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# Implementing Bayesian predictive procedures: The $K$ -prime and $K$ -square distributions

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## Abstract

The implementation of Bayesian predictive procedures under standard normal models is considered. Two distributions are of particular interest, the  $K$ -prime and  $K$ -square distributions. They also give exact inferences for simple and multiple correlation coefficients. Their cumulative distribution functions can be expressed in terms of infinite series of multiples of incomplete beta function ratios, thus adequate for recursive calculations. Efficient algorithms are provided. To deal with special cases where possible underflows may prevent recurrence to work properly, a simple solution is proposed which results in a procedure which is intermediate between two classes of algorithms. Some examples of applications are given.

*Key words:* Predictive distribution, Bayesian approach, Incomplete beta function

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## 1 Introduction

Bayesian predictive probabilities give statistical users a particularly useful device to answer essential questions such as: “how big should be the experiment to have a reasonable chance of demonstrating a given conclusion?” “given the current data, what is the chance that the final result will be in some sense conclusive, or on the contrary inconclusive?” Traditional frequentist procedures (e.g., sample size determination via power calculation), being conditional to

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the parameters, are carried out under a subset of parameter values, whereas Bayesian predictive probabilities, which consider all possible parameter values (conditionally to the data in hand), give them direct and natural answers. Some relevant references are Baum et al (1989), Spiegelhalter et al (1994), Johns and Andersen (1999), Lecoutre (2001), Lecoutre (2008), Berry (2005), Dmitrienko and Wang (2006), Grouin et al (2007). In particular, from a pilot study, the predictive probabilities on credible limits give a useful summary to help in the determination of the sample size of an experiment. If the power approach and the predictive approach sometimes result in relatively similar sample sizes (for instance, see Inoue et al, 2005), in general, the predictive approach requires a larger sample size. This can be considered the price to pay to avoid assumptions about parent effect size and variance.

Recently, the Association for Psychological Science has recommended that articles published in Psychological Science and their other journals report the “probability of replicating an effect”, denoted  $p_{rep}$  (Killeen, 2005) rather than the traditional  $p$ -value.  $p_{rep}$  is defined as the predictive probability of finding an effect of the same sign in a replication.

The above procedures are frequently used in the case of comparison of means for which the traditional procedures are the  $t$  and ANOVA  $F$  tests. For sample size determination, considering an unknown variance is often seen as an unnecessary sophistication. However, this requires that the sample sizes to be determined are relatively high. The probability of replication  $p_{rep}$  – such as it now appears in Psychological Science – and its extensions frequently involve small sample sizes, but solutions in use assume a known variance (Lecoutre et al, 2008). One hundred years after Student’s famous article (Student, 1908), one can hardly be satisfied with this unnecessary restriction.

The aim of this article is to contribute to implement predictive procedures that relax the assumption of a known variance. These procedures involve the  $K$ -prime and  $K$ -square distributions that have been introduced in Lecoutre (1984). They can be characterized as mixtures of the classical noncentral  $t$  and noncentral  $F$  distributions respectively (Lecoutre, 1999). In particular, the predictive distributions of the  $t$  test statistic and the associate limits of interval estimates under standard normal models, assuming a conjugate prior, is a  $K$ -prime distribution. The extension to ANOVA  $F$  tests involves the  $K$ -square distribution. Moreover, the  $K$ -prime and  $K$ -square respectively include as particular cases the distributions of the sample correlation coefficient and of the sample multiple correlation coefficient, allowing exact inferences about these two coefficients.

This article provide efficient algorithms for the calculation of the cumulative distribution functions (cdfs) of these distributions. These cdfs can be expressed in terms of infinite series of multiples of incomplete beta function ratios, thus

adequate for recursive calculations. More precisely, both imply the general form

$$\sum_{j=0}^{\infty} s^j g_j H_j(x), \tag{1}$$

with

$$s = \pm 1, \quad 0 \leq g_j \leq 1 \quad \forall j, \quad \sum_{j=0}^{\infty} g_j = 1$$

and where  $H_j(x)$  involves only the incomplete beta function.

