A two-sample test for comparison of long memory parameters

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Abstract

We construct a two-sample test for comparison of long memory parameters based on ratios of two rescaled variance (V/S) statistics studied in [Giraitis L., Leipus, R., Philippe, A., 2006. A test for stationarity versus trends and unit roots for a wide class of dependent errors. Econometric Theory 21, 989–1029]. The two samples have the same length and can be mutually independent or dependent. In the latter case, the test statistic is modified to make it asymptotically free of the long-run correlation coefficient between the samples. To diminish the sensitivity of the test on the choice of the bandwidth parameter, an adaptive formula for the bandwidth parameter is derived using the asymptotic expansion in [Abadir, K., Distaso, W., Giraitis, L., 2009. Two estimators of the long-run variance: Beyond short memory. Journal of Econometrics 150, 56–70]. A simulation study shows that the above choice of bandwidth leads to a good size of our comparison test for most values of fractional and ARMA parameters of the simulated series.

Keywords: Long memory, two-sample test for comparison of memory parameters, long-run covariance, rescaled variance, bandwidth choice

1 Introduction

Long memory is one of the most widely discussed "stylized facts" of financial time series (see, e.g., Teyssière and Kirman (2007)). In real data, long memory can be confused with short memory, unit roots, trends, structural changes, heavy tails and other features. Various tests for long memory have been developed in the literature. See Lo (1991), Kwiatkowski et al. (1992), Robinson (1994), Lobato and Robinson (1998), Giraitis et al. (2003, 2006), Surgailis et al. (2008). Most of these results pertain to the case of a single sample.

A natural extension of one-sample test about unknown long memory parameter d is two-sample testing for comparison of respective memory parameters d_1 and d_2. In particular, such test can be useful for the memory propagation (from durations to counts and realized volatility), question discussed in Deo et al. (2009). Several studies compare estimates of long memory parameter from different foreign exchange data and other sources (Cheung (1993), Soofi et al. (2006), Casas and Gao (2008)). Two-sample testing is also related to the change-point problem of the memory parameter discussed in Beran and Terrin (1996), Horváth (2001).

The present paper constructs a test for testing the null hypothesis $d_1 = d_2$ that long memory parameters $d_i \in [0, 1/2)$ of two samples of length n, taken from respective stationary processes X_i, $i = 1, 2$, are equal, against the alternative $d_1 \neq d_2$. The test statistic, T_n, is defined as a sum

$$T_n = \frac{V_1/S_{11,q}}{V_2/S_{22,q}} + \frac{V_2/S_{22,q}}{V_1/S_{11,q}}, \tag{1.1}$$

of two ratios of V/S, or rescaled variance, statistics $V_1/S_{11,q}$ and $V_2/S_{22,q}$ computed from samples $(X_1(1), \ldots, X_1(n))$ and $(X_2(1), \ldots, X_2(n))$. Here, V_i is the empirical variance of partial sums of X_i and $S_{ii,q}$ is the Newey-West or HAC estimator of the long-run variance of X_i. The V/S statistic was developed in Giraitis et al. (2003, 2006) following the works of Lo (1991) and Kwiatkowski et al. (1992) on related R/S type statistics. In particular, from Giraitis et al. (2006) one easily derives the asymptotic null distribution T of the statistic T_n under the condition that the two samples are independent. It is also easy to show that for $d_1 \neq d_2$, one of the ratios in (1.1) tends to infinity and the other one to zero, meaning that the test is consistent against the alternative $d_1 \neq d_2$.

However, independence of the two samples is too restrictive and may be unrealistic in financial data analysis since price movements of different assets are usually correlated and susceptible to common macroeconomic shocks. In order to eliminate the eventual dependence between samples, a modification $\tilde{T}_n$ of (1.1) is proposed, which uses residual observations $(\tilde{X}_1(1), \ldots, \tilde{X}_1(n))$, obtained by regressing partial sums of X_1 on partial sums of X_2. The modified statistic $\tilde{T}_n$ is shown to have the same limit null distribution T as if the two samples were independent.

It is well-known that a major problem in applications of the rescaled variance and related statistics is the choice of the bandwidth parameter q. The present paper contributes to this problem by providing an adaptive formula in (6.32) for "optimal" q which depends not only on the (common) memory parameter d but also on the difference between estimated short memory (AR) components of the spectrum of the sampled series. The derivation of the last result uses the expansion of the HAC estimator in Abadir et al. (2009). A simulation study confirms that using this choice of bandwidth leads to a good size of our comparison tests for most values of fractional and ARMA parameters of the simulated series.

Several interesting open problems were suggested by Referee and Associated Editor. The assumption of stationarity can be very restrictive for applications. We expect that our results can be extended to values of d_i outside the interval $[0, 1/2)$. Another possibility for future research is development of similar procedures to test equal memory for more than two series.

The plan of the paper is as follows. Section 2 formulates the settings of the paper (Assumptions $\mathrm{A}(d_1, d_2)$ and $\mathrm{B}(d_1, d_2)$) and derives the limit of the test statistics T_n and $\tilde{T}_n$ and the rejection region of the null hypothesis $d_1 = d_2$. Assumption $\mathrm{A}(d_1, d_2)$ guarantees the existence of long-run (co)variances and the consistency of the HAC estimators. Assumption $\mathrm{B}(d_1, d_2)$ specifies the joint limit behavior of partial sums of X_1 and X_2 as given by bivariate fractional Brownian motion. The last process is defined by means of stochastic integral representation as in Chung (2002). The test procedures are then presented in detail and a brief study focus on the asymptotic power of the tests T_n and $\tilde{T}_n$. Section 3 verifies Assumptions $\mathrm{A}(d_1, d_2)$ and $\mathrm{B}(d_1, d_2)$ for bivariate moving average (X_1, X_2). Section 4 assesses the performance of the tests T_n and $\tilde{T}_n$, by simulating bivariate FARIMA samples with various fractional and autoregressive/moving average parameters. Conclusions are given in Section 5. The Appendix contains auxiliary results and derivations.

Notation. Below, $\to_p, \to_{\mathrm{law}}, \to_{D[0,1]}$ and $\to_{\mathrm{fdd}}$ $(=_{\mathrm{fdd}})$ stand for the convergence in probability, the weak convergence of random variables, the weak convergence of random elements in the Skorohod space $D[0, 1]$, and the weak convergence (equality) of finite dimensional distributions, respectively. Relation '$\sim$' means that the ratio of both sides tends to 1.

