z' = \mu_k(z)$ . Then,  $\Phi_{y' \rightarrow z'} = \Phi_{y \rightarrow z}$ .*

*Proof.* As  $y'$  generates  $\mathbb{Z}_+(y') = \mathbb{Z}_+(y)$ , it is enough to look at this for the  $y'_{i,\rho}$  for  $(i, \rho) \in \text{irr}$ . By definition,

$$\Phi_{y' \rightarrow z'}(y'_{i,\rho}) = z'_i$$

If  $i = k$ , then

$$\Phi_{y \rightarrow z}(y'_{i,\rho}) = \Phi_{y \rightarrow z}(y_{i,\rho}^{-1}) = z_i^{-1} = z'_i.$$

If  $i \neq k$ , then

$$\begin{aligned} \Phi_{y \rightarrow z}(y'_{i,\rho}) &= \Phi_{y \rightarrow z}\left(y_{i,\rho} y_k^{[\rho \otimes_{E_i} A_{ik}]} (1 + y_k)^{[\rho \otimes_{E_i} A_{ki}^*] - [\rho \otimes_{E_i} A_{ik}]}\right) \\ &= z_i \prod_{\sigma \in C_k} \left[ z_k^{[\rho \otimes_{E_i} A_{ik}] \sigma} (1 + z_k)^{[\rho \otimes_{E_i} A_{ki}^*] \sigma - [\rho \otimes_{E_i} A_{ik}] \sigma} \right] \\ &= z_i \left[ z_k^{\dim_K(\rho \otimes_{E_i} A_{ik})} (1 + z_k)^{\dim_K(\rho \otimes_{E_i} A_{ki}^*) - \dim_K(\rho \otimes_{E_i} A_{ik})} \right] \\ &= z_i \left[ z_k^{\dim_{E_i} A_{ik}} (1 + z_k)^{\dim_{E_i} A_{ki}^* - \dim_{E_i} A_{ik}} \right] \\ &= z_i \left[ z_k^{\max(0, b_{ki})} (1 + z_k)^{-b_{ki}} \right] = z'_i \end{aligned}$$

(here we use the fact that every considered irreducible representation is of dimension 1, as the considered groups are commutative and  $K$  is algebraically closed).  $\square$

To make the relation with  $F$ -polynomials and  $\mathbf{g}$ -vectors in cluster algebras, we need the following adaptation of [DWZ1, lemma 5.2]:

**Proposition 9.14.** *Let  $(A, S, X, V)$  be a GSPDR such that  $(A, S)$  is non-degenerate. Let  $k \in I$ . Denote  $(A', S', X', V') = \mu_k(A, S, X, V)$ . Suppose also that the extended  $Y$ -seed  $(y', (A', S'))$  is obtained from  $(y, (A, S))$  by the mutation at  $k$ . Denote  $\mathbf{g}_{X,V} = (g_i)_{i \in I}$ ,  $\mathbf{g}_{X',V'} = (g'_i)_{i \in I}$ ,  $\mathbf{h}_{X,V} = (h_i)_{i \in I}$  and  $\mathbf{h}_{X',V'} = (h'_i)_{i \in I}$ . Then*

- (i)  $\mathbf{g}_{X,V} = \mathbf{h}_{X,V} - \mathbf{h}_{X',V'}$ ;
- (ii) one has

$$(y_k + 1)^{h_k} F_X(y) = (y'_k + 1)^{h'_k} F_{X'}(y')$$

where

$$(y_k + 1)^{h_k} = \prod_{i \in \text{irr}_k} (y_{(k,i)} + 1)^{h_{ki}};$$

(iii) for any  $j \in I$ ,

$$g'_j = \begin{cases} -g_j & \text{if } j = k \\ g_j + [g_k \otimes_{E_k} A_{kj}] - [h_k \otimes_{E_k} A_{kj}] + [h_k \otimes_{E_k} A_{jk}^*] & \text{if } j \neq k. \end{cases}$$

*Proof.* (i) By definition, for  $i \in I$ ,  $g_i = [\ker \gamma_i] - [X(i)] + [V(i)]$ ,  $h_i = -[\ker \beta_i]$  and  $h'_i = -[\ker \beta'_i]$  (where  $\beta'$  is the analogous of  $\beta$  for  $(X', V')$ ). So it is enough to prove that

$$[\ker \gamma_i] + [V_i] + [\ker \beta_i] = [X(i)] + [\ker \beta'_i].$$

From the definition of  $\beta'_i$  given in section 8, it is easy to see that  $\ker \beta'_i \simeq \ker(\gamma_i)/\text{im}(\beta_i) \oplus V_i$ . And, therefore, the searched equality reduces to

$$[\text{im } \beta_i] + [\ker \beta_i] = [X(i)]$$

which is obvious.

(ii) We follow the proof of [DWZ1, lemma 5.2]. Let  $\mathbf{e} \in C$  and  $\mathbf{e}'$  its projection in  $\bigoplus_{i \neq k} C_i$ . Let  $X_0 = X\overline{E}_k$  which is a  $\overline{E}_k\mathcal{P}(A, S)\overline{E}_k$ -module. For any  $\overline{E}_k\mathcal{P}(A, S)\overline{E}_k$ -submodule  $W$  of  $X_0$ , one can define

$$W_{\text{in}}(k) = W \otimes_{\overline{E}_k} AE_k \subset X_{\text{in}}(k) \quad \text{and} \quad W_{\text{out}}(k) = W \otimes_{\overline{E}_k} A^*E_k \subset X_{\text{out}}(k)$$

which are well defined because  $(A, S)$  has no loop (and therefore  $X_{\text{in}} = X \otimes_{\overline{E}_k} AE_k$  and  $X_{\text{out}} = X \otimes_{\overline{E}_k} A^*E_k$ ).