Dealing with a related problem, the Applied Statistics algorithm AS 278 developed for the psi-square distribution (Lecoutre, Guigues and Poitevineau, 1992) could be adapted to match the present cdfs. However, AS 278 is a Method 1 recursive algorithm, in the terms of Benton and Krishnamoorthy (2003): accumulation is simply done from index 0 (which maximizes  $H_j(x)$ ) until a convergence criterion is met. In some cases, especially when the non-centrality parameter of the distribution is large, it can lead to an exceedingly large number of iterations, and consequently to unacceptable execution time and loss of precision. Frick (1990) proposed an improvement that consists in starting iterations at an index such that the resulting truncation error is negligible, but this does not solve the problem.

Yet, the present cdfs are of the general class considered by Benton and Krishnamoorthy (2003) and, as such, are good candidates for what they called Method 2 class of algorithms. Essentially, this Method 2 is a both backward and forward recursive algorithm. As these authors assume  $\{g_j\}_j$  to be the dominant series in general (this is discussed in section 4), the starting index for iterations, say  $k$ , is chosen so that  $g_k$  is a maximum, which reduces the above mentioned problems.

Obviously, the best method would be to start iterations at the index (between 0 and  $k$ ) which maximizes the product  $g_j H_j(x)$  and not only one of the terms. However, this is not easy to determine in general when no analytic solution is available. Numerical determination would be time consuming and thus would overcome the benefit of an optimal starting point (inasmuch as it should be calculated for every  $x$ ). We return to this concern in section 4.

Therefore, we present in the next two sections a Method 2 class of algorithms applied respectively to the  $K$ -prime and  $K$ -square cdfs, but of general use as far as the general form (1) is concerned. In section 4 we compare the two methods and we discuss some remaining problems and propose, in some cases, a simple modification which leads to an algorithm that is intermediate between

Method 1 and Method 2. Some examples of applications of these cdfs are given in section 5 and section 6 is devoted to some concluding remarks.

## 2 $K$ -prime distribution

Technical characterizations of the  $K$ -prime distribution can be found in Lecoutre (1999). This distribution is written  $K'_{q,r}(a)$  where  $q, r$  are degrees of freedom parameters and  $a$  is a noncentrality parameter.

Particular cases of the  $K$ -prime distributions are:

$$\begin{aligned} a = 0 & : K'_{q,r}(0) \equiv t_r \text{ (usual } t \text{ distribution),} \\ q = \infty & : K'_{\infty,r}(a) \equiv t'_r(a) \text{ (noncentral } t \text{ distribution),} \\ r = \infty & : K'_{q,\infty}(a) \equiv \Lambda'_q(a) \text{ (lambda-prime distribution),} \\ q = \infty, r = \infty & : K'_{\infty,\infty}(a) \equiv N(a, 1) \text{ (normal distribution).} \end{aligned}$$

This cdf has the following properties:

$$\begin{aligned} \Pr(K'_{q,r}(a) < x) &= \Pr(K'_{r,q}(x) > a), \\ \Pr(K'_{q,r}(-a) < -x) &= \Pr(K'_{q,r}(x) > a), \\ \Pr(K'_{q,r}(a) < 0) &= \Pr(\Lambda'_q(a) < 0) = \Pr(t_q > a). \end{aligned}$$

Several cases are to be distinguished for the cdf:

If  $a > 0$  and  $x < 0$

$$\begin{aligned} \Pr(K'_{q,r}(a) < x) &= \Pr(K'_{q,r}(a) < 0) - \Pr(x < K'_{q,r}(a) < 0) \\ &= \Pr(t_q > a) - \sum_{j=0}^{\infty} (-1)^j g_j H_j(x), \end{aligned}$$

where

$$\begin{aligned} g_j &= \frac{1}{2} \frac{\Gamma(\frac{q+j}{2})}{\Gamma(1 + \frac{j}{2})\Gamma(\frac{q}{2})} \left(\frac{q}{q+a^2}\right)^{\frac{q}{2}} \left(\frac{a^2}{q+a^2}\right)^{\frac{j}{2}}, \\ H_j(x) &= I_{x^2/(r+x^2)}\left(\frac{j+1}{2}, \frac{r}{2}\right), \end{aligned}$$

and  $I_z$  is the incomplete beta function

$$I_z(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^z t^{a-1} (1-t)^{b-1} dt.$$

