Higher-order spin effects in the dynamics of compact binaries II. Radiation field

Luc Blanchet[a], Alessandra Buonanno[b,c,a] and Guillaume Faye[a]

[a] *$\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$, Institut d'Astrophysique de Paris,*
UMR 7095 CNRS Université Pierre & Marie Curie,
98[bis] boulevard Arago, 75014 Paris, France
[b] *Department of Physics, University of Maryland, College Park, MD 20742*
[c] *AstroParticule et Cosmologie (APC), UMR 7164-CNRS,*
11, place Marcellin Berthelot, 75005 Paris, France

Abstract

Motivated by the search for gravitational waves emitted by binary black holes, we investigate the gravitational radiation field of point particles with spins within the framework of the multipolar-post-Newtonian wave generation formalism. We compute: (i) the spin-orbit (SO) coupling effects in the binary's mass and current quadrupole moments one post-Newtonian (1PN) order beyond the dominant effect, (ii) the SO contributions in the gravitational-wave energy flux and (iii) the secular evolution of the binary's orbital phase up to 2.5PN order. Crucial ingredients for obtaining the 2.5PN contribution in the orbital phase are the binary's energy and the spin precession equations, derived in paper I of this series. These results provide more accurate gravitational-wave templates to be used in the data analysis of rapidly rotating Kerr-type black-hole binaries with the ground-based detectors LIGO, Virgo, GEO 600 and TAMA300, and the space-based detector LISA.

PACS numbers: 04.30.-w, 04.25.-g

I. INTRODUCTION

The aim of this paper is to derive the spin-orbit coupling terms in the gravitational radiation field of compact binary systems one post-Newtonian (1PN) [1] order beyond the dominant effect. Paper I of this series [1] dealt with the problem of the spin-orbit contributions in the compact binary equations of motion at 1PN relative order.

Our motivation is the on going search for gravitational waves (GWs) emitted by inspiralling binary systems of *spinning*, and possibly maximally spinning, black holes in the network of detectors LIGO (Laser Interferometer Gravitational Wave Observatory), Virgo, GEO 600 and TAMA300, and the future search with the space-based detector LISA. When the Kerr black holes are maximally spinning (or close to maximal), the GW templates need to take into account the effects of spins, not only for an accurate parameter estimation [2–7], but also for a successful detection [8–19]. Furthermore, spin effects should be included at PN orders beyond the currently known dominant spin-orbit and spin-spin terms. The contributions of spins are added to the templates developed for the case of non-spinning binary black holes or neutron stars, and which are currently known at 3.5PN order [20–23]. The spins represent some of the possible effects depending on the internal structure of the bodies which can be numerically important in the LIGO/Virgo bandwidth. This is true even if we observe only the inspiral phase of moderate-mass black holes with individual mass less than 10 $M_\odot$. Within the PN formalism the compact objects are treated as point particles. It is then natural to model spinning black holes as point particles with spins.

The equations of motion including the spin-orbit (SO) effect were obtained in paper I at 1PN relative order, which corresponds formally to the 2.5PN order beyond the Newtonian force law in the case of maximally rotating compact objects. Paper I essentially confirmed the equations of motion derived previously by Tagoshi, Ohashi and Owen [24]. Furthermore, paper I derived the complete set of Noetherian conserved integrals of motion at that order (namely, 2.5PN for the spins). In the present paper, we tackle the problem of the gravitational radiation field at the same 2.5PN order, using the multipolar PN wave generation formalism of Refs. [25–30]. More precisely, we shall compute here the SO contributions in the compact binary's *mass-type* and *current-type* quadrupole moments, both of them with 1PN relative accuracy. These moments, together with some easily computed higher multipole moments which necessitate only the lowest-order precision, are necessary to compute the total GW energy flux $\mathcal{F}$. The computation of the *current* quadrupole moment was previously attempted in Ref. [31], but we shall point out two important flaws in that reference (see below for details). Our result for the current quadrupole moment is substantially different from that of [31]. Concerning the *mass* quadrupole moment, it is computed here for the first time. Having in hand the total energy flux, using the center-of-mass energy E computed in paper I, we deduce (by energy balance arguments) the equation of secular evolution of the binary's orbital frequency. The latter is the crucial ingredient needed to build GW templates for spinning compact binaries.

To describe particles with spins we use the formalism originally developed in Ref. [32–35] (an effective field theory scheme has recently been proposed [36–38]). This formalism has already been successfully applied to the problem of spinning compact binaries in Refs. [24, 31, 39–43], and, in the test-mass limit case, in Refs. [44, 45]. In particular, Kidder, Will and

[1] By nPN we refer to the terms of relative order $(v/c)^{2n}$ where v is the binary's orbital velocity and c the speed of light.

Wiseman derived in Ref. [39, 40] the lowest-order spin-coupling effects — at 1.5PN order in the case of maximal Kerr black holes —, and the first spin-spin effect, quadratic in the spins — appearing at 2PN order —, in the equations of motion and the gravitational radiation field. As we shall see below we find complete agreement at that order with their results. The present paper together with paper I extend therefore the works [39, 40] to include the next-order spin effects. Since those effects, of 2.5PN order, are linear in the spins (the next-order spin-spin term coming along at 3PN order), we complete the derivation of all the spin contributions in the GW form up to 2.5PN order.

The paper is organized as follows. In Sec. II we review the general formalism for wave generation from arbitrary PN sources. In Sec. III we compute the multipole moments of compact binary systems at the lowest PN level in the spins (which means 1.5PN for mass moments, and 0.5PN for current ones). Sec. IV constitutes the core of the paper. We compute there the mass and current quadrupole moments at 1PN relative order, *i.e.* 2.5PN and 1.5PN for the mass and current types, respectively. In particular, the crucial contribution of non-compact-support terms (which are sourced by the gravitational field itself) is obtained. The final results for the multipole moments and the GW flux are presented in Sec. V, and in Sec. VI, we reduce those results to the physically relevant case of quasi-circular orbits. In Sec. VII, we express the main equations defining the binary evolution and the GW signal in terms of spin variables with constant magnitude, which generalize those used in Refs. [39, 40]. In Sec. VIII, relying notably on the results of paper I, we obtain the secular evolution of the orbital frequency in the case of circular orbits and discuss some implications for ground-based and space-based detectors. Section IX summarizes our main conclusions.

All the notations and conventions are the same as in paper I. Notably, in order to explicitly display the powers of $1/c$ appropriate for maximally spinning compact objects — in which case the spin variable is formally of the order 0.5PN $\sim 1/c$ —, it is convenient to adopt as basic spin variable a quantity having the dimension of an angular momentum multiplied by c, namely $S = c\,S^{\text{true}}$ (see paper I for discussion). The precise definition of the spin contravariant vector we use and the supplementary condition it satisfies are given in Sec. II of paper I. See the same reference for the details concerning the stress-energy tensor of spinning point particles.

II. POST-NEWTONIAN SOURCE MULTIPOLE MOMENTS

We start with a brief review of the PN multipole moment formalism at the basis of this approach (full details can be found in Refs. [21, 28–30]). This formalism is valid for general localized material sources, satisfying the usual PN requirements of weak self-gravity, slow motion and weak internal stresses. In particular, the size of the source a has to be small with respect to the typical (reduced) wavelength $\lambda\!\!\!^{-}$ of the gravitational radiation this source produces, *i.e.* $a/\lambda\!\!\!^{-} = \mathcal{O}(\epsilon)$, with $\epsilon \sim v/c$ being the slowness PN parameter. We shall abbreviate it as $\epsilon = 1/c$ and denote the PN remainder terms by $\mathcal{O}(1/c^n)$ henceforth.

Let $x^\mu = (c\,t, \mathbf{x})$ be an harmonic coordinate system covering the whole material source. We pose $h^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}$ where $g^{\mu\nu}$ and g are the inverse and the determinant of the usual covariant metric $g_{\mu\nu}$, and where $\eta^{\mu\nu}$ denotes an auxiliary Minkowskian background metric (Greek letters represent space-time indices; our signature is +2). The Einstein field equations, relaxed by the harmonic-coordinate condition, $\partial_\nu h^{\mu\nu} = 0$, take the form

$$\Box\, h^{\mu\nu} = \frac{16\pi G}{c^4}\,\tau^{\mu\nu} \equiv \frac{16\pi G}{c^4}\,|g|\,T^{\mu\nu} + \Lambda^{\mu\nu}\,[h]\,, \tag{2.1}$$

where $\Box \equiv \eta^{\mu\nu}\partial_{\mu\nu}$ is the flat d'Alembertian, and where the second equality defines the stress-energy *pseudo tensor* $\tau^{\mu\nu}$ of the matter and gravitational fields in harmonic coordinates. Here $T^{\mu\nu}$ is the stress-energy tensor of the matter fields and $\Lambda^{\mu\nu}\,[h]$ represents the gravitational source term, namely a complicated non-linear functional of $h^{\rho\sigma}$ and its space-time derivatives $\partial_\lambda h^{\rho\sigma}$ (see *e.g.* [21] for the expression). We shall see that $\Lambda^{\mu\nu}$ gives a crucial contribution to the multipole moments at the relative 1PN order in the spins. The stress-energy pseudo tensor is conserved by virtue of the harmonic-coordinate condition,

$$\partial_\nu h^{\mu\nu} = 0 \quad\Longrightarrow\quad \partial_\nu \tau^{\mu\nu} = 0\,. \tag{2.2}$$

The multipole moments of the source are generated by the components of the pseudo tensor $\tau^{\mu\nu}$ or, more precisely, of its formal PN expansion $\overline{\tau}^{\mu\nu} \equiv \mathrm{PN}[\tau^{\mu\nu}]$. In this sense, the formalism is physically valid for PN sources only. The PN expansion $\overline{\tau}^{\mu\nu}$ has a special structure which can be matched to the exterior multipolar field of a PN source, allowing one to define an appropriate notion of PN multipole moments [28–30]. It is convenient to define (Latin letters representing space indices)

$$\Sigma \equiv c^{-2}\left[\overline{\tau}^{00} + \overline{\tau}^{ii}\right] \quad \text{where } \overline{\tau}^{ii} = \delta_{ij}\overline{\tau}^{ij}\,, \tag{2.3a}$$
$$\Sigma_i \equiv c^{-1}\,\overline{\tau}^{0i}\,, \tag{2.3b}$$
$$\Sigma_{ij} \equiv \overline{\tau}^{ij}\,. \tag{2.3c}$$

The mass-type moments $I_L(t)$ and current-type ones $J_L(t)$ are referred to as the *source* multipole moments in order to distinguish them from the so-called *radiative* moments, seen at infinity and generally denoted $U_L(t)$ and $V_L(t)$. They are given by [2]

$$\begin{aligned} I_L(t) &= \mathop{\mathrm{FP}}_{B=0} \int d^3\mathbf{x}\,|\mathbf{x}|^B \int_{-1}^{1} dz \left\{ \delta_\ell(z)\,\hat{x}_L\,\Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\,\delta_{\ell+1}(z)\,\hat{x}_{iL}\,\dot{\Sigma}_i \right. \\ &\qquad \left. + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\,\delta_{\ell+2}(z)\,\hat{x}_{ijL}\ddot{\Sigma}_{ij} \right\}(\mathbf{x}, t + z|\mathbf{x}|/c)\,, \end{aligned} \tag{2.4a}$$

$$\begin{aligned} J_L(t) &= \mathop{\mathrm{FP}}_{B=0}\, \varepsilon_{ab\langle i_\ell} \int d^3\mathbf{x}\,|\mathbf{x}|^B \int_{-1}^{1} dz \left\{ \delta_\ell(z)\,\hat{x}_{L-1\rangle a}\,\Sigma_b \right. \\ &\qquad \left. - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\,\delta_{\ell+1}(z)\,\hat{x}_{L-1\rangle ac}\,\dot{\Sigma}_{bc} \right\}(\mathbf{x}, t + z|\mathbf{x}|/c)\,. \end{aligned} \tag{2.4b}$$

These expressions are general, in the sense that they are formally valid at any PN order, so that there are no remainder terms $\mathcal{O}(c^{-n})$ involved. The dots indicate the time derivatives

[2] In our notation, $L \equiv i_1 \cdots i_\ell$ represents a multi-index composed of ℓ multipolar indices $i_1, \cdots, i_\ell$, and $x_L \equiv x_{i_1} \cdots x_{i_\ell}$ stands for the product of ℓ spatial vectors $x^i \equiv x_i$. We denote the symmetric-trace-free (STF) projection by means of a hat over the tensor symbol, $\hat{x}_L \equiv \mathrm{STF}(x_{i_1} \cdots x_{i_\ell})$, or of brackets $\langle\rangle$ surrounding the indices, $\hat{x}_L \equiv x_{\langle L\rangle}$. When the multi-indices L are summed up (dummy indices), we omit to write the ℓ summation symbols from 1 to 3 over their indices.

and ε_{abc} is the Levi-Civita symbol in 3 dimensions with $\varepsilon_{123} = 1$. Notice the peculiar feature that, besides the usual spatial integration, the moments involve an extra integral over a variable z defining a "cone" of integration $u = t + z|\mathbf{x}|/c$; hence the sources $\Sigma_{\mu\nu}$ depend on the point $(\mathbf{x}, u)$ as indicated. The "weighting" function associated with the z-integral, $\delta_\ell(z) = \frac{(2\ell+1)!!}{2^{\ell+1}\ell!}\,(1-z^2)^\ell$, is normalized in such a way that $\int_{-1}^{1} dz\,\delta_\ell(z) = 1$. When performing explicitly the PN expansion of the moments, the z-integration is to be transformed into an infinite *local* series (in the sense that z is integrated out), which constitutes the basis of the practical evaluation of the multipole moments. Namely, we have

$$\int_{-1}^{1} dz\ \delta_\ell(z)\,\Sigma(\mathbf{x}, t + z|\mathbf{x}|/c) = \sum_{k=0}^{+\infty} \frac{(2\ell+1)!!}{(2k)!!(2\ell+2k+1)!!} \left(\frac{|\mathbf{x}|}{c}\frac{\partial}{\partial t}\right)^{2k} \Sigma(\mathbf{x}, t)\,. \tag{2.5}$$

The above expression is then inserted into the right-hand side (RHS) of Eqs. (2.4), and truncated to fit with the PN order of the calculation.

A crucial finite part (FP) procedure is involved in the definition of the source multipole moments (2.4). It consists of (i) multiplying the integrand of the moments by a regularization factor $|\mathbf{x}|^B$ where B is a complex number,[3] (ii) performing the Laurent expansion when B tends to the "physical" value $B = 0$, and (iii) picking up its finite part FP, namely the coefficient of the zeroth power of B. The finite part regularization is therefore equivalent to removing the poles B^{-1}, B^{-2}, ... (in the analytically continued B-dependent integral) before taking the limit $B \to 0$. The FP procedure is needed to compute the non-linear contributions to the moments (generated by the gravitational source term $\Lambda^{\mu\nu}$), which have a non-compact support extending up to spatial infinity. Notice that no assumption nor physical restriction (in principle) is involved, in the latter FP procedure, which has been proved [28–30] to yield the correct expression of the multipole moments for general extended PN sources. It is precisely the FP that guaranties this within the present formalism; the divergent terms B^{-1}, B^{-2}, ... have no direct physical significance. Such FP when $B \to 0$ is to be carefully distinguished from the self-field regularization (*e.g.*, Hadamard's or dimensional regularization) which is to be invoked when treating the application of the general formalism to singular point-particle sources.

