Clifford bundles: a unifying framework for images, vector fields and orthonormal frame fields regularization.

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CLIFFORD BUNDLES: A COMMON FRAMEWORK FOR IMAGES, VECTOR FIELDS AND ORTHONORMAL FRAME FIELDS REGULARIZATION *

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Abstract. The aim of this paper is to present a new framework for regularization by diffusion. The methods we develop in the sequel can be used to smooth multichannel images, multichannel image sequences (videos), vector fields and orthonormal frame fields in any dimension.[1]
From a mathematical viewpoint, we deal with vector bundles over Riemannian manifolds and so-called generalized Laplacians. Sections are regularized from heat equations associated to generalized Laplacians, the solutions being approximated by convolutions with kernels. Then, the behaviour of the diffusion is determined by the geometry of the vector bundle, i.e. by the metric of the base manifold and by a connection on the vector bundle. For instance, the heat equation associated to the Laplace-Beltrami operator can be considered from this point of view for applications to images and videos regularization. The main topic of this paper is to show that this approach can be extended in several ways to vector fields and orthonormal frame fields by considering the context of Clifford algebras. We introduce Clifford-Beltrami and Clifford-Hodge operators as generalized Laplacians on Clifford bundles over Riemannian manifolds. Laplace-Beltrami diffusion appears as a particular case of diffusion for degree 0 sections (functions). Dealing with base manifolds of dimension 2, applications to multichannel images, 2D vector fields and orientation fields regularization are presented.

Key words. regularization, heat equations, Clifford algebras, vector bundles, differential geometry.

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1. Introduction. Most multichannel image regularization methods are based on PDEs of the form

$$\frac{\partial I^i_t}{\partial t} = \sum_{j,k=1}^{2} f_{jk}\frac{\partial^2 I^i_t}{\partial j\,\partial k} + (\text{parts of order} \leq 1)$$

of initial condition $I\colon (x_1,x_2) \longmapsto (I^1(x_1,x_2),\cdots,I^n(x_1,x_2))$ a n-channels image, where f_{jk} are real functions. We refer to [28] for an overview on related works. From a geometric viewpoint, the set of right terms, for $i = 1\cdots n$, may be viewed as a second-order differential operator acting on sections of a vector bundle over a Riemannian manifold called a **generalized Laplacian** H [4]. As a consequence, it ensures existence and uniqueness of a kernel $K_t(x,y,H)$, called the **heat kernel of** H, generating the solution of the heat equation

$$\frac{\partial I_t}{\partial t} + HI_t = 0 \tag{1.1}$$

from a 'convolution' with the initial condition I. A generalized Laplacian H on a vector bundle E over a Riemannian manifold X is determined by three pieces of data: the metric g of the base manifold X that determines the second order part, a connection on E that determines the first order part, and a zero-order operator F that determines the zero order part. For instance, the so-called oriented Laplacians can

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[1]Preliminary form of the paper in [3].

be viewed as generalized Laplacians [1]. Similarly, on the vector bundle $C^\infty(X,g)$ of smooth functions on a Riemannian manifold (X,g), there is a canonical generalized Laplacian, the scalar Laplacian, which corresponds to the Laplace-Beltrami operator up to a sign. Considering each component I^k of a multichannel image as a function over a well-chosen Riemannian manifold, the heat equation associated to the scalar Laplacian leads to the Beltrami flow of Sochen et al. in the context of image regularization [23],[25]. The aim of this paper is to extend the Beltrami flow on $C^\infty(X,g)$ to vector fields and oriented orthonormal frame fields (i.e. $\mathrm{SO}(n)$-valued fields) on (X,g).

Regularization of non-scalar fields such as vector fields and manifold-valued fields has been widely investigated in the last few years. Concerning vector fields, we may refer to [9],[28], where the smoothing is made through the smoothing of a corresponding multichannel image and where the anisotropy is controlled at each point by the orientation of the vector field. More precisely, an oriented Laplacian is applied on each component of the image. The regularization of $\mathrm{SO}(n)$-valued fields has been treated in [13] in the context of principal bundles using a geometric flow on the $\mathrm{SO}(n)$ group, and in [29] using a minimization problem with orthogonal constraints. For $n=2$, this is the problem of orientations regularization, treated in [18],[22],[27]. More generally, regularization of symmetric definite positive matrice fields were proposed in [14],[21] and of Stiefel manifolds in [6],[11], all of them using geometric flows on manifolds.

In [6],[9],[28],[29], the behaviour of the flow is determined by the field itself. In [11],[13],[14],[21], it is determined by the choice of a Riemannian metric on the manifold too. In this paper, we make use of an additional geometric structure, called a connexion. Roughly speaking, a connection is a way to differentiate sections of a vector bundle. An example of connexion is the Levi-Cevita connection on the tangent bundle of a Riemannian manifold. By the use of heat equations on vector bundles in this paper, the behaviour of the diffusion is completely determined by the geometry of the vector bundle where the fields are considered as sections. By geometry of vector bundle, we mean a metric on the base manifold and a connection on the vector bundle. More precisely, the metric of the base manifold determines the anisotropy of the diffusion, whereas the connection determines the data averaged by the diffusion by the use of the transport parallel map.

In this paper, we extend the Beltrami flow to vector fields and orthonormal frame fields by extending the Laplace-Beltrami operator to vector fields and generators of orthonormal frame fields, i.e. fields with values in the Lie algebra $\mathfrak{so}(n)$ of $\mathrm{SO}(n)$. For this, we consider Clifford bundles, that are vector bundles where the fibers are endowed with a Clifford algebra structure. Clifford algebras framework [7],[16] finds a wide range of applications in computer science [24]. Application of Clifford bundles to image processing was introduced in [2] where the Di Zenzo's gradient, devoted to multichannel image segmentation, is generalized using covariant derivatives instead of usual derivatives.

We show that Clifford bundles provide a common framework to treat functions, vector fields, orthonormal frame fields and their generators over manifolds. We also give a general method to construct generalized Laplacians on Clifford bundles such that the subsequent heat equation gives a tool to regularize functions, vector fields and orthonormal frame fields. Even though we only treat the case of $\mathrm{SO}(n)$-valued fields in this paper, we expect that the method we propose allows to treat and regularize fields with values in connected components of much more Lie groups. Indeed, a lot of Lie groups and their Lie algebras have representations in Clifford algebras [10],[16].

This paper is organized as follows. In Section 2, we treat the problem of approximations of heat equations solutions. We discuss both continuous and discrete settings. We treat the particular case of the scalar Laplacian, and relate it with the Laplace-Beltrami operator. In Section 3, we first introduce Clifford bundles, and show that functions, vector fields and generators of orthonormal frame fields can be viewed as sections of a Clifford bundle. Moreover, we show that rotation fields can be lifted to sections of a Clifford bundle called spinor fields. Then, we show how to construct generalized Laplacians leading to functions, vector fields and orthonormal frame fields regularization from connections compatible with metrics of vector bundles. Dealing with particular Clifford bundles, we construct Clifford-Beltrami and Clifford-Hodge operators. In Section 4, we consider base manifolds of dimension 2. We compare diffusions provided by Clifford-Beltrami and Clifford-Hodge flows in the context of image processing with applications to image, 2D vector fields and orientation fields regularization. The first Appendix is devoted to differential geometry of vector bundles. We give definitions of notions used throughtout the paper. In the second one, we introduce heat equations on vector bundles. In the last Appendix, we detail the construction of Clifford algebras and give their main properties. We also explain how orthogonal transformations can be represented in the Clifford algebras context using the spinor group $\mathrm{Spin}(n)$.

2. Approximation of generalized heat equations solutions.

2.1. The continuous setting. Generally speaking, for arbitrary base manifold (X,g) and vector bundle E, there is no explicit formula for the heat kernel $K_t(x,y,H)$ of a generalized Laplacian H on E. As a consequence, there is no explicit formulae for solutions of the corresponding heat equation.

However, there exist kernels $K_t^N(x,y,H)$, for $N\in\mathbb{N}$, approximating the heat kernel of H on normal neighborhoods of the base manifold, for small t. Based on these results, solutions of generalized heat equations may be approximated for small t by the convolution of the initial condition with such kernels.

In this paper, we approximate the heat kernel of H by the kernel $K_t^0(x,y,H)$ defined by

$$\left(\frac{1}{4\pi t}\right)^{m/2} e^{-d(x,y)^2/4t}\,\Psi(d(x,y)^2)\,\tau(x,y)\,J(x,y)^{-1/2}$$

where m is the dimension of the base manifold X, and d stands for the geodesic distance on X related to the Levi-Cevita connection on (TX,g). Ψ is a function such that the term $\Psi(d(x,y)^2)$ equals 1 if y is inside a normal neighborhood of x and 0 otherwise. Hence we may assume that y is inside a normal neighborhood of x. The map $\tau(x,y)$ is the parallel transport map on E related to the connection ∇^E such that $H=\Delta^E+F$ (see Appendix B) along the unique geodesic joining y and x. Indeed, on normal neighborhoods, there is a unique geodesic joining any point to the origin. At last, J are the Jacobians of the coordinates changes from usual coordinates systems to normal coordinates systems.

Hence, we approximate the solution $e^{-tH}u$ of the heat equation $\partial u_t/\partial t + Hu_t = 0$, of initial condition $u_0 = u$, at $x \in X$ by $k_t^0 u(x) = (1/4\pi t)^{m/2} \times$

$$\int_X e^{-d(x,y)^2/4t}\, \Psi(d(x,y)^2)\, \tau(x,y)\, u(y)\;\; J(x,y)^{-1/2}\, dy$$

Remark: Whereas the heat kernel $K_t(x,y,H)$ of a generalized Laplacian H is unique, there is no one-one correspondance between H and the kernel $K_t^0(x,y,H)$ approximating its heat kernel. Indeed, $K_t^0(x,y,H)$ only depends of the Riemannian metric g on X and the connection ∇^E on E such that $H = \Delta^E + F$. Then given F_1 a zero-order operator on E and the generalized Laplacian $H_1 = \Delta^E + F_1$, we have

$$K_t^0(x,y,H) = K_t^0(x,y,H_1)$$

2.2. The discrete setting. For the purpose of applications to image processing, we discretize the computation of $k_t^0 u$. We proceed by discrete convolutions with masks.

Considering the standard Laplacian Δ on $\mathbb{R}^2$, the gaussian kernel $G_t(x,y)$ is the heat kernel $K_t(x,y,\Delta)$ of Δ. This means that the solution u_t of the heat equation $\partial u_t/\partial t = \Delta u_t$ of initial condition $u_0 = u$ is given by the convolution of u with the Gaussian kernel

$$e^{-t\Delta}u(x) = \left(\frac{1}{4\pi t}\right) \int_{\mathbb{R}^2} e^{-\|x-y\|^2/4t}\, u(y)\, dy$$

The Gaussian kernel has an infinite support and satisfies the property

$$\left(\frac{1}{4\pi t}\right) \int_{\mathbb{R}^2} e^{-\|x-y\|^2/4t}\;\, dy = 1$$

In practice, the discrete Gaussian kernel is truncated such that $G_t((i,j),(k,l)) = 0$ if the pixel (k,l) is outside a given neighborhood $\mathcal{N}_{(i,j)}$ of the pixel (i,j) (e.g. 5×5 neighborhood), and normalized inside $\mathcal{N}_{(i,j)}$, i.e.

$$G_t((i,j),(k,l)) = \frac{(1/4\pi t)\; e^{-\|(i,j)-(k,l)\|^2/4t}}{\sum_{\mathcal{N}_{(i,j)}} (1/4\pi t)\; e^{-\|(i,j)-(m,n)\|^2/4t}}$$

Hence, the discrete approximation of $e^{-t\Delta}u$ at a pixel (i,j) is given by the discrete convolution of u with such a mask. More precisely, we have

$$(e^{-t\Delta}u)(i,j) \simeq \sum_{(k,l)\in\mathcal{N}_{(i,j)}} G_t((i,j),(k,l))\; u(k,l)$$

More generally, the kernel

$$Q_t(x,y) = \left(\frac{1}{4\pi t}\right) e^{-d(x,y)^2/4t}\, \Psi(d(x,y)^2)\, J(x,y)^{-1/2}$$

on (X, g) may be viewed as the Riemannian counterpart of the Gaussian kernel on $\mathbb{R}^2$. Indeed, dealing with normal coordinates around a point x, they take both the form

$$\left(\frac{1}{4\pi t}\right) e^{-\|\mathbf{y}\|^2/4t}$$

where $\mathbf{y}$ are the normal coordinates of y.

Based on this observation, we propose to adapt the above method, devoted to compute a discrete approximation of $e^{-t\Delta}u$ on $\mathbb{R}^2$, in order to compute more generally a discrete approximation of $k_t^0 u$ on (X, g) of dimension 2. This generalization requires the construction of discrete geodesic curves and normal neighborhoods.

For this, we use the following definitions and results on differential geometry of manifolds (see Appendix A and [15] for more details).

1. Let X be a manifold equipped with a connection ∇ on its tangent bundle TX. The curve γ on X is said to be **autoparallel** if it satisfies $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

2. Normal neighborhoods may be characterized as neighborhoods where each point is joined to the origin by a unique autoparallel curve.