If  $a > 0$  and  $x > 0$

$$\begin{aligned} \Pr(K'_{q,r}(a) < x) &= \Pr(K'_{q,r}(a) < 0) + \Pr(0 < K'_{q,r}(a) < x) \\ &= \Pr(t_q > a) + \sum_{j=0}^{\infty} g_j H_j(x). \end{aligned}$$

If  $a < 0$ , we reduce to the above cases using

$$\Pr(K'_{q,r}(a) < x) = 1 - \Pr(K'_{q,r}(-a) < -x).$$

If  $a = 0$ , we simply have

$$\Pr(K'_{q,r}(0) < x) = \Pr(t_r < x).$$

Hence, the cdf of the  $K$ -prime involves the calculation of the cdf of the usual Student's  $t$  distribution and a series of the general form (1). The case where  $a$  and  $x$  are of a different sign is an unfavorable one, since the series is then alternate. Therefore, in the algorithm, the even and odd terms of the series should be accumulated separately in order to minimize the number of subtractions.

The forward and backward recurrence relations for the cdf are straightforward. For the  $H_j$ 's (the incomplete beta function) we have

$$\begin{aligned} H_{j+2} &= H_j - \frac{\Gamma(\frac{j+r+1}{2})}{\Gamma(\frac{j+3}{2})\Gamma(\frac{r}{2})} \left(\frac{x^2}{r+x^2}\right)^{\frac{j+1}{2}} \left(\frac{r}{r+x^2}\right)^{\frac{r}{2}}, \\ H_{j-2} &= H_j + \frac{\Gamma(\frac{j+r-1}{2})}{\Gamma(\frac{j+1}{2})\Gamma(\frac{r}{2})} \left(\frac{x^2}{r+x^2}\right)^{\frac{j-1}{2}} \left(\frac{r}{r+x^2}\right)^{\frac{r}{2}} \end{aligned}$$

and for the  $g_j$  coefficients

$$\begin{aligned} g_{j+2} &= \frac{q+j}{j+2} \frac{a^2}{q+a^2} g_j, \\ g_{j-2} &= \frac{j}{q+j-2} \frac{q+a^2}{a^2} g_j. \end{aligned}$$

From the forward recurrence relation, it is straightforward that imposing  $g_{j+2} < g_j$  leads to  $j > a^2(q-2)/q - 2$ . Thus, the starting point for iterations, say  $k$ , is taken as the mode of the  $g_j$ 's, i.e.  $k = \lceil a^2(q-2)/q \rceil - 1$ , where  $\lceil \cdot \rceil$  denotes the integer part.

Given the parameters,  $H_j(x)$  is a decreasing function of  $j$ . Thus, when stopping the calculations at step  $j$ , the truncation error ( $E_t$ ) is bounded by:

while  $j < k$

$$\begin{aligned}
E_t &\leq H_0(x) \sum_{i=0}^{k-j-1} g_i + H_k(x) \sum_{i=k+j+1}^{\infty} g_i \\
&\leq H_0(x) \sum_{i=0}^{k-j-1} g_i + H_0(x) \sum_{i=k+j+1}^{\infty} g_i \\
&\leq H_0(x) \left[ 1 - \sum_{i=k-j}^{k+j} g_i \right] \tag{2}
\end{aligned}$$

and when  $j \geq k$

$$E_t \leq H_{k+j}(x) \left[ 1 - \sum_{i=0}^{k+j} g_i \right].$$

Benton and Krishnamoorthy (2003) used  $E_t \leq 1 - \sum_{i=k-j}^{k+j} g_i$  instead of (2), so that the calculation of  $H_0(x)$  was avoided. We think that the relaxation of the stopping rule compensates for the increased execution time due to one call to the incomplete beta function.