We emphasize that the expressions (2.4) constitute *a priori* only a *definition* of the source multipole moments. The point is that such definition is fully related to the physical asymptotic wave form at infinity from the source, which is computed using the multipolar post-*Minkowskian* formalism [25, 26]. In particular, the *radiative* multipole moments $U_L(t)$ and $V_L(t)$ which parameterize the asymptotic wave form are given by some non-linear functionals of the source moments (2.4), made of many non-linear interactions between them, including the famous GW tails corresponding to the interaction between these moments and the total monopole mass M of the source. The tails have been computed within this approach in Refs. [46, 47]. However, for deriving the spin terms at 1PN relative order, all these non-linear multipole interactions, notably all the tails, are negligible (see further discussion below). It is therefore sufficient to consider only the source multipole moments of Eqs. (2.4).

[3] Generally the regularization factor is taken to be $|\mathbf{x}/r_0|^B$ where r_0 denotes some arbitrary constant length scale. Here we set $r_0 = 1$ for convenience.

III. LOWEST-ORDER SPIN EFFECTS IN THE MULTIPOLE MOMENTS

The multipole moments discussed above can be applied to any source. Here, we specialize them to binary systems of point-particles with spins. The formalism to describe particles with spins was developed in Refs. [32–35], and constitutes the basis of most subsequent computations in this field [24, 31, 39–41, 44, 45]. We reviewed this formalism in paper I and refer the reader to this paper for details and notation. The spin contribution (marked by the underneath label S) to the stress-energy tensor of the particles reads

$$\underset{\rm S}{T}^{\mu\nu}(t,\mathbf{x}) = -\frac{1}{c}\,\nabla_\rho\left[S_1^{\rho(\mu}\,v_1^{\nu)}\frac{\delta(\mathbf{x}-\mathbf{y}_1)}{\sqrt{-g_1}}\right] + 1\leftrightarrow 2\,, \tag{3.1}$$

where δ is the Dirac three-dimensional delta-function, ∇_ρ denotes the covariant derivative, $v_1^\mu(t) = (c, v_1^i)$ with $v_1^i = dy_1^i/dt$ being the coordinate velocity of particle 1, g_1 is the determinant of the metric evaluated at the location of particle 1 (following Hadamard's self-field regularization), and $1 \leftrightarrow 2$ means the same expression as preceding, but for particle 2. The anti-symmetric spin tensor $S_1^{\mu\nu}(t)$ is introduced in Sec. II of paper I. The covariant four-vector S_μ^1 is defined by $S_1^{\mu\nu} = -\frac{1}{\sqrt{-g_1}}\varepsilon^{\mu\nu\rho\sigma}u_\rho^1 S_\sigma^1$; it is transverse to the particle's four-velocity, $S_\mu^1 u_1^\mu = 0$. All results below are expressed in terms of some particular space-like contravariant spin variables for each of the particles, namely S_1^i and S_2^i, the definition of which can be found in Eq. (2.19) of paper I.

Similarly to the quantities $\Sigma_{\mu\nu}$ introduced in Eq. (2.3), we define the following matter-source densities, depending on the components of the spin stress-energy tensor (3.1):

$$\underset{\rm S}{\sigma} \equiv c^{-2}\left[\underset{\rm S}{T}^{00} + \underset{\rm S}{T}^{ii}\right] \quad \text{with } \underset{\rm S}{T}^{ii} = \delta_{ij}\,\underset{\rm S}{T}^{ij}\,, \tag{3.2a}$$

$$\underset{\rm S}{\sigma}_i \equiv c^{-1}\,\underset{\rm S}{T}^{0i}\,, \tag{3.2b}$$

$$\underset{\rm S}{\sigma}_{ij} \equiv \underset{\rm S}{T}^{ij}\,. \tag{3.2c}$$

They are such that their "non-spin" counterparts (say ${}_{\rm NS}\sigma_{\mu\nu}$) admit a finite non-zero limit when $c \to +\infty$. They read:

$$\underset{\rm S}{\sigma} = -\frac{2}{c^3}\,\varepsilon_{ijk}\,v_1^i\,S_1^j\,\partial_k\delta_1 + 1\leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^5}\right)\,, \tag{3.3a}$$

$$\underset{\rm S}{\sigma}_i = -\frac{1}{2c}\,\varepsilon_{ijk}\,S_1^j\,\partial_k\delta_1 + 1\leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right)\,, \tag{3.3b}$$

$$\underset{\rm S}{\sigma}_{ij} = -\frac{1}{c}\,\varepsilon_{kl(i}\,v_1^{j)}\,S_1^k\,\partial_l\delta_1 + 1\leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right)\,, \tag{3.3c}$$

where we denote $\delta_1 \equiv \delta(\mathbf{x}-\mathbf{y}_1)$ and where $\partial_k\delta_1$ means the gradient of δ_1 with respect to the field point $\mathbf{x} = (x^k)$. As shown by Eqs. (3.3), the leading order of the vector and tensor densities ${}_{\rm S}\sigma_i$ and ${}_{\rm S}\sigma_{ij}$ is 0.5PN $\sim 1/c$. However, the scalar density ${}_{\rm S}\sigma$ starts at a higher level, being at least 1.5PN $\sim 1/c^3$. At leading order in the spins, the $\Sigma_{\mu\nu}$'s, which depend on the contributions of both matter and gravitational fields according to Eqs. (2.1) and (2.3), will reduce to their compact-support material parts, namely the ${}_{\rm S}\sigma_{\mu\nu}$'s given by Eqs. (3.3). Indeed, the non-compact support gravitational part, whose origin lies in the source term

$\Lambda^{\mu\nu}$ present in the RHS of the field equations (2.1), always appears at a sub-dominant level, $1/c^2$ beyond the leading PN order. Hence,

$$\underset{\mathrm{S}}{\Sigma} = \underset{\mathrm{S}}{\sigma} + \mathcal{O}\left(\frac{1}{c^5}\right) , \tag{3.4a}$$

$$\underset{\mathrm{S}}{\Sigma}_i = \underset{\mathrm{S}}{\sigma}_i + \mathcal{O}\left(\frac{1}{c^3}\right) , \tag{3.4b}$$

$$\underset{\mathrm{S}}{\Sigma}_{ij} = \underset{\mathrm{S}}{\sigma}_{ij} + \mathcal{O}\left(\frac{1}{c^3}\right) . \tag{3.4c}$$

The non-compact-support gravitational source terms play a role in our computations at the next-to-leading order only (see Sec. IV). We conclude that the dominant contribution to the multipole moments (2.4) due to the spins is given by

$$\underset{\mathrm{S}}{I}_L = \int d^3\mathbf{x} \left\{ \hat{x}_L \underset{\mathrm{S}}{\sigma} - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \hat{x}_{iL} \underset{\mathrm{S}}{\dot{\sigma}}_i \right\} + \mathcal{O}\left(\frac{1}{c^5}\right) , \tag{3.5a}$$

$$\underset{\mathrm{S}}{J}_L = \varepsilon_{ab\langle i_\ell} \int d^3\mathbf{x}\, \hat{x}_{L-1\rangle a} \underset{\mathrm{S}}{\sigma}_b + \mathcal{O}\left(\frac{1}{c^3}\right) . \tag{3.5b}$$

Thus, the dominant order is 1.5PN $\sim 1/c^3$ for spins in the mass-type moments I_L, but only 0.5PN $\sim 1/c$ in the current-type ones J_L. It is then evident (mathematically and physically) that the spin part of the current moments always dominate over that of the mass moments. We insert the explicit values (3.3) into Eqs. (3.5), integrate in a straightforward way (resorting to an integration by parts and using the properties of the delta-function) and get

$$\underset{\mathrm{S}}{I}_L = \frac{2\ell}{c^3(\ell+1)} \left[\ell\, v_1^i\, S_1^j\, \varepsilon_{ij\langle i_\ell}\, y_1^{L-1\rangle} - (\ell-1)\, y_1^i\, S_1^j\, \varepsilon_{ij\langle i_\ell}\, v_1^{i_{\ell-1}}\, y_1^{L-2\rangle} \right] + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^5}\right) , \tag{3.6a}$$

$$\underset{\mathrm{S}}{J}_L = \frac{\ell+1}{2c}\, y_1^{\langle L-1}\, S_1^{i_\ell\rangle} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right) . \tag{3.6b}$$

It is worth to mention that in this calculation, limited to the lowest PN order, the spins can be considered as constant since their time variations, as given by the precessional equations (see paper I), are always smaller by a factor $1/c^2$ at least.

IV. HIGHER-ORDER SPIN EFFECTS IN THE MULTIPOLE MOMENTS

For the present purpose, we need the spin contributions to the mass-quadrupole moment I_{ij} and the current-quadrupole moment J_{ij} one PN order beyond the leading terms obtained in Eqs. (3.6). This means at 2.5PN order for I_{ij} and at 1.5PN order for J_{ij}. As said previously, the non-linear gravitational source terms, with non-compact support, start playing a role at the 1PN relative order.[4] Therefore, they do make a net contribution to the spin

[4] The rule admits some exceptions, though. For instance, in the non-spinning part of the mass multipole moments I_L, one may have expected the non-linear non-compact gravitational terms to appear at 1PN

parts of both I_{ij} at 2.5PN order and J_{ij} at 1.5PN order. We can note here that the authors of Ref. [31] computed J_{ij} at 1.5PN order *but* neglected all the non-compact source terms in their calculation, thus obtaining an incorrect result.

We now reduce the multipole moments to the required PN order by inserting the expansion formula (2.5) into the general expressions (2.4). Neglecting PN terms of higher order, as indicated, we have

$$I_{ij} = \mathop{\mathrm{FP}}_{B=0} \int d^3\mathbf{x}\, |\mathbf{x}|^B \Big\{ \hat{x}_{ij}\,\Sigma + \frac{1}{14c^2}\hat{x}_{ij}|\mathbf{x}|^2\,\ddot{\Sigma} + \frac{1}{504c^4}\hat{x}_{ij}\,|\mathbf{x}|^4\,\dddot{\Sigma} - \frac{20}{21c^2}\,\hat{x}_{ijk}\,\dot{\Sigma}_k - \frac{10}{189c^4}\,\hat{x}_{ijk}\,|\mathbf{x}|^2\,\dddot{\Sigma}_k + \frac{5}{54c^4}\,\hat{x}_{ijkl}\ddot{\Sigma}_{kl}\Big\} + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.1a}$$

$$J_{ij} = \mathop{\mathrm{FP}}_{B=0}\, \varepsilon_{ab\langle i} \int d^3\mathbf{x}\, |\mathbf{x}|^B \Big\{ \hat{x}_{j\rangle a}\,\Sigma_b + \frac{1}{14c^2}\hat{x}_{j\rangle a}\,|\mathbf{x}|^2\,\ddot{\Sigma}_b - \frac{5}{28c^2}\,\hat{x}_{j\rangle ac}\,\dot{\Sigma}_{bc}\Big\} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{4.1b}$$

where FP denotes the essential process of extracting the finite part, as explained in Sec. II. We can further reduce Eqs. (4.1) by substituting the appropriate expressions of the source terms $\Sigma_{\mu\nu}$ as given by Eqs. (4.11) of Ref. [28]. For completeness, we list all the necessary expressions below:

$$\begin{aligned}\Sigma = {} & \left[1 + \frac{4V}{c^2} + \frac{2}{c^4}(\hat{W} + 4V^2)\right]\sigma - \frac{1}{\pi G c^2}\,\partial_i V \partial_i V \\ & + \frac{1}{\pi G c^4}\Big\{ -V\partial_t^2 V - 2V_i\partial_t\partial_i V - \hat{W}_{ij}\partial_{ij}V - \frac{1}{2}(\partial_t V)^2 \\ & \qquad + 2\partial_i V_j \partial_j V_i - \partial_i V \partial_i \hat{W} - \frac{7}{2} V\partial_i V \partial_i V \Big\} + \mathcal{O}\left(\frac{1}{c^6}\right),\end{aligned} \tag{4.2a}$$

$$\Sigma_i = \left[1 + \frac{4V}{c^2}\right]\sigma_i + \frac{1}{\pi G c^2}\left\{\partial_k V(\partial_i V_k - \partial_k V_i) + \frac{3}{4}\partial_t V \partial_i V\right\} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{4.2b}$$

$$\Sigma_{ij} = \sigma_{ij} + \frac{1}{4\pi G}\left\{\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V\right\} + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{4.2c}$$

where the material source densities are given by $\sigma = c^{-2}\,[T^{00} + T^{ii}]$, $\sigma_i = c^{-1}\,T^{0i}$ and $\sigma_{ij} = T^{ij}$. The non-compact support terms in Eqs. (4.2) are parameterized by a particular set of "elementary" potentials V, V_i, $\hat{W}_{ij}$, ..., which enter the harmonic-coordinate near zone metric at the 2PN order computed in Ref. [48].[5] Their complete expressions are given in Sec. III of paper I.

We can simplify Σ substantially by using some identities of the type $\partial_i A \partial_i B = \frac{1}{2}\left[\Delta\,(AB) - A\Delta B - B\Delta A\right]$, replacing the Laplacians ΔA and ΔB by their PN sources, and disregarding the pure Laplacian term $\frac{1}{2}\Delta\,(AB)$ because it makes zero contribution to the moment. This last point comes essentially from the fact that, after integration by parts, a pure Laplacian term in the moment sources is equivalent to a source term proportional to

order, but these terms turn out to be in the form of a pure divergence and can be integrated out to zero. As a result, the non-compact terms of the non-spinning parts contribute to I_L at 2PN order only (and at 1PN order to J_L). See Refs. [27, 28] for details.

[5] The potential called W_{ij} in Ref. [28] differs from the present $\hat{W}_{ij}$ (whose definition is given in paper I) according to the formula $W_{ij} = \hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}$, hence $W = W_{ii} = -\frac{1}{2}\hat{W}$.

some $\Delta\hat{x}_L = 0$. Beware that, because of the presence of the finite part FP, such a "proof" is not correct in general and, in fact, the pure Laplacian terms do generally contribute at high PN orders. However, for the terms under concern, merely at the 1PN relative order, the argument can be made rigorous and shown to work, so that we can indeed discard these pure Laplacians in the present computation (see Ref. [28] for the proof). Hence, we get the simpler formula,

$$\Sigma' = \sigma + \frac{4V}{c^4}\sigma_{ii} + \frac{1}{\pi G c^4}\left\{-2V_i\partial_t\partial_i V - \hat{W}_{ij}\partial_{ij}V - \frac{1}{2}(\partial_t V)^2 + 2\partial_i V_j \partial_j V_i\right\} + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.3}$$

which differs from Σ by pure Laplacian terms of the type $\sim \Delta\,(AB)$. We have checked that the two different forms Σ and Σ', Eqs. (4.2a) and (4.3), lead to the same final result.