3. Dealing with a Riemannian manifold X equipped with the Levi-Cevita connection on its tangent bundle, an autoparallel curve on X is called a **geodesic**.

Then we construct geodesic curves from (i, j) and a normal neighborhood $\mathcal{V}_{(i,j)}$ of (i, j) as follows.

Let us first fix a neighborhood $\mathcal{N}_{(i,j)}$ of the pixel (i, j) (e.g. the 5×5 neighborhood of (i, j)). For each $p, q \in \{-1, 0, 1\}$ such that $(p, q) \neq (0, 0)$, we construct the geodesic γ such that $\gamma(0) = (i, j)$ and $\dot{\gamma}(0) = (p, q)$ by solving a discrete counterpart of the parallel transport of $\dot{\gamma}$ along γ related to the Levi-Cevita connection on TX. Then, we state that a point $(k, l) \in \mathcal{N}_{(i,j)}$ belongs to $\mathcal{V}_{(i,j)}$ if it belongs to a unique geodesic from (i, j) and if it does not follow a point on this geodesic that belongs to several geodesics from (i, j).

Hence, we truncate $Q_t((i, j), (k, l))$ in such a way that $Q_t((i, j), (k, l)) = 0$ if (k, l) is outside $\mathcal{V}_{(i,j)}$, and we normalize it inside $\mathcal{V}_{(i,j)}$, i.e.

$$Q_t((i, j), (k, l)) = \frac{(1/4\pi t)\ e^{-d((i,j)-(k,l))^2/4t}}{\sum_{\mathcal{V}_{(i,j)}} (1/4\pi t)\ e^{-d((i,j)-(m,n))^2/4t}}$$

Therefore, the discrete approximation of $k_t^0 u$ at a pixel (i, j) is given by the discrete convolution of the map

$$(k, l) \longmapsto \tau((i, j), (k, l))\, u(k, l)$$

with a such a mask. More precisely, we have

$$k_t^0 u(i, j) \simeq \sum_{(k,l)\in\mathcal{V}_{(i,j)}} Q_t((i, j), (k, l))\, \tau((i, j),(k, l))u(k, l)$$

Remark: Let us consider the operator $k_t^1 u$ associated to the kernel $K_t^1(x,y,H)$ approximating the heat kernel of H. In the continuous setting, we have

$$
\begin{aligned}
(k_t^1 u)(x) &= \int_X K_t^1(x,y,H)u(y)dy \\
&= \left(\frac{1}{4\pi t}\right)\int_X e^{-d(x,y)^2/4t}\,\Psi(d(x,y)^2)\,(\tau(x,y)+t\Phi_1(x,y,H))\,u(y)\,J(x,y)^{-\frac{1}{2}}\,dy
\end{aligned}
$$

In the discrete setting, we obtain

$$
k_t^1 u(i,j) \simeq \sum_{(k,l)\in\mathcal{V}_{(i,j)}} Q_t((i,j),(k,l))\,[\tau((i,j),(k,l)) + t\,\Phi_1((i,j),(k,l),H)]\,u(k,l)
$$

Considering the neighborhood $\mathcal{V}_{i,j} := \{(i,j)\}$ of the pixel (i,j), we have

$$
k_t^1 u(i,j) \simeq u(i,j) + t\,\Phi_1((i,j),(i,j),H)\,u(i,j)
$$

Moreover, we have $\Phi_1((i,j),(i,j),H) = -J^{1/2}\circ H\circ J^{-1/2}((i,j),(i,j))$.

From the property

$J(x,y) = 1 + O(\|\mathbf{y}\|^2)$ where $\mathbf{y}$ are the normal coordinates of y around x,

(see [4] Chapter 1), we can assume in the discrete setting that $J=1$ near the diagonal. Therefore, it gives $\Phi_1((i,j),(i,j),H) = -H$, and

$$
k_t^1 u(i,j) \simeq u(i,j) - t\,Hu(i,j) \tag{2.1}
$$

Finally, we obtain the Euler scheme of the PDE

$$
\frac{\partial u_t}{\partial t} + Hu_t = 0, \qquad u_0 = u
$$

2.3. Example: the scalar/Beltrami Laplacian. The Laplace-Beltrami operator can be viewed as a generalized Laplacian called the scalar Laplacian.

DEFINITION 2.1. *Let E be a vector bundle of rank 1 over a Riemannian manifold (X,g), endowed with the connection $\nabla^E = d+\omega$ with $\omega\equiv 0$. The scalar Laplacian is the connection Laplacian on E associated to the connection ∇^E.*

In local frames $\partial_i := \partial/\partial x_i$ of TX and e_1 of E, it is defined by

$$
\Delta(fe_1) = -\sum_{ij} g^{ij}\Big(\partial_i\partial_j - \sum_k \Gamma_{ij}^k\partial_k\Big) f\,e_1 \tag{2.2}
$$

where $f\in C^\infty(X)$, and Γ_{ij}^k are the symbols of the Levi-Cevita connection of (TX,g) with respect to the frame ∂_i.

In the following Proposition, we determine the parallel transport map on E related to the connection ∇^E.

Proposition 2.2. *Let* (X, g)*,* E *and* ∇^E *of Definition 2.1,* γ *be a* C^1 *curve in* X *such that* $\gamma(0) = y$*. The parallel transport* Y *along* γ *of the vector* $Y_0 = Y_0^1 e_1(y)$ *is* $Y(t) = Y_0^1 e_1(\gamma(t))$*.*

Proof. The parallel transport of Y_0 along γ is the solution $Y(t) = Y^1(t)e_1(\gamma(t))$ of the differential equation

$$\begin{cases} \nabla^E_{\dot\gamma} Y(t) = 0 \\ \\ Y(0) = Y_0 \end{cases} \tag{2.3}$$

$$\begin{aligned} \nabla^E_{\dot\gamma} Y(t) &= \nabla^E_{\dot\gamma}(Y^1\, e_1)(t) \\ &= \frac{\partial Y^1}{\partial t}(t)\, e_1(\gamma(t)) \end{aligned}$$

Finally, we obtain an ordinary differential equation

$$\begin{cases} \dfrac{\partial Y^1}{\partial t}(t) & = & 0 \\ \\ Y^1(0) & = & Y_0^1 \end{cases}$$

from which we deduce the solution of (2.3). □

In other words, the parallel transport has no effect on coordinates. As a consequence, the kernel $K_t^0(x, y, \Delta)$ equals

$$\left(\frac{1}{4\pi t}\right)^{m/2} e^{-d(x,y)^2/4t} Id(x, y) \tag{2.4}$$

on normal neighborhoods of x, where $Id(x, y)\colon E_y \longrightarrow E_x$ acts as the Identity map on coordinates.

The Laplace-Beltrami operator Δ_g is the differential operator of order 2 acting on smooth functions on a Riemannian manifold (X, g), and defined by

$$\Delta_g(f) = \frac{1}{\sqrt{g}} \sum_{jk} \partial_j(\sqrt{g} g^{jk} \partial_k f)$$

Considering the set $C^\infty(X)$ of smooth functions on X as a trivial vector bundle of rank 1 over (X, g) endowed with the connection ∇^E determined by the 1-form $\omega = 0$, we have

$$\Delta_g = -\Delta$$

This makes the PDE's $\partial f/\partial t = \Delta_g f$ and $\partial f e_1/\partial t + \Delta\, f e_1 = 0$ be equivalent.

The PDE $\partial f/\partial t = \Delta_g f$ appears in the Beltrami framework in the context of image regularization [23],[25]. In [23], the solution is approximated by an Euler scheme of the form (2.1), whereas in [25] the solution is approximated by the convolution of the initial condition with a kernel of the form (2.4).

3. Clifford bundles over Riemannian manifolds: a common framework to regularize functions, vector fields, and oriented orthonormal frame fields.

3.1. Clifford bundles: a common framework to treat functions, vector fields and oriented orthonormal frame fields on manifolds. In this section, we show that we can consider functions, vector fields and generators of rotation fields on a manifold X as sections of a trivial Clifford bundle $Cl(E,h)$ over X. Moreover, we can also lift rotation fields to sections of $Cl(E,h)$ called spinor fields.

Definition 3.1 (Clifford bundle). *Let E be a vector bundle of rank n over a manifold X, and equipped with a metric h. The Clifford bundle $Cl(E,h)$ of (E,h) is the quotient bundle*

$$Cl(E,h) = \mathcal{T}(E)\Big/I(E,h)$$

where $\mathcal{T}(E)$ is the bundle whose fiber at $x \in X$ is the tensor algebra of E_x and $I(E,h)$ is the bundle whose fiber at $x \in X$ is the two-sided ideal $I(E_x,h_x)$ in $\mathcal{T}(E_x)$, generated by elements $v \otimes v + h_x(v,v)$ for $v \in E_x$.

We obtain a bundle of rank 2^n over X whose fibers are endowed with a Clifford algebra structure, and the fiberwise multiplication in $Cl(E,h)$ gives an algebra structure to the space $\Gamma(Cl(E,h))$ of sections of $Cl(E,h)$.

More precisely, let $(e_1,\cdots,e_n)$ be a local oriented orthonormal frame field of (E,h) on $\Omega \subset X$. Then, any section s of $Cl(E,h)$ takes the form

$$s = s_0\,1 + \sum_{\substack{k=1\cdots n\\ i_1<\cdots<i_k}} s_{i_1\cdots i_k} e_{i_1}\cdots e_{i_k}$$

on Ω, for some functions $s_0, s_{i_1\cdots i_k} \in C^\infty(\Omega)$.

Example: Let X be a manifold of dimension m endowed with a metric g on its tangent bundle TX. The couple (TX,g) induces a Clifford bundle $Cl(TX,g)$ over X of rank 2^m.

The graduation on Clifford algebras carry over Clifford bundles. We denote by $\Gamma(Cl(E,h)_k)$ the set of sections s of $Cl(E,h)$ such that $s(x)$ is a k-vector $\forall x \in X$.

Let $x \in X$. From the embeddings of $\mathbb{R}$ and E_x into the Clifford algebra $Cl(E_x,h(x))$, both functions on X and sections of E may be viewed as sections of the Clifford bundle $Cl(E,h)$. More precisely, we have the identifications

$$C^\infty(X) \simeq \Gamma(Cl(E,h)_0) \qquad \Gamma(E) \simeq \Gamma(Cl(E,h)_1)$$

Let us denote by $\Gamma(\mathrm{P\,SO}(E,h))$ the set of smooth oriented orthonormal frame fields of (E,h). On a local trivialization U of (E,h), we have the identifications

$$\Gamma(\mathrm{P\,SO}(E_{|U},h_{|U})) \simeq C^\infty(U,\mathrm{SO}(n)) \qquad \Gamma(Cl(E_{|U},h_{|U})_2) \simeq C^\infty(U,\mathfrak{so}(n))$$

the latter arising from the Lie algebra isomorphism $\mathfrak{spin}(n) \simeq \mathfrak{so}(n)$. We call such fields **generators of oriented orthonormal frame fields**.

From these identifications, functions, vector fields and generators of rotation fields on X may be viewed as sections of a trivial Clifford bundle $Cl(E,h) \simeq X \times \mathbb{R}_{n,0}$, respectively of degree 0,1,2.

Indeed, a vector-valued function $v\colon X \longrightarrow \mathbb{R}^n$ of components $(v_1, \cdots, v_n)$ may be viewed as a section V of the trivial vector bundle $(E, \| \|_2) = X \times \mathbb{R}^n$ of the form

$$V(x) = (x, (v_1(x), \cdots, v_n(x)))$$

and consequently as a section of the trivial Clifford bundle $Cl(E, \| \|_2) = X \times \mathbb{R}_{n,0}$ of the form

$$V(x) = (x, (0, v_1(x), \cdots, v_n(x), 0, \cdots, 0))$$

This construction may be generalized endowing the vector bundle E of a definite metric h, in such a way that v may be viewed as a section V of a trivial vector bundle $(E,h) \simeq X \times \mathbb{R}^n$. Let $(e_1, \cdots, e_n)$ be an oriented orthonormal frame field of E with respect to h, and ϕ be the global trivialization of (E,h) such that

$$V(x) = V_1(x)e_1(x) + \cdots + V_n(x)e_n(x) = \phi(x, (v_1(x), \cdots, v_n(x)))$$

where $(V_1(x), \cdots, V_n(x))$ are the components of the vector $(v_1(x), \cdots, v_n(x))$ in the basis $(e_1(x), \cdots, e_n(x))$. As a consequence, v may be viewed as a section of the trivial Clifford bundle $Cl(E,h) \simeq X \times \mathbb{R}_{n,0}$ of global trivialization Φ, of the form

$$V(x) = V_1(x)e_1(x) + \cdots + V_n(x)e_n(x) = \Phi(x, (0, v_1(x), \cdots, v_n(x), 0, \cdots, 0))$$

Then, we treat the case of rotation fields and their generators. Let $r\colon X \longrightarrow \mathrm{SO}(n)$ be a rotation field on X, considered as a section of the principal bundle $X\times \mathrm{SO}(n)$ over X. From the covering $\xi\colon\ \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$, r may be lifted to a spinor field s, i.e. a section of the principal bundle $X\times \mathrm{Spin}(n)$. By the embedding of $\mathrm{Spin}(n)$ into $\mathbb{R}_{n,0}$, we have $X\times \mathrm{Spin}(n) \subset X \times \mathbb{R}_{n,0}$. Hence we can consider s as a section of the trivial Clifford bundle $X \times \mathbb{R}_{n,0}$. Moreover, from the Lie algebra isomorphism $\mathfrak{so}(n) \simeq \mathbb{R}^2_{n,0}$, we can consider generators of rotation fields on X as sections $\tilde{s}$ of $X \times \mathbb{R}_{n,0}$ of the form

$$\tilde{s}(x) = (x, \sum_{i<j} a_{ij}(x)\, e_i e_j)$$

for some functions a_{ij}, where $e_i e_j \in \mathbb{R}^2_{n,0}$ are induced by an orthonormal basis $(e_1, \cdots, e_n)$ of $(\mathbb{R}^n, \| \|_2)$.

