

## ON PEELING PROCEDURE APPLIED TO A POISSON POINT PROCESS

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Main results of this paper were obtained together with Alexander Nagaev, with whom the first author had collaborated for more than 35 years, until Alexander's tragic death in 2005. Since then, we have gathered strength and finalised this paper, strongly feeling Alexander's absence - our memories of him will stay with us forever.

**ABSTRACT.** In the focus of our attention is the asymptotic properties of the sequence of convex hulls which arise as a result of a peeling procedure applied to the convex hull generated by a Poisson point process. Processes of the considered type are tightly connected with empirical point processes and stable random vectors. Results are given about the limit shape of the convex hulls in the case of a discrete spectral measure. We give some numerical experiments to illustrate the peeling procedure for a larger class of Poisson point processes. **Keywords :** Control measure ; convex hull ; limiting shape ; peeling ; Poisson point processes ; stable vectors.

### 1. INTRODUCTION

Consider a Poisson point process (p.p.p.)  $\pi = \pi_{\alpha, \nu}$  with points scattered over  $\mathbb{R}^d$ . Identify  $\mathbb{R}^d \setminus \{\mathbf{0}\}$  with  $\mathbb{R}_+^1 \times S^{d-1}$  where  $S^{d-1}$  is the unit sphere. Assume that the intensity measure of this process  $\mu$  is of the form

$$(1) \quad \mu = \theta \times \nu$$

where

- (1)  $\theta$  is the absolutely continuous measure on  $\mathbb{R}_+^1$  determined by the density function

$$\frac{d\theta}{d\lambda}(r) = \alpha r^{-\alpha-1}, \quad r > 0,$$

$\lambda$  is the Lebesgue measure in  $\mathbb{R}^1$ ,  $\alpha > 0$  is a parameter while

- (2)  $\nu$ , called the spectral density, is a bounded measure on the  $\sigma$ -algebra  $\mathcal{B}_{S^{d-1}}$  of the Borel subsets of  $S^{d-1}$ . Without loss of generality we assume that  $\nu(S^{d-1}) = 1$ . We denote by  $S_\nu$  the support of  $\nu$ .

The representation (1) means that for any Borel  $A \subset \mathbb{R}^1$  and  $E \subset S^{d-1}$

$$\mu \left\{ x \mid |x| \in A, e_x \in E \right\} = \theta(A)\nu(E).$$

where  $e_x = |x|^{-1}x$ , for all  $x \in \mathbb{R}^d$ .

We assume that the Poisson point process  $\pi_{\alpha, \nu}$  is non-unilateral. It means that  $\nu$  is supported by a set  $S_\nu \subset S^{d-1}$  such that the cone  $\text{cone}(S_\nu)$  generated by  $S_\nu$  coincides with  $\mathbb{R}^d$ .

Let  $B(\mathbf{0}, r)$  denote the ball of a radius  $r$  centred at the origin. It is easily seen that for any  $\delta > 0$

$$\mu(B(\mathbf{0}, \delta)) = \infty, \quad \mu(\mathbb{R}^d \setminus B(\mathbf{0}, \delta)) < \infty.$$

It implies that with probability 1 in any neighbourhood of the origin there are infinitely many points of  $\pi$  while  $\pi(\mathbb{R}^d \setminus B(\mathbf{0}, \delta))$  is finite.

The interest to the point processes controlled by (1) is explained by the following facts.

Let  $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)}$  be independent copies of a random vector  $\xi$  such that the function  $\mathbb{P}\{|\xi| > r\}$  regularly varies as  $r \rightarrow \infty$  with the exponent  $-\alpha$  and the measures  $(\nu_r)_r$  defined by

$$\nu_r(E) = \mathbb{P}\{e_\xi \in E : |\xi| > r\}, \quad E \in \mathcal{B}_{S^{d-1}},$$

weakly converge to  $\nu$  on  $B^c(\mathbf{0}, \tau)$  for any  $\tau > 0$ .

Consider the empirical point process  $\beta_n$  generated by  $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)}$  or, more precisely, by the random set

$$\zeta_1^{(n)} = \{b_n^{-1}\xi^{(1)}, \dots, b_n^{-1}\xi^{(n)}\} = \{\hat{\xi}^{(1)}, \dots, \hat{\xi}^{(n)}\}$$

where

$$b_n = \inf \{r : n\mathbb{P}(e_\xi \in E \mid |\xi| > r) \leq 1\}.$$

It is easily seen that the point process  $\beta_n$  weakly converges to  $\pi = \pi_{\alpha, \nu}$  (see e.g. [5], Prop. 3.21). Thus, each  $\pi_{\alpha, \nu}$  is a weak limit of a sequence of the empirical processes.

It can be easily established that  $\pi_{\alpha, \nu}$  admits the following representation

$$(2) \quad \pi_{\alpha, \nu} = \sum_{k=1}^{\infty} \delta_{\{\Gamma_k^{-1/\alpha} \epsilon_k\}}$$

where

- $\Gamma_k = \sum_{i=1}^k \gamma_i$ , the sequence  $(\gamma_k)_{k \in \mathbb{N}}$  is a sequence of i.i.d. random variable with common exponential distribution with mean equal to one.
- $(\epsilon_k)_{k \in \mathbb{N}}$  is a sequence of i.i.d. with common distribution  $\nu$
- $(\gamma_k)_{k \in \mathbb{N}}$  and  $(\epsilon_k)_{k \in \mathbb{N}}$  are supposed to be independent

(see e.g. [2]). It is worth recalling that the point processes  $\pi_{\alpha, \nu}$  naturally arise within the framework of the theory of stable distributions. For example, if  $\alpha \in (0, 1)$  then the series  $\zeta = \sum_{x^{(j)} \in \pi_{\alpha, \nu}} x^{(j)}$  converges a.s.. Furthermore,  $\zeta$  has the  $d$ -dimensional stable distribution (see e.g. [3], [6], [2]).