A. Compact–support contribution

Having now the general set up for our computation we consider first the compact-support part of the multipole moments, *i.e.* that part proportional to the material source densities $\sigma_{\mu\nu}$, and given by the first terms in Eqs. (4.2)–(4.3). For these terms we make two computations. The first one consists of (i) evaluating all the components of $T^{\mu\nu}$ to the correct PN order [extending thus Eqs. (3.3) and including the monopolar sources, which may give rise to spin terms through the metric], (ii) computing their time derivatives by making use of the usual replacement of accelerations by the equations of motion, and of the time-derivatives of the spins by the precessional equations, (iii) transforming the time-derivatives, when applied to delta-functions, into spatial ones using the formula $\partial_t\delta_1 = -v_1^i\partial_i\delta_1$, (iv) operating by parts the spatial derivatives of delta-functions to finally integrate thanks to the basic property of delta-functions.

Normally, such basic property of delta-functions reads $\int d^3\mathbf{x}\, F(\mathbf{x})\,\delta_1(\mathbf{x}) = (F)_1$, where $(F)_1$ is simply the value of the function at the point 1. When the function is regular, there is no problem and we have $(F)_1 = F(\mathbf{y}_1, t)$, for instance $(\hat{x}_L)_1 = \hat{y}_1^L$. However, when the function F is singular at the point 1 (*i.e.* when $\mathbf{x} \to \mathbf{y}_1$), a choice must be made for a "self-field" regularization, able to subtract the infinities in a consistent way. There are various possibilities. In the present work we adopt, following paper I, the Hadamard self-field regularization. At the order we are working (relative 1PN order) the various possible choices are equivalent. For instance, one can show that dimensional regularization would give the same result as Hadamard's regularization, essentially because at such low PN order there are no poles in the dimension of space [say, $\propto (d-3)^{-1}$], which correspond to logarithmic divergences in Hadamard's regularization. We then define $(F)_1$ to be given by the *partie finie* of the function F at point 1 in the sense of Hadamard (see *e.g.* [49] for a full account of this regularization). Suppose for example that $F = U\,\hat{x}_L$ where $U = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}$ is the Newtonian potential of point particles, singular at the location of the particles. Then we easily compute that Hadamard's partie finie is $(F)_1 = \frac{Gm_2}{r_{12}}\,\hat{y}_1^L$.

Our alternative computation of the compact-support terms in the moments is the same as the one performed by Owen *et al.* [31]. It consists of applying the following formula (derived in [31]),

$$\int d^3\mathbf{x}\, F(\mathbf{x},t)\, \underset{\rm S}{T}^{\mu\nu}(t,\mathbf{x}) = -\frac{d}{cdt}\left[S_1^{0(\mu}\, v_1^{\nu)}\, \frac{(F)_1}{\sqrt{-g_1}}\right] + S_1^{\rho(\mu}\, v_1^{\nu)}\, \frac{(\partial_\rho F)_1}{\sqrt{-g_1}}$$
$$+\left[\underset{1}{\Gamma}{}^{(\mu}_{\rho\sigma}\, S_1^{\nu)\rho}\, v_1^\sigma - \underset{1}{\Gamma}{}^{\rho}_{\sigma\rho}\, S_1^{\sigma(\mu}\, v_1^{\nu)}\right] \frac{(F)_1}{\sqrt{-g_1}} + 1 \leftrightarrow 2\,, \qquad (4.4)$$

which is valid for any function $F(\mathbf{x},t)$, and where $(F)_1$ and $(\partial_\rho F)_1$ have to be understood as the Hadamard partie finie of the function and its derivative. This formula is very useful but must be handled with care. In particular, when computing $(\partial_\rho F)_1$ in the second term of the RHS of (4.4), we notice that the gradient is to be taken *first*, and only then should one deduce the value at point 1. The result can be different if one permutes the order of operations. Suppose for instance that one is computing $(\partial_t F)_1$ where $F = \hat{x}_L$. Clearly, since $\partial_t\, \hat{x}_L = 0$ the result is zero. However, if one computes $\frac{d}{dt}(F)_1$ instead of $(\partial_t F)_1$, one obtains an incorrect non-zero result, which is equal in this case to $\frac{d}{dt}\left[\hat{y}_1^L\right] = \ell y_1^{\langle L-1} v_1^{i_\ell\rangle}$. We found that this error, *i.e.* computing $\frac{d}{dt}(\hat{x}_L)_1 = \ell y_1^{\langle L-1} v_1^{i_\ell\rangle}$ instead of $(\partial_t\, \hat{x}_L)_1 = 0$, was committed in the evaluation of the (compact-support part of the) current quadrupole moment J_{ij} in Ref. [31].

B. Non-compact–support contribution

We now derive the non-compact support part of the multipole moments. Inspection of the expressions (4.2)–(4.3) shows that we need only the elementary potentials V, V_i, $\hat{W}_{ij}$ (and $\hat{W} \equiv \hat{W}_{ii}$) at their *lowest* PN order in the spins. They read:

$$\underset{\rm S}{V} = -\frac{2G}{c^3}\, \varepsilon_{ijk}\, v_1^i\, S_1^j\, \partial_k\left(\frac{1}{r_1}\right) + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^5}\right)\,, \qquad (4.5a)$$

$$\underset{\rm S}{V}_i = -\frac{G}{2c}\, \varepsilon_{ijk}\, S_1^j\, \partial_k\left(\frac{1}{r_1}\right) + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right)\,, \qquad (4.5b)$$

$$\underset{\rm S}{\hat{W}}_{ij} = -\frac{G}{c}\, \varepsilon_{kl(i}\, v_1^{j)}\, S_1^k\, \partial_l\left(\frac{1}{r_1}\right) + \frac{G}{c}\, \delta_{ij}\, \varepsilon_{klm}\, v_1^k\, S_1^l\, \partial_m\left(\frac{1}{r_1}\right) + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right)\,, \qquad (4.5c)$$

$$\underset{\rm S}{\hat{W}} = \frac{2G}{c}\, \varepsilon_{klm}\, v_1^k\, S_1^l\, \partial_m\left(\frac{1}{r_1}\right) + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right)\,. \qquad (4.5d)$$

The latter spin contributions enter only the "non-spin" parts of Eqs. (4.2)–(4.3). We need now to perform the integrations in Eq. (4.1). The calculation becomes simple if we use the following trick. In the multipole moment's integrands we transform all the gradients evaluated at the field point $\mathbf{x}$ into gradients evaluated at the source points $\mathbf{y}_1$ or $\mathbf{y}_2$ using *e.g.* $\frac{\partial}{\partial x^i}(1/r_1) = -\frac{\partial}{\partial y_1^i}(1/r_1)$. Then, we put the source-type gradients $\frac{\partial}{\partial y_1^i}$ and $\frac{\partial}{\partial y_2^i}$ outside the integrals and we express the result solely in terms of the function

$$Y_L(\mathbf{y}_1,\mathbf{y}_2) \equiv -\frac{1}{2\pi} \underset{B=0}{\rm FP} \int d^3\mathbf{x}\, |\mathbf{x}|^B \frac{\hat{x}_L}{r_1 r_2}\,, \qquad (4.6a)$$

which is known to admit the analytically closed-form [28]

$$Y_L = \frac{r_{12}}{\ell+1} \sum_{p=0}^{\ell} y_1^{\langle L-P} y_2^{P\rangle}\,. \qquad (4.6b)$$

Thus, the closed-form expressions of the non-compact (NC) parts of the spin multipole moments (they depend on the function Y_L for $\ell = 2, 3$) are:

$$\begin{aligned}
\underset{\mathrm{S}}{I}{}^{(\mathrm{NC})}_{ij} &= \frac{2Gm_2}{c^5}\Bigg\{\varepsilon_{mnk}\, v_1^l\, S_1^m\, \underset{1}{\partial}_n\, \underset{2}{\partial}_{kl} Y_{ij} - \varepsilon_{mnp}\, v_1^m\, S_1^n\, \underset{1}{\partial}_p\, \underset{2}{\Delta}\, Y_{ij} \\
&\quad - \varepsilon_{kmn}\, v_2^l\, S_1^m\, \underset{1}{\partial}_n\, \underset{2}{\partial}_{kl} Y_{ij} - 2\varepsilon_{lmn}\, v_2^k\, S_1^m\, \underset{1}{\partial}_{kn}\, \underset{2}{\partial}_l Y_{ij} \\
&\quad + \frac{10}{21}\,\varepsilon_{lmn}\, S_1^m\, \frac{d}{dt}\left[\underset{1}{\partial}_{nk}\, \underset{2}{\partial}_l Y_{ijk}\right] - \frac{10}{21}\,\varepsilon_{kmn}\, S_1^m\, \frac{d}{dt}\left[\underset{1}{\partial}_{nl}\, \underset{2}{\partial}_l Y_{ijk}\right]\Bigg\} \\
&\quad + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right), \qquad (4.7\mathrm{a}) \\
\underset{\mathrm{S}}{J}{}^{(\mathrm{NC})}_{ij} &= \frac{Gm_2}{c^3}\varepsilon_{kl\langle i}\Big\{-\varepsilon_{kmn}\, S_1^m\, \underset{2}{\partial}_k\, \underset{1}{\partial}_{ln} Y_{j\rangle k} + \varepsilon_{lmn}\, S_1^m\, \underset{2}{\partial}_p\, \underset{1}{\partial}_{pn} Y_{j\rangle k}\Big\} \\
&\quad + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^5}\right), \qquad (4.7\mathrm{b})
\end{aligned}$$

where $\partial_{1k} \equiv \partial/\partial y_1^k$ and $\partial_{2k} \equiv \partial/\partial y_2^k$. The final computation of the moments using formula (4.6b) is straightforward.

V. RESULTS FOR THE MULTIPOLE MOMENTS AND FLUX

The expressions for the multipole moments ${}_{\mathrm{S}}I_{ij}$ and ${}_{\mathrm{S}}J_{ij}$, including both compact and non-compact contributions as computed in Sec. IV, are quite long if written in a general frame. They can be substantially simplified by going to the frame of the center-of-mass (CM). When working in the CM frame it is convenient to use the following spin variables:

$$\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2\,, \qquad (5.1\mathrm{a})$$

$$\mathbf{\Sigma} \equiv m\left[\frac{\mathbf{S}_2}{m_2} - \frac{\mathbf{S}_1}{m_1}\right]. \qquad (5.1\mathrm{b})$$

These spin variables were initially introduced by Kidder [40] except that here we denote $\mathbf{\Sigma}$ what he calls $\mathbf{\Delta}$. Mass parameters will be denoted by $m \equiv m_1 + m_2$, $\delta m \equiv m_1 - m_2$ and $\nu \equiv m_1\, m_2/m^2$ for a mass ratio such that $\nu = 1/4$ for equal masses and $\nu \to 0$ in the test-mass limit.

The CM frame is defined by the nullity of the binary's dipole moment or equivalently the CM vector $\mathbf{G}$. At 2.5PN order including spin effects, it can easily be determined using the vector $\mathbf{G}$ evaluated in paper I. However, here we need only the lowest order term (1.5PN in the spins) together with the 1PN non-spin correction; the 2.5PN term in the spins cancels out. To the needed order we have (see *e.g.* Eq. (5.13) in [24])

$$\mathbf{y}_1 = \left[\frac{m_2}{m} + \frac{\nu}{2c^2}\frac{\delta m}{m}\left(v^2 - \frac{G\,m}{r}\right)\right]\mathbf{x} + \frac{\nu}{m\,c^3}\,\mathbf{v}\times\mathbf{\Sigma}\,, \qquad (5.2\mathrm{a})$$

$$\mathbf{y}_2 = \left[-\frac{m_1}{m} + \frac{\nu}{2c^2}\frac{\delta m}{m}\left(v^2 - \frac{G\,m}{r}\right)\right]\mathbf{x} + \frac{\nu}{m\,c^3}\,\mathbf{v}\times\mathbf{\Sigma}\,, \qquad (5.2\mathrm{b})$$

which gives the CM positions of the particles, $\mathbf{y}_1$ and $\mathbf{y}_2$, in terms of the relative position and velocity, $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$ and $\mathbf{v} = d\mathbf{x}/dt = \mathbf{v}_1 - \mathbf{v}_2$ (we pose $r = |\mathbf{x}|$ and $v^2 = \mathbf{v}\cdot\mathbf{v}$).

A. The multipole moments

Our final result for the spin part of the mass-quadrupole moment at 2.5PN order (1PN order beyond the dominant SO term), for general orbits and in the CM frame, reads

$$
\begin{aligned}
\underset{\mathrm{S}}{I}_{ij} &= \frac{\nu}{c^3}\bigg\{\frac{8}{3}\,x^{\langle i}(\mathbf{v}\times\mathbf{S})^{j\rangle} - \frac{4}{3}\,v^{\langle i}(\mathbf{x}\times\mathbf{S})^{j\rangle} \\
&\quad + \frac{8}{3}\,\frac{\delta m}{m}\,x^{\langle i}(\mathbf{v}\times\mathbf{\Sigma})^{j\rangle} - \frac{4}{3}\,\frac{\delta m}{m}\,v^{\langle i}(\mathbf{x}\times\mathbf{\Sigma})^{j\rangle}\bigg\} \\
&+ \frac{\nu}{c^5}\bigg\{\left[\frac{5}{3}+\frac{2}{7}\nu\right]\frac{Gm}{r^3}\,\frac{\delta m}{m}\,(xv)\,x^{\langle i}(\mathbf{x}\times\mathbf{\Sigma})^{j\rangle} \\
&\quad + \left(\left[\frac{7}{3}+4\nu\right]\frac{Gm}{r} + \left[\frac{26}{21}-\frac{116}{21}\nu\right]v^2\right)\frac{\delta m}{m}\,x^{\langle i}(\mathbf{v}\times\mathbf{\Sigma})^{j\rangle} \\
&\quad + \left[\frac{31}{21}+\frac{19}{21}\nu\right]\frac{Gm}{r^3}\,(xv)\,x^{\langle i}(\mathbf{x}\times\mathbf{S})^{j\rangle} \\
&\quad + \left(\left[\frac{25}{7}+\frac{55}{21}\nu\right]\frac{Gm}{r} + \left[\frac{26}{21}-\frac{26}{7}\nu\right]v^2\right)x^{\langle i}(\mathbf{v}\times\mathbf{S})^{j\rangle} \\
&\quad + \left(\left[-4-\frac{2}{7}\nu\right]\frac{Gm}{r} + \left[-\frac{6}{7}+\frac{64}{21}\nu\right]v^2\right)\frac{\delta m}{m}\,v^{\langle i}(\mathbf{x}\times\mathbf{\Sigma})^{j\rangle} \\
&\quad + \left[\frac{10}{21}-\frac{8}{21}\nu\right]\frac{\delta m}{m}\,(xv)\,v^{\langle i}(\mathbf{v}\times\mathbf{\Sigma})^{j\rangle} \\
&\quad + \left(\left[-\frac{26}{3}-\frac{2}{3}\nu\right]\frac{Gm}{r} + \left[-\frac{6}{7}+\frac{18}{7}\nu\right]v^2\right)v^{\langle i}(\mathbf{x}\times\mathbf{S})^{j\rangle} \\
&\quad + \left[\frac{10}{21}-\frac{10}{7}\nu\right](xv)\,v^{\langle i}(\mathbf{v}\times\mathbf{S})^{j\rangle} \\
&\quad + \left(\left[\frac{52}{21}-\frac{10}{7}\nu\right](S,x,v) + \left[\frac{62}{21}-\frac{18}{7}\nu\right]\frac{\delta m}{m}\,(\Sigma,x,v)\right)\frac{Gm}{r^3}\,x^{\langle i}x^{j\rangle} \\
&\quad + \left(\left[-\frac{5}{21}+\frac{5}{7}\nu\right](S,x,v) + \left[-\frac{5}{21}-\frac{4}{7}\nu\right]\frac{\delta m}{m}\,(\Sigma,x,v)\right)v^{\langle i}v^{j\rangle} \\
&\quad + \left(\left[-\frac{8}{3}+\frac{16}{3}\nu\right](xS) + \left[-\frac{8}{3}+\frac{8}{3}\nu\right]\frac{\delta m}{m}\,(x\Sigma)\right)\frac{Gm}{r^3}x^{\langle i}(\mathbf{x}\times\mathbf{v})^{j\rangle} \\
&\quad + \left(\left[\frac{4}{3}-4\nu\right](vS) + \left[\frac{4}{3}-\frac{8}{3}\nu\right]\frac{\delta m}{m}\,(v\Sigma)\right)v^{\langle i}(\mathbf{x}\times\mathbf{v})^{j\rangle}\bigg\} \\
&+ \mathcal{O}\left(\frac{1}{c^7}\right).
\end{aligned} \tag{5.3}
$$