If we denote by *exp* the exponential map of $\mathrm{Spin}(n)$, the section s given by

$$s(x) = (x, exp(\sum_{i<j} a_{ij}(x)\, e_i e_j))$$

is a lifting of the rotation field r, i.e. we have

$$r(x) = (x, \xi \circ exp(\sum_{i<j} a_{ij}(x)\, e_i e_j))$$

Let us detail the construction of the section $\tilde{s}$. From the matrix representation of SO(n), we can construct generators of rotations $r(x), x \in X$, by the computation of $log \circ r(x)$ where $log\colon$ SO(n) $\longrightarrow \mathfrak{so}(n)$ is the logarithm map of the Lie group SO(n). It gives a field of antisymmetric matrices on X. Then the Lie algebra isomorphism $\mathfrak{so}(n) \longrightarrow \mathbb{R}^2_{n,0}$ maps

$$log \circ r(x) = \sum_{i<j} 2a_{ij}(x)E_{ij} \qquad \text{to} \qquad \sum_{i<j} a_{ij}(x)\, e_ie_j$$

We identify the last expression with the section

$$\tilde{s}(x) = (x, \sum_{i<j} a_{ij}(x)\, e_ie_j))$$

of $X \times \mathbb{R}_{n,0}$.

Remark: For $n \geq 3$, the lifting of the rotation field r to a spinor field s is not unique, due to the 2-sheeted covering Spin(n) $\longrightarrow$ SO(n). Moreover, as the exponential map of the group SO(n) is not injective for any n, the field $\tilde{s}$ generating the field r is not unique.

Example: The case $n = 2$.
The lifting of a rotation field r to a spinor field s is unique from the isomorphism SO(2) $\simeq$ Spin(2). Indeed, the field θ lifts to $cos(\theta) + sin(\theta)e_1e_2$. For any $k \in \mathbb{Z}$, the field

$$\tilde{s} = (\theta + 2k\pi)e_1e_2$$

is a generator of the field θ.

At last, any scalar field $f\colon X \longrightarrow \mathbb{R}$ may be viewed as a section F of a trivial vector bundle $Cl(E, h) \simeq X \times \mathbb{R}_{n,0}$ of the form

$$F(x) = f(x)1(x)$$

In this paper, we are interested in the regularization of functions, vector fields and rotation fields over manifolds. By the above constructions, we can apply the context of heat equations on Clifford bundles to regularize these fields. By this approach, we can control the anisotropy of the regularization by the choice of a Riemannian metric of the base manifold X and the data we regularize by the choice of a connection on the Clifford bundle $Cl(E, h)$, for a well-chosen metric h.

Through the regularization of sections of degree 0, resp. 1, we can regularize functions and vector fields over X. We propose to regularize rotation fields through the regularization of their generators, i.e. sections of degree 2 of $Cl(E, \| \|_2) = X \times \mathbb{R}_{n,0}$. Indeed, as mentioned above, we can construct a generator $\tilde{s} \in \Gamma(Cl(E, \| \|_2)_2)$ of a rotation field r on X. The regularization of $\tilde{s}$ provides sections $\tilde{s}_{t,t\geq 0}$ of the form

$$\tilde{s}_t(x) = (x, \sum_{i<j} a_{ij}(t, x)e_ie_j)$$

Then computing

$$r_t(x) = (x, \xi \circ exp(\sum_{i<j} a_{ij}(t, x)e_ie_j))$$

we obtain rotation fields $r_{t,t\geq 0}$ on X regularizing the initial field r.

Finally, this general method only requires that the regularization process preserves the structure of function, vector field and generator of rotation field. Hence we are looking for heat equations preserving the graduation of Clifford bundles. More precisely, we are looking for PDEs of the form

$$\frac{\partial s_t}{\partial t} + Hs_t = 0$$

where H is a generalized Laplacian on $Cl(E,h)$ preserving the graduation, i.e.
$H\colon \Gamma(Cl(E,h)_k) \longrightarrow \Gamma(Cl(E,h)_k)$.

In the sequel, we show that such differential operators can be constructed from connections preserving the graduation. These connections can be constructed from connections on E compatible the metric h.

3.2. Connections preserving the graduation. Let E be a vector bundle of rank n equipped with a metric h over a manifold X.

PROPOSITION 3.2. *A connection ∇^E on E compatible with the metric h induces a connection ∇^C on the Clifford bundle $Cl(E,h)$ preserving the graduation, i.e. a connection satisfying* $\forall k \in \{0 \cdots n\}$

$$\nabla^E \varphi \in \Gamma(T^*X \otimes Cl(E,h)_k) \quad \textit{if} \quad \varphi \in \Gamma(Cl(E,h)_k)$$

Proof. Any connection ∇^E on E may be extended in a unique way to a connection ∇ on $\mathcal{T}(E)$ by linearity and postulation of the Leibniz rule.
For $U \in \Gamma(TX)$ and $V \in \Gamma(E)$, we have

$$\begin{aligned} \nabla_U (V \otimes V - h(V,V)) &= \frac{1}{2}\Big[(\nabla_U V + V) \otimes (\nabla_U V + V) - h(\nabla_U V + V, \nabla_U V + V)\Big] \\ &\quad -\frac{1}{2}\Big[(\nabla_U V - V) \otimes (\nabla_U V - V) - h(\nabla_U V - V, \nabla_U V - V)\Big] \end{aligned} \tag{3.1}$$

We prove (3.1) by developping both right and left terms of the equality.

For the right term we obtain

$$\nabla_U V \otimes V + V \otimes \nabla_U V - 2h(\nabla_U V, V)$$

Developping the left term we have

$$\nabla_U (V \otimes V - h(V,V)) = \nabla_U V \otimes V + V \otimes \nabla_U V - d_U h(V,V)$$

However, by definition of a connexion compatible with the metric of a vector bundle, we have for any $U \in \Gamma(TX)$ and $W, Z \in \Gamma(E)$

$$d_U h(W,Z) = h(\nabla_U W, Z) + h(W, \nabla_U Z)$$

Then, taking $W = Z = V$, it gives

$$d_U h(V,V) = 2h(\nabla_U V, V)$$

and the equality (3.1) is proved.

As the right term of (3.1) belongs to $\Gamma(I(E,h))$, it proves that ∇ preserves the $\Gamma(I(E,h))$.

We deduce that ∇ induces a connection ∇^C on $Cl(E,h)$. Indeed, for $a,b \in \Gamma(\mathcal{T}(E))$ in the same equivalence class (denoted by $\dot{a} = \dot{b}$), we have $\nabla_{U} b = \nabla_{U}(a + I_1)$ for some $I_1 \in \Gamma(I(E,h))$. Hence $\nabla_{U} b = \nabla_{U} a + I_2$ for some $I_2 \in \Gamma(I(E,h))$. Therefore $\dot{\widehat{\nabla(a)}} = \dot{\widehat{\nabla(b)}}$, and ∇^C is well-defined.

Let $\varphi, \psi \in \Gamma(Cl(E,h))$. By definition, there exist $a,b \in \Gamma(\mathcal{T}(E))$ such that $\varphi = \dot{a}$ and $\psi = \dot{b}$. Then,

$$\begin{aligned}
\nabla^C(\varphi\,\psi) &= \dot{\widehat{\nabla(a \otimes b)}} \\
&= \dot{\widehat{\nabla(a) \otimes b + a \otimes \nabla(b)}} \\
&= \dot{\widehat{\nabla(a) \otimes b}} + \dot{\widehat{a \otimes \nabla(b)}} \\
&= \dot{\widehat{\nabla(a)}}\,\dot{b} + \dot{a}\,\dot{\widehat{\nabla(b)}} \\
&= \nabla^C(\varphi)\psi + \varphi\nabla^C(\psi)
\end{aligned}$$

from which follows that ∇^C preserves the graduation. □

Remark: Dealing with the tangent bundle (TX, g) of a Riemannian manifold, the Levi-Cevita connection ∇ on (TX, g) is compatible with g. It follows that ∇ induces a connection ∇^C on $Cl(TX, g)$ preserving the graduation [5].

As a consequence, from a connection compatible with the metric h on E and a zero-order operator F on $Cl(E,h)$ preserving the graduation, we can construct a generalized Laplacian H on $Cl(E,h)$ preserving the graduation, by the formula (see Appendix B)

$$H = \Delta^C + F$$

where Δ^C is the connection Laplacian related to the connection ∇^C on $Cl(E,h)$ induced by the connection ∇^E.

By this method, we construct in the sequel two generalized Laplacians on Clifford bundles preserving the graduation, and restricting to the Laplace-Beltrami on functions (i.e. sections of degree 0), called Clifford-Beltrami and Clifford-Hodge operators.

3.3. The Clifford-Beltrami operator. Let X be a manifold of dimension m, endowed with a Riemannian metric g on its tangent bundle TX. Over the Riemannian manifold (X, g), we consider the vector bundle TX endowed with the metric $\tilde{g}$ given by the identity matrix I_m in local frames $(\partial/\partial x_1, \cdots, \partial/\partial x_m)$ induced by local coordinates system $(x_1, \cdots, x_m)$ on X. The vector bundle $(TX, \tilde{g})$ over X induces a Clifford bundle $Cl(TX, \tilde{g})$ over X. A local section s of $Cl(TX, \tilde{g})$ is of the form

$$s = s_0\,1 + \sum_{\substack{k=1\cdots m \\ i_1<\cdots<i_k}} s_{i_1\cdots i_k}\,\partial/\partial x_1 \cdots \partial/\partial x_k$$

for some functions $s_0, s_{i_1\cdots i_k} \in C^\infty(X)$.

The Levi-Cevita connection ∇ on $(TX, \tilde{g})$ is determined by $\omega \equiv 0$ by construction of $\tilde{g}$. From Section 3.2, it induces an algebra connection ∇^C on $Cl(TX, \tilde{g})$ satisfying

$$\nabla^C s = ds_0\, 1 + \sum_{\substack{k=1\cdots m \\ i_1<\cdots<i_k}} ds_{i_1\cdots i_k}\, \partial/\partial x_1 \cdots \partial/\partial x_k$$

In fact, ∇^C corresponds in this particular case to the differentation of components on $\Gamma(Cl(TX, \tilde{g}))$.

Then we construct the connection Laplacian Δ^C on $Cl(TX, \tilde{g})$ related to ∇^C. It takes the form

$$\Delta^C(s) = -\Delta_g s_0\, 1 - \sum_{\substack{k=1\cdots m \\ i_1<\cdots<i_k}} \Delta_g s_{i_1\cdots i_k}\, \partial/\partial x_1 \cdots \partial/\partial x_k$$

It consists in the action of minus the Laplace-Beltrami operator related to g on each component of a section. In this paper, we call this operator the **Clifford-Beltrami operator**.

As the Clifford-Beltrami operator is a generalized Laplacian, we may consider the corresponding heat equation, whose solution are provided by the convolution of the initial conditions with the heat kernel $K_t(x, y, \Delta^C)$ of Δ^C.

DEFINITION 3.3. *Let $s \in \Gamma(Cl(TX, \tilde{g}))$. The* ***Clifford-Beltrami flow*** *of s is the solution $s_{t,\, t\geq 0}$ of the heat equation*

$$\frac{\partial s_t}{\partial t} + \Delta^C s_t = 0, \quad s_0 = s$$

Then, as the Clifford-Beltrami operator preserves the graduation on $Cl(TX, \tilde{g})$, it preserves the structures of function, vector field and generator of orthonormal frame field. As a consequence, the Clifford-Beltrami flow provides a common tool to regularize functions, vector fields and oriented orthonormal frame fields.

3.4. The Clifford-Hodge operator. Let (X, g) be a Riemannian manifold. We denote by $\bigwedge T^*X$ the vector bundle of differential forms on (X, g). The vector space isomorphism (C.2) between $\bigwedge V^*$ and $Cl(V, Q)$ carries over vector bundles. It follows a canonical vector bundle isomorphism between $\bigwedge T^*X$ and $Cl(TX, g)$ that maps differential forms of degree k to sections of degree k of $Cl(TX, g)$.

The Hodge Laplacian Δ is a generalized Laplacian acting on differential forms on a Riemannian manifold (X, g). It is defined by

$$\Delta = d\delta + \delta d$$

where d is the exterior derivative operator and δ its formal adjoint [8].
In particular, when applied to 0-forms, i.e. functions, Δ corresponds to minus the Laplace-Beltrami operator.