2 Construction of tests and its properties

2.1 Assumptions and main results

Let $((X_1(t), X_2(t)),\ t \in \mathbb{Z})$ be a bivariate covariance stationary process, viz., a sequence of random vectors $(X_1(t), X_2(t)) \in \mathbb{R}^2$ such that $\mathrm{E}X_i(t) = \mu_i$ and

$$\mathrm{cov}(X_i(t), X_j(t + h)) = \gamma_{ij}(h)$$

do not depend on $t \in \mathbb{Z}$ for any $h \in \mathbb{Z}, i, j = 1, 2$. Introduce the popular Bartlett-kernel estimator of the long-run (co)variance:

$$S_{ij,q} = \sum_{h=-q}^{q} \left(1 - \frac{|h|}{q+1}\right) \hat{\gamma}_{ij}(h), \qquad (2.1)$$

where

$$\hat{\gamma}_{ij}(h) \;=\; n^{-1}\begin{cases}\sum_{t=1}^{n-h}(X_i(t)-\bar{X}_i)(X_j(t+h)-\bar{X}_j), & h\ge 0,\\ \sum_{t=1-h}^{n}(X_i(t)-\bar{X}_i)(X_j(t+h)-\bar{X}_j), & h\le 0,\end{cases} \tag{2.2}$$

$\bar{X}_i = n^{-1}\sum_{t=1}^n X_i(t)$. The estimator in (2.1) is also called the heteroskedasticity and autocorrelation consistent (HAC). Also define

$$S^{\circ}_{ij,q} \;=\; \sum_{h=-q}^{q}\left(1-\frac{|h|}{q+1}\right)\hat{\gamma}^{\circ}_{ij}(h), \tag{2.3}$$

where

$$\hat{\gamma}^{\circ}_{ij}(h) \;=\; n^{-1}\begin{cases}\sum_{t=1}^{n-h}(X_i(t)-\mu_i)(X_j(t+h)-\mu_j), & h\ge 0,\\ \sum_{t=1-h}^{n}(X_i(t)-\mu_i)(X_j(t+h)-\mu_j), & h\le 0.\end{cases} \tag{2.4}$$

Note $\hat{\gamma}_{ij}(h)=\hat{\gamma}_{ji}(-h),\ \hat{\gamma}^{\circ}_{ij}(h)=\hat{\gamma}^{\circ}_{ji}(-h),\ S_{12,q}=S_{21,q},\ S^{\circ}_{12,q}=S^{\circ}_{21,q}$.

<u>Assumption A(d_1,d_2)</u> There exist $d_i\in[0,1/2), i=1,2$ such that for any $i,j=1,2$ the following limits exist

$$\begin{aligned} c_{ij} &= \lim_{n\to\infty}\frac{1}{n^{1+d_i+d_j}}\mathrm{E}\left(\sum_{t=1}^{n}(X_i(t)-\mu_i)\right)\left(\sum_{s=1}^{n}(X_j(s)-\mu_j)\right)\\ &= \lim_{n\to\infty}\frac{1}{n^{1+d_i+d_j}}\sum_{t,s=1}^{n}\gamma_{ij}(t-s). \end{aligned} \tag{2.5}$$

Moreover,

$$\frac{\sum_{k,l=1}^{q}\hat{\gamma}_{ij}(k-l)}{\sum_{k,l=1}^{q}\gamma_{ij}(k-l)} \;\to_p\; 1 \tag{2.6}$$

as $q\to\infty, n\to\infty, n/q\to\infty$.

Remark 2.1 The asymptotic constant c_{ij} is called the long-run covariance of X_i and X_j. Condition (2.6) is similar to Giraitis et al. (2006, Assumption A.2). It guarantees the consistency of the HAC estimator (see below).

Proposition 2.2 *Let Assumption $A(d_1, d_2)$ hold. Then, as $q \to \infty, n \to \infty, n/q \to \infty$,*

$$q^{-d_i-d_j} S_{ij,q} \quad \to_p \quad c_{ij}, \quad q^{-d_i-d_j} S^{\circ}_{ij,q} \ \to_p \ c_{ij} \quad (i,j = 1,2). \tag{2.7}$$

Moreover,

$$\frac{1}{q}(S_{ij,q} - S^{\circ}_{ij,q}) \ = \ -(\bar{X}_i - \mu_i)(\bar{X}_j - \mu_j)\Big(1 + o_p(1)\Big). \tag{2.8}$$

Assumption $\mathrm{B}(d_1, d_2)$, below, specifies the joint limit of partial sums of X_1 and X_2. It is similar to Giraitis et al. (2006, Assumption A.1). The limit process (bivariate fractional Brownian motion) is defined through a stochastic integral representation (2.9) similarly as in Chung (2002, (6)). Equivalently, this process can be defined through the covariance function defined in (6.1).

Definition 2.3 *A nonanticipative bivariate fractional Brownian motion (bi-fBm) with memory parameters $d_i \in (-1/2, 1/2)$, $i = 1,2$, is a Gaussian process $B = ((B_1(s), B_2(s)), s \in \mathbb{R})$ admitting the following representation for $i = 1,2$*

$$B_i(t) \ = \ \begin{cases} c(d_i) \int_{\mathbb{R}} \Big((t-x)_+^{d_i} - (-x)_+^{d_i}\Big) W_i(\mathrm{d}x), & \textit{if } d_i \neq 0, \\ W_i(0,t], & \textit{if } d_i = 0, \end{cases} \tag{2.9}$$

$W = ((W_1(\mathrm{d}x), W_2(\mathrm{d}x)),\ x \in \mathbb{R})$ is a $2-$dimensional Gaussian independently scattered white noise with real components, zero mean and covariance matrix

$$\mathrm{E}W_i(\mathrm{d}x)W_j(\mathrm{d}x) = \mathrm{d}x \begin{cases} 1, & i = j, \\ \rho_W, & i \neq j \end{cases}, \tag{2.10}$$

for some $\rho_W \in [-1,1]$, and the constants $c(d_i)$ are determined by condition $\mathrm{E}B_i^2(1) = 1$ so that

$$c^2(d_i) = \Big(\int ((1-x)_+^{d_i} - (-x)_+^{d_i})^2 \mathrm{d}x\Big)^{-1} = \frac{\cos(d_i\pi)}{\mathrm{B}(d_i+1, d_i+1)},$$

where $\mathrm{B}(p,q)$ is the beta function and $x_+ = \max(x, 0)$.

Remark 2.4 The nonanticipative bi-fBm is a particular case of general bi-fBm having the stochastic representation

$$X(t) \ = \ \int_{\mathbb{R}} \left\{ \left((t-x)_+^D - (-x)_+^D\right) A_+ + \left((t-x)_-^D - (-x)_-^D\right) A_- \right\} \widetilde{W}(\mathrm{d}x),$$

where $D = \mathrm{diag}(d_1, d_2)$, $x_- = \max(-x, 0)$, A_+, A_- are real 2×2 matrices and $\widetilde{W}(\mathrm{d}x) = (\widetilde{W}_1(\mathrm{d}x), \widetilde{W}_2(\mathrm{d}x))$, $x \in \mathbb{R}$, is an independently scattered Gaussian white noise with zero mean, unit variance and independent components; see Didier and Pipiras (2010), also Lavancier et al. (2009, (1.6)). The representation in (2.9) corresponds to the matrices $A_+A_+^* = \begin{pmatrix} c(d_1)^2 & c(d_1)c(d_2)\rho \\ c(d_1)c(d_2)\rho & c(d_2)^2 \end{pmatrix}$, $A_- = 0$.

Remark 2.5 In the sequel by bi-fBm we mean the nonanticipative process in (2.9). A bi-fBm has stationary increments and the self-similarity property:

$$(\lambda^{-d_1-.5}B_1(\lambda t), \lambda^{-d_2-.5}B_2(\lambda t)) \ =_{\rm fdd} \ (B_1(t), B_2(t))$$

for any $\lambda > 0$. Lavancier et al. (2009) showed that these properties essentially determine the covariance function in (6.1)-(6.3), up to a choice of constants g_{ij} and g_i defined in (6.2) and (6.4) (see Section 6.1).