The scalar product of ordinary Euclidean vectors is indicated by parenthesis, *e.g.* $(vS) = \mathbf{v}\cdot\mathbf{S}$, the cross product by the usual cross symbol, $(\mathbf{x}\times\mathbf{\Sigma})^i = \varepsilon^{ijk}x^j\Sigma^k$, and the mixed product of three vectors by $(S,x,v) = \mathbf{S}\cdot(\mathbf{x}\times\mathbf{v}) = \varepsilon^{ijk}S^ix^jv^k$. We recall also that the STF projection is denoted using carets surrounding indices, *i.e.* $\langle ij\rangle$. Next, the spin part of the current quadrupole moment at 1.5PN order (also 1PN order beyond the leading term) is

$$
\underset{\mathrm{S}}{J}_{ij} = \frac{\nu}{c}\bigg\{-\frac{3}{2}\,x^{\langle i}\Sigma^{j\rangle}\bigg\}
$$

$$
\begin{aligned}
&+\ \frac{\nu}{c^3}\Bigg\{\left[\frac{3}{7}-\frac{16}{7}\nu\right](xv)\,v^{\langle i}\Sigma^{j\rangle}+\frac{3}{7}\frac{\delta m}{m}(xv)\,v^{\langle i}S^{j\rangle}\\
&\qquad+\left(\left[\frac{27}{14}-\frac{109}{14}\nu\right](v\Sigma)+\frac{27}{14}\frac{\delta m}{m}(vS)\right)x^{\langle i}v^{j\rangle}\\
&\qquad+\left(\left[-\frac{11}{14}+\frac{47}{14}\nu\right](x\Sigma)-\frac{11}{14}\frac{\delta m}{m}(xS)\right)v^{\langle i}v^{j\rangle}\\
&\qquad+\left(\left[\frac{19}{28}+\frac{13}{28}\nu\right]\frac{Gm}{r}+\left[-\frac{29}{28}+\frac{143}{28}\nu\right]v^2\right)x^{\langle i}\Sigma^{j\rangle}\\
&\qquad+\left(\left[-\frac{4}{7}+\frac{31}{14}\nu\right](x\Sigma)-\frac{29}{14}\frac{\delta m}{m}(xS)\right)\frac{Gm}{r^3}x^{\langle i}x^{j\rangle}\\
&\qquad+\left(-\frac{1}{14}\frac{Gm}{r}-\frac{2}{7}v^2\right)\frac{\delta m}{m}x^{\langle i}S^{j\rangle}\Bigg\}\\
&+\ \mathcal{O}\left(\frac{1}{c^5}\right)\,.
\end{aligned}
\tag{5.4}
$$

Notice that the 1.5PN current quadrupole moment ${}_{\rm S}J_{ij}$ was also computed in Ref. [31], see Eq. (4.18) there. However our result (5.4) differs from their result. There are two reasons for this discrepancy. The main reason is that Ref. [31] completely neglected the non-compact support terms, which originate from the non-linearities of the Einstein field equations *via* the term $\Lambda^{\mu\nu}$ in Eq. (2.1), and physically represent the gravitational field acting as a source for the multipole moment. As we have seen in Sec. IV these terms are not negligible. Their contribution to the 1.5PN order current moment ${}_{\rm S}J_{ij}$ has been computed in Eq. (4.7b). The second reason for the difference between Eq. (5.4) and the result of [31] is a computational error in [31] when they apply the integration formula (4.4) for computing the compact-support terms. We have already commented upon this error after Eq. (4.4) above. These two errors fully account for the discrepancy between our result and the one of Eq. (4.18) in Ref. [31].

Next, in order to derive the GW flux at 2.5PN order, we need the spin parts in the mass octupole and current octupole moments, but only at the lowest order in the spins. They can be obtained from our previous computation leading to Eqs. (3.6), after the CM reduction. We find [6]

$$
\begin{aligned}
\underset{\rm S}{I}_{ijk} &= \frac{\nu}{c^3}\Bigg\{-\frac{9}{2}\frac{\delta m}{m}x^{\langle i}x^j(\mathbf{v}\times\mathbf{S})^{k\rangle}-\frac{3}{2}(3-11\nu)\,x^{\langle i}x^j(\mathbf{v}\times\boldsymbol{\Sigma})^{k\rangle}\\
&\qquad+3\frac{\delta m}{m}x^{\langle i}v^j(\mathbf{x}\times\mathbf{S})^{k\rangle}+3\,(1-3\nu)\,x^{\langle i}v^j(\mathbf{x}\times\boldsymbol{\Sigma})^{k\rangle}\Bigg\}+\mathcal{O}\left(\frac{1}{c^5}\right)\,, &\text{(5.5a)}\\
\underset{\rm S}{J}_{ijk} &= \frac{\nu}{c}\Bigg\{2\,x^{\langle i}x^jS^{k\rangle}+2\frac{\delta m}{m}x^{\langle i}x^j\Sigma^{k\rangle}\Bigg\}+\mathcal{O}\left(\frac{1}{c^3}\right)\,. &\text{(5.5b)}
\end{aligned}
$$

[6] The result for ${}_{\rm S}I_{ijk}$ agrees with the one given by Eq. (4.17) in Ref. [31]. However, we notice a misprint in the second term of Eq. (4.17) in [31], in which x^{jk} should read x^jv^k.

B. The gravitational-wave energy flux

With all these moments, Eqs. (5.3)–(5.4) and (5.5), and only with those, we can compute the 2.5PN spin part of the GW flux. Indeed, recall that the spins start at 1.5PN order in the mass moments and at 0.5PN order in the current ones, so one can easily see that in higher multipoles spins will enter the flux at higher PN order. On the other hand, one can check that it is not necessary to include the effects of tails of GWs, and more generally of any non-linear multipole interaction. Indeed, the tails give a correction to each of the source-type multipole moments I_L and J_L at the relative order 1.5PN $\sim 1/c^3$ (see *e.g.* [28]). For the mass quadrupole I_{ij} the spin itself is at order 1.5PN so the tail will arise only at order 3PN $\sim 1/c^6$ in the flux. For the current quadrupole J_{ij} the spin is at 0.5PN but J_{ij} comes in the flux at 1PN order, so again we see that the corresponding tail will only be at 3PN order in the flux. In conclusion, for this problem it is sufficient to express the flux solely in terms of the source multipole moments I_L and J_L; all multipole interactions built in the radiative moments seen at infinity, namely U_L and V_L, are negligible. Furthermore, as we have seen only four multipolar contributions are important for this application. Therefore (*cf.* Eq. (4.28) in [28])

$$\begin{aligned}\mathcal{F} &= \frac{G}{c^5}\left\{\frac{1}{5}\dddot{I}_{ij}\dddot{I}_{ij}+\frac{1}{c^2}\left[\frac{1}{189}\ddddot{I}_{ijk}\ddddot{I}_{ijk}+\frac{16}{45}\dddot{J}_{ij}\dddot{J}_{ij}\right]+\frac{1}{84c^4}\ddddot{J}_{ijk}\ddddot{J}_{ijk}\right\}\\ &\quad+\ \text{terms not contributing to the spins at 2.5PN order}\,. \end{aligned}\tag{5.6}$$

In order to compute time derivatives of the moments (5.3)–(5.4) and (5.5), we must be careful at including the non-spin terms of I_{ij} and J_{ij} since these terms will generate by order reduction of the accelerations some new spin terms at 2.5PN order. In particular we need for this problem the non-spin part of the mass quadrupole I_{ij} at 1PN order, which is given by [50, 51],

$$\begin{aligned}I_{ij} = m\,\nu\Bigg\{&\left(1+\frac{1}{c^2}\left[\left(\frac{29}{42}-\frac{29}{14}\nu\right)v^2+\left(-\frac{5}{7}+\frac{8}{7}\nu\right)\frac{G\,m}{r}\right]\right)x^{\langle i}x^{j\rangle}\\ &+\frac{r^2}{c^2}\left(\frac{11}{21}-\frac{11}{7}\nu\right)v^{\langle i}v^{j\rangle}+\frac{(xv)}{c^2}\left(-\frac{4}{7}+\frac{12}{7}\nu\right)x^{\langle i}v^{j\rangle}\Bigg\}+\mathcal{O}\left(\frac{1}{c^4}\right)\,.\end{aligned}\tag{5.7}$$

The current quadrupole moment is also needed at 1PN order,

$$\begin{aligned}J_{ij} = -\nu\,\delta m\,x^k v^l\,\varepsilon^{kl\langle i}\Bigg\{&x^{j\rangle}\left(1+\frac{1}{c^2}\left[\left(\frac{27}{14}+\frac{15}{7}\nu\right)\frac{Gm}{r}+\left(\frac{13}{28}-\frac{17}{7}\nu\right)v^2\right]\right)\\ &+\frac{1}{c^2}v^{j\rangle}(xv)\left(\frac{5}{28}-\frac{5}{14}\nu\right)\Bigg\}+\mathcal{O}\left(\frac{1}{c^4}\right)\,.\end{aligned}\tag{5.8}$$

Note that for both I_{ij} and J_{ij} there are some contributions at 1.5PN order which depends on the spin variables and are generated from the Newtonian term evaluated in the CM; these contributions have been included in the results (5.3)–(5.4) above.

Finally, we obtain for the flux (in the general orbit case but in the CM frame) the structure

$$\begin{aligned}\mathcal{F} = \frac{8}{15}\frac{G^3m^4\nu^2}{c^5r^4}\Bigg\{&f_{\rm N}+\frac{1}{c^2}f_{\rm 1PN}+\frac{1}{c^3}\left[f_{\rm 1.5PN}+\underset{\rm S}{f}_{\rm 1.5PN}\right]+\frac{1}{c^4}\left[f_{\rm 2PN}+\underset{\rm SS}{f}_{\rm 2PN}\right]\\ &+\frac{1}{c^5}\left[f_{\rm 2.5PN}+\underset{\rm S}{f}_{\rm 2.5PN}\right]+\mathcal{O}\left(\frac{1}{c^6}\right)\Bigg\}\,.\end{aligned}\tag{5.9}$$

The non-spin pieces $f_{\rm N}$, $f_{\rm 1PN}$, $f_{\rm 1.5PN}$ and $f_{\rm 2.5PN}$ are already known and we shall need below the Newtonian and 1PN terms which are given by [50, 52]

$$f_{\rm N} = 12v^2 - 11(nv)^2\,, \tag{5.10a}$$

$$\begin{aligned} f_{\rm 1PN} =& \left(\frac{785}{28} - \frac{213}{7}\nu\right) v^4 + \left(-\frac{1487}{14} + \frac{696}{7}\nu\right)(nv)^2 v^2 + \left(\frac{2061}{28} - \frac{465}{7}\nu\right)(nv)^4 \\ &+ \left(-\frac{680}{7} + \frac{40}{7}\nu\right)\frac{G\,m}{r}v^2 + \left(\frac{734}{7} - \frac{30}{7}\nu\right)\frac{G\,m}{r}(nv)^2 + \left(\frac{4}{7} - \frac{16}{7}\nu\right)\left(\frac{G\,m}{r}\right)^2 . \end{aligned} \tag{5.10b}$$

Notice also that the non-spin terms $f_{\rm 1.5PN}$ and $f_{\rm 2.5PN}$ include the contributions of GW tails. Here we do not deal with the spin-spin (SS) term at 2PN order which is given in Refs. [39, 40]. We obtain the SO coupling part at 1.5PN order as

$$\begin{aligned} \underset{\rm S}{f}_{\rm 1.5PN} =& \frac{(S,n,v)}{m\,r}\left[78(nv)^2 - 8\frac{Gm}{r} - 80v^2\right] \\ &+ \frac{(\Sigma,n,v)}{m\,r}\left[51(nv)^2 + 4\frac{Gm}{r} - 43v^2\right]\frac{\delta m}{m}\,, \end{aligned} \tag{5.11}$$

and for this part we find perfect agreement with Kidder *et al.* [39, 40]. Finally, for the next-order SO part our result is

$$\begin{aligned} \underset{\rm S}{f}_{\rm 2.5PN} =& \frac{(S,n,v)}{m\,r}\left[(nv)^4\left(-\frac{2244}{7} + \frac{3144}{7}\nu\right) + \frac{G^2m^2}{r^2}\left(\frac{972}{7} + \frac{166}{7}\nu\right)\right. \\ &+ \frac{Gm}{r}(nv)^2\left(-\frac{2866}{7} + \frac{170}{7}\nu\right) + (nv)^2v^2\left(\frac{3519}{7} - \frac{5004}{7}\nu\right) \\ &\left. + \frac{Gm}{r}v^2\left(\frac{3504}{7} - 20\nu\right) + v^4\left(-\frac{1207}{7} + \frac{1810}{7}\nu\right)\right] \\ &+ \frac{(\Sigma,n,v)}{m\,r}\left[(nv)^4\left(-\frac{7941}{28} + \frac{2676}{7}\nu\right) + \frac{G^2m^2}{r^2}\left(-\frac{109}{7} + 18\nu\right)\right. \\ &+ \frac{Gm}{r}(nv)^2\left(-\frac{6613}{14} + \frac{1031}{7}\nu\right) + (nv)^2v^2\left(\frac{2364}{7} - \frac{3621}{7}\nu\right) \\ &\left. + \frac{Gm}{r}v^2\left(\frac{4785}{14} - 65\nu\right) + v^4\left(-\frac{2603}{28} + \frac{1040}{7}\nu\right)\right]\frac{\delta m}{m}\,. \end{aligned} \tag{5.12}$$

VI. REDUCTION TO QUASI CIRCULAR ORBITS

From now on we assume that when the binary enters the frequency bandwidth of the LIGO/Virgo/LISA detectors the orbit has been circularized by the gravitational radiation reaction effect. By circular orbit we mean an orbit which is circular when the gradual radiation reaction inspiral can be neglected, and when the effects of spins are averaged over time. With such proviso there is a well defined notion of a circular orbit (see Refs. [39, 40] and paper I).