For  $\mathbf{r}, \mathbf{s} \in C_k$ , define  $Z_{\mathbf{e}', \mathbf{r}, \mathbf{s}}(X)$  to be the subvariety of  $\text{Gr}_{\mathbf{e}'}(X_0)$  consisting of the  $W$  satisfying

- $[\alpha_k(W_{\text{in}}(k))] = \mathbf{r}$ ;
- $[\beta_k^{-1}(W_{\text{out}}(k))] = \mathbf{s}$ ;
- $\alpha_k(W_{\text{in}}(k)) \subset \beta_k^{-1}(W_{\text{out}}(k))$ .

Define also the variety

$$\tilde{Z}_{\mathbf{e}, \mathbf{r}, \mathbf{s}}(X) = \{W \in \text{Gr}_{\mathbf{e}}(X) \mid W\overline{E}_k \in Z_{\mathbf{e}', \mathbf{r}, \mathbf{s}}(X)\}$$

so that, by an easy computation,  $\tilde{Z}_{\mathbf{e}, \mathbf{r}, \mathbf{s}}(X)$  is a fiber bundle over  $Z_{\mathbf{e}', \mathbf{r}, \mathbf{s}}(X)$  with fiber  $\text{Gr}_{e_k - \mathbf{r}}(\mathbf{s} - \mathbf{r})$  (where, by abuse of notation, we identify  $\mathbf{s} - \mathbf{r} \geq 0$  with any of its representatives in  $\text{mod } E_k$ , and  $\text{Gr}_{e_k - \mathbf{r}}(\mathbf{s} - \mathbf{r}) = \emptyset$  if  $e_k - \mathbf{r}$  or  $\mathbf{s} - \mathbf{r}$  are not nonnegative). Hence, using the easy fact that  $\text{Gr}_{\mathbf{e}}(X)$  is the disjoint union of the  $\tilde{Z}_{\mathbf{e}, \mathbf{r}, \mathbf{s}}(X)$ , we obtain, as every considered irreducible representation is of dimension 1,

$$\chi(\text{Gr}_{\mathbf{e}}(X)) = \sum_{\mathbf{r}, \mathbf{s} \in C_k} \binom{\mathbf{s} - \mathbf{r}}{e_k - \mathbf{r}} \chi(Z_{\mathbf{e}', \mathbf{r}, \mathbf{s}}(X)).$$

where, for any  $\mathbf{r}_1, \mathbf{r}_2 \in C_k$ ,

$$\binom{\mathbf{r}_1}{\mathbf{r}_2} = \prod_{\rho \in \text{ind}_k} \binom{\mathbf{r}_{1, \rho}}{\mathbf{r}_{2, \rho}}$$

Then, substituting this expression in the definition of  $F_X$ , we obtain:

$$\begin{aligned}
F_X(y) &= \sum_{\mathbf{e} \in C} \left[ \sum_{\mathbf{r}, \mathbf{s} \in C_k} \binom{\mathbf{s} - \mathbf{r}}{e_k - \mathbf{r}} \chi(Z_{\mathbf{e}', \mathbf{r}, \mathbf{s}}(X)) \right] y^{\mathbf{e}} \\
&= \sum_{\substack{\mathbf{e}' \in \bigoplus_{i \neq k} C_i \\ \mathbf{r}, \mathbf{s} \in C_k}} \chi(Z_{\mathbf{e}', \mathbf{r}, \mathbf{s}}(X)) y^{\mathbf{e}'} \sum_{e_k \in C_k} \binom{\mathbf{s} - \mathbf{r}}{e_k - \mathbf{r}} y_k^{e_k} \\
&= \sum_{\substack{\mathbf{e}' \in \bigoplus_{i \neq k} C_i \\ \mathbf{r}, \mathbf{s} \in C_k}} \chi(Z_{\mathbf{e}', \mathbf{r}, \mathbf{s}}(X)) y^{\mathbf{e}' + \mathbf{r}} (1 + y_k)^{\mathbf{s} - \mathbf{r}}.
\end{aligned}$$

Now, as in [DWZ1], we have easily that

$$Z_{\mathbf{e}', \mathbf{r}, \mathbf{s}}(X) = Z_{\mathbf{e}', \bar{\mathbf{r}}, \bar{\mathbf{s}}}(X')$$

where

$$\bar{\mathbf{r}} = \left[ \mathbf{e}' \otimes_{\overline{E}_k} A^* E_k \right] - h_k - \mathbf{s} \quad \text{and} \quad \bar{\mathbf{s}} = \left[ \mathbf{e}' \otimes_{\overline{E}_k} A E_k \right] - h'_k - \mathbf{r}.$$

Using this, one gets

$$\begin{aligned}
(1 + y'_k)^{h'_k} F_{X'}(y') &= \sum_{\substack{\mathbf{e}' \in \bigoplus_{i \neq k} C_i \\ \bar{\mathbf{r}}, \bar{\mathbf{s}} \in C_k}} \chi(Z_{\mathbf{e}', \bar{\mathbf{r}}, \bar{\mathbf{s}}}(X')) y'^{\mathbf{e}' + \bar{\mathbf{r}}} (1 + y'_k)^{h'_k + \bar{\mathbf{s}} - \bar{\mathbf{r}}} \\
&= \sum_{\substack{\mathbf{e}' \in \bigoplus_{i \neq k} C_i \\ \mathbf{r}, \mathbf{s} \in C_k}} \chi(Z_{\mathbf{e}', \mathbf{r}, \mathbf{s}}(X)) y'^{\mathbf{e}'} y_k^{-\bar{\mathbf{s}} - h'_k} (1 + y_k)^{h'_k + \bar{\mathbf{s}} - \bar{\mathbf{r}}} \\
&= \sum_{\substack{\mathbf{e}' \in \bigoplus_{i \neq k} C_i \\ \mathbf{r}, \mathbf{s} \in C_k}} \chi(Z_{\mathbf{e}', \mathbf{r}, \mathbf{s}}(X)) y^{\mathbf{e}' + \mathbf{r}} (1 + y_k)^{h_k + \mathbf{s} - \mathbf{r}} \\
&= (1 + y_k)^{h_k} F_X(y)
\end{aligned}$$