Under the identification between $\bigwedge T^*X$ and $Cl(TX,g)$, the Hodge Laplacian can also be applied to $\Gamma(Cl(TX,g))$, and consequently to vector fields and generators of orthonormal frame fields. In the context of Clifford bundles, we call this operator the **Clifford-Hodge operator**. It is defined as the square of a first order differential operator called the **Dirac operator**.

Definition 3.4. *Let ∇^C be the connection on $Cl(TX,g)$ induced by the Levi-Cevita connection on (TX,g). Let $(e_1,\cdots,e_m)$ be a local oriented orthormal frame field of TX with respect to g. The Dirac operator is the first-order differential operator $D\colon \Gamma(Cl(TX,g)) \longrightarrow \Gamma(Cl(TX,g))$ defined locally by*

$$D\sigma = \sum_{i=1}^{m} e_i \, \nabla^C_{e_i}\sigma \tag{3.2}$$

This definition is independant of the choice of the local oriented orthonormal frame field.

By the Bochner identity, the Clifford-Hodge operator D^2 is a generalized Laplacian on $Cl(TX,g)$ preserving the graduation. Indeed, we have

$$D^2 = \Delta^C + \sum_{i<j} e_i e_j R_{e_i,e_j}$$

where $\sum_{i<j} e_i e_j R_{e_i,e_j}$ is a zero-order operator preserving the graduation. The term R_{e_i,e_j} is called the **curvature transformation** of ∇^C associated to e_i and e_j [19].

As the operator D^2 is a generalized Laplacian, we may consider the corresponding heat equation, whose solution are provided by the convolution of the initial conditions with the heat kernel $K_t(x,y,D^2)$ of D^2.

Definition 3.5. *Let $s \in \Gamma(Cl(X,g))$. The **Clifford-Hodge flow** of s is the solution $s_{t,t\geq 0}$ of the heat equation*

$$\frac{\partial s_t}{\partial t} + D^2 s_t = 0, \quad s_0 = s$$

Then, as the Clifford-Hodge operator preserves the graduation on $Cl(TX,g)$, it preserves the structures of function, vector field and generator of orthonormal frame field. As a consequence, the Clifford-Hodge flow provides a common tool to regularize functions, vector fields and oriented orthonormal frame fields.

4. Applications. In this paper, we approximate the solution of the heat equation associated to a generalized Laplacian H by the convolution of the initial condition with the kernel $K_t^0(x,y,H)$ approximating the heat kernel of H near the diagonal. This kernel is determined by the tranport parallel map associated to the connection ∇^E such that $H = \Delta^E + F$ (see Appendix B), and by geodesic distances on the base manifold. In this Section we compute parallel transport maps on $Cl(TX,\tilde{g})$ and $Cl(TX,g)$ related to the Clifford-Beltrami and Clifford-Hodge operators for base manifolds of dimension 2. Then we present applications in the context of image processing.

4.1. Clifford-Beltrami operator and subsequent parallel transport map. We denote by $(\partial_{x_1}, \partial_{x_2})$ the frame field $(\partial/\partial x_1, \partial/\partial x_2)$. From Section 3.3, we have

$$\nabla^C_{\partial_{x_1}} 1 = 0 \qquad \nabla^C_{\partial_{x_2}} 1 = 0$$

$$\begin{aligned} \nabla^C_{\partial_{x_1}} \partial_{x_1} &= 0 \qquad & \nabla^C_{\partial_{x_2}} \partial_{x_1} &= 0 \\ \nabla^C_{\partial_{x_1}} \partial_{x_2} &= 0 \qquad & \nabla^C_{\partial_{x_2}} \partial_{x_2} &= 0 \end{aligned}$$

$$\nabla^C_{\partial_{x_1}} \partial_{x_1} \partial_{x_2} = 0 \qquad \nabla^C_{\partial_{x_2}} \partial_{x_1} \partial_{x_2} = 0$$

Then, for $\varphi = \varphi_1 1 + \varphi_2 \partial_{x_1} + \varphi_3 \partial_{x_2} + \varphi_4 \partial_{x_1} \partial_{x_2}$ we have

$$\Delta^C \varphi = -(\Delta_g \varphi_1) 1 - (\Delta_g \varphi_2) \partial_{x_1} - (\Delta_g \varphi_3) \partial_{x_2} - (\Delta_g \varphi_4) \partial_{x_1} \partial_{x_2}$$

Proposition 4.1 (Parallel transport on $Cl(TX, \tilde{g})$). *Let X be a manifold of dimension 2. Let ∇^C be the connection on $Cl(TX, \tilde{g})$ induced by the Levi-Cevita connection on $(TX, \tilde{g})$. Let $Y_0 = Y_0^1 1(y) + Y_0^2 \partial_{x_1}(y) + Y_0^3 \partial_{x_2}(y) + Y_0^4 \partial_{x_1} \partial_{x_2}(y) \in Cl(TX, \tilde{g})_y$, and γ be a C^1 curve in X such that $\gamma(0) = y$. The parallel transport Y of Y_0 along γ is*

$$Y(t) = Y_0^1 1(\gamma(t)) + Y_0^2 \partial_{x_1}(\gamma(t)) + Y_0^3 \partial_{x_2}(\gamma(t)) + Y_0^4 \partial_{x_1} \partial_{x_2}(\gamma(t))$$

Proof. The parallel transport of Y_0 along γ is the solution $Y(t) = Y_1(t) 1(\gamma(t)) + Y_2(t) \partial_{x_1}(\gamma(t)) + Y_3(t) \partial_{x_2}(\gamma(t)) + Y_4(t) \partial_{x_1} \partial_{x_2}(\gamma(t))$ of the differential equation

$$\begin{cases} \nabla^C_{\dot{\gamma}} Y(t) = 0 \\ \\ Y(0) = Y_0 \end{cases} \tag{4.1}$$

$$\nabla^C_{\dot{\gamma}} Y(t) = \nabla^C_{\dot{\gamma}} Y_1 1 + Y_2 \partial_{x_1} + Y_3 \partial_{x_2} + Y_4 \partial_{x_1} \partial_{x_2} \ (t)$$

$$\begin{aligned} &= \frac{\partial Y_1}{\partial t}(t) 1(t) + Y_1(t)(\dot{\gamma_1}(t) \nabla^C_{\partial_{x_1}} 1 (t) + \dot{\gamma_2}(t) \nabla^C_{\partial_{x_2}} 1 (t)) \\ &\quad + \frac{\partial Y_2}{\partial t}(t) \partial_{x_1}(t) + Y_2(t)(\dot{\gamma_1}(t) \nabla^C_{\partial_{x_1}} \partial_{x_1} (t) + \dot{\gamma_2}(t) \nabla^C_{\partial_{x_2}} \partial_{x_1} (t)) \\ &\quad + \frac{\partial Y_3}{\partial t}(t) \partial_{x_2}(t) + Y_3(t)(\dot{\gamma_1}(t) \nabla^C_{\partial_{x_1}} \partial_{x_2} (t) + \dot{\gamma_2}(t) \nabla^C_{\partial_{x_2}} \partial_{x_2} (t)) \\ &\quad + \frac{\partial Y_4}{\partial t}(t) \partial_{x_1} \partial_{x_2}(t) + Y_4(t)(\dot{\gamma_1}(t) \nabla^C_{\partial_{x_1}} \partial_{x_1} \partial_{x_2} (t) + \dot{\gamma_2}(t) \nabla^C_{\partial_{x_2}} \partial_{x_1} \partial_{x_2} (t)) \\ &= \frac{\partial Y_1}{\partial t}(t) 1(t) \\ &\quad + \frac{\partial Y_2}{\partial t}(t) \partial_{x_1}(t) \\ &\quad + \frac{\partial Y_3}{\partial t}(t) \partial_{x_2}(t) \\ &\quad + \frac{\partial Y_4}{\partial t}(t) \partial_{x_1} \partial_{x_2}(t) \end{aligned}$$

Finally, we obtain four ordinary differential equations on $\mathbb{R}$ of the form $\partial Y_i / \partial t = 0$ for $i = 1 \cdots 4$, from which we deduce the parallel transport of Y_0 along γ. $\square$

4.2. Clifford-Hodge operator and subsequent parallel transport map. Let us denote by $\Gamma_{ij}^{k'}$ the Levi-Cevita connection's symbols of (TX,g) with respect to a local orthonormal frame field (e_1,e_2). By Section 3.2, we have

$$\nabla^C_{e_1} 1 = 0 \qquad \nabla^C_{e_2} 1 = 0$$

$$\begin{aligned} \nabla^C_{e_1} e_1 &= \Gamma_{11}^{2'} e_2 & \nabla^C_{e_1} e_2 &= -\Gamma_{11}^{2'} e_1 \\ \nabla^C_{e_2} e_1 &= \Gamma_{21}^{2'} e_2 & \nabla^C_{e_2} e_2 &= -\Gamma_{21}^{2'} e_1 \end{aligned}$$

$$\nabla^C_{e_1} e_1e_2 = 0 \qquad \nabla^C_{e_2} e_1e_2 = 0$$

Proposition 4.2. *Let (X,g) be a Riemannian manifold of dimension 2. Let (e_1,e_2) be a local oriented orthonormal frame field of (TX,g) and $\varphi = \varphi_1 1 + \varphi_2 e_1 + \varphi_3 e_2 + \varphi_4 e_1e_2 \in \Gamma(Cl(TX,g))$. Then*

$$\begin{aligned} D^2(\varphi_1 1) &= \Big(-d^2_{e_1,e_1}\varphi_1 - d^2_{e_2,e_2}\varphi_1 - \Gamma_{21}^{2'} d_{e_1}\varphi_1 + \Gamma_{11}^{2'} d_{e_2}\varphi_1 \Big) 1 \\ &= -\Delta_g(\varphi_1) 1 \end{aligned}$$

$$\begin{aligned} D^2(\varphi_2 e_1 + \varphi_3 e_2) = \Big(& -d^2_{e_1,e_1}\varphi_2 - d^2_{e_2,e_2}\varphi_2 - \Gamma_{21}^{2'} d_{e_1}\varphi_2 + \Gamma_{11}^{2'} d_{e_2}\varphi_2 + 2\,\Gamma_{11}^{2'} d_{e_1}\varphi_3 \\ & + 2\,\Gamma_{21}^{2'} d_{e_2}\varphi_3 + \varphi_2(d_{e_2}\Gamma_{11}^{2'} - d_{e_1}\Gamma_{21}^{2'}) + \varphi_3(d_{e_1}\Gamma_{11}^{2'} + d_{e_2}\Gamma_{21}^{2'}) \Big) e_1 \\ & + \Big(-d^2_{e_1,e_1}\varphi_3 - d^2_{e_2,e_2}\varphi_3 - \Gamma_{21}^{2'} d_{e_1}\varphi_3 + \Gamma_{11}^{2'} d_{e_2}\varphi_3 - 2\,\Gamma_{11}^{2'} d_{e_1}\varphi_2 \\ & - 2\,\Gamma_{21}^{2'} d_{e_2}\varphi_2 + \varphi_2(-d_{e_2}\Gamma_{21}^{2'} - d_{e_1}\Gamma_{11}^{2'}) + \varphi_3(-d_{e_1}\Gamma_{21}^{2'} + d_{e_2}\Gamma_{11}^{2'}) \Big) e_2 \end{aligned}$$

$$\begin{aligned} D^2(\varphi_4 e_1e_2) &= \Big(-d^2_{e_1,e_1}\varphi_4 - d^2_{e_2,e_2}\varphi_4 - \Gamma_{21}^{2'} d_{e_1}\varphi_4 + \Gamma_{11}^{2'} d_{e_2}\varphi_4 \Big) e_1e_2 \\ &= -\Delta_g(\varphi_4) e_1e_2 \end{aligned}$$

Proof. We obtain $D^2(\varphi)$ from (3.2) and the relations above defining ∇^C in the frame (e_1,e_2). Then we simplify the expression using some properties of the Levi-Cevita connection:
(i) In an orthonormal frame, its symbols satisfy $\Gamma_{ij}^{k'} = -\Gamma_{ik}^{j'}$.
(ii) $[e_1,e_2] = -\Gamma_{11}^{2'} e_1 - \Gamma_{21}^{2'} e_2$. □

Proposition 4.3 (Parallel transport on $Cl(TX,g)$). *Let (X,g) be a Riemannian manifold of dimension 2. Let (e_1,e_2) be a local oriented orthonormal frame of TX, and $Y_0 = Y_0^1 1(y) + Y_0^2 e_1(y) + Y_0^3 e_2(y) + Y_0^4 e_1e_2(y) \in Cl(TX,g)_y$. Let γ be a C^1 curve in X such that $\gamma(0) = y$. The parallel transport Y of Y_0 along γ is*

$$\begin{aligned} Y(t) = Y_0^1\, 1(\gamma(t)) &+ Re\Big[exp\Big(i\int_0^t \dot\gamma_1(s)\,\Gamma_{11}^{2'}(s) + \dot\gamma_2(s)\,\Gamma_{21}^{2'}(s)\ ds\Big)(Y_0^2 + i\,Y_0^3)\Big] e_1(\gamma(t)) \\ &+ Im\Big[exp\Big(i\int_0^t \dot\gamma_1(s)\,\Gamma_{11}^{2'}(s) + \dot\gamma_2(s)\,\Gamma_{21}^{2'}(s)\ ds\Big)(Y_0^2 + i\,Y_0^3)\Big] e_2(\gamma(t)) + Y_0^4\, e_1e_2(\gamma(t)) \end{aligned}$$