In the focus of our interest is the sequence of convex hulls that arise from the peeling procedure introduced in [1]. In what follows by  $C(A)$ ,  $A \subset \mathbb{R}^d$ , we denote the convex hull generated by  $A$ . Let  $C$  be a convex set. By  $\text{ext } C$  we denote the set of the extreme points of  $C$ . If  $C$  is a convex polyhedron then  $\text{ext } C$  is the finite set of its vertices.

It convenient to start with the binomial process  $(\zeta_1^{(n)})$ . Let  $C_1^{(n)} = C(\zeta_1^{(n)})$ . If the measure  $\mathbb{P}_\xi$ , the distribution of  $\xi$ , has no atoms then a.s.  $C_1^{(n)}$  is a polyhedron and, furthermore,

$$\zeta_1^{(n)} \cap \partial C_1^{(n)} = \text{ext } C_1^{(n)}.$$

Define

$$\varsigma_2^{(n)} = \varsigma_1^{(n)} \setminus \text{ext } C_1^{(n)}, \quad C_2^{(n)} = C(\varsigma_2^{(n)}),$$

then

$$\varsigma_3^{(n)} = \varsigma_2^{(n)} \setminus \text{ext } C_2^{(n)}, \quad C_3^{(n)} = C(\varsigma_3^{(n)})$$

and so on. Obviously, the sequence of the so-built non empty convex hulls  $C_1^{(n)}, C_2^{(n)}, \dots, C_k^{(n)}, \dots$  is finite and its length is random.

**Definition 1.** We say that the underlying distribution  $P_\xi$  and corresponding to it spectral measure  $\nu$  are non-unilateral if the minimal closed cone containing  $S_\nu$  coincides with  $\mathbb{R}^d$ .

If the underlying distribution  $P_\xi$  is non-unilateral then for any fixed  $k$

$$\mathbf{0} \in \text{int } C_k^{(n)}, \quad \inf \left\{ |x| \mid x \in \partial C_k^{(n)} \right\} > 0 \quad \text{a.s.}$$

for  $n$  sufficiently large. In [1], it was shown that if  $0 < \alpha < 2$  and  $\nu$  is non-unilateral then  $(C_0^{(n)}, C_1^{(n)}, \dots, C_k^{(n)})$ , as  $n \rightarrow \infty$ , converge in distribution to  $(C_0, C_1, \dots, C_k)$  for any fixed  $k$ . Consequently, the sequence  $\#\{\text{ext } C_k^{(n)}\}$  is bounded in probability as  $n$  tends to infinity.

In order to learn how  $C_k^{(n)}$  relates to  $C_k$  when  $k = k_n \rightarrow \infty$  we need, first, to learn how  $C_k$  behaves as  $k \rightarrow \infty$ . It should be noted that  $C_k^{(n)}$  can be regarded as the multi-dimensional analogue of the order statistics. So, the asymptotic properties of  $C_{k_n}^{(n)}$  are of great interest from the view-point of mathematical statistics.

We generalize now the construction of the peeling sequence to infinite set.

Let  $\varsigma = \varsigma_1$  denote the set of points of  $\pi$  or, in other words, let  $\varsigma$  support the random measure  $\pi$ . We may apply to  $\varsigma$  the same peeling procedure as in the case of the finite set  $\varsigma_1^{(n)}$ . As a result we obtain the sequence of sets  $\varsigma_1, \varsigma_2, \dots$ , the sequence of their convex hulls  $C_1, C_2, \dots$  and the sets of extreme points  $\text{ext } C_k$ ,  $k = 1, 2, \dots$ . Furthermore, a.s.  $\varsigma_{k+1} = \varsigma_k \setminus \text{ext } C_k$ . If  $\nu$  is non-unilateral then  $\mathbf{0} \in C_k$  a.s. for all  $k$ . Furthermore,  $C_k$  is a.s. a polyhedron and

$$\varsigma_k \cap \partial C_k = \text{ext } C_k.$$

Intuitively, we expect that the asymptotic behaviour of  $C_n$  is rather regular. It is convenient to state our basic conjecture in the following way:

**Denote**

$$(3) \quad \hat{C}_n = \rho_n^{-1} C_n, \quad \text{where } \rho_n = \max_{x \in C_n} |x|.$$

**If  $\nu$  is non-unilateral then there exists a non-random set  $\hat{C}$  such that**

$$\lim_{n \rightarrow \infty} d_H(\hat{C}_n, \hat{C}) = 0 \quad \text{a.s.}$$

**Definition 2.**  $\hat{C}$  (if it exists) is called the limit shape of the sequence  $\hat{C}_n$ .

It is easy to show that if such  $\hat{C}$  exists then it is certainly non-random. Indeed, labelling the points of  $\varsigma$  in the descending order of their distances from the origin we obtain a sequence  $x^{(1)}, x^{(2)}, x^{(3)}, \dots$  such that a.s.

$$|x^{(1)}| > |x^{(2)}| > |x^{(3)}| > \dots$$

It is worth noting that the joint distribution of  $|x^{(1)}|, |x^{(2)}|, \dots, |x^{(n)}|$  is absolutely continuous with the density of the form

$$(4) \quad p_n(r_1, r_2, \dots, r_n) = \alpha^n (r_1 r_2 \dots r_n)^{-\alpha-1} e^{-\nu(S^{d-1})r_1^{-\alpha}} \mathbb{1}_{\{r_1 > r_2 > \dots > r_n\}}.$$