*Stopping rule:* Stop when  $E_t$  becomes lower than a predetermined absolute error bound.

### 3 $K$ -square distribution

Technical characterizations of the  $K$ -square distribution can be found in Lecoutre (1999). This distribution is written  $K_{p,q,r}^2(a^2)$  where  $p, q, r$  are degrees of freedom parameters and  $a^2$  is a noncentrality parameter.

Particular cases of the  $K$ -square distribution are:

$$\begin{aligned}
a = 0 &: K_{p,q,r}^2(0) \equiv F_{p,r} \text{ (usual } F \text{ distribution),} \\
q = \infty &: K_{p,\infty,r}^2(a^2) \equiv F'_{p,r}(a^2) \text{ (noncentral } F \text{ distribution),}
\end{aligned}$$

$r = \infty : K_{p,q,\infty}^2(a^2) \equiv \Lambda_{p,q}^2(a^2)$  (lambda-square or alternate chi-square distribution),  
 $q = \infty, r = \infty : K_{p,\infty,\infty}^2(a^2) \equiv (1/p)\chi_p^2(a^2)$  (noncentral chi-square distribution).

For the cdf,  $s = 1$  in (1) and we simply have

$$\Pr(K_{p,q,r}^2(a^2) < x) = \sum_{j=0}^{\infty} g_j H_j(x),$$

with

$$g_j = \frac{\Gamma(\frac{q}{2} + j)}{\Gamma(j+1)\Gamma(\frac{q}{2})} \left(\frac{q}{q+a^2}\right)^{\frac{q}{2}} \left(\frac{a^2}{q+a^2}\right)^j$$

and

$$H_j(x) = I_{px/(r+px)}\left(\frac{p}{2} + j, \frac{r}{2}\right), \quad x > 0,$$

The recurrence relations for the incomplete beta function now write

$$\begin{aligned}
 H_{j+1} &= H_j - \frac{\Gamma(p/2 + r/2 + j)}{\Gamma(p/2 + j + 1)\Gamma(r/2)} \left(\frac{px}{r+px}\right)^{p/2+j} \left(\frac{r}{r+px}\right)^{r/2}, \\
 H_{j-1} &= H_j + \frac{\Gamma(p/2 + r/2 + j - 1)}{\Gamma(p/2 + j)\Gamma(r/2)} \left(\frac{px}{r+px}\right)^{p/2+j-1} \left(\frac{r}{r+px}\right)^{r/2}
 \end{aligned}$$

and for the  $g_j$  coefficients

$$\begin{aligned}
 g_{j+1} &= \frac{q/2 + j}{j+1} \frac{a^2}{q+a^2} g_j, \\
 g_{j-1} &= \frac{j}{q/2 + j - 1} \frac{q+a^2}{a^2} g_j.
 \end{aligned}$$

The coefficients  $g_j$  are the probabilities of obtaining the value  $j$  for a variate following a negative binomial distribution with parameters  $q/(q+a^2)$  and  $q/2$ . The mode is  $[a^2(q-2)/(2q)]$  (where  $[\cdot]$  denotes the integer part), hence the starting index for iterations. The stopping rule is the same as in the case of the  $K$ -prime.

## 4 Limitations and possible improvements

Drawbacks of Method 1 algorithms (possible underflows and an exceeding number of iterations) led to the development of Method 2 algorithms. In Method 1, the iterations start at index  $j = 0$  which maximizes  $H_j(x)$ , while in Method 2 they start at index  $j = k$  which maximizes  $g_j$ .

In Tables 1 and 2, we compare the number of iterations for these two methods, as applied respectively to the  $K$ -prime and  $K$ -square cdfs for various situations and with a precision set to  $10^{-4}$ . The ten first examples in Table 2 correspond to those in Table 1 of Benton and Krishnamoorthy (2003) for the distribution of the square of the sample multiple correlation coefficient. More precisely, the correspondence is as follows: the sampling distribution of the multiple correlation  $R^2$ , involving a sample of  $n$  independent observations from a  $m$ -variate normal population with square multiple correlation coefficient  $\rho^2$ , is such that

$$\frac{n-m}{r-1} \frac{R^2}{1-R^2} \mid \rho^2 \sim K_{m-1, n-1, n-m}^2 \left( (m-1) \frac{\rho^2}{1-\rho^2} \right).$$

One last example has been added, corresponding to  $r^2 = 0.33$ ,  $\rho^2 = 0.50$ ,  $m = 5$ ,  $n = 100$ .