Proof. The parallel transport of Y_0 along γ is the solution $Y(t) = Y_1(t)\,1(\gamma(t)) + Y_2(t)\,e_1(\gamma(t)) + Y_3(t)\,e_2(\gamma(t)) + Y_4(t)\,e_1e_2(\gamma(t))$ of the differential equation

$$\left\{\begin{array}{l} \nabla^C_{\dot\gamma}\,Y(t) = 0 \\ \\ Y(0) = Y_0 \end{array}\right. \tag{4.2}$$

$$\begin{aligned}
\nabla^C_{\dot\gamma}\,Y(t) &= \nabla^C_{\dot\gamma} Y_1\,1 + Y_2\,e_1 + Y_3\,e_2 + Y_4\,e_1e_2\,(t) \\
&= \frac{\partial Y_1}{\partial t}(t)\,1(t) + Y_1(t)(\dot\gamma_1(t)\nabla^C_{e_1}\,1\,(t) + \dot\gamma_2(t)\nabla^C_{e_2}\,1\,(t)) \\
&\quad + \frac{\partial Y_2}{\partial t}(t)\,e_1(t) + Y_2(t)(\dot\gamma_1(t)\nabla^C_{e_1}\,e_1\,(t) + \dot\gamma_2(t)\nabla^C_{e_2}\,e_1\,(t)) \\
&\quad + \frac{\partial Y_3}{\partial t}(t)\,e_2(t) + Y_3(t)(\dot\gamma_1(t)\nabla^C_{e_1}\,e_2\,(t) + \dot\gamma_2(t)\nabla^C_{e_2}\,e_2\,(t)) \\
&\quad + \frac{\partial Y_4}{\partial t}(t)\,e_1e_2(t) + Y_4(t)(\dot\gamma_1(t)\nabla^C_{e_1}\,e_1e_2\,(t) + \dot\gamma_2(t)\nabla^C_{e_2}\,e_1e_2\,(t)) \\
&= \frac{\partial Y_1}{\partial t}(t)\,1(t) \\
&\quad + \frac{\partial Y_2}{\partial t}(t)\,e_1(t) + Y_2(t)(\dot\gamma_1(t)\,\Gamma^{2'}_{11}(t)\,e_2(t) + \dot\gamma_2(t)\,\Gamma^{2'}_{21}(t)\,e_2(t)) \\
&\quad + \frac{\partial Y_3}{\partial t}(t)\,e_2(t) + Y_3(t)(-\dot\gamma_1(t)\,\Gamma^{2'}_{11}(t)\,e_1(t) - \dot\gamma_2(t)\,\Gamma^{2'}_{21}(t)\,e_1(t)) \\
&\quad + \frac{\partial Y_4}{\partial t}(t)\,e_1e_2(t)
\end{aligned}$$

Finally, we obtain a differential equation on $\mathbb{R}^4$

$$\begin{pmatrix} dY_1/dt \\ dY_2/dt \\ dY_3/dt \\ dY_4/dt \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \dot\gamma_1\,\Gamma^{2'}_{11} + \dot\gamma_2\,\Gamma^{2'}_{21} & 0 \\ 0 & -\,\dot\gamma_1\,\Gamma^{2'}_{11} - \dot\gamma_2\,\Gamma^{2'}_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}$$

of initial condition $Y_1(0) = Y_0^1$, $Y_2(0) = Y_0^2$, $Y_3(0) = Y_0^3$ and $Y_4(0) = Y_0^4$.

It leads to $Y_1(t) = Y_0^1$, $Y_4(t) = Y_0^4$ and a differential equation on $\mathbb{R}^2$

$$\begin{pmatrix} dY_2/dt \\ dY_3/dt \end{pmatrix} = \begin{pmatrix} 0 & \dot\gamma_1\Gamma^{2'}_{11} + \dot\gamma_2\Gamma^{2'}_{21} \\ -\dot\gamma_1\Gamma^{2'}_{11} - \dot\gamma_2\Gamma^{2'}_{21} & 0 \end{pmatrix} \begin{pmatrix} Y_2 \\ Y_3 \end{pmatrix} \tag{4.3}$$

of initial condition $Y_2(0) = Y_0^2$ and $Y_3(0) = Y_0^3$.

Under the identification between $\mathbb{R}^2$ and $\mathbb{C}$, (4.3) becomes

$$\left\{\begin{array}{rcl} \partial(Y_2 + i\,Y_3)/\partial t &=& i\,(\dot\gamma_1\,\Gamma^{2'}_{11} + \dot\gamma_2\,\Gamma^{2'}_{21})(Y_2 + i\,Y_3) \\ \\ Y_2(0) + i\,Y_3(0) &=& Y_0^2 + i\,Y_0^3 \end{array}\right. \tag{4.4}$$

The solution of (4.4) is

$$Y_2(t) + i\,Y_3(t) = exp\Big(i\int_0^t \dot{\gamma_1}(s)\,\Gamma_{11}^{2'}(s) + \dot{\gamma_2}(s)\,\Gamma_{21}^{2'}(s)\ ds\Big)(Y_0^2 + i\,Y_0^3) \tag{4.5}$$

from which follows the parallel transport of Y_0 along γ. □

4.3. The particular context of images. Let us consider a n-channels image $I\colon (x_1,x_2) \longmapsto (I^1(x_1,x_2),\cdots,I^n(x_1,x_2))$ defined on a domain $\Omega \subset \mathbb{R}^2$. I determines a surface S embedded in $\mathbb{R}^{n+2}$ parametrized by

$$\varphi\colon (x_1,x_2) \longmapsto (x_1,x_2,I^1(x_1,x_2),\cdots,I^n(x_1,x_2))$$

Then we endow $\mathbb{R}^{n+2}$ of a metric h of matrix representation $diag(1,1,h_1,\cdots,h_n)$ where $h_1,\cdots,h_n$ are positive functions. We denote by g the metric on S induced by h. This construction makes the couple (S,g) be a Riemannian manifold of dimension 2 of global chart (Ω,φ).

On the trivial vector bundle TS, the natural global frame is $(\partial/\partial x_1, \partial/\partial x_2)$ induced by the cartesian coordinates system (x_1,x_2) on Ω. However, the Clifford-Hodge operator D^2 on $Cl(TS,g)$ is defined with respect to an oriented orthonormal frame field of (TS,g) (see Section 3.4). In what follows, we construct an oriented orthonormal frame field (e_1,e_2) of (TS,g) and compute the transformation of Levi-Cevita connection's symbols with respect to the frame change from $(\partial/\partial x_1,\partial/\partial x_2)$ to (e_1,e_2).

Proposition 4.4 (Positively oriented orthonormal basis of T_pS).
Let $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ *be the matrix representation of g at p in the basis $(\partial/\partial x_1,\partial/\partial x_2)(p)$. Let λ^+ and λ^- ($\lambda^+ \geq \lambda^-$) be the two eigenvalues of the induced endomorphism. Then a positively oriented orthonormal basis (e_1,e_2) of T_pS may be constructed from eigenvectors, distinguishing four cases:*

(i) if $F \neq 0$, take $(e_1,e_2) =$

$$\left(\begin{pmatrix} \dfrac{\lambda^+ - G}{\sqrt{\lambda^+}\sqrt{F^2+(\lambda^+-G)^2}} \\[2ex] \dfrac{F}{\sqrt{\lambda^+}\sqrt{F^2+(\lambda^+-G)^2}} \end{pmatrix}, sign(F)\begin{pmatrix} \dfrac{\lambda^- - G}{\sqrt{\lambda^-}\sqrt{F^2+(\lambda^--G)^2}} \\[2ex] \dfrac{F}{\sqrt{\lambda^-}\sqrt{F^2+(\lambda^--G)^2}} \end{pmatrix}\right)$$

in the basis $(\partial/\partial x_1,\partial/\partial x_2)(p)$.

(ii) if $F = 0$ and $E > G$, take

$$(e_1,e_2) = \Big(\frac{\partial/\partial x_1(p)}{\sqrt{E}}, \frac{\partial/\partial x_2(p)}{\sqrt{G}}\Big)$$

(iii) if $F = 0$ and $E < G$, take

$$(e_1,e_2) = \Big(\frac{\partial/\partial x_2(p)}{\sqrt{G}}, -\frac{\partial/\partial x_1(p)}{\sqrt{E}}\Big)$$

(iv) if $F=0$ and $E=G$, the whole space T_pS is eigenspace. Then for any θ, the two vectors

$$\left(\begin{pmatrix} \dfrac{\cos(\theta)}{\sqrt{E}} \\[2ex] \dfrac{\sin(\theta)}{\sqrt{E}} \end{pmatrix}, \begin{pmatrix} \dfrac{-\sin(\theta)}{\sqrt{E}} \\[2ex] \dfrac{\cos(\theta)}{\sqrt{E}} \end{pmatrix}\right)$$

in the basis $(\partial/\partial x_1, \partial/\partial x_2)(p)$ form a positively oriented orthonormal basis of T_pS.

Proof. As unit eigenvectors, they form an orthonormal basis of T_pS. The orientation is clearly positive in the cases $(ii),(iii)$ and (iv). For the case (i), one just need to compute the 2-form

$$\omega = \Big[\Big(\frac{\lambda^+ - G}{\sqrt{\lambda^+}\sqrt{F^2+(\lambda^+-G)^2}}\Big)dx_1 + \Big(\frac{F}{\sqrt{\lambda^+}\sqrt{F^2+(\lambda^+-G)^2}}\Big)dx_2\Big]\wedge$$

$$\Big[\Big(\frac{\lambda^- - G}{\sqrt{\lambda^-}\sqrt{F^2+(\lambda^--G)^2}}\Big)dx_1 + \Big(\frac{F}{\sqrt{\lambda^-}\sqrt{F^2+(\lambda^--G)^2}}\Big)dx_2\Big]$$

Then

$$\omega = \frac{F(\lambda^+-\lambda^-)}{\sqrt{\lambda^+\lambda^-}\sqrt{F^2+(\lambda^+-G)^2}\sqrt{F^2+(\lambda^--G)^2}}\,dx_1\wedge dx_2$$

and the sign of the scalar term is given by the sign of F. □

Following this construction for each $p\in S$, we obtain a positively oriented orthonormal frame field (e_1,e_2) of (TS,g), where e_1 is the unit vector field of highest variations (eigenvectors associated to the eigenvalues λ^+) and e_2 the unit vector field of lowest variations (eigenvectors associated to the eigenvalues λ^-).

By the antisymmetry property of its symbols $\Gamma_{ij}^{k'}$ in an orthonormal frame field, the Levi-Cevita connection is entirely determined by the symbols $\Gamma_{11}^{2'}$ and $\Gamma_{21}^{2'}$ in such frames. In the next proposition, we determine the expressions of these two symbols in a frame field (v_1,v_2) in function of the symbols Γ_{ij}^k of the connection in the frame field $(\partial/\partial x_1, \partial/\partial x_2)$.