Let  $(\eta, \varepsilon), (\eta_1, \varepsilon^{(1)}), (\eta_2, \varepsilon^{(2)}), \dots$  be i.i.d. with common distribution

$$P\{\eta > r, \varepsilon \in E\} = e^{-r} \nu(E).$$

According to (2), we have

$$(5) \quad \{x^{(j)}\}_{j=1}^\infty \stackrel{d}{=} \{\varepsilon^{(j)}(\eta_1 + \dots + \eta_j)^{1/\alpha}\}_{j=1}^\infty,$$

which implies that the event  $\{\lim_{n \rightarrow \infty} \hat{C}_n \text{ exists}\}$  belongs to the  $\sigma$ -algebra  $\mathcal{I}$  of the events invariant with respect to all finite permutations of the random vectors  $(\eta_1, \varepsilon^{(1)}), (\eta_2, \varepsilon^{(2)}), \dots$ . By the Hewitt-Savage zero-one law  $\mathcal{I}$  is trivial. Since the limit set  $\hat{C} = \lim_{n \rightarrow \infty} \hat{C}_n$  is  $\mathcal{I}$ -measurable we conclude that  $\hat{C}$  is constant with probability 1.

Now we give a first example where the existence of the limit shape is proved.

**Example 1.** Let  $S_\nu$  consist of  $d+1$  unit vectors  $e^{(1)}, \dots, e^{(d+1)}$  such that  $\text{cone}(e^{(1)}, \dots, e^{(d+1)})$  coincides with  $\mathbb{R}^d$ .  $\pi_{\alpha, \nu}$  is decomposed on  $d+1$  one-dimensional independent p.p.p. of the form  $(x_k^{(i)} = |x_k^{(i)}| e^{(i)})$ . Since  $\nu$  is non-unilateral the points  $x_k^{(i)}, i = 1, 2, \dots, d+1$ , serve as vertices of  $C_k, k = 1, 2, \dots$ . Moreover,  $|x_k^{(i)}| \stackrel{d}{=} (\nu_i)^{1/\alpha} (\eta_1 + \dots + \eta_k)^{1/\alpha}$  and  $\rho_n n^{1/\alpha} \rightarrow t^+ = \max_{1 \leq i \leq d+1} (\nu_i)^{1/\alpha}$ , a.s.. Then the limit shape  $\hat{C}$  is the convex polyhedron with vertices  $v^{(i)} = (t_i/t^+) e^{(i)}$  and  $t_i = (\nu_i)^{1/\alpha}, i = 1, 2, \dots, d+1$ .

If  $\#(S_\nu) > d+1$  then the situation becomes much more complicated. Theorem 1 and 2 proved below deal with a case where a non-unilateral  $\nu$  is supported by a finite number of unit vectors.

Intuitively, we expect that, say, in case of  $\nu$  uniformly distributed over  $S^{d-1}$  the unit ball arises as the limit shape. However, it is not easy at all to prove this formally. The authors tried to verify the credibility of this conjecture using the Monte Carlo simulation. Obviously, the representation (2) provides a basis for such a simulation. The results of simulation presented below make this conjecture very credible.

It should be emphasised that the basic goal of the present paper is to draw attention to new and interesting problems of stochastic geometry. So far, little or nothing is known about the peels no matter what point process they concern.

The paper is organised as follows. In Section 2, we obtained a partial result on the limit shape of the convex hulls  $C_k(\pi_{\alpha, \nu})$  when the spectral measure of the process  $\pi_\alpha$  is atomic. Section 4 contains some numerical experiments.

## 2. ALMOST SURE CONVERGENCE OF THE PEELING

In this section we assume that the spectral measure  $\nu$  of the process  $\pi_{\alpha, \nu}$  is atomic, i.e. it is supported by a finite number of the points  $e^{(1)}, \dots, e^{(l)}$  belonging to the unit sphere  $S^{d-1}$ .

Furthermore, it is also assumed that  $\text{cone}\{e^{(1)}, \dots, e^{(l)}\} = \mathbb{R}^d$ . Denote by  $\nu_i = \nu(\{e^{(i)}\}), i = 1, 2, \dots, l$ , the atoms of  $\nu$ .

It implies that the considered point process is a superposition of the one-dimensional independent Poisson point processes defined on the rays

$$\mathcal{L}_i = \{x \mid x = te^{(i)}, t > 0\}, i = 1, \dots, l.$$

If a Borel set  $A \subset \mathcal{L}_i$  then

$$(6) \quad \mu(A) = \nu_i \alpha \int_A r^{-\alpha-1} dr.$$

**Definition 3.** Let  $A = \{a^{(1)}, \dots, a^{(m)}\}$  be a finite set where  $a^{(i)} \in \mathbb{R}^d$  for  $i = 1, \dots, m$ , and  $m \geq d + 1$ . The set  $A$  is extreme if

$$\text{ext } C(A) = A.$$

**Theorem 1.** Let  $C_k(\pi_{\alpha, \nu})$  be the  $k$ -th convex hull of the Poisson point process  $\pi_{\alpha, \nu}$ . Denote by  $C_\infty$  the convex hull generated by  $A = \{\nu_1^{1/\alpha} e^{(1)}, \dots, \nu_l^{1/\alpha} e^{(l)}\}$ .

If  $A$  is extreme then as  $k \rightarrow \infty$

$$(7) \quad d_H \left( k^{1/\alpha} C_k(\pi_{\alpha, \nu}), C_\infty \right) = O \left( \sqrt{\frac{\text{llnk}}{k}} \right) \text{ a.s.}$$

where  $\text{llnk} = \ln \ln k$ . The polyhedron  $C_\infty$  determines the limit shape of the convex hulls  $C_k(\pi_{\alpha, \nu})$ .