Of course, as soon as both methods attain at least  $2k$  iterations, they return identical results as the same terms are summed up (for instance, this is the case in the fifth example of Table 1). As can be seen, relatively to Method 1, Method 2 can indeed reduce the number of iterations by a great amount: more than 60% in most of Table 2 examples. When the precision criterion is turned to  $10^{-12}$  (as in Benton and Krishnamoorthy, 2003), the gain diminishes, naturally, but is still about 40%. However, it is also obvious that Method 2 is not systematically better. This can be seen in the last example of Table 2, and is especially clear in the case of the  $K$ -prime distribution (Table 1) where the number of iterations can be increased by more than 1000%. It's not surprising that Method 1 performs better when the noncentrality parameter is small, but it also happens when this parameter is higher, as in the case of the second and third examples of Table 1.

More generally, whenever  $H_k(x)$  tends to zero quickly with respect to  $k$ , Method 1 algorithms perform better than Method 2 algorithms, because only the first terms of the series (1) contribute significantly to the sum. And when  $H_k(x)$  is still close to  $H_0(x)$ , Method 2 is likely to be quasi optimum.

Furthermore, with Method 2, it can happen that the initial recurrence increment for the  $H_j$ 's is too small with respect to the machine limit so that a

Table 1

Comparison between Methods 1 and 2 for the  $K$ -prime cdf algorithm.  $M_i$  is number of iterations for Method  $i$ . Gain is the gain, in percentage, of Method 1 over Method 2, a negative number indicates Method 1 performs better.

$x$	$q$	$r$	$a$	$\Pr(K'_{p,q,r}(a) < x)$	M1	M2	gain
1	5	20	10	0.0007	9	119	-1222%
11	5	20	50	0.0017	332	2999	-803%
40	50	50	50	0.0612	2892	4799	-60%
40	50	5	50	0.4277	4387	4799	-9%
50	50	20	30	0.5242	1844	1844	0%
40	100	5	50	0.1783	3644	3224	12%
45	100	10	40	0.6377	2499	2084	17%
65	1000	15	50	0.8820	3007	1052	65%

Table 2

Comparison between methods 1 and 2 for the  $K$ -square cdf algorithm.  $M_i$  is number of iterations for Method  $i$ . Gain is the gain, in percentage, of Method 1 over Method 2, a negative number indicates Method 1 performs better.

$x$	$p$	$q$	$r$	$a^2$	$\Pr(K^2_{p,q,r}(a^2) < x)$	M1	M2	gain
36	2	20	18	46.667	0.7771	57	57	0%
0.19444	4	11	7	4.7143	0.0126	3	3	0%
288	3	99	96	891	0.4382	618	598	3%
972	11	1199	1188	10791	0.4339	5953	1844	69%
795.2	5	999	994	3996	0.4661	2246	796	65%
475.2	5	599	594	2396	0.4562	1390	624	65%
715.2	5	899	894	3596	0.4643	2033	756	63%
202.909	11	1499	1488	2248.5	0.4297	1252	420	66%
216.545	11	1599	1588	2398.5	0.4319	1331	433	67%
223.364	11	1649	1638	2473.5	0.4330	1371	439	68%
11.6978	4	99	95	99	0.0063	47	90	-91%

zero is returned and recurrence is impossible: e.g., for the  $K$ -square cdf, this increment term is lower than  $10^{-307}$  when  $p = 10, q = 20, r = 30, a^2 = 500$  and  $x = 0.1$ . So, both methods are subject to underflows, whether through the  $g_j$ 's (Method 1) or whether through the  $H_j(x)$ 's (Method 2).