Note that each component B_i is a univariate fractional Brownian motion with variance $\mathrm{E}B_i^2(t) = |t|^{2d_i+1}$, $i = 1, 2$. Also note that $(B_1(1), B_2(1))$ has a bivariate Gaussian distribution with zero means, unit variances and the correlation coefficient $\rho = \mathrm{E}B_1(1)B_2(1) = \rho_W \kappa(d_1, d_2)$, where $\kappa(d_1, d_2)$ depends only on d_1, d_2. In the case $d_1 = d_2$, we have that $\rho = \rho_W$ and the process $(\tilde{B}_1(t) = B_1(t) - \rho B_2(t), t \in \mathbb{R})$ is a fractional Brownian motion with variance $\mathrm{E}\tilde{B}_1^2(t) = (1-\rho^2)|t|^{2d_1+1}$. Moreover, the processes $\tilde{B}_1$ and B_2 are *independent*. Indeed, from (2.9) it is immediate that $\tilde{B}_1$ has a similar stochastic representation with $W_1(\mathrm{d}x)$ replaced by $\tilde{W}_1(\mathrm{d}x) = W_1(\mathrm{d}x) - \rho W_2(\mathrm{d}x)$, with $\tilde{W}_1$ independent of W_2.

Assumption B(d_1, d_2) Assumption A(d_1, d_2) is satisfied and, moreover,

$$\begin{aligned}&\Big(n^{-d_1-(1/2)}\sum\nolimits_{t=1}^{[n\tau]}(X_1(t) - \mathrm{E}X_1(t)), n^{-d_2-(1/2)}\sum\nolimits_{t=1}^{[n\tau]}(X_2(t) - \mathrm{E}X_2(t))\Big)\\ &\to_{\rm fdd} \ (\sqrt{c_{11}}B_1(\tau), \sqrt{c_{22}}B_2(\tau)),\end{aligned} \tag{2.11}$$

where c_{ij} are the same as in (2.5), $c_{ii} > 0$, $i = 1, 2$ and (B_1, B_2) is a bi-fBm with memory parameters d_1, d_2 and the correlation coefficient $\rho = \mathrm{corr}(B_1(1), B_2(1)) = c_{12}/\sqrt{c_{11}c_{22}} \in [-1, 1]$.

Define the empirical variance of partial sums of X_i:

$$V_i \ = \ n^{-2}\sum_{k=1}^{n}\Big(\sum_{t=1}^{k}(X_i(t) - \bar{X}_i)\Big)^2 - n^{-3}\Big(\sum_{k=1}^{n}\sum_{t=1}^{k}(X_i(t) - \bar{X}_i)\Big)^2. \tag{2.12}$$

The following proposition obtains the limit distribution of the statistic T_n defined in (1.1).

Proposition 2.6 *Let Assumptions A(d_1, d_2) and B(d_1, d_2) be satisfied with some $d_1, d_2 \in [0, 1/2)$ and $\rho \in [-1, 1]$, and let $n, q, n/q \to \infty$.*
(i) If $d_1 = d_2 = d$ then

$$T_n \ \to_{\rm law} \ T \ = \ \frac{U_1}{U_2} + \frac{U_2}{U_1}, \tag{2.13}$$

where

$$U_i \ = \ \int_0^1 (B_i^0(\tau))^2\mathrm{d}\tau - \Big(\int_0^1 B_i^0(\tau)\mathrm{d}\tau\Big)^2, \quad i = 1, 2, \tag{2.14}$$

where $(B_i^0(\tau) = B_i(\tau) - \tau B_i(1), \tau \in [0,1])$, $i = 1, 2$ *are fractional Brownian bridges obtained from bivariate fBm* $((B_1(\tau), B_2(\tau)), \tau \in \mathbb{R})$ *with the same memory parameters* $d_1 = d_2 = d$ *and correlation coefficient* $\rho = \rho_W$ *(see Definition 2.3).*

(ii) If $d_1 \neq d_2$ *then*

$$T_n \to_p \infty. \tag{2.15}$$

Let $\tilde{V}_1, \tilde{S}_{11,q}$ be the statistics in (2.12), (2.1), respectively, where $X_1(t), t = 1, \cdots, n$ is replaced by

$$\tilde{X}_1(t) = X_1(t) - (S_{12,q}/S_{22,q})X_2(t), \qquad t = 1, \dots, n. \tag{2.16}$$

In particular, note

$$\tilde{S}_{11,q} = S_{11,q} - \frac{S_{12,q}^2}{S_{22,q}}. \tag{2.17}$$

Define

$$\tilde{T}_n = \frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}} + \frac{V_2/S_{22,q}}{\tilde{V}_1/\tilde{S}_{11,q}}. \tag{2.18}$$

Note, $\tilde{T}_n$ is obtained by replacing $V_1, S_{11,q}$ in the definition of T_n in (1.1) by the corresponding quantities $\tilde{V}_1, \tilde{S}_{11,q}$ as defined above.

In the following proposition, we prove that under the null hypothesis, the limit distribution of $\tilde{T}_n$ is free of ρ, contrary to T in (2.14). Note that the limit of $\tilde{T}_n$ coincides with (2.14) when $\rho = 0$. This occurs for example when the statistics T_n is calculated from two independent processes X_1, X_2.

Proposition 2.7 *Let Assumptions* $A(d_1, d_2)$ *and* $B(d_1, d_2)$ *be satisfied with some* $d_1, d_2 \in [0, 1/2)$ *and* $\rho \in (-1, 1)$, *and let* $n, q, n/q \to \infty$.

(i) If $d_1 = d_2 = d$ *then*

$$\tilde{T}_n \to_{\text{law}} \tilde{T} = \frac{\hat{U}_1}{U_2} + \frac{U_2}{\hat{U}_1}, \tag{2.19}$$

where $\hat{U}_1, U_2$ *are independent and have the same distribution in (2.14).*

(ii) If $d_1 > d_2$ *then*

$$\tilde{T}_n \to_p \infty. \tag{2.20}$$

(iii) If $d_1 < d_2$ *then*

$$\tilde{T}_n \to_p \frac{\rho^2}{1-\rho^2} + \frac{1-\rho^2}{\rho^2}. \tag{2.21}$$

Remark 2.8 The ratio $\hat{\beta} = S_{12,q}/S_{22,q}$ in (2.16) minimizes the sum of squares:

$$\sum_{i=1-q}^{n} \Big(\sum_{t=i\vee 1}^{(i+q)\wedge n} (X_1(t) - \bar{X}_1) - \beta \sum_{t=i\vee 1}^{(i+q)\wedge n} (X_2(t) - \bar{X}_2) \Big)^2.$$

Therefore, $\hat{\rho} = \hat{\beta}\sqrt{S_{22,q}/S_{11,q}}$ can be considered as the least squares estimate of the long-run correlation coefficient ρ between partial sums of the two samples.

2.2 Testing procedures

Let $t_\alpha(d)$ denote the upper $\alpha-$quantile of the r.v. $\tilde{T}$ defined in (2.19) (or T in (2.14) when $\rho = 0$), viz.

$$\alpha = \mathrm{P}(\tilde{T} > t_\alpha(d)), \qquad \alpha \in (0,1). \tag{2.22}$$

Let

$$\hat{d} = (\hat{d}_1 + \hat{d}_2)/2, \tag{2.23}$$

where $\hat{d}_i$ is an estimator of d_i satisfying

$$\hat{d}_i - d_i = o_p(1/\log n) \qquad (i = 1, 2). \tag{2.24}$$

Similarly to Giraitis et al. (2006, Lemma 2.1), it can be proved that the quantile function $t_\alpha(d)$ is continuous in $d \in [0, 1/2)$ for any $\alpha \in (0,1)$. Therefore, the estimated quantile $t_\alpha(\hat{d}) \to_p t_\alpha(d)$ as $n \to \infty$ and the asymptotic level of the tests associated to the critical regions in (2.25)-(2.26) is preserved by replacing $t_\alpha(d)$ by $t_\alpha(\hat{d})$.