**Remark 1.** If  $\sigma$  is uniformly distributed over its support in the sense that  $\nu_i = l^{-1}$ , then the total number of the vertices of  $C_\infty$  equals  $l$ . Furthermore, they lie on the sphere of the radius  $l^{-1/\alpha}$ . Loosely speaking, the convex hulls  $C_k(\pi_{\alpha, \nu})$  are getting round as  $k \rightarrow \infty$ .

If the condition  $A$  is extreme is omitted, we can state the following result :

**Theorem 2.** Let  $C_k(\pi_{\alpha, \nu})$  be the  $k$ -th convex hull of  $\pi_{\alpha, \nu}$ . Denote by  $C_\infty$  the convex hull generated by  $A = \{\nu_1^{1/\alpha} e^{(1)}, \dots, \nu_l^{1/\alpha} e^{(l)}\}$ . Then as  $k \rightarrow \infty$

$$(8) \quad d_H \left( k^{1/\alpha} C_k(\pi_{\alpha, \nu}), C_\infty \right) \rightarrow 0 \text{ a.s.}$$

**Remark 2.** Let  $f$  be a continuous homogeneous functional of a degree  $\gamma$  defined on convex sets. From Theorem 2, we get

$$k^{\frac{\gamma}{\alpha}} f(C_k(\pi_{\alpha, \nu})) \rightarrow f(C_\infty) \text{ a.s.}$$

In particular, if  $f(A)$  is the surface Lebesgue measure, i.e.  $f(A) = \lambda^{d-1}(\partial A)$ , then

$$(9) \quad f(C_k(\pi_{\alpha, \nu})) \sim \frac{f(C_\infty)}{k^{\frac{d-1}{\alpha}}}.$$

But if  $f(A) = \lambda^d(A)$ , then

$$(10) \quad f(C_k(\pi_{\alpha, \nu})) \sim \frac{f(C_\infty)}{k^{\frac{d}{\alpha}}}.$$

## 3. PROOF

**3.1. Auxiliary lemmas.** Let  $\eta_1, \eta_2, \dots$  be i.i.d. random variables with the standard exponential distribution, so that  $a = \mathbb{E}\eta_1 = \text{Var}\eta_1 = 1$ . Define the sums

$$(11) \quad \Gamma_n = \eta_1 + \dots + \eta_n.$$

By the law of the iterated logarithm there exists an a.s. finite random variable  $\kappa$  with values in  $\mathbb{N}$  such that for  $n \geq \kappa$

$$(12) \quad |n^{-1}\Gamma_n - 1| < 2\sqrt{\frac{\ln \ln n}{n}} \quad a.s..$$

Consider a function  $h(z) = z^{-1/\alpha}$ . If  $|z - 1| \leq 1/2$  then

$$|h(z) - h(1)| \leq L_\alpha |z - 1|, \quad L_\alpha < \infty.$$

Let  $n' = \min \left\{ n \mid 2\sqrt{\frac{\ln \ln n}{n}} < 1/2 \right\}$ . If  $n \geq \max(n', \kappa)$  then

$$|h(n^{-1}\Gamma_n) - h(1)| \leq 2L_\alpha \sqrt{\frac{\ln \ln n}{n}}$$

and, therefore, for  $n \geq \kappa = \kappa(\omega)$

$$(13) \quad \left| \Gamma_n^{-1/\alpha} - n^{-1/\alpha} \right| \leq 2L_\alpha \frac{\sqrt{\ln \ln n}}{n^{1/\alpha+1/2}} \quad a.s..$$

We call the *configuration* any countable set of points from  $\mathbb{R}^d$  such that for any  $\delta > 0$  there are a finite number of points belonging to the set that lie outside the ball  $\{x \mid |x| \leq \delta\}$ . So the point  $\mathbf{0}$  is the limit point of any configuration. We call a configuration  $\varsigma$  *non-unilateral* if all the convex hulls,  $C_k = C_k(\varsigma)$ ,  $k = 1, 2, \dots$ , generated by  $\varsigma$  contain  $\mathbf{0}$  as an interior point. It is evident that under the conditions of Theorem 1 the random measure  $\pi_{\alpha, \nu}$  is almost surely supported by a non-unilateral configuration  $\varsigma$ .

Denote by  $\text{int}(\varsigma)$  the set of the interior points of  $\varsigma$ , i.e.

$$\text{int}(\varsigma) = \{x \mid x \in \varsigma, x \notin \partial C_1(\varsigma)\}.$$

**Lemma 1.** *Let  $\varsigma_1, \varsigma_2 \in \mathcal{K}$  be such that  $\varsigma_1 \subset \varsigma_2$ , then for all  $k \in \mathbb{N}$ ,*

$$(14) \quad C_k(\varsigma_1) \subset C_k(\varsigma_2).$$

*Proof.* It is trivial that  $C_1(\varsigma_1) \subset C_1(\varsigma_2)$ .