Proposition 4.5. *Let (v_1,v_2) be a frame field such that $v_1 = a\,\partial/\partial x_1 + b\,\partial/\partial x_2$ and $v_2 = c\,\partial/\partial x_1 + d\,\partial/\partial x_2$. Then*

$\Gamma_{11}^{2'} = 1/(ad-bc)\times$

$$\Big(-ab\frac{\partial a}{\partial x_1} - a^2b\,\Gamma_{11}^1 - 2ab^2\,\Gamma_{12}^1 - b^2\frac{\partial a}{\partial x_2} - b^3\,\Gamma_{22}^1 + a^3\,\Gamma_{11}^2 + a^2\frac{\partial b}{\partial x_1} + 2a^2b\,\Gamma_{12}^2 + ab\frac{\partial b}{\partial x_2} + ab^2\,\Gamma_{22}^2\Big)$$

$\Gamma_{21}^{2'} = 1/(ad-bc)\times$

$$\Big(-bc\frac{\partial a}{\partial x_1} - acb\,\Gamma_{11}^1 - (bc+ad)b\,\Gamma_{12}^1 - bd\frac{\partial a}{\partial x_2} - b^2d\,\Gamma_{22}^1 + a^2c\,\Gamma_{11}^2 + ac\frac{\partial b}{\partial x_1} + (bc+ad)a\,\Gamma_{12}^2$$

$$+ad\frac{\partial b}{\partial x_2} + abd\,\Gamma_{22}^2\Big)$$

Proof. By definition, we have

$$\nabla_{v_1} v_1 = \Gamma_{11}^{1'} v_1 + \Gamma_{11}^{2'} v_2 \qquad \nabla_{v_2} v_1 = \Gamma_{21}^{1'} v_1 + \Gamma_{21}^{2'} v_2$$

With respect to the frame field $(\partial_{x_1}, \partial_{x_2}) := (\partial/\partial x_1, \partial/\partial x_2)$, we obtain

$$\begin{aligned}
\nabla_{v_1} v_1 &= \nabla_{a\partial_{x_1} + b\partial_{x_2}} a\,\partial_{x_1} + b\,\partial_{x_2} \\
&= a\nabla_{\partial_{x_1}} a\,\partial_{x_1} + a\nabla_{\partial_{x_1}} b\,\partial_{x_2} + b\nabla_{\partial_{x_2}} a\,\partial_{x_1} + b\nabla_{\partial_{x_2}} b\,\partial_{x_2} \\
&= a\Big[\frac{\partial a}{\partial x_1}\,\partial_{x_1} + a\Big(\Gamma_{11}^1\,\partial_{x_1} + \Gamma_{11}^2\,\partial_{x_2}\Big)\Big] + a\Big[\frac{\partial b}{\partial x_1}\,\partial_{x_2} + b\Big(\Gamma_{12}^1\,\partial_{x_1} + \Gamma_{12}^2\,\partial_{x_2}\Big)\Big] \\
&\quad +b\Big[\frac{\partial a}{\partial x_2}\,\partial_{x_1} + a\Big(\Gamma_{21}^1\,\partial_{x_1} + \Gamma_{21}^2\,\partial_{x_2}\Big)\Big] + b\Big[\frac{\partial b}{\partial x_2}\,\partial_{x_2} + b\Big(\Gamma_{22}^1\,\partial_{x_1} + \Gamma_{22}^2\,\partial_{x_2}\Big)\Big]
\end{aligned}$$

$$= \Big[a\frac{\partial a}{\partial x_1} + a^2\,\Gamma_{11}^1 + 2ab\,\Gamma_{12}^1 + b\frac{\partial a}{\partial x_2} + b^2\,\Gamma_{22}^1\Big]\partial_{x_1} + \Big[a\frac{\partial b}{\partial x_1} + a^2\,\Gamma_{11}^2 + 2ab\,\Gamma_{12}^2 + b\frac{\partial b}{\partial x_2} + b^2\,\Gamma_{22}^2\Big]\partial_{x_2}$$

Then, since

$$\partial_{x_1} = \frac{1}{ad-bc}(d\,v_1 - b\,v_2) \qquad \partial_{x_2} = \frac{1}{ad-bc}(-c\,v_1 + a\,v_2)$$

we obtain

$$\begin{aligned}
\nabla_{v_1} v_1 = \frac{1}{ad-bc}\Big[&ad\frac{\partial a}{\partial x_1} + a^2 d\,\Gamma_{11}^1 + 2abd\,\Gamma_{12}^1 + bd\frac{\partial a}{\partial x_2} + b^2 d\,\Gamma_{22}^1 - a^2 c\,\Gamma_{11}^2 - ac\frac{\partial b}{\partial x_1} \\
&-2abc\,\Gamma_{12}^2 - bd\frac{\partial b}{\partial x_2} - b^2 c\,\Gamma_{22}^2\Big] v_1 \\
+\frac{1}{ad-bc}\Big[&-ab\frac{\partial a}{\partial x_1} - a^2 b\,\Gamma_{11}^1 - 2ab^2\,\Gamma_{12}^1 - b^2\frac{\partial a}{\partial x_2} - b^3\,\Gamma_{22}^1 + a^3\,\Gamma_{11}^2 + a^2\frac{\partial b}{\partial x_1} \\
&+2a^2 b\,\Gamma_{12}^2 + ab\frac{\partial b}{\partial x_2} + ab^2\,\Gamma_{22}^2\Big] v_2
\end{aligned}$$

from which we deduce $\Gamma_{11}^{2'}$,

and

$$\begin{aligned}
\nabla_{v_2} v_1 = \frac{1}{ad-bc}\Big[&cd\frac{\partial a}{\partial x_1} + acd\,\Gamma_{11}^1 + (bc+ad)d\,\Gamma_{12}^1 + d^2\frac{\partial a}{\partial x_2} + bd^2\,\Gamma_{22}^1 - ac^2\,\Gamma_{11}^2 - c^2\frac{\partial b}{\partial x_1} \\
&-(bc+ad)c\,\Gamma_{12}^2 - cd\frac{\partial b}{\partial x_2} - cbd\,\Gamma_{22}^2\Big] v_1 \\
+\frac{1}{ad-bc}\Big[&-bc\frac{\partial a}{\partial x_1} - acb\,\Gamma_{11}^1 - (bc+ad)b\,\Gamma_{12}^1 - bd\frac{\partial a}{\partial x_2} - b^2 d\,\Gamma_{22}^1 + a^2 c\,\Gamma_{11}^2 \\
&+ac\frac{\partial b}{\partial x_1} + (bc+ad)a\,\Gamma_{12}^2 + ad\frac{\partial b}{\partial x_2} + abd\,\Gamma_{22}^2\Big] v_2
\end{aligned}$$

from which we deduce $\Gamma_{21}^{2'}$. □

From this proposition, we deduce the expression of the Levi-Cevita's connection symbols in the frame field (e_1, e_2).

4.4. Experiments. We apply the Clifford-Beltrami and Clifford-Hodge flows in the context of image processing. The base manifold X we consider is the surface S embedded in $\mathbb{R}^5$ parametrized by the graph of a color image $I = (I^1, I^2, I^3)$ given with its RGB components. We endow S with the Riemannian metric g induced by the metric h on $\mathbb{R}^5$, for some functions h_i (see Section 4.3). Then we construct the trivial Clifford bundles $Cl(TS, \tilde{g}) = S \times \mathbb{R}_{2,0}$ and $Cl(TS, g) \simeq S \times \mathbb{R}_{2,0}$ over (S, g).

Clifford-Beltrami flow in $\Gamma(Cl(TS,\tilde{g})_0)$ and Clifford-Hodge flow in $\Gamma(Cl(TS,g)_0)$ can be applied to regularize the color image I. Indeed we can consider each component I^i of I as a section of degree 0 of the Clifford bundles $Cl(TS,\tilde{g})$ and $Cl(TS,g)$, of the form $I^i 1$. Then the heat equations associated to Clifford-Beltrami Δ^C and Clifford-Hodge D^2 operators lead in both cases to the following 3 PDEs

$$\frac{\partial I_t^i}{\partial t} = \Delta_g I_t^i, \quad I_0^i = I^i$$

Finally, we obtain the 3 PDEs of Beltrami framework of Sochen et al. in the context of color image regularization (see e.g. [23],[25]).

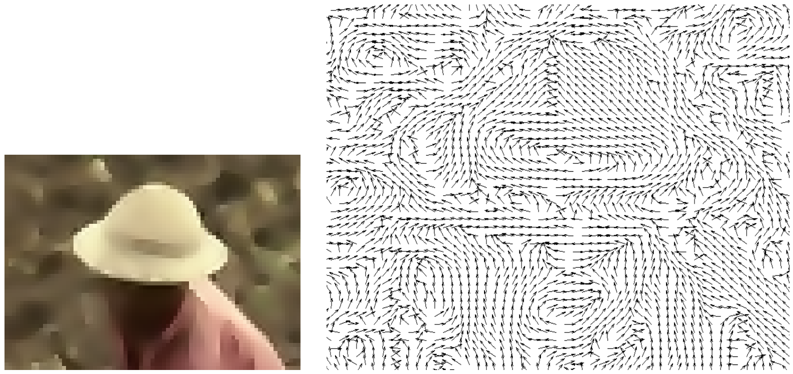
Fig. 4.1 is an illustration of color image regularization induced by Clifford-Beltrami and Clifford-Hodge flows. It is given by the computations of $k_t^0(x, y, \Delta^C)I^i 1$ and $k_t^0(x, y, D^2)I^i 1$, $i = 1, 2, 3$, where convolutions are done on 5x5 neighborhoods (see Section 2). Fig. 4.1(a) is taken from the Berkeley image segmentation database [20]. Fig. 4.1(b) is the result of the diffusions after 10 iterations for $t = 0.2$, and g induced by $h_1 = h_2 = h_3 = 0.01$. We obtain a smoothing of the initial image on regions of low color variations whereas high edges are preserved.

(a) Original image (b) Clifford-Beltrami/Hodge flows of functions

FIG. 4.1. *Color image regularization*

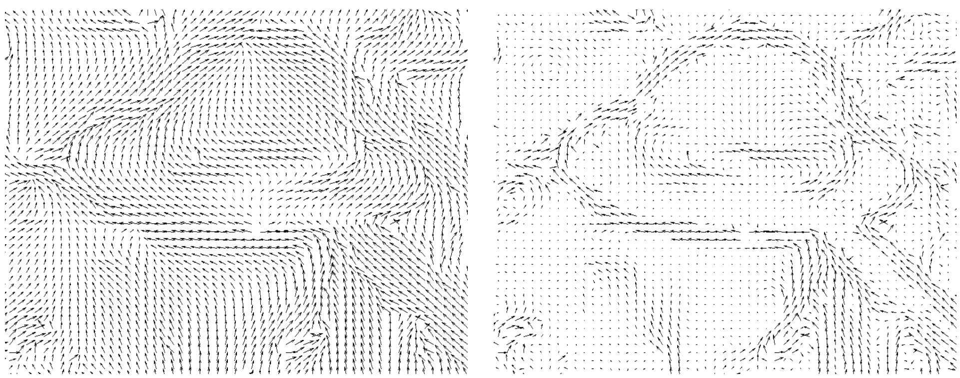
Clifford-Beltrami flow in $\Gamma(Cl(TS,\tilde{g})_1)$ and Clifford-Hodge flow in $\Gamma(Cl(TS,g)_1)$ can be applied to regularize vector fields related to I. Indeed, let $v = (v_1, v_2)$ be a vector field on Ω. By the global chart φ of S, it can be considered as a tangent vector field on S. Then, v is the section $v_1\partial/\partial x_1 + v_2\partial/\partial x_2$ of $Cl(TS,\tilde{g})$ and the section $\tilde{v}_1 e_1 + \tilde{v}_2 e_2$ of $Cl(TS,g)$ where (e_1, e_2) is the oriented orthonormal frame field of (TS, g) constructed in Section 4.3. Then we consider the heat equations related to Δ^C on $Cl(TS,\tilde{g})$ and D^2 on $Cl(TS,g)$ of initial condition $v_0 = v$

$$\frac{\partial v_t}{\partial t} + \Delta^C v_t = 0 \qquad\qquad \frac{\partial v_t}{\partial t} + D^2 v_t = 0$$



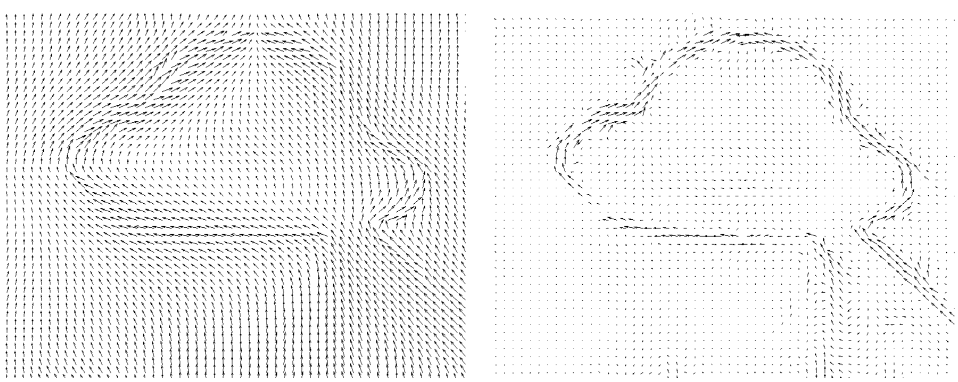
(a) (b)

Fig. 4.2. *Unit vector field of edge orientations*



(a) Clifford-Beltrami flow of vector field (b) Clifford-Hodge flow of vector field

Fig. 4.3. *Unit vector field of edge orientations regularization for* $h_i = 0.1$



(a) Clifford-Beltrami flow of vector field (b) Clifford-Hodge flow of vector field

Fig. 4.4. *Unit vector field of edge orientations regularization for* $h_i = 0.01$

Fig. 4.2(b) is the unit vector field v indicating edge orientations in an area around the hat of Fig. 4.2(a). We compare Clifford-Beltrami and Clifford-Hodge flows of v for two different metrics g on S. Results are given by the computations of $k_t^0(x, y, \Delta^C)v$ and $k_t^0(x, y, D^2)v$, where convolutions are done on 5x5 neighborhoods. Fig. 4.3 shows results for $t = 0.3$ after 200 iterations where g is induced by $h_i = 0.1$. Fig. 4.4 shows results for $t = 0.3$ after 200 iterations where g is induced by $h_i = 0.01$.