Testing the equality of the memory parameters in the case of independent samples. We wish to test the null hypothesis $d_1 = d_2$ against the alternative $d_1 \neq d_2$ under the assumption that X_1 and X_2 are independent. The decision rule at $\alpha-$level of significance is the following: we reject the null hypothesis when

$$T_n > t_\alpha(\hat{d}). \tag{2.25}$$

The consistency of this test is ensured by Proposition 2.6 (ii).

Testing the equality of the memory parameters in the case of possibly dependent samples. We wish to test the null hypothesis $d_1 = d_2$ against the alternative $d_1 > d_2$ in the general case when X_1 and X_2 are possibly dependent. The decision rule at $\alpha-$level of significance is the following: we reject the null hypothesis when

$$\tilde{T}_n > t_\alpha(\hat{d}). \tag{2.26}$$

The consistency of this test is ensured by Proposition 2.7 (ii).

Remark 2.9 For testing $d_1 = d_2$ against $d_1 < d_2$, the samples X_1 and X_2 should be exchanged in the statistic (2.18).

Remark 2.10 As noted by the referee, an undesirable feature of the testing procedure in (2.26) is that a rejection might occur not only when $d_1 > d_2$ but also when $d_1 < d_2$ (see Proposition 2.7 (iii)). To alleviate this feature, in (2.26) one can use the statistic

$$\tilde{T}_n^+ = \frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}}$$

instead of $\tilde{T}_n$. Note $\tilde{T}_n = \tilde{T}_n^+ + (\tilde{T}_n^+)^{-1}$. Under the assumptions of Proposition 2.7, the limit distribution of $\tilde{T}_n^+$ can easily be obtained from the proof of this proposition. In particular, if $d_1 = d_2 = d$, then

$$\tilde{T}_n^+ \to_{\text{law}} \tilde{T}^+ = \frac{\hat{U}_1}{U_2},$$

where $\hat{U}_1, U_2$ are independent and have the same distribution in (2.14). From the proof of Proposition 2.7, it also follows easily that $\tilde{T}_n^+ \to_p \infty$ $(d_1 > d_2)$ and $\tilde{T}_n^+ \to_p \rho^2/(1-\rho^2)$ $(d_1 < d_2)$, i.e., $\tilde{T}_n^+$ does not explode to infinity for $d_1 < d_2$ and $|\rho|$ not very close to 1.

2.3 Asymptotic behavior of the power function

We discuss in this section the asymptotic power of the testing procedures in (2.25) and (2.26).

For testing $d_1 = d_2$ against $d_1 > d_2$, we have proved that the rejection probability of the null hypothesis tends to 1, i.e., that

$$\mathrm{P}(\tilde{T}_n > t_\alpha(\hat{d})) \to 1, \tag{2.27}$$

for any $\alpha \in (0,1)$.

Section 4 provides finite sample rejection frequencies of the null hypothesis for some choices of parameters d_1, d_2 and some bivariate FARIMA models. A natural question in this context is to estimate the convergence rate in (2.27), or the decay rate of the probability $\mathrm{P}(\tilde{T}_n \le a)$, as a function of a, n, d_1, d_2, ρ and (possibly) other quantities of the model assumptions.

From the proof of Proposition 2.7 (ii), see (6.14) below, we have that for $d_1 > d_2$, the normalized statistic $\tilde{T}_n$ has a limit distribution,

$$(q/n)^{2(d_1-d_2)}\tilde{T}_n = (q/n)^{2(d_1-d_2)}\frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}} + o_p(1) \to_{\text{law}} \frac{U_1}{(1-\rho^2)U_2} = \Lambda, \tag{2.28}$$

say, where $U_i, i = 1,2$ are defined in (2.14). Therefore, we can expect that the probability $\mathrm{P}(\tilde{T}_n \le a)$ decays as the probability $\mathrm{P}(\Lambda \le a(q/n)^{2(d_1-d_2)})$, when $n \to \infty$. The decay rate of $\mathrm{P}(\Lambda \le x)$, $x \to 0$ is unknown, even for independent U_1, U_2, but in principle can be estimated from Monte-Carlo experiments. It is also plausible that r.v. Λ has a bounded probability density near $x = 0$ and so the above discussion suggests a decay rate $\mathrm{P}(\tilde{T}_n \le a) = O((q/n)^{2(d_1-d_2)})$.

However, the above argument is heuristic; in particularly, the replacement of $\tilde{T}_n$ by $(q/n)^{2(d_2-d_1)}\Lambda$ is not rigorously justified. It is clear that in order to correctly assess the probability $\mathrm{P}(\tilde{T}_n \le a)$, it is necessary to control from above the probabilities of *high* values of $\tilde{V}_1$ and $S_{22,q}$ in the denominator of the ratio $\frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}} = \frac{\tilde{V}_1 S_{22,q}}{V_2 \tilde{S}_{11,q}}$ and the probabilities of *small* values of $V_2, \tilde{S}_{11,q}$ in the numerator of the last ratio. While the former probabilities can be controlled by the Markov inequality, direct estimation of the latter probabilities is difficult and is replaced by an assumption on the distribution functions of these r.v.'s, see (2.29) below.

Proposition 2.11 *Let Assumption $A(d_1, d_2)$ be satisfied with $0 \le d_2 < d_1 < 1/2$. Moreover, assume that the distribution functions of the normalized r.v.'s $n^{-2d_1}V_1$ and $q^{-2d_2}S_{22,q}$ satisfy the following condition: for any $M > 0, a_0 > 0$ there exists a constant K such that the inequalities*

$$\mathrm{P}(n^{-2d_1}V_1 \le a) \le Ka, \qquad \mathrm{P}(q^{-2d}S_{22,q} \le a) \le Ka \tag{2.29}$$

are satisfied for any $n > M, q > M, n/q > M$ and any $0 < a \le a_0$. Then there exists a constant K_1, independent of n, q, a, and such that

$$\mathrm{P}(T_n \le a) \ \le \ K_1 a^{1/4}\Big(\frac{q}{n}\Big)^{(d_1-d_2)/2}, \qquad \mathrm{P}(\tilde{T}_n \le a) \ \le \ K_1 a^{1/4}\Big(\frac{q}{n}\Big)^{(d_1-d_2)/2} \tag{2.30}$$

hold for all $n, q, n/q$ sufficiently large and any $a \ge 0$ from a compact set.

The proof of the above proposition is given in Section 6.2. Note that condition (2.29) is implied by the existence of bounded probability densities of $n^{-2d_1}V_1$ and $q^{-2d_2}S_{22,q}$. Also note that the assumptions of Proposition 2.11 refer to *marginal* distributions of V_1 and $S_{22,q}$ only, and do not impose a restriction on the joint distribution of the four statistics in the definition of T_n and $\tilde{T}_n$.