- (iii) As  $g_k = h_k - h'_k$ ,  $g'_k = -g_k$ . If  $j \neq k$ , the equality we want to prove becomes, using again  $g_k = h_k - h'_k$ ,

$$[\ker \gamma'_j] - [\ker \beta'_k \otimes_{E_k} A_{kj}] = [\ker \gamma_j] - [\ker \beta_k \otimes_{E_k} A_{jk}^*]$$

and, up to a possible exchange of  $(A, S, X, V)$  and  $(A', S', X', V')$ , we can suppose that  $A_{kj} = 0$  (because  $A$  is 2-acyclic) and therefore, we have to prove that

$$[\ker \gamma'_j] = [\ker \gamma_j] - [\ker \beta_k \otimes_{E_k} A_{jk}^*].$$

Let

$$(\tilde{A}, \tilde{S}, \tilde{X}, \tilde{V}) = \tilde{\mu}_k(A, S, X, V)$$

in such a way that  $(A', S')$  is right-equivalent to  $(\tilde{A}, \tilde{S})_{\text{red}}$ . In this setting, one will prove that

$$[\ker \tilde{\gamma}_j] = [\ker \gamma_j] - [\ker \beta_k \otimes_{E_k} A_{jk}^*].$$

We can decompose

$$X_{\text{out}}(j) = X \otimes_E A^* E_j = X(k) \otimes_{E_k} A_{jk}^* \oplus X \overline{E}_k \otimes_{\overline{E}_k} \overline{E}_k A^* E_j$$

and we get

$$\tilde{X}_{\text{out}}(j) = X_{\text{out}}(k) \otimes_{E_k} A_{jk}^* \oplus X \overline{E}_k \otimes_{\overline{E}_k} \overline{E}_k A^* E_j$$

and

$$\tilde{X}_{\text{in}}(j) = \tilde{X}(k) \otimes_{E_k} \tilde{A}_{kj} \oplus X_{\text{in}}(j) = X'(k) \otimes_{E_k} A_{jk}^* \oplus X_{\text{in}}(j).$$

Along these decompositions, one has:

$$\gamma_j = \left( \psi \circ (\beta_k \otimes_{E_k} A_{jk}^*) \quad \eta \right) \quad \text{and} \quad \tilde{\gamma}_j = \begin{pmatrix} \alpha'_k \otimes_{E_k} A_{jk}^* & 0 \\ \psi & \eta \end{pmatrix}$$

where  $\psi : X_{\text{out}}(k) \otimes_{E_k} A_{jk}^* \rightarrow X_{\text{in}}(j)$  and  $\eta : X \overline{E}_k \otimes_{\overline{E}_k} \overline{E}_k A^* E_j \rightarrow X_{\text{in}}(j)$  are two  $E_j$ -modules morphisms (basically speaking, these two morphisms encode the part of  $\gamma_j$  which is not modified by the mutation at  $k$ ). Using definitions of section 8, we get easily that  $\ker \alpha'_k = \text{im } \beta_k$  and we get an exact sequence of  $E_j$ -modules:

$$0 \rightarrow \ker \beta_k \otimes_{E_k} A_{jk}^* \oplus \{0\} \rightarrow \ker \gamma_i \xrightarrow{f} \ker \tilde{\gamma}_i \rightarrow 0$$

where, along the previous decompositions

$$f(u, v) = ((\beta_k \otimes_{E_k} A_{jk}^*)u, v).$$

This short exact sequence implies that

$$[\ker \tilde{\gamma}_j] = [\ker \gamma_j] - [\ker \beta_k \otimes_{E_k} A_{jk}^*].$$

To finish, it remains to prove that  $[\ker \tilde{\gamma}_j] = [\ker \gamma'_j]$ . The proof is the same than in [DWZ1].  $\square$

**Definition 9.15.** For any GSPDR  $(A, S, X, V)$ , we define in the following way the reduced  $\mathbf{g}$ -vectors,  $\mathbf{h}$ -vectors and  $F$ -polynomials:

- for  $i \in I$ , let  $\check{\mathbf{g}}_{X,V} = (\check{g}_i)_{i \in I}$  defined by  $\check{g}_i = \dim_K g_i$  where  $(g_i)_{i \in I} = \mathbf{g}_{X,V}$ ;
- for  $i \in I$ , let  $\check{\mathbf{h}}_{X,V} = (\check{h}_i)_{i \in I}$  defined by  $\check{h}_i = \dim_K h_i$  where  $(h_i)_{i \in I} = \mathbf{h}_{X,V}$ ;
- $\check{F}_X = \Phi_{Y \rightarrow Z}(F_X)$  where  $(Y_i)_{i \in \text{irr}}$  and  $(Z_i)_{i \in I}$  are families of indeterminates.

**Corollary 9.16.** Let  $(A, S, X, V)$  be a GSPDR such that  $(A, S)$  is non-degenerate and locally free. Let  $k \in I$ . Denote

$$(A', S', X', V') = \mu_k(A, S, X, V).$$

Suppose also that the  $Y$ -seed  $(z', B(A'))$  is obtained from  $(z, B(A))$  by the mutation at  $k$ . Denote  $\check{\mathbf{g}}_{X,V} = (\check{g}_i)_{i \in I}$ ,  $\check{\mathbf{g}}_{X',V'} = (\check{g}'_i)_{i \in I}$ ,  $\check{\mathbf{h}}_{X,V} = (\check{h}_i)_{i \in I}$  and  $\check{\mathbf{h}}_{X',V'} = (\check{h}'_i)_{i \in I}$ . We also denote by  $(b_{ij})_{i,j \in I}$  the coefficients of  $B(A)$ . Then

- (i)  $\forall i \in I, \check{g}_i = \check{h}_i - \check{h}'_i$ ;
- (ii) one has

$$(z_k + 1)^{\check{h}_k} \check{F}_X(z) = (z'_k + 1)^{\check{h}'_k} \check{F}_{X'}(z');$$

- (iii) for any  $j \in I$ ,

$$\check{g}'_j = \begin{cases} -\check{g}_j & \text{if } j = k \\ \check{g}_j + \max(0, b_{jk})\check{g}_k - b_{jk}\check{h}_k & \text{if } j \neq k; \end{cases}$$