For circular orbits the orbital frequency ω is linked to the distance r between particles in harmonic coordinates by a relativistic extension of Kepler's law, which has already been given in paper I for what concerns the SO effects. Let us write it again here, but let us also add to it, for the benefit of potential users of these formulas, all the non-spin contributions up to the 2.5PN order, following known results from the literature (*e.g.* [21] and references therein). The 3PN and 3.5PN non-spin terms, computed in Refs. [20, 23], can be added straightforwardly if necessary. However, for convenience in this paper, we shall not display the non-linear spin-spin (SS) terms. Thus, all formulas of this Section will be complete up to 2.5PN order at *linear* order in the spins (*i.e.* but for the SS contributions). We have

$$\begin{aligned}\omega^2 = \frac{G\,m}{r^3}\Bigg\{&1+\gamma\,(-3+\nu)+\gamma^2\left(6+\frac{41}{4}\nu+\nu^2\right)\\&+\frac{\gamma^{3/2}}{G\,m^2}\left[-5S_\ell-3\frac{\delta m}{m}\Sigma_\ell\right]\\&+\frac{\gamma^{5/2}}{G\,m^2}\left[\left(\frac{39}{2}-\frac{23}{2}\nu\right)S_\ell+\left(\frac{21}{2}-\frac{11}{2}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right]+\mathcal{O}\left(\frac{1}{c^6}\right)\Bigg\}\,,\end{aligned}\tag{6.1}$$

in which $\gamma\equiv\frac{G\,m}{r\,c^2}=\mathcal{O}\left(c^{-2}\right)$ denotes the harmonic-coordinate PN parameter. We recognize the lowest-order (1.5PN $\sim\gamma^{3/2}$) spin-orbit term and its 1PN correction at the 2.5PN $\sim\gamma^{5/2}$ level. Here, as in paper I, we introduce an orthonormal triad $\{\mathbf{n},\boldsymbol{\lambda},\boldsymbol{\ell}\}$ defined by $\mathbf{n}=\mathbf{x}/r$, $\boldsymbol{\ell}=\mathbf{L}_\mathrm{N}/|\mathbf{L}_\mathrm{N}|$ where $\mathbf{L}_\mathrm{N}\equiv\mu\,\mathbf{x}\times\mathbf{v}$ denotes the Newtonian angular momentum, and $\boldsymbol{\lambda}=\boldsymbol{\ell}\times\mathbf{n}$. The quantities S_ℓ and Σ_ℓ in Eq. (6.1) are the components of the spin vectors (5.1) perpendicular to the orbital plane, namely $S_\ell\equiv\mathbf{S}\cdot\boldsymbol{\ell}$ and $\Sigma_\ell\equiv\boldsymbol{\Sigma}\cdot\boldsymbol{\ell}$. The relation (6.1) can be inverted to give γ in terms of an alternative PN parameter x, directly related to the orbital frequency through $x\equiv\left(\frac{G\,m\,\omega}{c^3}\right)^{2/3}=\mathcal{O}\left(c^{-2}\right)$. As usual, it is better to express the PN formulas in terms of the frequency-dependent PN parameter x rather than γ because they are invariant under a large class of gauge transformations. Hence,

$$\begin{aligned}\gamma = x\Bigg\{&1+x\left(1-\frac{\nu}{3}\right)+x^2\left(1-\frac{65}{12}\nu\right)\\&+\frac{x^{3/2}}{G\,m^2}\left[\frac{5}{3}S_\ell+\frac{\delta m}{m}\Sigma_\ell\right]\\&+\frac{x^{5/2}}{G\,m^2}\left[\left(\frac{13}{3}+\frac{2}{9}\nu\right)S_\ell+\left(3-\frac{\nu}{3}\right)\frac{\delta m}{m}\Sigma_\ell\right]+\mathcal{O}\left(\frac{1}{c^6}\right)\Bigg\}\,.\end{aligned}\tag{6.2}$$

The SO term at order 1.5PN $\sim x^{3/2}$ is in agreement with Eq. (16) in [39].

The reduction of the GW flux $\mathcal{F}$, given by Eqs. (5.11)–(5.12), to circular orbits is straightforward, but care has to be taken from the fact that the *non-spin* parts of the flux at Newtonian and 1PN orders yield crucial contributions to the SO terms for circular orbits [beside the ones given by straightforward reduction of Eqs. (5.11)–(5.12)]. Such contributions are generated by replacement of Eq. (6.1) into the 1PN flux given for general orbits by Eq. (5.10). Finally, we obtain

$$\begin{aligned}\mathcal{F} =&\frac{32}{5}\frac{c^5}{G}\gamma^5\,\nu^2\Bigg\{1+\gamma\left(-\frac{2927}{336}-\frac{5}{4}\nu\right)+4\pi\gamma^{3/2}\\&+\gamma^2\left(\frac{293383}{9072}+\frac{380}{9}\nu\right)+\pi\gamma^{5/2}\left(-\frac{25663}{672}-\frac{125}{8}\nu\right)\\&+\frac{\gamma^{3/2}}{G\,m^2}\left[-\frac{37}{3}S_\ell-\frac{25}{4}\frac{\delta m}{m}\Sigma_\ell\right]\end{aligned}$$

$$+\frac{\gamma^{5/2}}{G\,m^2}\left[\left(\frac{17897}{168}+23\nu\right)S_\ell+\left(\frac{6253}{112}+\frac{277}{24}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right]+\mathcal{O}\left(\frac{1}{c^6}\right)\Bigg\}\,, \tag{6.3}$$

or, equivalently, in terms of the PN parameter x,

$$\begin{aligned}\mathcal{F}=&\frac{32}{5}\frac{c^5}{G}\,x^5\,\nu^2\Bigg\{1+x\left(-\frac{1247}{336}-\frac{35}{12}\nu\right)+4\pi x^{3/2}\\&+x^2\left(-\frac{44711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\right)+\pi x^{5/2}\left(-\frac{8191}{672}-\frac{583}{24}\nu\right)\\&+\frac{x^{3/2}}{G\,m^2}\left[-4S_\ell-\frac{5}{4}\frac{\delta m}{m}\Sigma_\ell\right]\\&+\frac{x^{5/2}}{G\,m^2}\left[\left(-\frac{23}{4}+\frac{245}{9}\nu\right)S_\ell+\left(-\frac{33}{16}+\frac{37}{4}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right]+\mathcal{O}\left(\frac{1}{c^6}\right)\Bigg\}\,.\end{aligned} \tag{6.4}$$

All the non-spin terms are included up to 2.5PN order. Notice in particular the non-spin terms, proportional to π, which are at the same 1.5PN and 2.5PN orders as the SO effects; these terms are due to GW tails [20, 21]. [7] For the leading SO term at order 1.5PN $\sim x^{3/2}$ we find perfect agreement with Eq. (17b) in [39]. We also check that the result agrees in the test-mass limit $\nu\to 0$ with the black-hole perturbation calculation of Tagoshi *et al.* [53] [see Eq. (G19) there].

The reduction of the center-of-mass energy E (computed in Sec. VII of paper I) to circular orbits is straightforward, and we simply report here the final result, completing it by the known non-spin terms [54]. We have

$$\begin{aligned}E=&-\frac{\mu\,c^2\,\gamma}{2}\Bigg\{1+\gamma\left(-\frac{7}{4}+\frac{\nu}{4}\right)+\gamma^2\left(-\frac{7}{8}+\frac{49}{8}\nu+\frac{\nu^2}{8}\right)\\&+\frac{\gamma^{3/2}}{G\,m^2}\left[3S_\ell+\frac{\delta m}{m}\Sigma_\ell\right]\\&+\frac{\gamma^{5/2}}{G\,m^2}\left[(7-4\nu)\,S_\ell+(3-2\nu)\,\frac{\delta m}{m}\Sigma_\ell\right]+\mathcal{O}\left(\frac{1}{c^6}\right)\Bigg\}\,,\end{aligned} \tag{6.5}$$

or, equivalently,

$$\begin{aligned}E=&-\frac{\mu\,c^2\,x}{2}\Bigg\{1+x\left(-\frac{3}{4}-\frac{\nu}{12}\right)+x^2\left(-\frac{27}{8}+\frac{19}{8}\nu-\frac{\nu^2}{24}\right)\\&+\frac{x^{3/2}}{G\,m^2}\left[\frac{14}{3}S_\ell+2\frac{\delta m}{m}\Sigma_\ell\right]\\&+\frac{x^{5/2}}{G\,m^2}\left[\left(13-\frac{49}{9}\nu\right)S_\ell+\left(5-\frac{8}{3}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right]+\mathcal{O}\left(\frac{1}{c^6}\right)\Bigg\}\,.\end{aligned} \tag{6.6}$$

Alternatively, in terms of the single-spin variables the spin-dependent part of the above equation reads

$$\underset{\mathrm{S}}{E}=-\frac{\mu\,c^2\,x}{2}\sum_{i=1,2}\chi_i\,\kappa_i\Bigg\{x^{3/2}\left[\frac{8}{3}\frac{m_i^2}{m^2}+2\nu\right]$$

[7] For the non-spin tail term at 2.5PN order we take into account the published Erratum to [20].

$$+x^{5/2}\left[\frac{m_i^2}{m^2}\left(8-\frac{25}{9}\nu\right)+\nu\left(5-\frac{8}{3}\nu\right)\right]\Bigg\}\,, \tag{6.7}$$

where we denote by $\kappa_i = \hat{\mathbf{S}}_i \cdot \boldsymbol{\ell}$ for $i = 1, 2$ the orientation of the spins with respect to the Newtonian angular momentum, and by χ_i their magnitude defined in the standard way by $\mathbf{S}_i = G\, m_i^2\, \chi_i\, \hat{\mathbf{S}}_i$.

Assuming non-precessing orbits, we list in Table I the energy and the frequency at the so-called innermost circular orbit (ICO) [55]. The ICO is defined by the minimum of the center-of-mass energy for circular orbits expressed as a function of the orbital frequency ω, and is computed from Eq. (6.6).

		$m\,\omega_{\rm ICO}$	$E_{\rm ICO}/m$
1PN		0.522	-0.0405
	$\kappa_i = 0$	0.522	-0.0405
1.5PN	$\kappa_i = +1$	$-$	$-$
	$\kappa_i = -1$	0.111	-0.0163
	$\kappa_i = 0$	0.137	-0.0199
2PN	$\kappa_i = +1$	0.318	-0.0390
	$\kappa_i = -1$	0.0733	-0.0130
	$\kappa_i = 0$	0.137	-0.0199
2.5PN	$\kappa_i = +1$	$-$	$-$
	$\kappa_i = -1$	0.060	-0.0117
	$\kappa_i = 0$	0.129	-0.0193
3PN	$\kappa_i = +1$	$-$	$-$
	$\kappa_i = -1$	0.059	-0.0116

TABLE I: Energy and angular frequency at the ICO for equal-mass ($\nu = \frac{1}{4}$) binary systems. The spins are maximal ($\chi_i = 1$) and have different orientations ($\kappa_i = 0, \pm 1$). In three cases, indicated by a dash, there is no ICO, *i.e.* the energy function admits no real minimum. Spin-spin effects at 2PN order are included.

The orbital angular momentum (computed in paper I) in the case of circular orbits reads

$$\begin{aligned}
\mathbf{L} = \frac{G\,m^2}{c}\,\nu\,\gamma^{-1/2}\Bigg\{\boldsymbol{\ell}\Bigg[&(1+2\gamma)+\left(-3\,S_\ell-\Sigma_\ell\,\frac{\delta m}{m}\right)\frac{\gamma^{3/2}}{G\,m^2}\\
&+\left(\frac{5}{2}-\frac{9}{2}\nu\right)\gamma^2+\left(\left(-\frac{59}{8}+\frac{25}{8}\nu\right)S_\ell+\left(-\frac{27}{8}+\frac{3}{2}\nu\right)\frac{\delta m}{m}\,\Sigma_\ell\right)\frac{\gamma^{5/2}}{G\,m^2}\Bigg]\\
+\frac{\gamma^{3/2}}{Gm^2}\,\boldsymbol{\lambda}&\left[-\frac{3}{2}\,S_\lambda-\frac{1}{2}\,\Sigma_\lambda\,\frac{\delta m}{m}+\left(\left(\frac{3}{8}-\frac{61}{8}\nu\right)S_\lambda+\left(-\frac{9}{8}-\frac{15}{4}\nu\right)\frac{\delta m}{m}\,\Sigma_\lambda\right)\gamma\right]\\
+\frac{\gamma^{3/2}}{Gm^2}\,\mathbf{n}&\left[\frac{5}{2}\,S_n+\frac{3}{2}\,\Sigma_n\,\frac{\delta m}{m}+\left(\left(-\frac{13}{8}-\frac{13}{8}\nu\right)S_n+\left(\frac{15}{8}-\frac{3}{4}\nu\right)\frac{\delta m}{m}\,\Sigma_n\right)\gamma\right]\Bigg\}\,,
\end{aligned} \tag{6.8}$$

or equivalently

$$\mathbf{L} = \frac{G\,m^2}{c}\,\nu\,x^{-1/2}\Bigg\{\boldsymbol{\ell}\Bigg[1+\left(\frac{3}{2}+\frac{\nu}{6}\right)x+\left(-\frac{23}{6}S_\ell-\frac{3}{2}\,\frac{\delta m}{m}\,\Sigma_\ell\right)\frac{x^{3/2}}{G\,m^2}$$

$$+\left(\frac{27}{8}-\frac{19\nu}{8}+\frac{\nu^2}{24}\right)x^2+\left(\left(-\frac{77}{8}+\frac{259}{72}\nu\right)S_\ell+\left(-\frac{33}{8}+\frac{7}{4}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right)\frac{x^{5/2}}{G\,m^2}$$
$$+\frac{x^{3/2}}{Gm^2}\boldsymbol{\lambda}\left[-\frac{3}{2}S_\lambda-\frac{1}{2}\Sigma_\lambda\frac{\delta m}{m}+\left(\left(-\frac{9}{8}-\frac{57}{8}\nu\right)S_\lambda+\left(-\frac{13}{8}-\frac{43}{12}\nu\right)\frac{\delta m}{m}\Sigma_\lambda\right)x\right]$$
$$+\frac{x^{3/2}}{Gm^2}\mathbf{n}\left[\frac{5}{2}S_n+\frac{3}{2}\Sigma_n\frac{\delta m}{m}+\left(\left(\frac{7}{8}-\frac{59}{24}\nu\right)S_n+\left(\frac{27}{8}-\frac{5}{4}\nu\right)\frac{\delta m}{m}\Sigma_n\right)x\right]\Bigg\}\,. \tag{6.9}$$

For future use we give here the precessional equations evaluated in paper I, but reduced to circular orbits:

$$\frac{d\mathbf{S}}{dt}=\nu\,\omega\left\{x\left[-4S_\lambda-2\frac{\delta m}{m}\Sigma_\lambda\right]\mathbf{n}+x\left[3S_n+\frac{\delta m}{m}\Sigma_n\right]\boldsymbol{\lambda}\right.$$
$$+x^2\left[\left(1-\frac{20}{3}\nu\right)S_\lambda+\left(-2-\frac{10}{3}\nu\right)\frac{\delta m}{m}\Sigma_\lambda\right]\mathbf{n}$$
$$\left.+x^2\left[(9-11\nu)\,S_n-\frac{16}{3}\nu\frac{\delta m}{m}\Sigma_n\right]\boldsymbol{\lambda}\right\}+\mathcal{O}\left(\frac{1}{c^6}\right)\,, \tag{6.10a}$$
$$\frac{d\boldsymbol{\Sigma}}{dt}=\omega\left\{x\left[(-2+4\nu)\,\Sigma_\lambda-2\frac{\delta m}{m}S_\lambda\right]\mathbf{n}+x\left[(1-\nu)\,\Sigma_n+\frac{\delta m}{m}S_n\right]\boldsymbol{\lambda}\right.$$
$$+x^2\left[\left(-2+\frac{17}{3}\nu+\frac{20}{3}\nu^2\right)\Sigma_\lambda+\left(-2-\frac{10}{3}\nu\right)\frac{\delta m}{m}S_\lambda\right]\mathbf{n}$$
$$\left.+x^2\left[\left(\frac{11}{3}\nu+\frac{31}{3}\nu^2\right)\Sigma_n-\frac{16}{3}\nu\frac{\delta m}{m}S_n\right]\boldsymbol{\lambda}\right\}+\mathcal{O}\left(\frac{1}{c^6}\right)\,. \tag{6.10b}$$

We recall our notation (5.1) for the spin variables. We denote by S_n, Σ_n and S_λ, Σ_λ the components of the spins along the vectors $\mathbf{n}$ and $\boldsymbol{\lambda}$ respectively. As before we have neglected the SS terms.