A tempting solution, when  $H_k(x)$  is too small, would be to choose a modified index, say  $k'$ , such that  $H_{k'}(x)$  reaches a predetermined value (i.e. one

markedly above the machine limit); unfortunately, such an inversion of the beta cdf involves an iterative procedure and so is to be discarded on grounds of speed efficiency. An alternative solution is to lower  $k$  by some amount. This amount will depend, among others, on  $x$ . Given the parameters and  $j$ ,  $H_j(x)$  is an increasing function of the argument of the incomplete beta function, say  $z$ , that is itself a function of  $x$ . The lower  $H_j(x)$ , the more  $k$  has to be lowered. Thus, for sake of simplicity and as a first attempt, we propose to use the identity function on  $z$  so that  $k$  is simply lowered by multiplying it by the argument of the incomplete beta function ( $px/(px+r)$  for the  $K$ -square and  $x^2/(x^2+r)$  for the  $K$ -prime).

This modification avoids underflows in the preceding example. Furthermore, it sometimes permits to reduce the number of iterations. Thus, it could be introduced as soon as  $H_k(x)$  is below some arbitrary threshold (e.g., when  $H_k(x)/H_0(x) < 0.01$ ) and not only when a true underflow occurs. For instance, for the distribution  $K_{10,80,200}^2(500)$ , when  $x$  takes the values 35, 30, 20, and 10, the number of iterations is always 390 (for a precision of  $10^{-4}$ ), while when turning to the modified starting index, it drops respectively to 309, 291, 243 and 163. In the first example of Table 1, the modification leads to 9 iterations (instead of 119 with the unmodified version), just as Method 1. Obviously, it is not relevant when  $H_j(x)$  diminishes rather slowly with  $j$ , which is the case for the Table 2 examples, except the last one. In that last example, the modification leads again to the same number of iterations as Method 1 (47). Another example of reduction of iterations, concerning the  $K$ -prime distribution, is given in the next section.

Therefore, we could finally suggest the following tactic:

- (1) Calculate  $g_0H_0(x)$  and  $g_kH_k(x)$  and choose as the starting index (0 or  $k$ ) the one which leads to the maximum.
- (2) If 0 is chosen and recurrence is impossible, try  $k$ .
- (3) If  $k$  is chosen and recurrence is impossible (or if  $H_k(x)$  is very small compared to  $H_0(x)$ ), multiply it by the argument of the incomplete beta function (this can be repeated).

## 5 Examples of applications

### 5.1 Predictive probabilities

Suppose a simple two-sample experiment was designed to compare a new drug with a placebo. For this purpose, the investigators used a two-sample  $t$  test with equal numbers of subjects  $n_1 = 10$  in each group, in order to test

$H_0 : \delta = 0$  against the alternative  $H_1 : \delta > 0$ . Let us denote by  $m_1$  the sample mean difference in the current data and by  $s_1$  the pooled estimate of the common standard deviation  $\sigma$ . The observed  $t$  statistic was  $T_1 = 1.10$ , hence  $p = 0.143$  (one-tailed).

Let us consider a conjugate prior for  $(\mu, \sigma^2)$ , characterized by

$$\mu|\sigma^2 \sim N\left(m_0, \frac{2}{n_0}\sigma^2\right) \text{ and } \sigma^2 \sim s_0^2 \left(\frac{\chi_{q_0}^2}{q_0}\right)^{-1}.$$

Lecoutre (1999) demonstrated that the predictive distribution of the  $t$  test statistic for  $n$  future observations is a  $K$ -prime distribution

$$t \sim \sqrt{1 + \frac{n}{n_0}} K'_{q_0, 2n-2} \left( \frac{T_0}{\sqrt{1 + \frac{n_0}{n}}} \right) \quad \text{where } T_0 = \frac{m_0}{s_0} \sqrt{n_0/2}.$$

As a particular case, here the prior is the posterior distribution from the available data, starting with the usual noninformative prior  $p(\mu, \sigma^2) \propto 1/\sigma^2$ , hence  $m_0 = m_1$ ,  $s_0 = s_1$ ,  $n_0 = n_1$  and  $q_0 = 2n_1 - 2$ . We get for a replication ( $n = n_1 = 10$ ) the predictive distribution

$$t \sim \sqrt{2} K'_{18,18} \left( \frac{T_1}{\sqrt{2}} \right),$$

that only depends on the observed  $t$  test statistic  $T_1 = 1.10$  and its associated degrees of freedom.