3 Application to bivariate linear processes with long memory.

In this section we specify Assumptions $\mathrm{A}(d_1, d_2)$ and $\mathrm{B}(d_1, d_2)$ to a class of bivariate linear models $(X_1(t), X_2(t)),\ t \in \mathbb{Z}$ as given by

$$X_i(t) \ = \ \sum_{k=0}^{\infty} \psi_{i1}(k)\xi_1(t-k) + \sum_{k=0}^{\infty} \psi_{i2}(k)\xi_2(t-k), \qquad i = 1, 2, \tag{3.1}$$

where $\psi_{ij}(k)$ are real coefficients with $\sum_{k=0}^{\infty} \psi_{ij}^2(k) < \infty$ and $(\xi_1(t), \xi_2(t))$, $t \in \mathbb{Z}$ is a bivariate (weak) white noise with nondegenerate covariance matrix $(\rho_{\xi,ij})_{i,j=1,2}$. In other words,

$(\xi_1(t), \xi_2(t))$, $t \in \mathbb{Z}$ is a sequence of random vectors with zero mean $\mathrm{E}\xi_1(t) = \mathrm{E}\xi_2(t) = 0$ and covariances

$$\mathrm{E}\xi_i(t)\xi_j(s) \;=\; \begin{cases} \rho_{\xi,ij}, & t = s, \\ 0, & t \neq s. \end{cases} \tag{3.2}$$

Without loss of generality, below we shall assume $\rho_{\xi,11} = \rho_{\xi,22} = 1,\ \rho_{\xi,12} = \rho_{\xi,21} = \rho_\xi \in (-1, 1)$.

<u>Assumption $\widetilde{\mathrm{A}}(d_{ij})$</u> $(X_1(t), X_2(t))$ is a bivariate linear covariance stationary process as in (3.1) with coefficients $\psi_{ij}(k)$ satisfying the following conditions:

- If $d_{ij} \in (0, 1/2)$

  $$\psi_{ij}(k) \;=\; (\alpha_{ij} + o(1))\,|k|^{d_{ij}-1} \quad (k \to \infty)$$

  where $\alpha_{ij} \neq 0$ are some numbers, $i, j = 1, 2$.

- If $d_{ij} = 0$

  $$\sum_{k=0}^{\infty} |\psi_{ij}(k)| < \infty, \qquad \alpha_{ij} = \sum_{k=0}^{\infty} \psi_{ij}(k).$$

<u>Assumption $\widetilde{\mathrm{B}}(d_{ij})$</u> Assumption $\widetilde{\mathrm{A}}(d_{ij})$ is satisfied and, moreover, $(\xi_1(t), \xi_2(t))$, $t \in \mathbb{Z}$ is a sequence of i.i.d. random vectors.

Proposition 3.1 *(i) Let $(X_1(t), X_2(t))$ satisfy Assumption $\widetilde{A}(d_{ij})$. Then the limits c_{ij} in Assumption $A(d_1, d_2)$, (2.5) exist, with*

$$d_i = \max\{d_{i1}, d_{i2}\} \in [0, 1/2) \qquad (i = 1, 2). \tag{3.3}$$

(ii) Let $(X_1(t), X_2(t))$ satisfy Assumption $\widetilde{B}(d_{ij})$ and there exists $\delta > 0$ such that $\mathrm{E}|\xi_i(t)|^{2+\delta} < \infty$ $(i = 1, 2)$. Then $(X_1(t), X_2(t))$ satisfies Assumptions $A(d_1, d_2)$ and $B(d_1, d_2)$, with d_i as defined in (3.3). Moreover, the finite-dimensional convergence in Assumption $B(d_1, d_2)$, (2.11) extends to the functional convergence in the Skorohod space $D[0, 1]$.

Remark 3.2 Proposition 3.1 (ii) complements Chung (2002, Th.1), who discussed convergence of partial sums of $K-$variate linear processes to $K-$variate fractional Brownian motion under slightly different assumptions. Proposition 3.1 (i) and Proposition 2.2 (2.7) also complement the result in Abadir et al. (2009) about consistency of the HAC estimator for linear processes, by relaxing the 4th moment condition on the noise in the case $d_i \in [1/4, 1/2)$.

Let us consider some parametric examples of bivariate linear processes. Hereafter, we denote by L the backward shift i.e. $LX(t) = X(t-1)$.

Example 3.3 Let $a_{ij} \in \mathbb{R}$ $(i = 1, 2)$ be some constants, and let

$$X_i(t) \quad = \quad (1 - L)^{-d_i}(a_{i1}\xi_1(t) + a_{i2}\xi_2(t)) \qquad (i = 1, 2) \tag{3.4}$$

be FARIMA$(0, d_i, 0)$ processes, with possibly different parameters $d_i \in (0, 1/2)$. This process satisfies Assumption $\widetilde{\mathrm{A}}(d_{ij})$ with $d_{ij} = d_i$ and $\alpha_{ij} = \Gamma(d_i)^{-1}a_{ij}$ $(i, j = 1, 2)$.

If $(\xi_i(t), \xi_2(t))$ form a sequence of i.i.d. vectors as in Assumption $\widetilde{\mathrm{B}}(d_{ij})$, partial sums of $(X_1(t), X_2(t))$ converge to a bivariate fBm (B_{1,d_1}, B_{2,d_2}).

The limiting fBm has independent components if and only if the noise has uncorrelated components, i.e., if $\mathrm{E}(a_{11}\xi_1(t) + a_{12}\xi_2(t))(a_{21}\xi_1(t) + a_{22}\xi_2(t)) = 0$. For $d_1 = d_2$, the last condition is equivalent to the uncorrelatedness of the components of the process: $\mathrm{E}X_1(t_1)X_2(t_2) = 0$ $(t_1, t_2 \in \mathbb{Z})$.

Example 3.4 Consider the following system of linear equations:

$$\begin{aligned}(1 - L)^{d'_{11}}X_1(t) + \beta(1 - L)^{d'_{12}}X_2(t) &= \xi_1(t),\\ (1 - L)^{d'_{22}}X_2(t) &= \xi_2(t),\end{aligned}$$

where $d'_{ij} \in [0, 1/2), \beta \in \mathbb{R}$ are parameters, $d'_{22} + d'_{12} - d'_{12} < 1/2$ and where $(\xi_1(t), \xi_2(t)), t \in \mathbb{Z}$ are as in (3.1). A covariance stationary solution of the above equation is given by

$$\begin{aligned}X_2(t) &= (1 - L)^{-d'_{22}}\xi_2(t),\\ X_1(t) &= (1 - L)^{-d'_{11}}\xi_1(t) - \beta(1 - L)^{d'_{12} - d'_{11} - d'_{22}}\xi_2(t).\end{aligned}$$

Then $(X_1(t), X_2(t))$ satisfies Assumption $\widetilde{\mathrm{A}}(d_{ij})$ with $d_{11} = d'_{11}, d_{12} = d'_{22} + d'_{11} - d'_{12}, d_{21} = 0$ and $d_{22} = d'_{22}$.

Assume $(\xi_1(t), \xi_2(t))$ is a sequence of i.i.d. vectors satisfying the conditions in Proposition 3.1 (ii) and let $\beta \neq 0$. There are three cases $d'_{22} > d'_{12}$, $d'_{22} < d'_{12}$ and $d'_{22} = d'_{12}$ leading to $d_{11} < d_{12}$, $d_{11} > d_{12}$ and $d_{11} = d_{12}$, respectively. In each of these cases we can determine the memory parameter d_i of $(X_i(t)), i = 1, 2$ and the limiting bi-fBm in Assumption $\mathrm{B}(d_1, d_2)$, together with the correlation coefficient ρ.