To compare easily with previous results in the literature, we have also computed the precessing equations for the spin variables $\mathbf{S}_1$ and $\mathbf{S}_2$. They read

$$\frac{d\mathbf{S}_1}{dt}=\omega\,\nu\,x\left\{S_{1n}\,\boldsymbol{\lambda}\left(2+\frac{m_2}{m_1}\right)-2S_{1\lambda}\,\mathbf{n}\left(1+\frac{m_2}{m_1}\right)\right.$$
$$+x\,S_{1n}\,\boldsymbol{\lambda}\left(9\frac{m_1^2}{m^2}+\frac{37}{3}\frac{m_1\,m_2}{m^2}+\frac{11}{3}\frac{m_2^2}{m^2}\right)$$
$$\left.+x\,S_{1\lambda}\,\mathbf{n}\left(3\frac{m_1}{m}-\frac{7}{3}\frac{m_2}{m}-2\frac{m_2^2}{m\,m_1}\right)\right\}+\mathcal{O}\left(\frac{1}{c^6}\right)\,, \tag{6.11a}$$
$$\frac{d\mathbf{S}_2}{dt}=\omega\,\nu\,x\left\{S_{2n}\,\boldsymbol{\lambda}\left(2+\frac{m_1}{m_2}\right)-2S_{2\lambda}\,\mathbf{n}\left(1+\frac{m_1}{m_2}\right)\right.$$
$$+x\,S_{2n}\,\boldsymbol{\lambda}\left(9\frac{m_2^2}{m^2}+\frac{37}{3}\frac{m_1\,m_2}{m^2}+\frac{11}{3}\frac{m_1^2}{m^2}\right)$$
$$\left.+x\,S_{2\lambda}\,\mathbf{n}\left(3\frac{m_2}{m}-\frac{7}{3}\frac{m_1}{m}-2\frac{m_1^2}{m\,m_2}\right)\right\}+\mathcal{O}\left(\frac{1}{c^6}\right)\,, \tag{6.11b}$$

where $S_{1\lambda}, S_{1n}$ and $S_{2\lambda}, S_{2n}$ are the projections of the single-spin variables along $\boldsymbol{\lambda}$ and $\mathbf{n}$. We notice that with our choice of spin variables $\mathbf{S}_1$ and $\mathbf{S}_2$ the magnitude of the spin is not constant even when restricting Eqs. (6.11) to 1.5PN order. In Sec. VII we define some alternative spin variables $\mathbf{S}_1^{\rm c}$ and $\mathbf{S}_2^{\rm c}$, such that the magnitude of these spin vectors remains constant, *i.e.* $\mathbf{S}_i^{\rm c}\cdot d\mathbf{S}_i^{\rm c}/dt=0$ with $i=1,2$. The spin vectors $\mathbf{S}_i^{\rm c}$ agree with Kidder's [40] spin variables at the 1PN order, and generalize them to the next 2PN order. The main advantage of the definition $\mathbf{S}_i^{\rm c}$ is that the precession equations can be then written in the form $d\mathbf{S}_i^{\rm c}/dt=\mathbf{\Omega}_i\times\mathbf{S}_i^{\rm c}$, where $\mathbf{\Omega}_i$ are the precession angular frequency vectors (given in Sec. VII).

VII. SPIN VARIABLES WITH CONSTANT MAGNITUDE

In this paper and paper I we found convenient to use some specific spin variables $\mathbf{S}_1$ and $\mathbf{S}_2$, defined in Sec. II of paper I. However, as discussed in paper I, other papers in the literature use a definition of the spin variables different from ours. For example, the spin-precession equations at 1PN order in Ref. [40] read

$$\frac{d\mathbf{S}_1^{\rm c}}{dt} = \omega\,\nu\,x\,(\boldsymbol{\ell}\times\mathbf{S}_1^{\rm c})\left(2+\frac{3}{2}\frac{m_2}{m_1}\right)=\omega\,\nu\,x\,[S_{1n}^{\rm c}\,\boldsymbol{\lambda}-S_{1\lambda}^{\rm c}\,\mathbf{n}]\left(2+\frac{3}{2}\frac{m_2}{m_1}\right), \quad (7.1a)$$

$$\frac{d\mathbf{S}_2^{\rm c}}{dt} = \omega\,\nu\,x\,(\boldsymbol{\ell}\times\mathbf{S}_2^{\rm c})\left(2+\frac{3}{2}\frac{m_1}{m_2}\right)=\omega\,\nu\,x\,[S_{2n}^{\rm c}\,\boldsymbol{\lambda}-S_{2\lambda}^{\rm c}\,\mathbf{n}]\left(2+\frac{3}{2}\frac{m_1}{m_2}\right), \quad (7.1b)$$

where the superscript c stands for constant; in fact, the spin variables $\mathbf{S}_1^{\rm c}$, $\mathbf{S}_2^{\rm c}$ are such that their norm or magnitude remains constant. Indeed one can readily check from Eqs. (7.1) that $\mathbf{S}_i^{\rm c}\cdot d\mathbf{S}_i^{\rm c}/dt=0$ with $i=1,2$. Our spin variables are related at the 1PN order to the constant-spin ones (in the center-of-mass and for circular orbits) as

$$\mathbf{S}_1^{\rm c}=\left(1+\frac{Gm_2}{c^2r}\right)\mathbf{S}_1-\frac{m_2^2}{2c^2m^2}\,S_{1\lambda}\,r^2\,\omega^2\,\boldsymbol{\lambda}\,, \quad (7.2a)$$

$$\mathbf{S}_2^{\rm c}=\left(1+\frac{Gm_1}{c^2r}\right)\mathbf{S}_2-\frac{m_1^2}{2c^2m^2}\,S_{2\lambda}\,r^2\,\omega^2\,\boldsymbol{\lambda}\,. \quad (7.2b)$$

We can check that by taking the time derivative of the RHS of Eqs. (7.2), plugging in Eqs. (6.11) at 1PN order, we recover Eqs. (7.1). Note that the total angular momentum is invariant, since

$$\mathbf{J}=\mathbf{L}+\frac{1}{c}\mathbf{S}_1+\frac{1}{c}\mathbf{S}_2=\mathbf{L}^{\rm c}+\frac{1}{c}\mathbf{S}_1^{\rm c}+\frac{1}{c}\mathbf{S}_2^{\rm c}\,. \quad (7.3)$$

Let us now define, at the 2PN order, in a general frame and for general orbits, some spin variables reducing to $\mathbf{S}_1^{\rm c}$ and $\mathbf{S}_2^{\rm c}$ at the 1PN order, and such that the magnitude of these spins remains constant. We shall still denote the latter 2PN spins as $\mathbf{S}_1^{\rm c}$, $\mathbf{S}_2^{\rm c}$; thus, we shall have, at the 2PN order, $\mathbf{S}_i^{\rm c}\cdot d\mathbf{S}_i^{\rm c}/dt=0$ with $i=1,2$. First of all we find that the new spin variables are related to the ones used in previous sections (and in the whole of paper I) by

$$\begin{aligned}\mathbf{S}_1^{\rm c}=\mathbf{S}_1&+\frac{1}{c^2}\bigg[-\frac{1}{2}(v_1S_1)\mathbf{v}_1+\frac{Gm_2}{r_{12}}\mathbf{S}_1\bigg]\\&+\frac{1}{c^4}\bigg[\mathbf{n}_{12}\frac{Gm_2}{r_{12}}(n_{12}S_1)\Big(-4\frac{Gm_1}{r_{12}}+\frac{1}{2}\frac{Gm_2}{r_{12}}\Big)+\frac{1}{2}\frac{Gm_2}{r_{12}}\mathbf{S}_1\Big(-(n_{12}v_2)^2+\frac{Gm_1}{r_{12}}\Big)\\&\qquad+\mathbf{v}_1\bigg(-\frac{1}{8}(v_1S_1)v_1^2+\frac{Gm_2}{r_{12}}\Big(-\frac{5}{2}(v_1S_1)+4(v_2S_1)\Big)\bigg)+2\frac{Gm_2}{r_{12}}\mathbf{v}_2(v_2S_1)\bigg]\,,\end{aligned} \quad (7.4)$$

together with the expression for $\mathbf{S}_2^{\rm c}$ obtained by exchanging all the particle's labels $1 \leftrightarrow 2$. In Eq. (7.4) the notation is exactly the same as in paper I. The main advantage of such definition (7.4) is that the precession equations can now be written into the form

$$\frac{d\mathbf{S}_1^{\rm c}}{dt} = \boldsymbol{\Omega}_1 \times \mathbf{S}_1^{\rm c}\,, \tag{7.5a}$$

$$\frac{d\mathbf{S}_2^{\rm c}}{dt} = \boldsymbol{\Omega}_2 \times \mathbf{S}_2^{\rm c}\,, \tag{7.5b}$$

showing that the spins precess around the directions of $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$, and at the rates $|\boldsymbol{\Omega}_1|$ and $|\boldsymbol{\Omega}_2|$. The precession angular frequency vectors $\boldsymbol{\Omega}_1$, $\boldsymbol{\Omega}_2$ can be computed up to the 2PN order by using the precession equations (6.1)–(6.3) of paper I. We find

$$\begin{aligned}
\boldsymbol{\Omega}_1 = {} & \frac{G m_2}{c^2 r_{12}^2}\left[\frac{3}{2}\mathbf{n}_{12}\times\mathbf{v}_1 - 2\mathbf{n}_{12}\times\mathbf{v}_2\right] \\
& + \frac{G m_2}{c^4 r_{12}^2}\bigg[\mathbf{n}_{12}\times\mathbf{v}_1\Big(-\frac{9}{4}(n_{12}v_2)^2 + \frac{1}{8}v_1^2 - (v_1v_2) + v_2^2 + \frac{7}{2}\frac{Gm_1}{r_{12}} - \frac{1}{2}\frac{Gm_2}{r_{12}}\Big) \\
& \qquad + \mathbf{n}_{12}\times\mathbf{v}_2\Big(3(n_{12}v_2)^2 + 2(v_1v_2) - 2v_2^2 + \frac{Gm_1}{r_{12}} + \frac{9}{2}\frac{Gm_2}{r_{12}}\Big) \\
& \qquad + \mathbf{v}_1\times\mathbf{v}_2\Big(3(n_{12}v_1) - \frac{7}{2}(n_{12}v_2)\Big)\bigg]\,,
\end{aligned} \tag{7.6}$$

together with $1 \leftrightarrow 2$. In the center-of-mass frame we obtain

$$\begin{aligned}
\boldsymbol{\Omega}_1 = \frac{Gm}{r^2c^2}\bigg\{ & \frac{3}{4} + \frac{\nu}{2} + \frac{1}{c^2}\bigg[\frac{Gm}{r}\Big(-\frac{1}{4} - \frac{3}{8}\nu + \frac{\nu^2}{2}\Big) \\
& \qquad + \Big(-\frac{3}{2}\nu + \frac{3}{4}\nu^2\Big)(nv)^2 + \Big(\frac{1}{16} + \frac{11}{8}\nu - \frac{3}{8}\nu^2\Big)v^2\bigg] \\
& + \frac{\delta m}{m}\bigg(-\frac{3}{4} + \frac{1}{c^2}\bigg[\frac{Gm}{r}\Big(\frac{1}{4} - \frac{\nu}{8}\Big) - \frac{3}{2}\nu(nv)^2 + \Big(-\frac{1}{16} + \frac{\nu}{2}\Big)v^2\bigg]\bigg)\bigg\}\,\mathbf{n}\times\mathbf{v}\,.
\end{aligned} \tag{7.7}$$

For circular orbits,

$$\boldsymbol{\Omega}_1 = \frac{c^3 x^{5/2}}{Gm}\bigg\{\frac{3}{4} + \frac{\nu}{2} + x\Big(\frac{9}{16} + \frac{5}{4}\nu - \frac{\nu^2}{24}\Big) + \frac{\delta m}{m}\bigg[-\frac{3}{4} + x\Big(-\frac{9}{16} + \frac{5}{8}\nu\Big)\bigg]\bigg\}\,\boldsymbol{\ell}\,. \tag{7.8}$$

Note that to obtain $\boldsymbol{\Omega}_2$ we simply have to change $\delta m \to -\delta m$. Recall that all these expressions are valid at linear order in the spins (excluding the SS terms); this means that with this approximation $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$ are independent of the spins.

Finally let us express some of the main results of this paper in terms of the spin variables with constant magnitude. For the spin-dependent part of the circular-orbit energy [Eqs. (6.6) or (6.7)] we get

$$\begin{aligned}
\underset{\rm S}{E} = -\frac{\mu\,c^2\,x}{2}\bigg\{ & 1 + \frac{x^{3/2}}{G\,m^2}\left[\frac{14}{3}S_\ell^{\rm c} + 2\frac{\delta m}{m}\Sigma_\ell^{\rm c}\right] \\
& + \frac{x^{5/2}}{G\,m^2}\left[\left(11 - \frac{61}{9}\nu\right)S_\ell^{\rm c} + \left(3 - \frac{10}{3}\nu\right)\frac{\delta m}{m}\Sigma_\ell^{\rm c}\right] + \mathcal{O}\left(\frac{1}{c^6}\right)\bigg\}\,,
\end{aligned} \tag{7.9}$$

where $S^{\rm c}_\ell \equiv \mathbf{S}^{\rm c}\cdot\boldsymbol{\ell}$ and $\Sigma^{\rm c}_\ell \equiv \boldsymbol{\Sigma}^{\rm c}\cdot\boldsymbol{\ell}$, with $\mathbf{S}^{\rm c} = \mathbf{S}^{\rm c}_1 + \mathbf{S}^{\rm c}_2$ and $\boldsymbol{\Sigma}^{\rm c} = \frac{m}{m_2}\mathbf{S}^{\rm c}_2 - \frac{m}{m_1}\mathbf{S}^{\rm c}_1$. Using the energy (7.9) as function of the constant spin variables, we have computed the ICO. For an equal-mass binary with spins anti-aligned with the orbital angular momentum, at 2.5PN (3PN) order, we get $E_{\rm ICO}/m = -0.0122$ ($E_{\rm ICO}/m = -0.0119$) and $m\omega_{\rm ICO} = 0.064$ ($m\omega_{\rm ICO} = 0.061$) to be compared with the numbers listed in Table I. The difference is not negligible.