(iv) if  $F_X \in \mathbb{Q}_+(Y)$ , then  $\check{F}_X \in \mathbb{Q}_+(Z)$ . Then  $\check{\mathbf{h}}_X$  and  $\check{F}_X$  are related by the following formula:

$$Z_0^{\check{\mathbf{h}}_X} = \check{F}_X|_{\text{Trop}(Z_0)} \left( Z_{0,i}^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0, -b_{ji})} \right)_{i \in I}.$$

*Proof.* The points (i) and (iii) are immediate consequences of proposition 9.14. To prove (ii), it is enough to apply  $\Phi_{y \rightarrow z}$  to the analogous identity in proposition 9.14 (for any extended free  $Y$ -seed  $(y, (A, S))$ ) and then apply proposition 9.13. For (iv), remark that for any  $(i, \rho) \in \text{irr}$ ,

$$\Phi_{Y_0 \rightarrow Z_0} \left( Y_{0,i,\rho}^{-1} Y_0^{[\rho \otimes_{E_i} E_i A^*]} \right) = Z_{0,i}^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0, -b_{ji})}$$

is independent of  $\rho$  and therefore, it is easy to see that

$$\begin{aligned} & \check{F}_X|_{\text{Trop}(Z_0)} \left( Z_{0,i}^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0, -b_{ji})} \right)_{i \in I} \\ &= \Phi_{Y_0 \rightarrow Z_0} \left( F_X|_{\text{Trop}(Y_0)} \left( Y_{0,i,\rho}^{-1} Y_0^{[\rho \otimes_{E_i} E_i A^*]} \right)_{(i,\rho) \in I} \right) \\ &= \Phi_{Y_0 \rightarrow Z_0} \left( Y_0^{\mathbf{h}_X} \right) = Z_0^{\check{\mathbf{h}}_X} \end{aligned}$$

using proposition 9.6.  $\square$

In [FZ2], (see also [DWZ1, §2]), Fomin and Zelevinsky defined the notions of the  $F$ -polynomials and the  $\mathbf{g}$ -vectors associated to a sequence of mutation. More precisely, for a skew-symmetrizable matrix  $B$  (which will play the role of an initial seed), a sequence of indices  $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^n$  and  $k \in I$ , they define a polynomial  $F_{k;\mathbf{i}}^B \in \mathbb{Z}[Z_i]_{i \in I}$  and a vector  $\mathbf{g}_{k;\mathbf{i}}^B \in \mathbb{Z}^I$ .

**Definition 9.17.** Let  $(A, S)$  be a non-degenerate GSP and  $\mathbf{i} = (i_1, \dots, i_n)$  be in  $I^n$  and  $V$  an  $E$ -module. We denote

$$\left( A_{V;\mathbf{i}}^{A,S}, S_{V;\mathbf{i}}^{A,S}, X_{V;\mathbf{i}}^{A,S}, V_{V;\mathbf{i}}^{A,S} \right) = \mu_{i_1} \mu_{i_2} \dots \mu_{i_n} (\mu_{i_n} \dots \mu_{i_2} \mu_{i_1} (A, S), 0, V).$$

Remark that  $\left( A_{V;\mathbf{i}}^{A,S}, S_{V;\mathbf{i}}^{A,S} \right)$  is right-equivalent to  $(A, S)$ .

Thus, we can adapt theorem [DWZ1, theorem 5.1]:

**Theorem 9.18.** Let  $(A, S)$  be a non-degenerate locally free GSP. Let  $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^n$ ,  $k \in I$  and  $\rho \in \text{irr}_k$ . Then

$$\mathbf{g}_{k;\mathbf{i}}^{B(A)} = \check{\mathbf{g}}_{X_{\rho;\mathbf{i}}^{A,S}, V_{\rho;\mathbf{i}}^{A,S}} \quad \text{and} \quad F_{k;\mathbf{i}}^{B(A)} = \check{F}_{X_{\rho;\mathbf{i}}^{A,S}}.$$

*Proof.* With corollary 9.16, it is the same proof as in [DWZ1].  $\square$

We get also this following, analogous to [DWZ1, corollary 5.3]:

**Corollary 9.19.** In the situation of theorem 9.18, suppose that  $F_{k;\mathbf{i}}^{B(A)} \neq 1$ , hence  $X_{\rho;\mathbf{i}}^{A,S} \neq \{0\}$  and  $V_{\rho;\mathbf{i}}^{A,S} = \{0\}$  (see proposition 8.8). Let  $x_{k;\mathbf{i}}^{B(A)}$  be the corresponding cluster variable in the coefficient-free cluster algebra. In other terms

$$\left( \left( x_{i;\mathbf{i}}^{B(A)} \right)_{i \in I}, B' \right) = \mu_{i_n} \dots \mu_{i_2} \mu_{i_1} ((x_i)_{i \in I}, B(A)).$$

Then we have the following cluster character formula:

$$x_{k;\mathbf{i}}^{B(A)} = \prod_{i \in I} x_i^{-d_i} \sum_{\mathbf{e} \in C} \chi(\mathrm{Gr}_{\mathbf{e}}(X)) \prod_{i \in I} x_i^{-\mathrm{rk} \gamma_i + \sum_{j \in I} (\max(0, b_{ij}) e_j + \max(0, -b_{ij})(d_j - e_j))}$$

where  $X = X_{\rho;\mathbf{i}}^{A,S}$ ,  $d_i = \dim_K X(i)$  and  $e_i = \dim_K \mathbf{e}_i$ .