4 A simulation study

In this section we assess the finite-sample performance of our procedures to test $d_1 = d_2$ versus $d_1 > d_2$ and provide a practical recommendation for the choice of the bandwidth parameter q.

The memory parameters d_1 and d_2 are estimated with the help of the adaptive version of the FEXP estimator (see Iouditsky *et al.* (2001)), which in practice turns out to be less sensitive to the short memory part of the long memory process as compared to other estimators. The bandwidth parameter is chosen according to the adaptive formula (6.32) derived in Appendix (see Section 6.3). The optimisation of the bandwidth is realised under the null hypothesis in order to ensure a good size to the test procedure. The choice of q depends on $\hat{d} = \frac{1}{2}(\hat{d}_1 + \hat{d}_2)$ as in the expansion obtained by Abadir et al. (2009, (2.14)), but also takes into account the short memory spectrum, in a form of certain coefficient.

The simulated samples are independent or dependent Gaussian FARIMA processes with different fractional and autoregressive/moving average parameters. The 5% quantile function in (2.22) was approximated from extensive Monte-Carlo experiments by

$$t_{5\%}(d) \approx 3.7d^2 + 8.6d + 5.2. \tag{4.1}$$

Independent samples. Table 1-3 concerns the case of independent samples, the test procedure is based on T_n. Tables 1 and 2 report the percentages of rejection of the null hypothesis $d_1 = d_2$ of the test $T_n > t_{5\%}(\hat{d})$ from 1000 replications of *independent* FARIMA$(1, d, 0)$ samples of size $n \in \{1024,\ 4096\}$, for five values of $d_i \in \{0,\ .1,\ .2,\ .3,\ .4\}$ and three values $a_i \in \{0,\ .4,\ .8\}$ of the autoregressive parameter. Recall from Proposition 2.6 that for independent samples T_n and $\tilde{T}_n$ have the same limiting distribution. We can see from both tables that the T_n test has fairly good size for most values of d_i and a_i. Table 3 provides the mean values of $\hat{q}$. Since these values are rather scattered across the table and the rejection frequency is very sensitive to the choice of q, the general impression from Tables 1-3 is that the adaptive choice of q in (6.32) was necessary. We also note that the power of the test decreases with increase of $|a_1 - a_2|$. The last fact can possibly be explained by the bias induced by the AR part in the FEXP estimator of d_i.

Dependent samples. Table 4 reports the performance of the test $\tilde{T}_n > t_{5\%}(\hat{d})$ on *dependent* samples as in Example 3.3, with $a_{11} = a_{22} = 1 - p,\ a_{12} = a_{21} = p$, where $p \in [0, 1/2)$ is a parameter. In other words, X_i are FARIMA(0,d_i,0) processes with mutually correlated innovations. The asymptotic correlation coefficient ρ between normalized partial sums of X_1 and X_2 is proportional to $2p(1-p)/(p^2 + (1-p)^2)$ and so ρ increases monotonically from 0 to 1 as p increases from 0 to 1/2. Since $p = 0$ corresponds to independent samples, the results in Table 4 can be compared with those for $a_1 = a_2 = 0$ in Tables 1 and 2. It appears that both tests T_n and $\tilde{T}_n$ perform similarly and that the long-run parameter ρ is well estimated to be zero.

The purpose of Tables 5 and 6 is twofold. Firstly, we want to evaluate the performance of $\tilde{T}_n$ on independent samples. Secondly, we want to assess the robustness of the adaptive formula (6.32) for bandwidth based on AR approximation of the short memory part with respect to other short memory specifications. To this end, we generate a FARIMA$(3, d, 0)$ process with polynomial AR function $1+0.7x^3$ (Table 5) and a FARIMA(0,d,2) process with polynomial MA function $1-(1/6)x+(1/6)x^2$ (Table 6). Together with the zero hypothesis rejection frequencies, Tables 5 and 6 also provide the (averaged) values of the adaptive estimator of the bandwidth q. One can immediately recognize that the estimated values of

q in Table 5 are much greater than the corresponding values on Table 6; nevertheless the size of the $\tilde{T}_n$ test is respected in both tables. One can conclude from Table 6 that the adaptive formula for q works rather well even if the FAR model (on which this formula is based) is misspecified.

n=1024		a_2=0					a_2=0.4					a_2=0.8				
	$d_1\backslash d_2$	0	.1	.2	.3	.4										
a_1=0	0	4.1														
	.1	12	5.0													
	.2	36	13	3.8												
	.3	64	32	9.3	3.2											
	.4	84	55	26	7.4	3.6										
	$d_1\backslash d_2$	0	.1	.2	.3	.4	0	.1	.2	.3	.4					
a_1=.4	0	5.8					4.5									
	.1	9.1	4.5				13	3.7								
	.2	21	7.4	4.0			35	9.6	4.9							
	.3	45	20	6.6	3.7		59	33	12	3.7						
	.4	66	35	17	4.2	3.4	82	58	26	9.1	2.5					
	$d_1\backslash d_2$	0	.1	.2	.3	.4	0	.1	.2	.3	.4	0	.1	.2	.3	.4
a_1=.8	0	3.8					4.9					2.2				
	.1	3.4	3.6				4.0	4.4				9.1	4.5			
	.2	13	3.7	3.9			13	4.4	4.8			28	9.9	3.8		
	.3	26	10	3.6	5.2		32	14	4.0	4.4		56	31	11	3.2	
	.4	45	25	9.6	4.5	5.9	55	32	13	4.1	5.8	80	55	31	9.5	3.5

Table 1: Frequency of rejection (in percentages) of the null hypothesis of the test $T_n > c_{5\%}(\hat{d})$. The samples are simulated following FAR(1,d_i) models. For fixed a_1, a_2, each cell contains a triangular array of dimension 5x5 corresponding to the different parameters (d_1, d_2) with $d_i \in \{0,\ .1,\ .2,\ .3,\ .4\}$ (i=1,2) and $d_1 \geq d_2$. The sample size is $n = 1024$. The estimation is based on 1000 replications.

n=4096		a_2=0					a_2=0.4					a_2=0.8				
	$d_1\backslash d_2$	0	.1	.2	.3	.4										
a_1=0	0	5.1														
	.1	23	4.1													
	.2	59	20	4.8												
	.3	86	49	14	4.3											
	.4	96	77	42	11	2.9										
	$d_1\backslash d_2$	0	.1	.2	.3	.4	0	.1	.2	.3	.4					
a_1=.4	0	5.2					3.5									
	.1	11	4.2				20	5.7								
	.2	35	11	4.8			57	19	4.2							
	.3	70	34	11	5.2		84	54	16	4.0						
	.4	88	64	30	10	3.3	95	78	48	14	3.0					
	$d_1\backslash d_2$	0	.1	.2	.3	.4	0	.1	.2	.3	.4	0	.1	.2	.3	.4
a_1=.8	0	5.4					5.2					4.9				
	.1	7.4	3.8				9.0	4.3				20	5.5			
	.2	25	9.0	4.3			30	10	4.8			42	15	3.8		
	.3	53	24	7.5	4.5		59	26	10	3.8		84	47	16	4.5	
	.4	75	44	21	6.2	4.2	82	50	25	8.2	5.3	95	79	50	16	4.2