We also computed the spin-dependent part of the orbital angular momentum and the flux in terms of the spin variables with constant magnitude,

$$\begin{aligned}
\mathop{\mathbf{L}^{\rm c}}_{\rm S} &= \frac{G\,m^2}{c}\,\nu\,x^{-1/2}\Bigg\{\boldsymbol{\ell}\Bigg[\left(-\frac{35}{6}S^{\rm c}_\ell - \frac{5}{2}\frac{\delta m}{m}\Sigma^{\rm c}_\ell\right)\frac{x^{3/2}}{G\,m^2}\\
&\quad + \left(\left(-\frac{77}{8}+\frac{427}{72}\nu\right)S^{\rm c}_\ell + \left(-\frac{21}{8}+\frac{35}{12}\nu\right)\frac{\delta m}{m}\Sigma^{\rm c}_\ell\right)\frac{x^{5/2}}{G\,m^2}\Bigg]\\
&\quad + \frac{x^{3/2}}{Gm^2}\boldsymbol{\lambda}\Bigg[-3\,S^{\rm c}_\lambda - \Sigma^{\rm c}_\lambda\frac{\delta m}{m} + \left(\left(-\frac{7}{2}+3\nu\right)S^{\rm c}_\lambda + \left(-\frac{1}{2}+\frac{4}{3}\nu\right)\frac{\delta m}{m}\Sigma^{\rm c}_\lambda\right)x\Bigg]\\
&\quad + \frac{x^{3/2}}{Gm^2}\mathbf{n}\Bigg[\frac{1}{2}S^{\rm c}_n + \frac{1}{2}\Sigma^{\rm c}_n\frac{\delta m}{m} + \left(\left(\frac{11}{8}-\frac{19}{24}\nu\right)S^{\rm c}_n + \left(\frac{11}{8}-\frac{5}{12}\nu\right)\frac{\delta m}{m}\Sigma^{\rm c}_n\right)x\Bigg]\Bigg\}\,, \qquad (7.10)\\
\mathop{\mathcal{F}}_{\rm S} &= \frac{32}{5}\frac{c^5}{G}x^5\,\nu^2\Bigg\{1 + \frac{x^{3/2}}{G\,m^2}\left[-4S^{\rm c}_\ell - \frac{5}{4}\frac{\delta m}{m}\Sigma^{\rm c}_\ell\right]\\
&\quad + \frac{x^{5/2}}{G\,m^2}\left[\left(-\frac{9}{2}+\frac{272}{9}\nu\right)S^{\rm c}_\ell + \left(-\frac{13}{16}+\frac{43}{4}\nu\right)\frac{\delta m}{m}\Sigma^{\rm c}_\ell\right] + \mathcal{O}\left(\frac{1}{c^6}\right)\Bigg\}\,. \qquad (7.11)
\end{aligned}$$

VIII. PHASE EVOLUTION AND ACCUMULATED NUMBER OF GW CYCLES

In this Section we compute the time evolution of the binary's orbital frequency ω, which results from the gravitational radiation reaction damping force. Instead of computing directly the radiation reaction force, we use the standard energy balance argument

$$\mathcal{F} = -\frac{dE}{dt}\,, \qquad (8.1)$$

where $\mathcal{F}$ is the total emitted GW energy flux computed in Sec. V, and E denotes the binary's center-of-mass energy, namely the integral of the motion associated with the conservative part of the equations of motion, and which has been computed in paper I. Using $\mathcal{F}[x]$ and $E[x]$ expressed in terms of the spin variables with constant magnitude by Eqs. (7.9) and (7.11), we deduce $\dot\omega$ from (8.1), which is equivalent to

$$\frac{\dot\omega}{\omega} = -\frac{3}{2x}\left(\frac{dE\,[x]}{dx}\right)^{-1}\mathcal{F}\,[x]\,. \qquad (8.2)$$

Notice that for this calculation it is crucial to use the variables associated with the constant magnitude spins, $\mathbf{S}^{\rm c}$ and $\boldsymbol{\Sigma}^{\rm c}$ (rather than our original spin variables $\mathbf{S}$ and $\boldsymbol{\Sigma}$ [8]), since the spin variables $\mathbf{S}^{\rm c}$ and $\boldsymbol{\Sigma}^{\rm c}$ are secularly constant, *i.e.* do not evolve by gravitational radiation

[8] We are grateful to A. Gopakumar and G. Schäfer for pointing out this fact to us.

reaction. A proof that $\mathbf{S}^{\rm c}$ and $\mathbf{\Sigma}^{\rm c}$ are secularly constant (to the considered order) can be found in Ref. [56]. Hence the balance equation in the form of Eq. (8.2) gives directly the secular evolution of the orbital frequency.

During this computation the standard PN approximation is applied, *i.e.* we expand both the numerator and the denominator of Eq. (8.2) in the usual PN way, and finally express the result as a Taylor series in x. Other ways of addressing the computation, using particular PN resummation techniques, can be found in Refs. [14, 57] and references therein. We give the end result for the parameter $\xi \equiv \dot{\omega}/\omega^2$, which can be viewed as the dimensionless adiabatic parameter associated with the gradual inspiral, and which is dominantly of 2.5PN order (namely, the order of radiation reaction). In the final result, as everywhere else, the SO effects are at the 1.5PN and 2.5PN orders beyond the dominant approximation. We get

$$\begin{aligned}\frac{\dot{\omega}}{\omega^2} =& \frac{96}{5}\,\nu\,x^{5/2}\left\{1+x\left(-\frac{743}{336}-\frac{11}{4}\nu\right)+4\pi x^{3/2}\right.\\&+x^2\left(\frac{34103}{18144}+\frac{13661}{2016}\nu+\frac{59}{18}\nu^2\right)+\pi x^{5/2}\left(-\frac{4159}{672}-\frac{189}{8}\nu\right)\\&+\frac{x^{3/2}}{G\,m^2}\left[-\frac{47}{3}S^{\rm c}_\ell-\frac{25}{4}\frac{\delta m}{m}\Sigma^{\rm c}_\ell\right]\\&\left.+\frac{x^{5/2}}{G\,m^2}\left[\left(-\frac{5861}{144}+\frac{1001}{12}\nu\right)S^{\rm c}_\ell+\left(-\frac{809}{84}+\frac{281}{8}\nu\right)\frac{\delta m}{m}\Sigma^{\rm c}_\ell\right]\right\}\,. \end{aligned} \tag{8.3}$$

If necessary the non-spin contributions at orders 3PN and 3.5PN can be straightforwardly added [20, 23].

Equations (7.5), (7.8), (7.9), (7.11) and (8.3) with non-spin terms added through 3.5PN order and spin-spin terms included, together with the equation describing the rate of change of the orbital angular-momentum direction (deduced from $\dot{\mathbf{L}}^{\rm c} = -\frac{1}{c}\dot{\mathbf{S}}^{\rm c}_1 - \frac{1}{c}\dot{\mathbf{S}}^{\rm c}_2$ at leading order) and the radiation field (see, e.g., Sec. C and Appendix B in Ref. [40]), can be solved semi-analytically, for special spin and mass configurations, or numerically. They provide more accurate templates than currently used in the literature [8–19] and would need to be implemented for the search of GWs from spinning, precessing binaries.

In the general case, taking into account the effect of precession of the orbital plane induced by spin modulations, the GW phase $\Phi_{\rm GW}$ is given by $\Phi_{\rm GW} = \phi_{\rm GW} + \delta\phi_{\rm GW}$, where $\phi_{\rm GW}$ is the "carrier phase", defined by $\phi_{\rm GW} = 2\phi$ with $\phi = \int \omega\, dt$, and $\delta\phi_{\rm GW}$ is a standard precessional correction, arising from the changing orientation of the orbital plane. The precessional correction $\delta\phi_{\rm GW}$ can be computed by standard methods using numerical integration, see Ref. [9]. Thus, the carrier phase $\phi_{\rm GW}$ constitutes the main theoretical output to be provided for the templates, and can directly be computed numerically from our main result, Eq. (8.3). In absence of orbital-plane's precession, *e.g.*, for spins aligned or anti-aligned with the orbital angular momentum, the GW phase reduces to $\phi_{\rm GW}$, and the latter can be obtained by integrating *analytically* Eq. (8.3). We get

$$\begin{aligned}\phi =& \phi_0 - \frac{1}{32\nu}\left\{x^{-5/2}+x^{-3/2}\left(\frac{3715}{1008}+\frac{55}{12}\,\nu\right)+\frac{x^{-1}}{Gm^2}\left(\frac{235}{6}S^{\rm c}_\ell+\frac{125}{8}\frac{\delta m}{m}\Sigma^{\rm c}_\ell\right)\right.\\&-10\pi\,x^{-1}+x^{-1/2}\left(\frac{15293365}{1016064}+\frac{27145}{1008}\,\nu+\frac{3085}{144}\,\nu^2\right)+\pi\,\ln x\left(\frac{38645}{1344}-\frac{65}{16}\,\nu\right)\\&\left.+\frac{\ln x}{Gm^2}\left[\left(-\frac{554345}{2016}-\frac{55}{8}\,\nu\right)S^{\rm c}_\ell+\left(-\frac{41745}{448}+\frac{15}{8}\,\nu\right)\frac{\delta m}{m}\Sigma^{\rm c}_\ell\right]\right\}\,,\end{aligned} \tag{8.4}$$

where ϕ_0 denotes some constant phase. In terms of the single-spin variables $\mathbf{S}_1^{\mathrm{c}}$ and $\mathbf{S}_2^{\mathrm{c}}$, the spin-dependent part of the above equation reads

$$\underset{\mathrm{S}}{\phi} = -\frac{1}{32\nu} \sum_{i=1,2} \chi_i^{\mathrm{c}}\, \kappa_i^{\mathrm{c}} \left\{ \left(\frac{565}{24} \frac{m_i^2}{m^2} + \frac{125}{8}\nu \right) x^{-1} \right.$$
$$\left. + \left[\left(-\frac{732985}{4032} - \frac{35}{4}\nu \right) \frac{m_i^2}{m^2} + \left(-\frac{41745}{448} + \frac{15}{8}\nu \right) \nu \right] \ln x \right\} , \tag{8.5}$$

where χ_i^{c} and κ_i^{c} are defined by $\mathbf{S}_i^{\mathrm{c}} = G\, m_i^2\, \chi_i^{\mathrm{c}}\, \hat{\mathbf{S}}_i^{\mathrm{c}}$ and $\kappa_i^{\mathrm{c}} = \hat{\mathbf{S}}_i^{\mathrm{c}} \cdot \boldsymbol{\ell}$. The number of accumulated GW cycles between some minimal and maximal frequencies is

$$\mathcal{N}_{\mathrm{GW}} = \frac{\phi_{\max} - \phi_{\min}}{\pi} . \tag{8.6}$$

We list in Tables II and III the number of accumulated GW cycles (8.6) for typical binary masses in the most sensitive frequency band of ground-based and space-based detectors. For comparison we also show the contribution due to spin-spin terms at 2PN order evaluated in Ref. [39], as well as those due to the non-spin 3PN and 3.5PN orders computed in [20, 23]. We denote $\xi^{\mathrm{c}} = \hat{\mathbf{S}}_1^{\mathrm{c}} \cdot \hat{\mathbf{S}}_2^{\mathrm{c}}$. From the Tables II and III we deduce two important results of this paper. First, we see that at 2.5PN order, if spins are maximal, *i.e.* $\chi_i^{\mathrm{c}} = 1$, the number of GW cycles due to spin couplings is comparable to the number of GW cycles due to non-spin terms. Secondly, we find that for small mass-ratio binaries, the number of GW cycles due to linear spins at 2.5PN order can be much larger than the number of GW cycles due to spin-spin terms at 2PN order. These results thus show that the 2.5PN spin terms evaluated in the present paper have to be included in the GW templates if we want to extract accurately the binary parameters.

TABLE II: Post-Newtonian contributions to the number of GW cycles (8.6) accumulated from $\omega_{\min} = \pi \times 10\,\mathrm{Hz}$ to $\omega_{\max} = \omega_{\mathrm{ISCO}} = 1/(6^{3/2}\, m)$ for binaries detectable by LIGO and Virgo. For comparison, we add the contributions of spin-spin terms at 2PN order (we denote $\xi^{\mathrm{c}} = \hat{\mathbf{S}}_1^{\mathrm{c}} \cdot \hat{\mathbf{S}}_2^{\mathrm{c}}$) and non-spin terms at 3PN and 3.5PN orders.

	$(10+1.4)M_\odot$	$(10+10)M_\odot$	$(1.4+1.4)M_\odot$
Newtonian	3577	601	16034
1PN	+213	+59.3	+441
1.5PN	$-181 + 114\,\kappa_1^{\mathrm{c}}\,\chi_1^{\mathrm{c}} + 11.8\,\kappa_2^{\mathrm{c}}\,\chi_2^{\mathrm{c}}$	$-51.4 + 16.0\,\kappa_1^{\mathrm{c}}\,\chi_1^{\mathrm{c}} + 16.0\,\kappa_2^{\mathrm{c}}\,\chi_2^{\mathrm{c}}$	$-211 + 65.7\,\kappa_1^{\mathrm{c}}\,\chi_1^{\mathrm{c}} + 65.7\,\kappa_2^{\mathrm{c}}\,\chi_2^{\mathrm{c}}$
2PN	$+9.8 - 4.4\,\kappa_1^{\mathrm{c}}\,\kappa_2^{\mathrm{c}}\,\chi_1^{\mathrm{c}}\,\chi_2^{\mathrm{c}} + 1.5\,\xi^{\mathrm{c}}\,\chi_1^{\mathrm{c}}\,\chi_2^{\mathrm{c}}$	$+4.1 - 3.3\,\kappa_1^{\mathrm{c}}\,\kappa_2^{\mathrm{c}}\,\chi_1^{\mathrm{c}}\,\chi_2^{\mathrm{c}} + 1.1\,\xi^{\mathrm{c}}\,\chi_1^{\mathrm{c}}\,\chi_2^{\mathrm{c}}$	$+9.9 - 8.0\,\kappa_1^{\mathrm{c}}\,\kappa_2^{\mathrm{c}}\,\chi_1^{\mathrm{c}}\,\chi_2^{\mathrm{c}} + 2.8\,\xi^{\mathrm{c}}\,\chi_1^{\mathrm{c}}\,\chi_2^{\mathrm{c}}$
2.5PN	$-20 + 33.9\,\kappa_1^{\mathrm{c}}\,\chi_1^{\mathrm{c}} + 2.9\,\kappa_2^{\mathrm{c}}\,\chi_2^{\mathrm{c}}$	$-7.1 + 5.7\,\kappa_1^{\mathrm{c}}\,\chi_1^{\mathrm{c}} + 5.7\,\kappa_2^{\mathrm{c}}\,\chi_2^{\mathrm{c}}$	$-11.7 + 9.3\,\kappa_1^{\mathrm{c}}\,\chi_1^{\mathrm{c}} + 9.3\,\kappa_2^{\mathrm{c}}\,\chi_2^{\mathrm{c}}$
3PN	+2.3	+2.2	+2.6
3.5PN	−1.8	−0.8	−0.9

The number of accumulated GW cycles can be a useful diagnostic to understand the importance of spin effects, but taken alone it provides incomplete information. First, $\mathcal{N}_{\mathrm{GW}}$ is related only to the number of orbital cycles of the binary within the orbital plane, but it does not reflect the precession of the plane, which modulates the wave form in both amplitude and phase. These modulations are important effects. In fact, it has been shown [13, 14, 16] that neither the standard non-spinning-binary templates (which do not have built-in modulations) nor the original Apostolatos templates [10] (which add only modulations to the

TABLE III: Post-Newtonian contributions to the number of GW cycles (8.6) accumulated until $\omega_{\rm max} = \omega_{\rm ISCO} = 1/(6^{3/2}\, m)$ over one year of integration, for binaries detectable by LISA. For comparison, we add the contribution of spin-spin terms at 2PN order (we denote $\xi^{\rm c} = \hat{\mathbf{S}}_1^{\rm c} \cdot \hat{\mathbf{S}}_2^{\rm c}$) and non-spin terms at 3PN and 3.5PN orders.