We can compute the probability of finding a positive mean in a replication (Killeen's  $p_{rep}$ ) as

$$Pr\left(K'_{18,18}\left(\frac{1.10}{\sqrt{2}}\right) > 0\right) = 0.777.$$

We can also compute the predictive probability of a significant replication. For instance we find the probability 0.334 that the one-tailed  $p$  value will be less than 0.05 (i.e.  $t > 1.734$ ):

$$Pr\left(\sqrt{2} K'_{18,18}\left(\frac{1.10}{\sqrt{2}}\right) > 1.734\right) = Pr\left(K'_{18,18}\left(\frac{1.10}{\sqrt{2}}\right) > 1.226\right) = 0.334.$$

The investigators generally largely underestimate this probability: see Lecoutre and Rouanet (1993), Lecoutre (2000). Note that there is also a non negligible probability of finding a significant effect in the negative direction:  $Pr(\sqrt{2} K'_{18,18}(1.10)/\sqrt{2} < -1.734) = 0.027$ .

An example of application to sample size determination in clinical trials from a pilot study is given in Grouin et al (2007).

The predictive probabilities for  $F$  ratios and usual standardized effect size measures in ANOVA designs can be computed from the  $K$ -square distribution. Let us consider for instance the data of a one-way design with  $g$  groups of equal sample sizes  $n_0$ . Let  $F_0$ , the observed ANOVA  $F$  ratio for the overall comparison of the means. Assuming before the experiment the usual non informative prior, the posterior predictive distribution for the  $F$  ratio in a future experiment with equal sample sizes  $n$  is a  $K$ -square distribution (Lecoutre, 1999):

$$F \sim \frac{1 + \frac{n}{n_0}}{g - 1} K_{g-1, gn_0 - g, gn - g}^2 \left( \frac{g - 1}{1 + \frac{n_0}{n}} F_0 \right).$$

## 5.2 Distributions of correlation coefficients

Other applications of the  $K$ -prime and  $K$ -square distributions are exact inferences for correlation coefficients. The sampling distribution of the correlation coefficient  $r$ , involving a sample of  $n$  independent observations from a bivariate normal population with population coefficient  $\rho$ , is such that

$$\sqrt{n - 2} \frac{r}{\sqrt{1 - r^2}} \mid \rho \sim K'_{n-1, n-2} \left( \sqrt{n - 1} \frac{\rho}{\sqrt{1 - \rho^2}} \right),$$

so that exact tests and confidence limits for  $\rho$  can be computed from the  $K$ -prime cdf. For instance, when  $n = 250$  and assuming  $\rho = 0.80$ , the probability to observe a sample  $r$  lower than 0.75 is 0.0227. If this is calculated using the standard Method 2 algorithm with a precision of  $10^{-12}$ , 860 iterations are required, whereas only 595 are needed with the modification proposed in section 4 (if the precision is set to  $10^{-6}$ , the numbers of iterations become respectively 546 and 502).

Moreover, in the Bayesian framework, assuming a uniform prior for  $\rho$ , the posterior distribution is also a  $K$ -prime distribution.

The sampling distribution of the multiple correlation  $R^2$  has been presented in section 4.

## 6 Concluding remarks

We presented an algorithm for two Bayesian predictive distributions involved in the designing of experiments and in the computation of “the probability of replication” under usual normal models. Furthermore, we used these distributions to compare two available methods for computing cdfs that are expressed as discrete mixtures of continuous distributions (the incomplete beta function in our case). If in many cases the two methods are likely to perform equally well, it appeared that none of them is systematically better, depending, among others, upon the particular functions involved in the cdfs, and that they both suffer a comparable problem: due to underflows, the starting index of iterations can be such that recurrence is impossible. Method 2 was proposed to avoid Method 1 underflows, and here we proposed to manage Method 2 underflows by lowering the starting index by a quantity which is the argument of the incomplete beta function. This is a tentative solution that can be viewed as a crude approach to the problem of finding the optimum starting index.