The role of the functions h_i is to control the anisotropy of the diffusion. Indeed, we have explained throughtout the paper that the anisotropy of the diffusion is determined by the metric g of the base manifold. In this context, g is determined by the functions h_i. Therefore, the more the functions h_i are low, the more the diffusion is isotropic, as it can be seen comparing Fig. 4.3 and Fig. 4.4. By the base manifold we have chosen for the applications, g measures color variations. It explains why the anisotropy of the diffusions is related with color variations of the image on Fig. 4.2(a). By definition, the Clifford-Beltrami flow consists in a Beltrami flow of each component of v. We see on Fig. 4.3(a) and Fig. 4.4(a) that it preserves the vector field on high color variations and smoothes it on low color variations. The Clifford-Hodge flow also preserves the vector field on high color variations, but it vanishes it on regions of low color variations, as it can be seen on Fig. 4.3(b) and Fig. 4.4(b).

Clifford-Beltrami flow in $\Gamma(Cl(TS, \tilde{g})_2)$ and Clifford-Hodge flow in $\Gamma(Cl(TS, g)_2)$ can be respectively applied to regularize rotation fields in $(TS, \tilde{g})$ and (TS, g). As we deal with vector bundle of rank 2, rotation fields can be given by unit vector fields. We propose to compare regularizations of unit vector fields depending on they are treated as vector fields or as rotation fields. By our choice of the unit vector field treated in this paper (the unit vector field of edge orientations), the Clifford-Hodge flow can not be considered in this application. Indeed, unit vector fields in the frame $(\partial/\partial x_1, \partial/\partial x_2)$ are not unit in the frame (e_1, e_2). Therefore, we are only concerned with the Clifford-Beltrami flow.

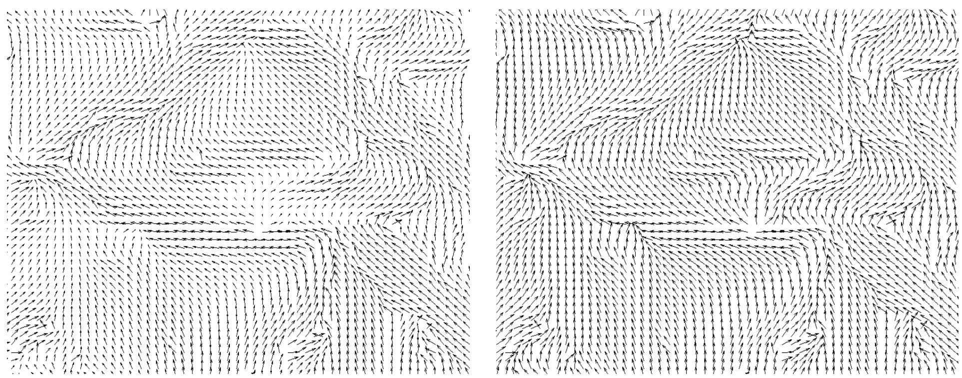
Let ψ be the rotation field given by $v = (cos\,\psi, sin\,\psi)$ on $\Omega \subset \mathbb{R}^2$, where v is the unit vector field of edge orientations (see Fig. 4.2(b)). Following the method of Section 3.1, we construct the section

$$\widetilde{\Psi}\colon x \longmapsto (x, \tilde{\psi}(x)\,\partial/\partial x_1\,\partial/\partial x_2) \in \Gamma(Cl(TS, \tilde{g})_2)$$

such that for each $x \in S$, $\tilde{\psi}(x)$ is an infinitesimal generator of the rotation $\psi(x)$. The 2π periodicity in the choice of $\tilde{\psi}$ is discussed in the Remark below. From Section 4.1, the Clifford-Beltrami flow of $\widetilde{\Psi}$ leads to

$$\frac{\partial \tilde{\psi}_t}{\partial t} = \Delta_g \tilde{\psi}_t, \quad \tilde{\psi}_0 = \tilde{\psi}$$

Remark: The 2π periodicity in the choice of $\tilde{\psi}$ may be removed by constructing $\widetilde{\Psi}$ not as a global section of $\Gamma(Cl(TS, \tilde{g}))$, but locally on small neighborhoods. Indeed, for $x_0 \in S$, we state $\widetilde{\psi}(x) = \psi(x) - 2\pi$ if $\psi(x) - \psi(x_0) \geq \pi$, $\widetilde{\psi}(x) = \psi(x) + 2\pi$ if $\psi(x) - \psi(x_0) < -\pi$ and $\widetilde{\psi}(x) = \psi(x)$ otherwise. By this construction $\tilde{\psi}$ takes values locally in $[\psi(x_0) - \pi, \psi(x_0) + \pi[$. Hence, it makes sense to the computation of $(k_t^0\widetilde{\Psi})(x_0)$. Then we extend this construction for each point in S. This method can be viewed as the Lie group counterpart of the choice of maximal normal coordinates systems on manifolds. Indeed, we always work in the tangent space of the neutral element and consider the exponential map of the Lie group.



(a) Clifford-Beltrami flow of vector fields

(b) Clifford-Beltrami flow of orthonormal frame fields

FIG. 4.5. *Unit vector field of edge orientations regularization for $h_i = 0.1$*

Fig. 4.5(b) shows the result of the diffusion for $t = 0.3$ after 200 iterations. We have constructed $\widetilde{\Psi}$ locally on 5x5 neighborhoods. Compared to the Clifford-Beltrami flow of edge orientations treated as a vector field (Fig. 4.5(a)), the regularization provides similar orientations but the unit norm of the initial unit vector field is preserved when it is treated as an oriented orthonormal frame field.

5. Conclusion. In this paper, we have proposed a new framework to treat scalar, vector and oriented orthonormal frame fields on manifolds, by considering Clifford bundles. We have shown that scalar and vector fields can be viewed as sections of Clifford bundles, respectively of degree 0 and 1, and that oriented orthonormal frame fields can be lifted to sections of Clifford bundles called spinor fields. We have also shown that sections of degree 2 can be identified with generators of orthonormal frame fields. In this paper, we were particulary concerned with the problem of regularization of these fields. Using the framework of heat equations associated to generalized Laplacians on vector bundles over Riemannian manifolds, we have shown that the behaviour of the regularization of these fields is determined by the choice of a connection on a Clifford bundle and a Riemannian metric on the base manifold. We have considered the Clifford-Beltrami and Clifford-Hodge flows generalizing the Beltrami flow to sections of Clifford bundles. Dealing with base manifolds of dimension 2, we have shown applications in the context of image processing. By the choice of different metrics and connections, different regularizations can be performed. Moreover, by the choice of other base manifolds and vector bundles, regularization of fields in other contexts can be envisaged. For instance, dealing with base manifolds of dimension 3, we can extend the applications presented in this paper to the context of video processing.

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Appendix A. Differential Geometry of Vector Bundles.

See [12],[15],[17],[26] for some references on differential geometry of manifolds and vector bundles.

Definition A.1. *A **smooth vector bundle of rank** n is a triplet (E, π, X) where X and E are two C^∞ manifolds, and $\pi\colon E \longrightarrow X$ is a surjective map such that the preimage $\pi^{-1}(x)$ of $x \in X$ is endowed with a structure of vector space of dimension n. X is called the **base manifold** and E the **total space** of the vector bundle. The set $\pi^{-1}(x)$ is called the **fiber** over x, and is denoted by E_x.*

*The vector bundle is said to be **locally trivial** if the following conditions hold: for each $x \in X$, there is a neighborhood U of x and a diffeomorphism $\phi\colon U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$ satisfying $\pi \circ \phi(x, f) = x$, and such that the map $\phi_x\colon \mathbb{R}^n \longrightarrow E_x$ is a vector space isomorphism. The couple (U, ϕ) is called a **local trivialization**.*

*The vector bundle is said to be **trivial** if there exists a diffeomorphism $\Phi\colon X \times \mathbb{R}^n \longrightarrow E$ such that $\pi \circ \Phi(x, f) = x$, and $\Phi_x\colon \mathbb{R}^n \longrightarrow E_x$ is a vector space isomorphism.*

Example: Let X be a C^∞ manifold of dimension m. The disjoint union of tangent spaces $TX := \bigsqcup T_xX$ for $x \in X$, is the total space of a vector bundle (TX, π, X) of rank m called the **tangent bundle of** X. Tangent space T_xX is the fiber over x.

Definition A.2. *A metric h on a vector bundle is the assigment of a scalar product h_x on each fiber $\pi^{-1}(x)$.*

Example: A Riemannian metric on a manifold is a definite positive metric on its tangent bundle.

Definition A.3. *A **section** of a smooth vector bundle (E, π, X) is a differentiable map $S\colon X \longrightarrow E$ such that $\pi \circ S = Id_X$.*

Let $(f_1, \cdots, f_n)$ be a basis of $\mathbb{R}^n$. In a local trivialization (U, ϕ) of (E, π, X), any section may be written

$$S(x) = \sum_{i=1}^{n} s_i(x)\phi(x, f_i)$$

for some functions $s_i \in C^\infty(X)$.

*The set $\{\phi(\cdot, f_1), \cdots, \phi(\cdot, f_n)\}$ is called a **local frame** of (E, π, X).* The set of sections of (E, π, X) is denoted by $\Gamma(E)$.

Example: Tangent vector fields on X are the sections of the tangent bundle (TX, π, X).

Definition A.4. *A **connection** on (E, π, X) is a map $\nabla^E\colon \Gamma(TX) \times \Gamma(E) \longrightarrow \Gamma(E)$ satisfying the following axioms:*

-$\nabla^E_{fu+gv}Y = f\nabla^E_u Y + g\nabla^E_v Y$
-$\nabla^E_u fY = (d_u f)Y + f\nabla^E_u Y$

for $f, g \in C^\infty(X)$, $u, v \in \Gamma(TX)$ *and* $Y \in \Gamma(E)$.

Hence, a connection on (E, π, X) *may be written as* $d+\omega$, *where* d *is the differentation of components and* $\omega \in \Gamma(T^*X \otimes End(E))$.

In local frames $(e_1, \cdots, e_n)$ *of* E *and* $(u_1, \cdots, u_m)$ *of* TX, *a connection is determined by* $n^2 \times m$ *functions* Υ_{ij}^k *such that*

$$\nabla_{u_i}^E e_j = \sum_{k=1}^n \Upsilon_{ij}^k e_k$$

Example: The **Levi-Cevita** connection is the connection on the tangent bundle of a Riemannian manifold (X, g) determined by the m^3 functions

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij})$$

with respect to the local frame $(\partial/\partial x_1, \cdots, \partial/\partial x_m)$ of TX given by a local coordinates system $(x_1, \cdots, x_m)$ of X.

Definition A.5. *Let* ∇^E *be a connection on a vector bundle* (E, π, X), *and* γ *a* C^1 *curve in* X *such that* $\gamma(0) = y$. *The* ***parallel transport*** *of* $Y_0 \in E_y$ *along the curve* γ *is the section* $Y(t)$ *that is solution of the following differential equation*

$$\begin{cases} \nabla_{\dot{\gamma}}^E Y(t) = 0 \\ Y(0) = Y_0 \end{cases}$$

A section Y *along a curve* γ *is* ***parallel*** *if* $\nabla_{\dot{\gamma}}^E Y(t) = 0$ *for each* t.

Example: Let (X, g) be a Riemannian manifold endowed with the Levi-Cevita connection ∇ on its tangent bundle. A **Geodesic curve** on X is a C^1 curve γ whose tangent vector field $\dot{\gamma}$ is parallel along γ, i.e. $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.

Appendix B. Heat Kernels of Generalized Laplacians.

We refer to [4] for more details on this part.

Definition B.1. *Let* E *be a vector bundle over a Riemannian manifold* (X, g), *endowed with a connection* ∇^E. *Let* ∇ *be the Levi-Cevita connection of* (TX, g). *To any pair of tangent vector fields* V *and* W *on* X, *we associate an invariant second derivative* $\nabla_{V,W}^2 : \Gamma(E) \longrightarrow \Gamma(E)$ *by setting*

$$\nabla_{V,W}^2 \varphi \equiv \nabla_V^E \nabla_W^E \varphi - \nabla_{\nabla_V W}^E \varphi$$

Then the ***connection Laplacian*** $\Delta^E : \Gamma(E) \longrightarrow \Gamma(E)$ *is defined by*

$$\Delta^E \varphi = -trace(\nabla_{\cdot,\cdot}^2 \varphi)$$

where trace denotes the contraction with the metric g.

In particular, if e_i is a local orthonormal frame of TX, the operator Δ^E is given by

$$\Delta^E = -\sum_i \left(\nabla^E_{e_i} \nabla^E_{e_i} - \nabla^E_{\nabla_{e_i} e_i} \right)$$

With respect to the local frame $\partial_i := \partial/\partial x_i$ defined by a local coordinates system of X, we have

$$\Delta^E = -\sum_{ij} g^{ij} \left(\nabla^E_{\partial_i} \nabla^E_{\partial_j} - \sum_k \Gamma^k_{ij} \nabla^E_{\partial_k} \right)$$

where the symbols Γ^k_{ij} are defined by $\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$.

Definition B.2. *Let E be a vector bundle over a Riemannian manifold (X, g). A* ***Generalized Laplacian*** *on E is a second-order differential operator $H\colon \Gamma(E) \longrightarrow \Gamma(E)$ that may be written*

$$H = \Delta^E + F$$

for some connection ∇^E and a zero-order operator F.

In particular, any connection Laplacian is a generalized Laplacian.