## 10. $\mathcal{E}$ -INVARIANT

The aim of this part is analogous to [DWZ1, §7, §8]. Let  $(A, S, X, V)$  and  $(A, S, X', V')$  be two GSPDRs with the same non-degenerate GSP. We denote:

$$\langle X, X' \rangle = \dim_K \mathrm{Hom}_{\mathcal{P}(A,S)}(X, X').$$

Define the three following integer functions:

$$\mathcal{E}^{\mathrm{inj}}(X, V; X', V') = \langle X, X' \rangle + ([X] | \mathbf{g}_{X', V'})$$

$$\mathcal{E}^{\mathrm{sym}}(X, V; X', V') = \mathcal{E}^{\mathrm{inj}}(X, V; X', V') + \mathcal{E}^{\mathrm{inj}}(X', V'; X, V)$$

$$\mathcal{E}(X, V) = \mathcal{E}^{\mathrm{inj}}(X, V; X, V) = \frac{\mathcal{E}^{\mathrm{sym}}(X, V; X, V)}{2}$$

where  $[X] \in C$  is the class of  $X$  seen as an  $E$ -module, and, for  $\mathbf{e}, \mathbf{e}' \in C$  (resp.  $\mathbf{e}, \mathbf{e}' \in C_k$  for  $k \in I$ ),

$$(\mathbf{e} | \mathbf{e}') = \sum_{\substack{i \in \mathrm{irr} \\ (\text{resp. } i \in \mathrm{irr}_k)}} \mathbf{e}_i \mathbf{e}'_i.$$

Then, we get, with the same proof as [DWZ1, theorem 7.1]:

**Theorem 10.1.** *We have, for any  $k \in I$ ,*

$$\begin{aligned} & \mathcal{E}^{\mathrm{inj}}(\mu_k(X, V); \mu_k(X', V')) - \mathcal{E}^{\mathrm{inj}}(X, V; X', V') \\ &= (\mathbf{h}_{\mu_k(X, V), k} | \mathbf{h}_{X', V', k}) - (\mathbf{h}_{X, V, k} | \mathbf{h}_{\mu_k(X', V'), k}). \end{aligned}$$

In particular,  $\mathcal{E}^{\mathrm{sym}}$  and  $\mathcal{E}$  are stable under mutations.

*Proof.* The only difference with [DWZ1] is that computations have to be done in the Grothendieck groups. Moreover, we have to worry about the skew-symmetrizability: with our convention, informally speaking, all  $b_{ik}$  should be replaced by  $-b_{ki}$  in the proof of [DWZ1]). For example,

$$\sum_{i \in I} \max(0, b_{ik}) \dim_K X(i)$$

in [DWZ1] will be replaced here by  $[X \otimes_E A^* E_k]$  whose dimension is

$$\sum_{i \in I} \max(0, -b_{ki}) \dim_K X(i)$$

if the GSP is locally free and  $B = B(A)$ . □

We get also the following analogous of [DWZ1, corollary 7.2]:

**Corollary 10.2.** *If  $(X, V)$  is obtained by a sequence of mutations from a negative decorated representation  $(\{0\}, V)$  then  $\mathcal{E}(X, V) = 0$ .*

We denote by  $A^{\text{op}}$  the  $(E, E)$ -bimodule whose underlying vector space is  $A$  and whose bimodule structure is given by  $g \cdot a^{\text{op}} \cdot h = (h^{-1} \cdot a \cdot g^{-1})^{\text{op}}$  if  $g \in \Gamma_i$  and  $h \in \Gamma_j$  for some  $i, j \in I$  and  $\text{op} : A \rightarrow A^{\text{op}}$  comes from the identity of  $A$ . It is then easy to extend  $\text{op}$  to an anti-isomorphism of algebras  $E\langle\langle A \rangle\rangle \rightarrow E\langle\langle A^{\text{op}} \rangle\rangle$ . Thus,  $(X^*, V^*)$  is a decorated representation of the GSP  $(A^{\text{op}}, S^{\text{op}})$  on the ring  $E$ , where for each  $i \in I$ ,  $X_i^*$  is contragredient to  $X_i$ ,  $V_i^*$  is contragredient to  $V_i$  and  $a^{\text{op}}$  acts on  $X^*$  as the transpose of  $a$  for every  $a \in A$ . Thus, one gets the analogous of [DWZ1, proposition 7.3]:

**Proposition 10.3.** *We have  $\mathcal{E}(X^*, V^*) = \mathcal{E}(X, V)$ .*

*Proof.* As for any  $i \in I$ , the characteristic of  $K$  does not divide  $\#\Gamma_i$ , we have an isomorphism of right  $E$ -modules

$$\begin{aligned} (X \otimes_E A)^* &\rightarrow X^* \otimes_E A^{*\text{op}} \simeq X^* \otimes_E A^{\text{op}*} \\ f &\mapsto \sum_{\substack{x \in \mathcal{B}_X \\ a \in \mathcal{B}_A}} f(x \otimes a) x^* \otimes a^{*\text{op}} \\ \left( x \otimes a \mapsto \sum_{i \in I} \sum_{g \in \Gamma_i} \frac{\varphi(xg)\psi(g^{-1}a)}{\#\Gamma_i} \right) &\leftarrow \varphi \otimes \psi^{\text{op}} \end{aligned}$$

which does not depend of the bases  $\mathcal{B}_X$  and  $\mathcal{B}_A$  of  $X$  and  $A$ . Thus, we have, as in [DWZ1],

$$\begin{aligned} \mathcal{E}(X, V) &= \langle X, X \rangle + ([X] \parallel [X \otimes_E A^*]) + \left( [X] \left| [V] - [X] - \left[ \bigoplus_{i \in I} \text{im } \gamma_i \right] \right. \right) \\ &= \langle X, X \rangle + ([X \otimes_E A] \parallel [X]) + \left( [X] \left| [V] - [X] - \left[ \bigoplus_{i \in I} \text{im } \gamma_i \right] \right. \right) \\ &= \langle X^*, X^* \rangle + ([X \otimes_E A]^* \parallel [X^*]) \\ &\quad + \left( [X^*] \left| [V^*] - [X^*] - \left[ \bigoplus_{i \in I} \text{im } \gamma_i^* \right] \right. \right) \\ &= \langle X^*, X^* \rangle + ([X^* \otimes_E A^{\text{op}*}] \parallel [X^*]) \\ &\quad + \left( [X^*] \left| [V^*] - [X^*] - \left[ \bigoplus_{i \in I} \text{im } \gamma_i^* \right] \right. \right) \\ &= \mathcal{E}(X^*, V^*) \end{aligned}$$

where we used that

$$\begin{aligned} ([X] \parallel [X \otimes_E A^*]) &= \dim_K \text{Hom}_E(X, X \otimes_E A^*) \\ &= \dim_K \text{Hom}_E(X \otimes_E A, X) = ([X \otimes_E A] \parallel [X]). \quad \square \end{aligned}$$