	$(10^6 + 10^6) M_\odot$	$(10^6 + 10^5) M_\odot$	$(10^5 + 10^5) M_\odot$
Newtonian	2267	4985	9570
1PN	$+134$	$+281$	$+323$
1.5PN	$-92.4 + 28.8\, \kappa_1^{\rm c}\, \chi_1^{\rm c} + 28.8\, \kappa_2^{\rm c}\, \chi_2^{\rm c}$	$-243 + 161\, \kappa_1^{\rm c}\, \chi_1^{\rm c} + 11.5\, \kappa_2^{\rm c}\, \chi_2^{\rm c}$	$-170 + 53\, \kappa_1^{\rm c}\, \chi_1^{\rm c} + 53\, \kappa_2^{\rm c}\, \chi_2^{\rm c}$
2PN	$+6.0 - 4.8\, \kappa_1^{\rm c}\, \kappa_2^{\rm c}\, \chi_1^{\rm c}\, \chi_2^{\rm c} + 1.7\, \xi^{\rm c}\, \chi_1^{\rm c}\, \chi_2^{\rm c}$	$+12.5 - 4.4\, \kappa_1^{\rm c}\, \kappa_2^{\rm c}\, \chi_1^{\rm c}\, \chi_2^{\rm c} + 1.5\, \xi^{\rm c}\, \chi_1^{\rm c}\, \chi_2^{\rm c}$	$+8.7 - 7.1\, \kappa_1^{\rm c}\, \kappa_2^{\rm c}\, \chi_1^{\rm c}\, \chi_2^{\rm c} + 2.4\, \xi^{\rm c}\, \chi_1^{\rm c}\, \chi_2^{\rm c}$
2.5PN	$-9.0 + 7.1\, \kappa_1^{\rm c}\, \chi_1^{\rm c} + 7.1\, \kappa_2^{\rm c}\, \chi_2^{\rm c}$	$-26.5 + 47.0\, \kappa_1^{\rm c}\, \chi_1^{\rm c} + 2.7\, \kappa_2^{\rm c}\, \chi_2^{\rm c}$	$-11.0 + 8.7\, \kappa_1^{\rm c}\, \chi_1^{\rm c} + 8.7\, \kappa_2^{\rm c}\, \chi_2^{\rm c}$
3PN	$+2.3$	$+2.3$	$+2.5$
3.5PN	-0.9	-2.3	-0.9

phase) can reproduce satisfactorily the detector response to the GWs emitted by precessing binaries. Modulations both in the phase and the amplitude of the wave form must be included [14, 15, 17, 19]. Second, even if two signals have $\mathcal{N}_{\rm GW}$ that differ by ~ 1, one can always shift their arrival times to obtain higher overlaps, but at the cost of introducing systematic errors in the binary parameters. To quantify the impact of the 2.5PN spin terms in detecting GWs from spinning, precessing binaries, one should evaluate the maximized overlap (fitting factor) between the 2.5PN template family and the 2PN template family used in Refs. [14, 15, 17, 19]. Those template families are defined by the GW signal computed along the binary evolution together with the spin and angular momentum precession equations. We expect that the maximized overlap between the 2.5PN and 2PN templates could be high, because the spins and the directional parameters entering the template families provide much leeway to compensate for non-trivial variations of the phasing (see *e.g.*, Table II in Ref. [15] where maximized overlaps between several PN templates were computed). This study goes beyond the goal of this paper and will be tackled in future work.

IX. CONCLUSIONS

Within the multipole-moment formalism developed in Refs. [25–30], we obtained the SO couplings, 1PN order beyond the dominant effect, in the binary's mass and current quadrupole moments, as well as in the GW energy flux. The current-quadrupole moment with SO couplings at 1.5PN order was derived in Ref. [31], but our result differs from the expression computed there for two reasons. (i) The authors of Ref. [31] neglected the non-compact support terms which originate from the non-linearities of the Einstein field equations and are not negligible at this order. (ii) Their result for the compact-support terms is affected by a computational error. The mass-quadrupole moment with SO couplings at 2.5PN order (including all compact and non-compact support terms) is computed here for the first time.

The binary's energy and the spin precession equations including SO couplings through 2.5PN order were computed in paper I. They were used to derive the secular evolution of the binary's orbital phase through 2.5PN order in the spins. We found that the 2.5PN terms give a relevant contribution to the number of accumulated GW cycles within the binary's orbital plane. In Tables II and III, we listed the number of GW cycles for typical binaries detectable with ground-based and space-based detectors, such as LIGO/Virgo and LISA. When spins are maximal, the SO contribution at 2.5PN order is comparable to that of the non-spin part at the same 2.5PN order. For some binary mass-configurations, the SO contribution at 2.5PN order can be larger than that of SS couplings at 2PN order.

In order to extract accurately the parameters of maximally or mildly spinning binaries with ground-based detectors of first generation, having typical signal-to-noise ratio (SNR) of the order of 10, we expect the spin corrections through 3.5PN order to be sufficient. With space-based detectors having SNR of the order of 10^2–10^3, we would need *a priori* to compute non-spin and spin corrections at much higher PN order (for parameter estimation). For what concerns the impact of the SO couplings at 2.5PN order on the actual detection, it would be relevant to evaluate the maximized overlaps between templates that include SO effects through 2.5PN order against templates that include SO and SS effects through 2PN order. We anticipate that, at the cost of introducing systematic errors in the estimation of the binary parameters, the maximized overlaps could be high. In fact, the binary and directional parameters may compensate variations in the PN phasing.

For future applications, we listed in Sec. VII the relevant equations defining the spinning dynamics and the GW phasing in terms of the constant spin variables. Such formulation is broadly used in the literature to define spinning, precessing templates for compact binaries [9–19, 39, 40].

Finally, we computed the contributions of the spin terms to the location of the innermost circular orbit in the case of black-hole binaries. The results for equal-mass objects with maximal spins are summarized in Table I. Spin couplings at 1.5PN and 2.5PN orders can give significant contributions to the energy and frequency at the ICO (and nearby).

Acknowledgments

We are grateful to Achamveedu Gopakumar and Gerhard Schäfer for pointing out to us that without calculating the secular effects to the spin precession equations, i.e., $[\dot{\mathbf{S}}_1]_{\rm sec}$ and $[\dot{\mathbf{S}}_2]_{\rm sec}$, it is not possible to derive from the balance equation the expression of $\dot{\omega}/\omega^2$ and compute the GW phasing $\phi(\omega, \mathbf{S}_1, \mathbf{S}_2, \boldsymbol{\ell})$.

---

[1] G. Faye, L. Blanchet, and A. Buonanno, Phys. Rev. D **74**, 104033 (2006), gr-qc/0605139.
[2] C. Cutler and E. Flanagan, Phys. Rev. D **49**, 2658 (1994).
[3] E. Poisson and C. Will, Phys. Rev. D **52**, 848 (1995).
[4] A. Krolàk, K. Kokkotas, and G. Schäfer, Phys. Rev. D **52**, 2089 (1995).
[5] C. Cutler, Phys. Rev. D **57**, 7089 (1998), gr-qc/9703068.
[6] A. Vecchio, Phys. Rev. D **70**, 042001 (2004), astro-ph/0304051.
[7] E. Berti, A. Buonanno, and C. Will, Phys. Rev. D **71**, 084025 (2005), gr-qc/0411129.
[8] C. Cutler, T. Apostolatos, L. Bildsten, L. Finn, E. Flanagan, D. Kennefick, D. Markovic, A. Ori, E. Poisson, G. Sussman, et al., Phys. Rev. Lett. **70**, 2984 (1993).
[9] T. Apostolatos, C. Cutler, G. Sussman, and K. Thorne, Phys. Rev. D **49**, 6274 (1994).
[10] T. Apostolatos, Phys. Rev. D **52**, 605 (1995).
[11] T. Apostolatos, Phys. Rev. D **54**, 2421 (1996).
[12] T. Apostolatos, Phys. Rev. D **54**, 2438 (1996).

[13] P. Grandclément, V. Kalogera, and A. Vecchio, Phys. Rev. D **67**, 042003 (2003), gr-qc/0207062.
[14] A. Buonanno, Y. Chen, and M. Vallisneri, Phys. Rev. D **67**, 104025 (2003), gr-qc/0211087.
[15] Y. Pan, A. Buonanno, Y. Chen, and M. Vallisneri, Phys. Rev. D **69**, 104017 (2004), gr-qc/0310034.
[16] P. Grandclément and V. Kalogera, Phys. Rev. D **67**, 082002 (2003), gr-qc/0405090.
[17] A. Buonanno, Y. Chen, Y. Pan, and M. Vallisneri, Phys. Rev. D **70**, 104003 (2004), erratum Phys. Rev. D **74**, 029902 (2006), gr-qc/0405090.
[18] P. Grandclément, M. Ihm, V. Kalogera, and K. Belczynski, Phys. Rev. D **69**, 102002 (2004), gr-qc/0312084.
[19] A. Buonanno, Y. Chen, Y. Pan, H. Tagoshi, and M. Vallisneri, Phys. Rev. D **72**, 084027 (2005), gr-qc/0508064.
[20] L. Blanchet, G. Faye, B. R. Iyer, and B. Joguet, Phys. Rev. D **65**, 061501(R) (2002), erratum Phys. Rev. D **71**, 129902(E) (2005), gr-qc/0105099.
[21] L. Blanchet, Living Rev. Rel. **9**, 4 (2006), gr-qc/0202016.
[22] K. Arun, L. Blanchet, B. R. Iyer, and M. S. Qusailah, Class. Quant. Grav. **21**, 3771 (2004), erratum Class. Quant. Grav. **22**, 3115 (2005), gr-qc/0404085.
[23] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer, Phys. Rev. Lett. **93**, 091101 (2004), gr-qc/0406012.
[24] H. Tagoshi, A. Ohashi, and B. Owen, Phys. Rev. D **63**, 044006 (2001), gr-qc/0010014.
[25] L. Blanchet and T. Damour, Phil. Trans. Roy. Soc. Lond. A **320**, 379 (1986).
[26] L. Blanchet, Proc. Roy. Soc. Lond. A **409**, 383 (1987).
[27] L. Blanchet and T. Damour, Annales Inst. H. Poincaré Phys. Théor. **50**, 377 (1989).
[28] L. Blanchet, Phys. Rev. D **51**, 2559 (1995), gr-qc/9501030.
[29] L. Blanchet, Phys. Rev. D **54**, 1417 (1996), erratum Phys. Rev. D **71**, 129904(E) (2005), gr-qc/9603048.
[30] L. Blanchet, Class. Quant. Grav. **15**, 1971 (1998), gr-qc/9801101.
[31] B. Owen, H. Tagoshi, and A. Ohashi, Phys. Rev. D **57**, 6168 (1998), gr-qc/9710134.
[32] A. Papapetrou, Proc. R. Soc. London A **209**, 248 (1951).
[33] B. Barker and R. O'Connell, Phys. Rev. D **12**, 329 (1975).
[34] B. Barker and R. O'Connell, Gen. Relativ. Gravit. **11**, 149 (1979).
[35] I. Bailey and W. Israel, Ann. Phys. **130**, 188 (1980).
[36] W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D **73**, 104029 (2006), hep-th/0409156.
[37] R. Porto, Phys. Rev. D **73**, 104031 (2006), gr-qc/0511061.
[38] R. Porto and I. Z. Rothstein, Phys. Rev. Lett. **97**, 021101 (2006), gr-qc/0604099.
[39] L. Kidder, C. Will, and A. Wiseman, Phys. Rev. D **47**, R4183 (1993).
[40] L. Kidder, Phys. Rev. D **52**, 821 (1995), gr-qc/9506022.
[41] H. Cho, Class. Quant. Grav. **15**, 2465 (1998), gr-qc/9703071.
[42] L. Gergely, Phys. Rev. D **61**, 024035 (1999), gr-qc/9911082.
[43] B. Mikóczi, M. Vasúth, and L. Gergely, Phys. Rev. D **71**, 124043 (2005), astro-ph/0504538.
[44] Y. Mino, M. Shibata, and T. Tanaka, Phys. Rev. D **53**, 622 (1996).
[45] T. Tanaka, Y. Mino, M. Sasaki, and Shibata, Phys. Rev. D **54**, 3762 (1996), gr-qc/9602038.
[46] L. Blanchet and T. Damour, Phys. Rev. D **46**, 4304 (1992).
[47] L. Blanchet, Class. Quant. Grav. **15**, 113 (1998), erratum Class. Quant. Grav. **22**, 3381 (2005), gr-qc/9710038.
[48] L. Blanchet, G. Faye, and B. Ponsot, Phys. Rev. D **58**, 124002 (1998), gr-qc/9804079.

[49] L. Blanchet and G. Faye, J. Math. Phys. **41**, 7675 (2000), gr-qc/0004008.
[50] L. Blanchet and G. Schäfer, Mon. Not. Roy. Astron. Soc. **239**, 845 (1989).
[51] L. Blanchet and B. R. Iyer, Phys. Rev. D **71**, 024004 (2005), gr-qc/0409094.
[52] R. Wagoner and C. Will, Astrophys. J. **210**, 764 (1976).
[53] H. Tagoshi, M. Shibata, T. Tanaka, and M. Sasaki, Phys. Rev. D **54**, 1439 (1996), arXiv:gr-qc/9603028.
[54] L. Blanchet and B. R. Iyer, Class. Quant. Grav. **20**, 755 (2003), gr-qc/0209089.
[55] L. Blanchet, Phys. Rev. D **65**, 124009 (2002), gr-qc/0112056.
[56] C. Will, Phys. Rev. D **71**, 084027 (2005), gr-qc/0502039.
[57] A. Buonanno, Y. Chen, and M. Vallisneri, Phys. Rev. D **67**, 024016 (2003), erratum Phys. Rev. D **74**, 029903 (2006), gr-qc/0205122.