Table 2: The same results as Table 1 for sample size $n = 4096$.

n=4096		a_2=0					a_2=0.4					a_2=0.8				
	$d_1\backslash d_2$	0	.1	.2	.3	.4										
a_1=0	0	3.2														
	.1	2.7	2.1													
	.2	2.3	2.0	1.7												
	.3	1.9	1.8	1.4	1.0											
	.4	1.8	1.5	1.0	0.5	0.3										
	$d_1\backslash d_2$	0	.1	.2	.3	.4	0	.1	.2	.3	.4					
a_1=0.4	0	11.2					5.4									
	.1	9.0	7.5				4.4	3.7								
	.2	7.5	6.3	5.3			3.6	3.0	2.7							
	.3	6.2	5.3	4.3	2.9		3.1	2.6	2.0	1.4						
	.4	5.3	4.4	2.9	1.8	1.0	2.6	2.0	1.4	0.8	0.4					
	$d_1\backslash d_2$	0	.1	.2	.3	.4	0	.1	.2	.3	.4	0	.1	.2	.3	.4
a_1=0.8	0	24.8					22.0					10.2				
	.1	19.9	16.2				17.5	14.1				8.2	6.7			
	.2	16.2	13.4	11.3			14.2	11.6	9.6			6.7	5.7	4.7		
	.3	13.5	11.3	8.8	5.9		11.6	9.7	7.6	4.9		5.6	4.6	3.4	2.3	
	.4	11.3	8.8	5.8	3.6	2.1	9.7	7.5	5.0	3.0	1.6	4.5	3.4	2.2	1.3	0.6

Table 3: The mean values on 1000 replications of $\hat{q}$ according to (6.32) for the simulations of Table 2.

		$n = 1028$					$n = 4096$				
	$d_1 \backslash d_2$	0	.1	.2	.3	.4	0	.1	.2	.3	.4
$p = 0$	0	4.3					5.6				
	.1	14	5.4				22	6.4			
	.2	38	13	3.7			62	17	5.7		
	.3	66	33	8.5	3.9		87	55	17	5.9	
	.4	83	57	27	7.5	3.4	97	83	45	13	3.8
	$d_1 \backslash d_2$	0	.1	.2	.3	.4	0	.1	.2	.3	.4
$p = .15$	0	5.8					5.8				
	.1	13	4.9				22	5.9			
	.2	40	10	6.2			64	21	6.3		
	.3	69	35	9.6	3.7		90	58	17	5.9	
	.4	84	61	26	6.0	3.0	98	83	50	14	4.1
	$d_1 \backslash d_2$	0	.1	.2	.3	.4	0	.1	.2	.3	.4
$p = .35$	0	5.4					4.9				
	.1	17	4.5				30	5.3			
	.2	54	14	4.7			84	26	5.3		
	.3	81	43	9.0	3.5		98	76	20	4.2	
	.4	95	74	30	7.2	2.9	100	95	60	13	3.1
	$d_1 \backslash d_2$	0	.1	.2	.3	.4	0	.1	.2	.3	.4
$p = .45$	0	6.2					5.5				
	.1	43	3.0				84	4.6			
	.2	84	28	3.6			98	63	5.9		
	.3	95	74	13	3.9		100	96	33	3.1	
	.4	97	90	46	6.8	3.2	100	99	85	14	4.4

Table 4: Frequency of rejection (in percentages) of the null hypothesis of the test $\tilde{T}_n > c_{5\%}(\hat{d})$. The samples are simulated following the model in (3.4). For fixed p, each cell contains a triangular array of dimension 5×5 corresponding to the different parameters (d_1, d_2) with $d_i \in \{0, .1, .2, .3, .4\}$ (i=1,2) and $d_2 \leq d_1$.

$d_1\backslash d_2$	0	.1	.2	.3	.4	$d_1\backslash d_2$	0	.1	.2	.3	.4
0	6.1					0	34.7				
.1	7.4	5.6				.1	28.4	24.0			
.2	21	8.2	4.0			.2	23.9	20.8	18.2		
.3	46	20	7.5	6.0		.3	20.8	18.3	16.0	13.0	
.4	64	39	18	6.0	4.5	.4	18.2	15.9	13.0	9.0	5.8

Table 5: [Left] Frequency of rejection (in percentages) of the null hypothesis of the test $\tilde{T}_n > c_{5\%}(\hat{d})$. The two samples X_1 and X_2 are independent. X_1 is simulated from FARIMA(3,d_1,0) model with polynomial AR function $1+.7x^3$ and X_2 from FARIMA(0,d_2,0) model. [Right] Adaptive estimation of the bandwidth parameter q. The sample size is 4096 and the statistics are evaluated from 1000 independent replications.

$d_1\backslash d_2$	0	.1	.2	.3	.4	$d_1\backslash d_2$	0	.1	.2	.3	.4
0	4.9					0	7.0				
.1	17	6.7				.1	6.1	5.2			
.2	48	16	4.9			.2	5.3	4.6	4.3		
.3	72	44	15	4.3		.3	4.8	4.2	3.8	2.8	
.4	88	71	35	11	4.3	.4	4.3	3.5	2.7	1.8	1.1

Table 6: [Left] Frequency of rejection (in percentages) of the null hypothesis of the test $\tilde{T}_n > c_{5\%}(\hat{d})$. The two samples X_1 and X_2 are independent. X_1 is simulated from FARIMA(0,d_1,2) model with polynomial MA function $1-(1/6)x+(1/6)x^2$ and X_2 from FARIMA(0,d_2,0) model. [Right] Adaptive estimation of the bandwidth parameter q. The sample size is 4096 and the statistics are evaluated from 1000 independent replications.

5 Concluding remarks

The paper constructs a two-sample test for comparison of long memory parameters $d_i \in [0, 1/2), i = 1, 2$ of covariance-stationary time series $X_i, i = 1, 2$ with discrete time. The test statistic, T_n, is defined as the sum of the ratio and the reciprocal ratio of the rescaled variance (V/S) statistics, computed for each sample, whose asymptotic and finite-sample behavior was studied in Giraitis et al. (2003, 2006). Under some assumptions which involve the existence of long-run covariances and the joint convergence of partial sums of X_1 and X_2 to a bivariate fractional Brownian motion, we derive the asymptotic null distribution of T_n. A modification $\tilde{T}_n$ of the test statistic T_n is introduced and shown to be asymptotically free of the long-run correlation coefficient between the two samples. The case when (X_1, X_2) form a bivariate linear process is discussed in detail. Simulation results using FARIMA samples with various fractional and autoregressive parameters show that the proposed tests have a good size for most values of fractional and autoregressive/moving average parameters. The robustness property of the test is largely due to our choice of bandwidth according to the adaptive formula in (6.32) which takes into account the estimated difference of short memory spectrum of the sampled processes. The derivation of the last formula uses the asymptotic expansion of the HAC estimator in Abadir et al. (2009).

6 Appendix. Proofs and auxiliary results

Subsection 6.1 provides alternative definition of bi-fBm by explicit cross-covariance function. Subsection 6.2 contains proofs of Propositions 2.2, 2.6, 2.7, 2.11 and 3.1. Subsection 6.3 is given to the derivation of the adaptive bandwidth formula in (6.32).