Hence, the following theorem has the same proof as [DWZ1, theorem 8.1] (note that all [DWZ1, §10] can be easily adapted in this case):

**Theorem 10.4.** *The  $\mathcal{E}$ -invariant satisfies*

$$\mathcal{E}(X, V) \geq \left( \left[ \bigoplus_{i \in I} \ker \beta_i \right] \left| \left[ \bigoplus_{i \in I} \frac{\ker \gamma_i}{\text{im } \beta_i} \right] \right. \right) + ([X] \parallel [V]).$$

Then, we obtain the analogous of [DWZ1, corollary 8.3]:

**Corollary 10.5.** *If  $\mathcal{E}(X, V) = 0$  then for each  $(k, \rho) \in \text{irr}$ ,*

- (i) *either  $[M_k]_\rho = 0$  or  $[V_k]_\rho = 0$ ;*
- (ii) *either  $[\ker \gamma_k]_\rho = 0$  or  $[\ker \gamma_k]_\rho = [\text{im } \beta_k]_\rho$ .*

## 11. APPLICATIONS TO CLUSTER ALGEBRAS

We conclude here that the following conjectures of [FZ2] are true for skew-symmetrizable integer matrix which can be obtained from a non-degenerate GSP with abelian groups. In particular, every matrix of the form  $DS$  where  $D$  is diagonal with integer coefficients and  $S$  is skew-symmetric with integer coefficients can be obtained in view of section 7. Every exchange matrix corresponding to the situation described in [Dem] (in particular every acyclic ones) can also be raised. Let  $B$  be such a skew-symmetrizable integer matrix. We suppose moreover that some  $(A, S)$  is fixed satisfying the hypothesis of section 9 such that  $B(A) = B$ .

**Proposition 11.1** ([FZ2, conjecture 5.4]). *For any  $\mathbf{i} \in I^n$  and  $k \in I$ ,  $F_{k;\mathbf{i}}^B$  has constant term 1.*

**Proposition 11.2** ([FZ2, conjecture 5.5]). *For any  $\mathbf{i} \in I^n$  and  $k \in I$ ,  $F_{k;\mathbf{i}}^B$  has a maximum monomial for divisibility order with coefficient 1.*

These first two are immediate, as in [DWZ1, §9].

**Proposition 11.3** ([FZ2, conjecture 7.12]). *For any  $\mathbf{i} \in I^n$ ,  $k \in I$ , we denote by  $k\mathbf{i}$  the concatenation of  $(k)$  and  $\mathbf{i}$ . Let  $j \in I$  and  $(g_i)_{i \in I} = \mathbf{g}_{j;\mathbf{i}}^B$  and  $(g'_i)_{i \in I} = \mathbf{g}_{j;k\mathbf{i}}^{\mu_k(B)}$ . Then we have, for any  $i \in I$ ,*

$$g'_i = \begin{cases} -g_i & \text{if } i = k; \\ g_i + \max(0, b_{ik})g_k - b_{jk} \min(g_k, 0) & \text{if } i \neq k. \end{cases}$$

*Proof.* We need here to add some trick to the proof of [DWZ1, §9]. Indeed, we need to prove, as in [DWZ1], that

$$\min(0, g_k) = h_k.$$

But what we obtain by using corollary 10.5 is

$$\min(0, g_{k,\rho}) = h_{k,\rho}$$

for any  $\rho \in \text{irr}_k$ . Moreover, we have, as seen before,

$$g_k = \sum_{\rho \in \text{irr}_k} g_{k,\rho} \quad \text{and} \quad h_k = \sum_{\rho \in \text{irr}_k} h_{k,\rho}$$

and therefore, what we need is equivalent to the fact that the  $g_{k,\rho}$  are of the same sign. We will prove this with an indirect method. Retaining the notation of definition 9.17, we get

$$X_{E_j;\mathbf{i}}^{A,S} = \sum_{\rho \in \text{irr}_j} X_{\rho;\mathbf{i}}^{A,S}$$

and therefore, by linearity of  $\mathbf{g}$ ,

$$\mathbf{g}_{X_{E_j;\mathbf{i}}^{A,S}} = \sum_{\rho \in \text{irr}_j} \mathbf{g}_{X_{\rho;\mathbf{i}}^{A,S}}.$$

Hence, we get:

$$(\#\Gamma_j)g_k = \dim_K \left[ \mathbf{g}_{X_{E_j;\mathbf{i}}}^{A,S} \right]_k.$$

In the same way,

$$(\#\Gamma_j)h_k = \dim_K \left[ \mathbf{h}_{X_{E_j;\mathbf{i}}}^{A,S} \right]_k.$$

Moreover, by an immediate induction using proposition 9.14, as  $[E_j]$  is the class of a free  $E_j$ -module,  $\left[ \mathbf{g}_{X_{E_j;\mathbf{i}}}^{A,S} \right]_k$  and  $\left[ \mathbf{h}_{X_{E_j;\mathbf{i}}}^{A,S} \right]_k$  are also free and therefore, their coefficients in term of the irreducible representations of  $E_k$  are of the same sign. Hence, we obtain, by adding these components

$$\min(0, (\#\Gamma_j)g_k) = (\#\Gamma_j)h_k$$

and the rest follows as in [DWZ1]. Note that it implies also that the  $g_{k,\rho}$  are of the same sign.  $\square$

The three following propositions have the same proof than in [DWZ1, §9]:

**Proposition 11.4** ([FZ2, conjecture 6.13]). *For any  $\mathbf{i} \in I^n$ , the vectors  $\mathbf{g}_{i;\mathbf{i}}^B$  for  $i \in I$  are sign-coherent. In other terms, for  $i, i', j \in I$ , the  $j$ -th components of  $\mathbf{g}_{i;\mathbf{i}}^B$  and  $\mathbf{g}_{i';\mathbf{i}}^B$  have the same sign.*

**Proposition 11.5** ([FZ2, conjecture 7.10(2)]). *For any  $\mathbf{i} \in I^n$ , the vectors  $\mathbf{g}_{i;\mathbf{i}}^B$  for  $i \in I$  form a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^I$ .*

**Proposition 11.6** ([FZ2, conjecture 7.10(1)]). *For any  $\mathbf{i}, \mathbf{i}' \in I^n$ , if we have*

$$\sum_{i \in I} a_i \mathbf{g}_{i;\mathbf{i}}^B = \sum_{i \in I} a'_i \mathbf{g}_{i;\mathbf{i}'}^B$$

*for some nonnegative integers  $(a_i)_{i \in I}$  and  $(a'_i)_{i \in I}$ , then there is a permutation  $\sigma \in \mathfrak{S}_I$  such that for every  $i \in I$ ,*

$$a_i = a'_{\sigma(i)} \quad \text{and} \quad a_i \neq 0 \Rightarrow \mathbf{g}_{i;\mathbf{i}}^B = \mathbf{g}_{\sigma(i);\mathbf{i}'}^B \quad \text{and} \quad a_i \neq 0 \Rightarrow F_{i;\mathbf{i}}^B = F_{\sigma(i);\mathbf{i}'}^B.$$

*In particular,  $F_{i;\mathbf{i}}^B$  is determined by  $\mathbf{g}_{i;\mathbf{i}}^B$ .*

## 12. AN EXAMPLE AND A COUNTEREXAMPLE

The aim of this part is to show an example where the technique shown in the previous sections works and a counterexample where there is no non-degenerate potential.

Suppose here that  $K = \mathbb{C}$ . We fix  $\Gamma_1 = \Gamma_2$  to be the trivial group and  $\Gamma_3 = \mathbb{Z}/2\mathbb{Z}$ . We take  $A_{12} = \mathbb{C}$  and  $A_{23} = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$ , the other  $A_{ij}$  vanishing. Then  $A$  is acyclic and therefore  $S = 0$  is a non-degenerate potential, in view of section 7. Moreover,

$$B(A) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}$$

which is of type  $C_3$ . Its exchange graph is given on figure 1 where the small dots ( $\cdot$ ) symbolize vertices with trivial group and big dots ( $\bullet$ ) symbolize

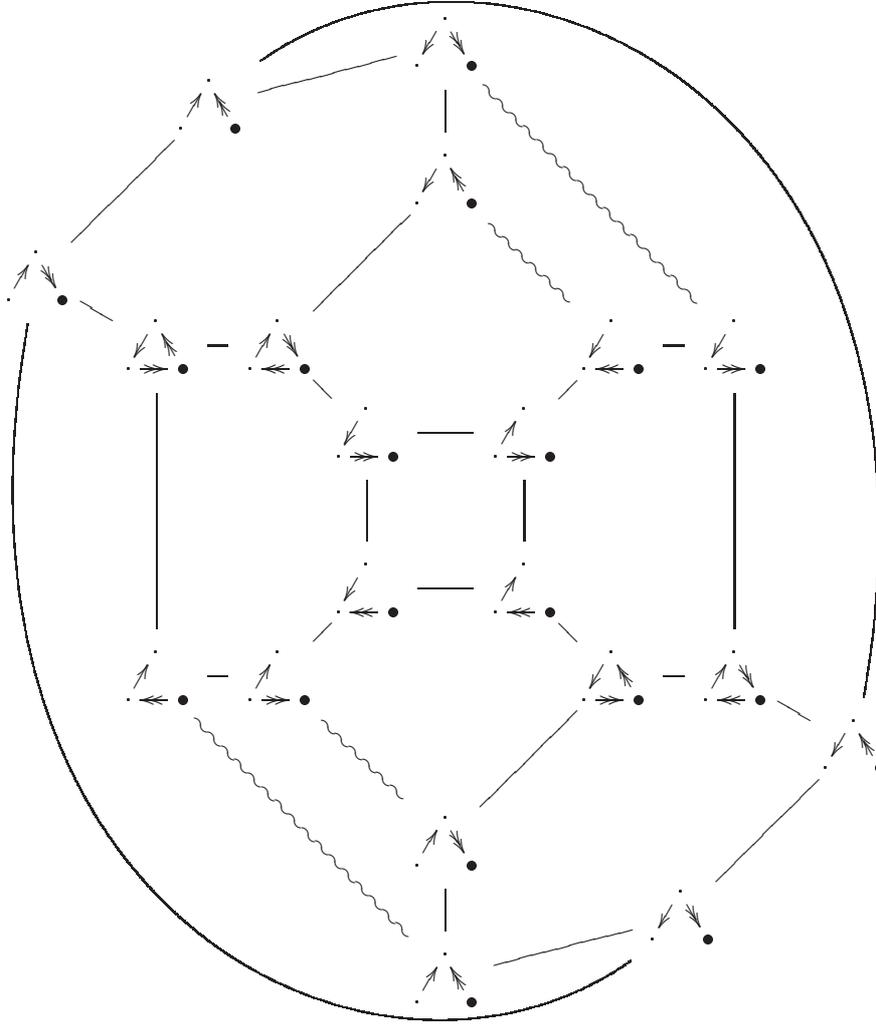


FIGURE 1. Exchange graph of type  $B_3$

vertices with group  $\mathbb{Z}/2\mathbb{Z}$ . Simple arrows symbolize  $\mathbb{C}$  and double arrows symbolize  $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$ . Thus,  $(A, S)$  will be symbolized by



Finally, wave lines ( $\sim$ ) symbolize mutations composed with the exchange of vertices 1 and 2.