Note that if  $x$  is an interior point of  $C_1(\varsigma_1)$ , i.e.  $x \in \text{int}(C_1(\varsigma_1))$ , then  $x$  is also an interior point of  $C_1(\varsigma_2)$ , therefore

$$\text{int}(C_1(\varsigma_1)) \subset \text{int}(C_1(\varsigma_2))$$

and this implies that

$$C_2(\varsigma_1) = C_1(\text{int}(C_1(\varsigma_1))) \subset C_1(\text{int}(C_1(\varsigma_2))) = C_2(\varsigma_2)$$

By induction, the lemma is proved.  $\square$

**Lemma 2.** *Let  $\mathcal{K}$  be the set of non-unilateral configurations such that no  $d+1$  points lie on the same hyperplane. Let  $\varsigma, \varsigma' \in \mathcal{K}$  be such that  $\varsigma' \subset \varsigma$  and  $\#(\varsigma' \setminus \varsigma) = m < \infty$ . Then we have, for all  $k \in \mathbb{N}$ ,*

$$(15) \quad C_{k+m}(\varsigma) \subset C_k(\varsigma') \subset C_k(\varsigma)$$

*Proof.* Since  $\varsigma, \varsigma' \in \mathcal{K}$  and  $\mathbf{0}$  is the only limit point of both configurations all  $C_k(\varsigma), k = 1, 2, \dots$ , are polyhedrons. Note that for all  $k, l \geq 1$

$$(16) \quad C_{k+1}(\varsigma) = C_1(\text{int}(C_k(\varsigma) \cap \varsigma))$$

and

$$(17) \quad C_{k+l}(\varsigma) = C_k(\text{int}(C_l(\varsigma) \cap \varsigma)).$$

First, let  $m = 1$ . Note that the inclusion  $C_1(\varsigma') \subset C_1(\varsigma)$  follows directly from the relation  $\varsigma' \subset \varsigma$ . Denote  $\{a\} = \varsigma \setminus \varsigma'$ . Consider two possible cases  $a \notin C_1(\varsigma')$  and  $a \in C_1(\varsigma')$  one after another.

Let  $a \notin C_1(\varsigma')$ . In this case  $a \in \partial C_1(\varsigma)$  i.e.  $C_1(\varsigma') \neq C_1(\varsigma)$ . It implies that  $\text{int}(\varsigma) \subset \varsigma'$ . Utilising (16) under  $k = 1$  yields  $C_2(\varsigma') \subset C_1(\varsigma)$ . Since the inclusion  $C_1(\varsigma') \subset C_1(\varsigma)$  is obvious we conclude that (15) holds for  $k = m = 1$ .

Further, let us make use of the induction by  $k$ . Assume that (15) holds for  $m = 1$  and all  $k \leq n$  and show that then it holds for  $m = 1$  and  $k = n + 1$ . By the induction assumption we have

$$(18) \quad C_{n+1}(\varsigma) \subset C_n(\varsigma') \subset C_n(\varsigma).$$

Since

$$\text{int}(C_n(\varsigma') \cap \varsigma') = \text{int}(C_n(\varsigma') \cap \varsigma)$$

we obtain, taking into account (16),

$$C_{n+l}(\varsigma') = C_1(\text{int}(C_n(\varsigma') \cap \varsigma)).$$

From the right hand side inclusion of (18), it follows that  $C_{n+1}(\varsigma') \subset C_{n+1}(\varsigma)$ . Further, from the left hand side inclusion of (18) we conclude that

$$\text{int}(C_{n+1}(\varsigma) \cap \varsigma) = \text{int}(C_n(\varsigma') \cap \varsigma').$$

Applying (17) yields  $C_{n+2}(\varsigma) \subset C_{n+1}(\varsigma')$ . Thus, (15) holds for  $k = n + 1$  and  $m = 1$ , i.e. the case  $a \notin C_1(\varsigma')$  is exhausted.

If  $a \in C_1(\varsigma')$ , then there exists an integer  $n_0$  such that

$$C_n(\varsigma') = C_n(\varsigma), \quad n = 1, 2, \dots, n_0$$

$$C_{n_0+1}(\varsigma') \neq C_{n_0+1}(\varsigma').$$

Furthermore,  $a \notin C_{n_0+1}(\varsigma')$ . Obviously, the relations (15) are trivial for  $m = 1$  and  $n = 1, 2, \dots, n_0$ . Hence, it remains to apply the above argument to the configurations  $C_{n_0+1}(\varsigma) \cap \varsigma$  and  $C_{n_0+1}(\varsigma') \cap \varsigma'$ . Thus, the lemma is proved for all  $k$  and  $m = 1$ .

Now, let  $m > 1$ , i.e.  $\varsigma \setminus \varsigma' = \{a_1, \dots, a_m\}$ .

Consider the configurations

$$\varsigma_0 = \varsigma, \quad \varsigma_1 = \varsigma \setminus \{a_1\},$$

$$\varsigma_2 = \varsigma \setminus \{a_1, a_2\}, \dots,$$

$$\varsigma_m = \varsigma \setminus \{a_1, \dots, a_m\} = \varsigma'.$$

Note that the neighbouring configurations differ by a single point. So, one may apply (18). Applying it yields

$$C_{k+m}(\varsigma) \subset C_{k+m-1}(\varsigma_1) \subset C_{k+m-2}(\varsigma_2) \subset \dots \subset C_k(\varsigma_m) = C_k(\varsigma') \subset C_k(\varsigma).$$

The lemma is proved.  $\square$

### 3.2. Proof of Theorem 1.

**Lemma 3.** *Let  $e^{(i)}$  for  $i = 1, \dots, l$  with  $l \geq d + 1$ , be unit vectors such that*

$$\text{cone} \{e^{(1)}, \dots, e^{(l)}\} = \mathbb{R}^d.$$

*If  $A = \{\nu_1^{1/\alpha} e^{(1)}, \dots, \nu_l^{1/\alpha} e^{(l)}\}$  is extreme then there exists  $r > 0$  and  $\varepsilon$  depending only on  $A$  and on the dimension  $d$  such that the set  $\{r_1 \nu_1^{1/\alpha} e^{(1)}, \dots, r_l \nu_l^{1/\alpha} e^{(l)}\}$  is extreme for all  $(r_1, \dots, r_l)$  such that  $|r_i/r - 1| < \varepsilon$ ,  $i = 1, \dots, l$ .*

*Proof.* The proof of this lemma is evident.  $\square$

*Proof of Theorem 1.* Let us label the points lying on the ray  $\mathcal{L}_i$  in the descending order of their norms. So, we have the sequence  $x_1^{(i)}, x_2^{(i)}, \dots$  such that a.s.  $|x_1^{(i)}| > |x_2^{(i)}| > \dots$ . Obviously, the sequences  $\{x_n^{(i)}, n \in \mathbb{N}\}$ ,  $1 \leq i \leq l$ , are jointly independent. Furthermore, from (6)

$$\{|x_n^{(i)}|\} \stackrel{d}{=} \{\nu_i^{1/\alpha} \Gamma_n^{-1/\alpha}\}$$

where  $\Gamma_n$  is defined as in (11).