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## References

- Baum, M., Houghton, J., and Abrams, K. R., 1989. Early stopping rules: clinical perspectives and ethical considerations. *Stat. Med.* 13, 1459-1469.
- Benton, D., Krishnamoorthy, K., 2003. Computing discrete mixtures of continuous distributions: noncentral chisquare, noncentral  $t$  and the distribution of the square of the sample multiple correlation coefficient. *Comput. Stat. Data An.* 43, 249-267.
- Berry, D., 2005. Introduction to Bayesian methods III: use and interpretation of Bayesian tools in design and analysis. *Clin. Trials.* 2, 295-300.
- Dmitrienko, A., Wang, M. D., 2006. Bayesian predictive approach to interim monitoring in clinical trials. *Stat. Med.* 25, 2178-2195.
- Frick, H., 1990. A remark on Algorithm AS 226: Computing noncentral beta probabilities. *Appl. Statist.* 36, 311-312.
- Grouin J.-M., Coste M., Bunouf P., Lecoutre B., 2007. Bayesian sample size determination in non-sequential clinical trials: Statistical aspects and some regulatory considerations. *Stat. Med.*, 26, 4914-4924.

- Inoue, L. Y. T., Berry, D. A., and Parmigiani, G., 2005. Relationship between Bayesian and frequentist sample size determination. *Am. Stat.* 59, 79-87.
- Johns, D. and Andersen, J. S., 1999. Use of predictive probabilities in phase II and phase III clinical trials. *J. Bioph. Stat.* 9, 67-79.
- Killeen, P. R., 2005. An alternative to null-hypothesis significance tests. *Psych. Sci.* 16, 345-353.
- Lecoutre, B., 1984. *L'Analyse Bayésienne des Comparaisons*. Lille (FR): Presses Universitaires de Lille.
- Lecoutre, B., 1999. Two useful distributions for Bayesian predictive procedures under normal models. *J. Statist. Plann. Inference* 79, 93-105.
- Lecoutre, B., 2001. Bayesian predictive procedure for designing and monitoring experiments. In *Bayesian Methods with Applications to Science, Policy and Official Statistics*, Luxembourg: Office for Official Publications of the European Communities, 301-310.
- Lecoutre, B., 2008. Bayesian methods for experimental data analysis. In C. R. Rao, J. Miller & D. C. Rao (Eds.), *Handbook of statistics: Epidemiology and Medical Statistics (Vol 27)*, Amsterdam: Elsevier, 775-812.
- Lecoutre, B., Guigues, J.-L., Poitevineau, J., 1992. Distribution of quadratic forms of multivariate Student variables. *Appl. Statist.* 41, 617-627.
- Lecoutre, B., Lecoutre, M.-P. and Poitevineau, J., 2008. Killeen's probability of replication and predictive probabilities: How to compute and use them. Submitted for publication.
- Lecoutre, M.-P., 2000. And... What about the researcher's point of view. In H. Rouanet, J.-M. Bernard, M.-C. Bert, B. Lecoutre, M.-P. Lecoutre & B. Le Roux, *New ways in statistical methodology: from significance tests to Bayesian inference* (2nd edition), 65-95, Bern (SW): Peter Lang.
- Lecoutre, M.-P. and Rouanet H., 1993. Predictive judgments in situations of statistical analysis. *Org. Behav. Hum. Dec. Proc.* 54, 45-56.
- Spiegelhalter, D. J., Freedman, L. S., and Parmar, M. K. B., 1994. Bayesian approaches to randomized trials. *J. R. Stat. Soc. A* 157, 357-416.
- Student, 1908. The probable error of a mean. *Biometrika* 6, 1-25.