Now, we will compute explicitly  $F_{3;213}^B$  and  $\mathfrak{g}_{3;213}^B$ . We will follow the construction of section 9. According to the exchange graph,

$$\mu_3\mu_1\mu_2(A, 0) = \left( \begin{array}{c} \cdot \\ \swarrow \searrow \\ \cdot \quad \bullet \end{array}, 0 \right) = (A', S').$$

Let  $\rho$  be one the two irreducible modules over  $\mathbb{Z}/2\mathbb{Z}$ . Then

$$\begin{aligned}\mu_3(A', S', 0, \rho) &= \left( \begin{array}{c} \cdot \\ \swarrow \downarrow \nearrow \\ \cdot \end{array}, 0, \begin{array}{c} 0 \\ \rho \end{array}, 0 \right) \\ \mu_1\mu_3(A', S', 0, \rho) &= \left( \begin{array}{c} \cdot \\ \swarrow \downarrow \nearrow \\ \cdot \end{array}, \dots, \begin{array}{c} \mathbb{C} \\ \downarrow \\ \rho \end{array}, 0 \right) \\ \mu_2\mu_1\mu_3(A', S', 0, \rho) &= \left( \begin{array}{c} \cdot \\ \swarrow \downarrow \nearrow \\ \cdot \end{array}, \dots, \begin{array}{c} \mathbb{C} \\ \swarrow \downarrow \searrow \\ \mathbb{C} \end{array} \rightarrow \rho, 0 \right)\end{aligned}$$

(the arrows are obvious) and therefore,

$$X_{\rho;213}^B = \begin{array}{c} \mathbb{C} \\ \downarrow \\ \mathbb{C} \rightarrow \rho \end{array}$$

which induces that:

$$F_{X_{\rho;213}^B} = 1 + Y_\rho + Y_2Y_\rho + Y_1Y_2Y_\rho$$

and therefore

$$\check{F}_{X_{\rho;213}^B} = 1 + Y_3 + Y_2Y_3 + Y_1Y_2Y_3.$$

Moreover,

$$\mathfrak{g}_{X_{\rho;213}^B} = \begin{pmatrix} 0 \\ 0 \\ -\rho \end{pmatrix}$$

and therefore

$$\check{\mathfrak{g}}_{X_{\rho;213}^B} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

It is easy to check by hand that these coincide with  $F_{3;213}^B$  and  $\mathfrak{g}_{3;213}^B$  obtained for example by formulas of [DWZ1, §2].

Let now  $B$  be the matrix defined by

$$B = \begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & -1 & -1 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We will show that there is no non-degenerate locally free GSP with mutation matrix  $B$ . Suppose that  $(I, (\Gamma_i), A, S)$  is a non-degenerate GSP with mutation matrix  $B$ . Then,  $\Gamma_1, \dots, \Gamma_5$  have the same cardinal which is two times the one of  $\Gamma_6$ . Applying  $\mu_3$  followed by  $\mu_5$  create 2-cycles and implies, in view of proposition 4.10, that

$$A_{23} \otimes_{E_3} A_{31} \simeq (A_{15} \otimes_{E_5} A_{52})^*.$$

In the same way, applying  $\mu_4$  followed by  $\mu_5$  implies that

$$A_{24} \otimes_{E_4} A_{41} \simeq (A_{15} \otimes_{E_5} A_{52})^* .$$

With the same type of argument, applying  $\mu_3$ ,  $\mu_4$  and  $\mu_6$  implies that

$$(A_{23} \otimes_{E_3} A_{31})^{\oplus 2} \simeq A_{24} \otimes_{E_4} A_{41} \oplus A_{23} \otimes_{E_3} A_{31} \simeq (A_{16} \otimes_{E_6} A_{62})^* .$$

As all considered groups are semisimple, it is easy to see that the  $(E_1, E_6)$ -bimodule  $A_{16}$  can be decomposed as a direct sum of the form

$$A_{16} = \bigoplus_{i=1}^m r_i \otimes_K s_i$$

where the  $r_i$  are irreducible left  $E_1$ -modules and the  $s_i$  are irreducible right  $E_6$ -modules. Moreover, the  $r_i \otimes_K s_i$  are irreducible bimodule and satisfy, because of  $B$ ,

$$\forall r \in \text{irr}_1, \sum_{i | r_i \simeq r} \dim_K s_i = \dim_K r \text{ and } \forall s \in \text{irr}_6, \sum_{i | s_i \simeq s} \dim_K r_i = 2 \dim_K s .$$

Thus, there are exactly two indices which can be supposed to be 1 and 2 such that  $s_1, s_2$  are trivial and  $r_1$  and  $r_2$  are of dimension 1 and appear only one time in the sequence  $(r_i)$ . In the same way,

$$A_{62} = \bigoplus_{i=1}^n t_i \otimes_K u_i$$

with

$$\forall t \in \text{irr}_6, \sum_{i | t_i \simeq t} \dim_K u_i = 2 \dim_K t \text{ and } \forall u \in \text{irr}_2, \sum_{i | u_i \simeq u} \dim_K t_i = \dim_K u .$$

Thus, there are exactly two indices which can be supposed to be 1 and 2 such that  $t_1, t_2$  are trivial and the  $u_1$  and  $u_2$  are of dimension 1 and appear only one time in the sequence  $(u_j)$ . Hence,

$$(A_{16} \otimes A_{62})^* = \bigoplus_{i=1}^m \bigoplus_{\substack{j=1 \\ s_i \simeq t_j^*}}^n (u_j^* \otimes_K r_i^*)^{\dim_K s_i}$$

contains  $u_1^* \otimes r_1^* \oplus u_1^* \otimes r_2^* \oplus u_2^* \otimes r_1^* \oplus u_2^* \otimes r_2^*$  as the only summands containing  $u_1^*, u_2^*, r_1^*$  and  $r_2^*$ . Finally,  $(A_{16} \otimes A_{62})^*$  can not be decomposed as a direct sum of two times the same bimodule, which is a contradiction.

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MAX PLANCK INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, D-53111 BONN  
*E-mail address:* `Laurent.Demonet@normalesup.org`