To any generalized Laplacian H on a vector bundle E over a compact manifold X, one may associate an operator $e^{-tH}\colon \Gamma(E) \longrightarrow \Gamma(E)$, for $t > 0$, with the property that $u_t(x) = e^{-tH} u(x)$ satisfies the **heat equation**

$$\frac{\partial u_t}{\partial t} + H u_t = 0, \qquad u_0 = u$$

We shall define e^{-tH} as an integral operator of the form

$$(e^{-tH} u)(x) = \int_X K_t(x, y, H) u(y) dy$$

where $K_t(x, y, H)\colon E_y \longrightarrow E_x$ is a linear map depending smoothly on x, y and t. It is called the **heat kernel of** H.

In the following theorem, we summarize some results on approximations of the heat kernel and solutions of the heat equation (see [4] p. 84).

Theorem B.3. *Let $x \in X$. We denote by $\mathbf{y^i}$ the normal coordinates of a point y in the injectivity radius of X at x, ∂_i the corresponding partial derivatives, and by $g_{ij}(\mathbf{y})$ the scalar product of ∂_i and ∂_j at $\mathbf{y}$. Moreover, we define*

$$J(x, y) = det(g_{ij}(\mathbf{y}))^{1/2} \qquad \text{for } y = exp_x(\mathbf{y})$$

Let ϵ chosen smaller than the injectivy radius of X. Let $\Psi\colon \mathbb{R}_+ \longrightarrow [0, 1]$ be a smooth function such that $\Psi(s) = 1$ if $s < \epsilon^2/4$ and $\Psi(s) = 0$ if $s > \epsilon^2$.
Let $\tau(x, y)\colon E_y \longrightarrow E_x$ be the parallel transport along the unique geodesic curve joining x and y, and $d(x, y)$ its length.

Then the kernels $K_t^N(x,y,H)$ *defined by*

$$\left(\frac{1}{4\pi t}\right)^{\frac{m}{2}} e^{-d(x,y)^2/4t}\ \Psi(d(x,y)^2)\sum_{i=0}^{N} t^i\Phi_i(x,y,H)J(x,y)^{-\frac{1}{2}},$$

where the sections Φ_i *are given by* $\Phi_0(x,y,H)=\tau(x,y)$ *and* $\tau(x,y)^{-1}\Phi_i(x,y,H)=$

$$-\int_0^1 s^{i-1}\tau(x_s,y)^{-1}(B_x.\Phi_{i-1})(x_s,y,H)ds$$

with B_x *be the operator* $J^{1/2}\circ H_x\circ J^{-1/2}$ *where* H_x *is the operator* H *with respect to the first variable,*

satisfy

1. For every $N>m/2$*, the kernel* $K_t^N(x,y,H)$ *is asymptotic to* $K_t(x,y,H)$*:*

$$\left\|\partial_t^k[\,K_t(x,y,H)-K_t^N(x,y,H)\,]\right\|_l = O(t^{N-m/2-l/2-k})$$

where $\|\,\|_l$ *is a norm on* C^l *sections.*

2. Denoting by k_t^N *the operator defined by*

$$(k_t^N u)(x)=\int_X K_t^N(x,y,H)u(y)dy$$

we have $\lim_{t\to 0}\|k_t^N u-u\|_l=0$ *for every* N*.*

Moreover, we have the following estimate:

$$\left\|e^{-tH}u-\sum_{i=0}^{k}\frac{(-tH)^i}{i!}u\right\|_j = O(t^{k+1})$$

which justifies the notation e^{-tH}*.*

Appendix C. Clifford algebras and the Lie group Spin(n).

C.1. Clifford algebras. We refer to [7],[16],[19] for more details on Clifford algebras.

Definition C.1 (Clifford algebra). *Let* V *be a vector space of finite dimension* n *over* $\mathbb{R}$ *equipped with a quadratic form* Q*. Formally speaking, the Clifford algebra* $Cl(V,Q)$ *of* (V,Q) *is the solution of the following universal problem. We search a couple* $(Cl(V,Q),i_Q)$ *where* $Cl(V,Q)$ *is an* $\mathbb{R}$*-algebra and* $i_Q : V \longrightarrow Cl(V,Q)$ *is* $\mathbb{R}$*-linear satisfying:*

$$(i_Q(v))^2 = Q(v).1$$

for all v *in* V *(1 denotes the unit of* $Cl(V,Q)$*) such that for each* $\mathbb{R}$*-algebra* A *and each* $\mathbb{R}$*-linear map* $f : V \longrightarrow A$ *with*

$$(f(v))^2 = Q(v).1$$

for all v in V (1 denotes the unit of A), then there exists a unique morphism

$$g : Cl(V,Q) \longrightarrow A$$

of $\mathbb{R}$-algebras such that $f = g \circ i_Q$.
The solution is unique up to isomorphisms and is given as the (non commutative) quotient

$$T(V)/(v \otimes v + Q(v).1)$$

of the tensor algebra of V by the two-sided ideal generated by $v \otimes v + Q(v).1$, where v belongs to V (see [19] for a proof).
It is well known that there exists a unique anti-automorphism t on $Cl(V,Q)$ such that

$$t(i_Q(v)) = i_Q(v)$$

for all v in V. It is called reversion and usually denoted by $x \longmapsto x^\dagger$, x in $Cl(V,Q)$. In the same way there exists a unique automorphism α on $Cl(V,Q)$ such that

$$\alpha(i_Q(v)) = -i_Q(v)$$

for all v in V. In this paper we write v for $i_Q(v)$ (according to the fact that i_Q embeds V in $Cl(V,Q)$).

When it is defined, we denote $\|x\| = \sqrt{xx^\dagger}$ and say that x is a unit if $xx^\dagger = \pm 1$.
Let V be a vector space on $\mathbb{R}$ of dimension n, Q a quadratic form on V and $(e_1, \cdots, e_n)$ an orthonormal basis of V with respect to Q. As a vector space $Cl(V,Q)$ is of dimension 2^n on $\mathbb{R}$ and a basis is given by the set

$$\{e_{i_1} e_{i_2} \cdots e_{i_k}, \quad i_1 < i_2 < \cdots < i_k, \quad k \in \{1, \ldots, n\}\}$$

and the unit 1. An element of degree k

$$\sum_{i_1 < \cdots < i_k} \alpha_{i_1 \ldots i_k} e_{i_1} e_{i_2} \cdots e_{i_k}$$

is called a k-vector. A 0-vector is a scalar and $e_1 e_2 \cdots e_n$ is called the pseudoscalar. We denote $\langle x \rangle_k$ the component of degree k of an element x of $Cl(V,Q)$.
The inner product of x_r of degree r and y_s of degree s is defined by

$$x_r \cdot y_s = \langle x_r y_s \rangle_{|r-s|}$$

if r and s are positive and by

$$x_r \cdot y_s = 0$$

otherwise.
The outer product of x_r of degree r and y_s of degree s is defined by

$$x_r \wedge y_s = \langle x_r y_s \rangle_{r+s} \tag{C.1}$$

These products extend by linearity on $Cl(V,Q)$. Clearly, if a and b are vectors of V, then the inner product of a and b coincides with the scalar product defined by Q.

Remark: For $Q \equiv 0$, the Clifford algebra $Cl(V,Q)$ corresponds to the exterior algebra $\bigwedge V$ of V. Indeed, in this case the product in the Clifford algebra is the outer product (C.1), and we have an algebra isomorphism between $Cl(V,Q)$ and $\bigwedge V$. For arbitrary quadratic form Q, there is a vector space isomorphism between $\bigwedge V$ and $Cl(V,Q)$, that maps the subspace $\bigwedge^k V$ to $Cl(V,Q)_k$. It follows a vector space isomorphism between the space $\bigwedge V^*$ of linear forms on V and the Clifford algebra $Cl(V,Q)$, that maps k-linear forms to k-vectors. More precisely, for $(e_1, \cdots, e_n)$ a orthonormal basis of (V,Q), we have the identifications

$$\begin{array}{ll} \text{k-form :} & \displaystyle\sum_{\substack{i_1<\cdots<i_k \\ 1\leq i_k\leq m}} \omega_{i_1\cdots i_k} e^{i_1}\wedge\cdots\wedge e^{i_k} \longleftrightarrow \sum_{\substack{i_1<\cdots<i_k \\ 1\leq i_k\leq m}} \omega_{i_1\cdots i_k}\, e_{i_1}\cdots e_{i_k} \\ & \\ \text{0-form :} & l \longleftrightarrow l\,1 \end{array} \tag{C.2}$$

In this paper, we deal in particular with the Clifford algebra of the Euclidean space $(\mathbb{R}^n, \|\,\|_2)$ denoted by $\mathbb{R}_{n,0}$. $\mathbb{R}^k_{n,0}$ is the subspace of elements of degree k and $\mathbb{R}^*_{n,0}$ is the group of elements that admit an inverse in $\mathbb{R}_{n,0}$.

C.2. The Lie groups Spin(n) and SO(n). The group Spin(n) is defined by

$$\mathrm{Spin}(n) = \left\{ \prod_{i=1}^{2k} a_i,\ a_i \in \mathbb{R}^1_{n,0},\ \|a_i\| = 1 \right\}$$

It is well known that Spin(n) is a connected compact Lie group that universally covers SO(n) ($n \geq 3$). Under the identification of $(\mathbb{R}^n, \|\,\|_2)$ and its embedding $\mathbb{R}^1_{n,0}$ into $\mathbb{R}_{n,0}$, the covering group is given by the map ξ

$$\begin{array}{ccccc} & \mathrm{Spin(n)} & \longrightarrow & \mathrm{SO}(n) & \\ \xi: & s & \longmapsto & R_s: & \begin{array}{ccc} \mathbb{R}^1_{n,0} & \longrightarrow & \mathbb{R}^1_{n,0} \\ x & \longmapsto & sxs^{-1} \end{array} \end{array}$$

For instance, one can verify that Spin(3) is the group

$$\{a1 + be_1e_2 + ce_2e_3 + de_3e_1,\ a^2+b^2+c^2+d^2 = 1\}$$

and is isomorphic to the group $\mathbb{H}^1$ of unit quaternions.
We have Spin(2)= $\{a1 + be_1e_2,\ a^2+b^2 = 1\}$. It follows the identifications

$$\mathrm{Spin}(2) \simeq S^1 \simeq \mathrm{SO}(2)$$

The Lie algebra $\mathfrak{spin}(n)$ of Spin(n) is $\mathbb{R}^2_{n,0}$ with Lie bracket

$$A \times B = AB - BA.$$

The exponential map $exp : \mathfrak{spin}(n) \longrightarrow \mathrm{Spin}(n)$ is onto and corresponds to the usual matrix exponential map. As a consequence, every spinor can be written as

$$S = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$$

for some 2-vector A.

From the covering group $\mathrm{Spin}(n) \longrightarrow SO(n)$, we have a Lie algebra isomorphism

$$\mathfrak{spin}(n) \simeq \mathfrak{so}(n)$$

Dealing with the matrix representation of the group $\mathrm{SO}(n)$, we have

$$\mathfrak{so}(n) \simeq \{n \times n \text{ skew symmetric matrices}\}$$

Denoting by $E_{ij}, i < j$ the elementary skew symmetric matrix such that $E_{ij}(i,j) = -1$ and $E_{ij}(j,i) = 1$, the Lie algebra isomorphism maps E_{ij} to $\dfrac{1}{2}e_i e_j$.

In the rest of this part, we show how rotations in $\mathbb{R}^n$ can be interpreted in the Clifford algebras formalism. From Hestenes and Sobczyk [16], we know that every A in $\mathbb{R}^2_{n,0}$ can be written as

$$A = A_1 + A_2 + \cdots + A_m$$

where $m \leq n/2$ and

$$A_j = \|A_j\| a_j b_j, \quad j \in \{1, \ldots, m\},$$

with

$$\{a_1, \ldots, a_m, b_1, \ldots, b_m\}$$

a set of orthonormal vectors. Thus

$$A_j A_k = A_k A_j = A_k \wedge A_j$$

whenever $j \neq k$ and

$$A_k^2 = -\|A_k\|^2 < 0$$

This means that the planes encoded by A_k and A_j are orthogonal and implies that

$$e^{A_1 + A_2 + \cdots + A_m} = e^{A_{\sigma(1)}} e^{A_{\sigma(2)}} \ldots e^{A_{\sigma(m)}}$$

for all σ in the permutation group $\mathfrak{S}(m)$. Actually, as A_k^2 is negative we have

$$e^{A_i} = \cos(\|A_i\|) + \sin(\|A_i\|) \frac{A_i}{\|A_i\|}$$

The corresponding rotation

$$R_i : x \longmapsto e^{-A_i} x e^{A_i}$$

acts in the oriented plane defined by A_i as a plane rotation of angle $2\|A_i\|$. The vectors orthogonal to A_i are invariant under R_i.
It then appears that any element R of $\mathrm{SO}(n)$ is a composition of commuting simple rotations, in the sense that they have only one invariant plane. The vectors left invariant by R are those of the orthogonal subspace to A. If $m = n/2$ this latter is trivial. The previous decomposition is not unique if $\|A_k\| = \|A_j\|$ for some j and k with $j \neq k$. In this case infinitely many planes are left invariant by R.

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