Let  $\varepsilon > 0$ . According to (13), there exists  $n_0 = n_0(\omega)$  such that for all  $i = 1, \dots, l$  and all  $n \geq n_0$

$$(19) \quad |x_n^{(i)} - \nu_i^{1/\alpha} n^{-1/\alpha}| \leq 2L_\alpha n^{-1/\alpha-1/2} \sqrt{\ln n}.$$

and

$$2L_\alpha n^{-1/2} \sqrt{\ln n} < \varepsilon.$$

Let the configuration  $\zeta'$  is formed by the points  $x_n^{(i)}$ ,  $n \geq n_0$ ,  $i = 1, \dots, l$  i.e.

$$\zeta' = \bigcup_{i=1}^l \{x_{n_0}^{(i)}, x_{n_0+1}^{(i)}, \dots\}.$$

Consider for all  $k \leq 1$

$$A_k^+ = \{(n_0 + k - 1)^{-1/\alpha} (1 + \varepsilon_k) \nu_1^{1/\alpha} e^{(1)}, \dots, (n_0 + k - 1)^{-1/\alpha} (1 + \varepsilon_k) \nu_l^{1/\alpha} e^{(l)}\}$$

and

$$A_k^- = \{(n_0 + k - 1)^{-1/\alpha} (1 - \varepsilon_k) \nu_1^{1/\alpha} e^{(1)}, \dots, (n_0 + k - 1)^{-1/\alpha} (1 - \varepsilon_k) \nu_l^{1/\alpha} e^{(l)}\}$$

where

$$\varepsilon_k = 2L_\alpha \sqrt{\frac{\ln(k + n_0 - 1)}{k + n_0 - 1}}.$$

By virtue of (19) the points  $x_{n_0}^{(1)}, \dots, x_{n_0}^{(l)}$  hit the layer  $C(A_1^+) \setminus C(A_1^-)$ . Then by Lemma 3 the convex hull  $C_1(\zeta')$  is the polyhedron and

$$\text{ext } C_1(\zeta') = \{x_{n_0}^{(1)}, \dots, x_{n_0}^{(l)}\}.$$

Similarly, the set

$$\{x_{n_0+1}^{(1)}, \dots, x_{n_0+1}^{(l)}\} \subset C(A_2^+) \setminus C(A_2^-)$$

and, therefore, it is extreme, i.e.

$$\text{ext } C(\{x_{n_0+1}^{(1)}, \dots, x_{n_0+1}^{(l)}\}) = \{x_{n_0+1}^{(1)}, \dots, x_{n_0+1}^{(l)}\}.$$

It is evident that

$$\text{ext } C_2(\zeta') = \{x_{n_0+1}^{(1)}, \dots, x_{n_0+1}^{(l)}\}.$$

Continuing in this way we obtain at the  $k$ -th convex hull  $C_k(\zeta')$  such that

$$\text{ext } C_k(\zeta') = \{x_{n_0+k-1}^{(1)}, \dots, x_{n_0+k-1}^{(l)}\} \subset C(A_k^+) \setminus C(A_k^-).$$

The last inclusion implies that

$$d_H \left( (k+n_0-1)^{1/\alpha} C_k(\zeta'), C_\infty \right) \leq \varepsilon_k,$$

where, we remind,  $C_\infty$  is the convex hull generated by  $A = \{\nu_1^{1/\alpha} e^{(1)}, \dots, \nu_l^{1/\alpha} e^{(l)}\}$ . From (15) it follows that

$$C_{k+m}(\zeta') \subset C_{k+m}(\pi_{\alpha,\nu}) \subset C_k(\zeta'), \quad \text{with } m = (n_0 - 1)l.$$

Therefore,

$$\text{ext } C_{k+m}(\pi_{\alpha,\nu}) \subset C(A_k^+) \setminus C(A_{k+m'}^-), \quad \text{with } m' = (n_0 - 1)(l - 1).$$

So, for all sufficiently large  $k$

$$d_H \left( (k+m)^{1/\alpha} C_{k+m}(\pi_{\alpha,\nu}), C_\infty \right) \leq 2\varepsilon_k.$$

Since  $m$  is fixed the theorem follows.  $\square$

**3.3. Proof of Theorem 2.** Let  $\epsilon$  be an arbitrary positive real. Hereafter, we denote  $A^{(\epsilon)}$  the  $\epsilon$ -neighbourhood of a set  $A$ ,

$$A^{(\epsilon)} = \{x : d(x, A) < \epsilon\}.$$

Let  $A_1 = A \cap \partial C(A) \stackrel{\text{def}}{=} \{\nu_j^{1/\alpha} e^{(j)}, j \in J\}$ , the set  $A_1$  is extreme. From the process  $\pi_{\alpha,\nu}$ , we construct a new p.p.p.  $\pi_1$  obtained by deleting all the points on the rays  $\mathcal{L}_j = \{x \mid x = te^{(j)}, t > 0\}$ ,  $j \in J$ . By Lemma 1, we have for all  $n \in \mathbb{N}$

$$(20) \quad C_n(\pi_1) \subset C_n(\pi_{\alpha,\nu})$$

Moreover,  $A_1$  is extreme and  $C_\infty = C(A) = C(A_1)$ , thus Theorem 1 ensures the convergence

$$(21) \quad d_H(n^{1/\alpha} C_n(\pi_1), C_\infty) \rightarrow 0 \text{ a.s.}$$

From (20) and (21), it exists  $n_1 \in \mathbb{N}$  such that for all  $n > n_1$

$$(22) \quad C_\infty \subset n^{1/\alpha} C_n(\pi_1)^{(\epsilon)} \subset n^{1/\alpha} C_n(\pi_{\alpha,\nu})^{(\epsilon)}.$$

It is easy to see that there exists  $\tilde{\nu}_i$ ,  $i \in I$  such that the set

$$A_2 = \{\nu_j^{1/\alpha} e^{(j)}, j \in J; ; \tilde{\nu}_i^{1/\alpha} e^{(i)}, i \in \{1, \dots, l\} \setminus J\}$$

is extreme and satisfies the following relation

$$(23) \quad C_\infty \subset C(A_2) \subset C_\infty^{(\epsilon)}.$$

From  $\pi_{\alpha,\nu}$  we construct a second p.p.p.  $\pi_2$  by adding the independent point processes  $(\tilde{\pi}_i)_{i \in J}$  verifying the following conditions

- the  $(\tilde{\pi}_i)_{i \in J}$  are independent of  $\pi_{\alpha,\nu}$ ;
- for each  $i \in J$  the spectral measure of  $\tilde{\pi}_i$  is supported by  $\mathcal{L}_i$  and the intensity measure is  $\tilde{\mu}_i(A) = (\tilde{\nu}_i - \nu_i)\alpha \int_A r^{-\alpha-1} d r$ .

According to Theorem 1 we have

$$(24) \quad d_H(n^{1/\alpha}C_n(\pi_2), C(A_2)) \rightarrow 0 \text{ a.s.}$$

and using Lemma 1, we get, for all  $n \in \mathbb{N}$ ,

$$(25) \quad C_n(\pi_{\alpha,\nu}) \subset C_n(\pi_2).$$

From (23),(24) and (25), it exists  $n_2 \in \mathbb{N}$  such that for all  $n > n_2$

$$(26) \quad n^{1/\alpha}C_n(\pi_{\alpha,\nu}) \subset n^{1/\alpha}C_n(\pi_2) \subset C(A_2)^{(\epsilon)} \subset C_\infty^{(2\epsilon)}.$$

According to (22) and (26), for all  $n \geq \max(n_1, n_2)$ , we have

$$n^{1/\alpha}C_n(\pi_{\alpha,\nu}) \subset C_\infty^{(2\epsilon)} \quad \text{and} \quad C_\infty \subset n^{1/\alpha}C_n(\pi_{\alpha,\nu})^{(2\epsilon)}.$$

By definition of  $d_H$ , this means

$$d_H(n^{1/\alpha}C_n(\pi_{\alpha,\nu}), C_\infty) \leq 2\epsilon,$$

and we get (8).

#### 4. SIMULATION AND CONJECTURES

We investigate using some simulations the limit shape and the asymptotic behaviour of basic functionals in the case of continuous spectral measure. Hereafter we consider the example of the uniform distribution as spectral measure.

The point processes  $\{x^{(j)}, j \in \mathbb{N}\}$  are simulated using the representation (5). Let  $C_{1,n}$  be the convex hull generated by the first  $n$  points  $x^{(1)}, x^{(2)}, \dots, x^{(n)}$  and  $\kappa_{n,1} = \min_{x \in \partial C_{1,n}} |x|$ . Since the points of the simulated p.p.p. are ordered by their distances from the origin, it is evident that

$$C_{1,n'} = C_1 \text{ with } n' = \min\{n : \kappa_{n,1} > |x^{(n+1)}|\}.$$

This fact is used to construct the successive convex hulls  $(C_k)_{k \in \mathbb{N}}$ .

Figure 1 gives an impression about the behaviour of the peels. The observed closeness of the peels to the unit circle also support our conjecture about the existence of the limit shape that is expected to be a circle.

It is of great interest to get impression about a possible behaviour of such basic functionals of the convex polygons  $C_k$ ,  $k = 1, 2, \dots$ , as the perimeter  $\mathcal{L}$ , the area  $\mathcal{A}$  and the total number of vertices  $\mathcal{N}$ . It seems evident that  $\mathcal{L}(C_k)$  and  $\mathcal{A}(C_k)$  tend to zero as  $k \rightarrow \infty$ . Intuitively, we expect that  $\mathcal{N}(C_k) \rightarrow \infty$  as  $k \rightarrow \infty$ . Figure 2[top] represents the logarithm of those functionals as function of  $\log(k)$ , calculated on simulated p.p.p. for different values of  $\alpha$ . The observed closeness of the points to straight lines makes it reasonable to expect that in a sense

$$(27) \quad \mathcal{L}(C_k) \asymp k^{-\gamma_l}, \mathcal{A}(C_k) \asymp k^{-\gamma_a}, \mathcal{N}(C_k) \asymp k^{\gamma_n}$$

with  $\gamma_l$  and  $\gamma_a$  depending on  $\alpha$  whereas it seems that  $\gamma_n$  does not depend on  $\alpha$ .

The next step consists in estimating the exponents and possibly the dependence on  $\alpha$ . Using independent replications of p.p.p., we estimate the three exponents defined in (27) for different values of  $\alpha$ . Figure 2 [bottom] represents the logarithm of the estimated exponents versus  $\log(\alpha)$ . For the three cases, the linear approximation seems reasonable. According to the estimated coefficients of the straight lines (see the equations in the caption of Figure 2), it looks very credible that the true values are

$$(28) \quad \gamma_l = \frac{3}{2\alpha}, \gamma_a = \frac{3}{\alpha} \text{ and } \gamma_n = \frac{1}{2}.$$

After the  $k$ -th iterative step of the peeling procedure, the number of deleted points should be the order of  $\sum_{j=1}^k \mathcal{N}(C_j) \asymp k^{\frac{3}{2}}$  and

$$\rho_k = \max_{x \in C_k} |x| \asymp \left(k^{\frac{3}{2}}\right)^{-1/\alpha}$$

Moreover we can expect that  $d_H(\rho_k^{-1}C_k(\pi_{\alpha,\nu}), C_\infty)$  converges to zero.

Using the arguments of Remark 2, this convergence would lead to  $\mathcal{L}(C_k) \asymp \rho_k \asymp k^{-3/(2\alpha)}$  and  $\mathcal{A}(C_k) \asymp \rho_k^2 \asymp k^{-3/\alpha}$ . These convergence rates are in agreement with the estimated values of  $\alpha_l$  and  $\alpha_a$  obtained in (28).

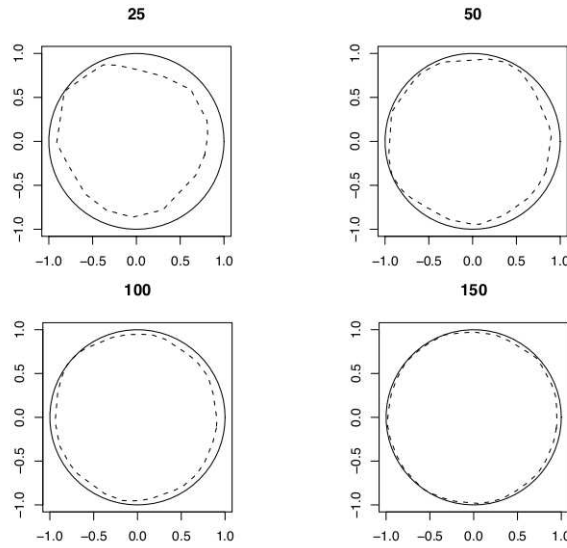


FIGURE 1. [dotted line] the normalized simulated shapes of  $\hat{C}_{25}$ ,  $\hat{C}_{50}$ ,  $\hat{C}_{100}$  and  $\hat{C}_{150}$  in the case  $\alpha = 3/2$  and the spectral measure is uniform. [solid line] the unit circle.

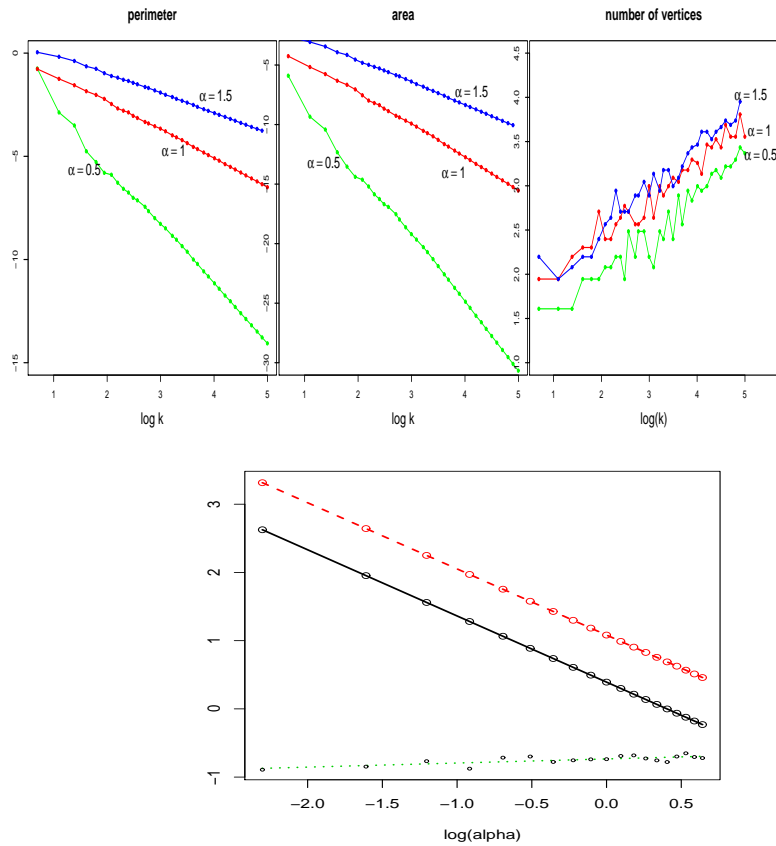


FIGURE 2. [TOP] Log-log representation of the values of  $\mathcal{L}(C_k)$ ,  $\mathcal{A}(C_k)$  and  $\mathcal{N}(C_k)$  as function of  $k$ . The functionals are calculated on simulated p.p.p. for different values of  $\alpha = 0.5; 1.0; 1.5$  and the uniform distribution as spectral measure. [BOTTOM] The estimated values of the logarithm of exponents defined in (27) versus  $\ln(\alpha)$  and the best linear fittings  $\ln \hat{\gamma}_a = -0.97 \ln \alpha + \ln 2.95$ ,  $\ln \hat{\gamma}_l = -0.97 \ln \alpha + \ln 1.48$  and  $\ln \hat{\gamma}_n = 0.06 \ln \alpha + \ln 0.48$ . Three exponents are estimated for each  $\alpha$  on 1000 independent replications.

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