6.1 Covariance function of bivariate fractional Brownian motion

From (2.9) and Samorodnitsky and Taqqu (2006) we have for any $s, t \in \mathbb{R}$

$$\mathrm{E}B_i(s)B_i(t) = \frac{1}{2}\{|s|^{2d_i+1} + |t|^{2d_i+1} - |t-s|^{2d_i+1}\}, \qquad i = 1, 2.$$

The analytic expression of cross-covariance $\mathrm{E}B_1(s)B_2(t)$ is derived in Lavancier et al. (2009). It takes a different form in the cases $d_1 + d_2 \neq 0$ and $d_1 + d_2 = 0$. Let

$$\psi(d_1, d_2) = \frac{\mathrm{B}(d_1+1, d_2+1)\sqrt{\cos(d_1\pi)\cos(d_2\pi)}}{\sqrt{\mathrm{B}(d_1+1, d_1+1)\mathrm{B}(d_2+1, d_2+1)}}.$$

Let $d_1 + d_2 \neq 0$ ($d_1, d_2 \in (-1/2, 1/2)$. Then

$$\mathrm{E}B_1(s)B_2(t) = \frac{\rho_W}{2}\Big\{g_{12}(s)|s|^{d_1+d_2+1} + g_{21}(t)|t|^{d_1+d_2+1} - g_{21}(t-s)|t-s|^{d_1+d_2+1}\Big\}, \tag{6.1}$$

where

$$g_{ij}(t) = \begin{cases} g_{ij}, & t > 0, \\ g_{ji}, & t < 0 \end{cases}$$

and where

$$g_{12} = \psi(d_1,d_2)\sin(d_1\pi)/\sin((d_1{+}d_2)\pi), \quad g_{21} = \psi(d_1,d_2)\sin(d_2\pi)/\sin((d_1{+}d_2)\pi). \quad (6.2)$$

In the case $d_1 + d_2 = 0$,

$$\begin{aligned} \mathrm{E}B_1(s)B_2(t) &= \frac{\rho_W}{2}\Big\{g_1(|s|+|t|-|t-s|) \\ &\quad + g_2(t\log|t| + s\log|s| - (t-s)\log|t-s|)\Big\}, \end{aligned} \quad (6.3)$$

where

$$g_1 = (1/2)\psi(d_1,d_2)(\cos(\pi d_1)+\cos(\pi d_2)), \quad g_2 = \psi(d_1,d_2)(d_2-d_1). \quad (6.4)$$

6.2 Proofs of the propositions

Proof of Proposition 2.2. The first relation in (2.7) is immediate from (2.5) and (2.6); see also Giraitis et al. (2006, (3.4)). Then, the second relation in (2.7) follows from (2.8), which is proved below.

Assume without loss of generality that $\mu_i = \mu_j = 0$. By Assumption A(d_1,d_2), there exists a constant C such that for any $n,h \ge 1$,

$$\mathrm{E}\Big(\sum_{t=n-h+1}^{n} X_i(t)\Big)^2 \le Ch^{1+2d_i}, \qquad i=1,2. \quad (6.5)$$

In particular, $\mathrm{E}\bar{X}_i^2 = O(n^{2d_i-1})$. Let $h \ge 1$. Then

$$\hat{\gamma}_{ij}(h) - \hat{\gamma}^{\circ}_{ij}(h) = -\bar{X}_i\bar{X}_j + \bar{X}_i\frac{1}{n}\sum_{t=n-h+1}^{n}X_j(t) + \bar{X}_j\frac{1}{n}\sum_{t=n-h+1}^{n}X_i(t) - \frac{h}{n}\bar{X}_i\bar{X}_j.$$

Clearly, (2.8) follows from $q/n \to 0$ and

$$\frac{1}{qn}|\bar{X}_i|\sum_{h=1}^{q}\Big|\sum_{t=n-h+1}^{n}X_j(t)\Big| = o_p(n^{d_i+d_j-3/2}). \quad (6.6)$$

By (6.5) and Cauchy-Schwarz inequality,

$$\begin{aligned} \mathrm{E}|\bar{X}_i|\sum_{h=1}^{q}\Big|\sum_{t=n-h+1}^{n}X_j(t)\Big| &\le \mathrm{E}^{1/2}\bar{X}_i^2\mathrm{E}^{1/2}\sum_{h=1}^{q}\Big|\sum_{t=n-h+1}^{n}X_j(t)\Big|^2 \\ &\le Cn^{d_i-1/2}\Big(\sum_{h=1}^{q}h^{1+2d_j}\Big)^{1/2} \\ &= qn\ o(n^{d_i+d_j-3/2}) \end{aligned}$$

since $q = o(n)$. This proves (6.6). □

Proof of Proposition 2.6. Both parts (i) and (ii) follow from the joint convergence

$$\Big((q/n)^{2d_1}V_1/S_{11,q},(q/n)^{2d_2}V_2/S_{22,q}\Big) \quad \to_{\rm law} \quad (U_1,U_2), \tag{6.7}$$

with U_1, U_2 as in (2.14). The last relation follows similarly as in Giraitis et al. (2006, Lemmas 7.1 and 7.2). □

Proof of Proposition 2.7. (i) Recall from Remark 2.4 that $\rho = \rho_W$. From (2.7), (2.17) and Assumptions A(d_1,d_2) and B(d_1,d_2) (with $d_1 = d_2 = d$) it follows that

$$\begin{aligned} q^{-2d}\tilde S_{11,q} &= q^{-2d}S_{11,q} - \frac{(q^{-2d}S_{12,q})^2}{q^{-2d}S_{22,q}} \\ &\to_p \ c_{11} - \frac{c_{12}^2}{c_{22}} \ = \ c_{11}(1-\rho^2) \end{aligned}$$

and

$$\begin{aligned} & n^{-d-(1/2)}\Big(\sum\nolimits_{t=1}^{[n\tau]}(\tilde X_1(t)-\bar{\tilde X}_1), \sum\nolimits_{t=1}^{[n\tau]}(X_2(t)-\bar X_2)\Big) \\ & \to_{\rm fdd} \ (\sqrt{c_{11}}(B_1(\tau)-\rho B_2(\tau)), \sqrt{c_{22}}B_2(\tau)) \\ & =_{\rm fdd} \ \Big(\sqrt{c_{11}(1-\rho^2)}\hat B_1(\tau), \sqrt{c_{22}}B_2(\tau)\Big), \end{aligned}$$

where $(\hat B_1(\tau) = (B_1(\tau)-\rho B_2(\tau))/\sqrt{1-\rho^2})$ is a fBm independent of $(B_2(\tau))$; see Remark 2.4 . Let $(\hat B_1^0(\tau) = \hat B_1(\tau) - \tau\hat B_1(1), \tau\in[0,1])$, $(B_2^0(\tau) = B_2(\tau)-\tau B_2(1), \tau\in[0,1])$ be respective fractional Brownian bridges.

These relations together with (2.7) imply similarly as in Giraitis et al. (2006) that

$$n^{-2d}\tilde V_1 \ \to_{\rm law} \ c_{11}(1-\rho^2)\Big(\int_0^1(\hat B_1^0(\tau))^2{\rm d}\tau - \Big(\int_0^1 \hat B_1^0(\tau){\rm d}\tau\Big)^2\Big), \tag{6.8}$$

$$n^{-2d}V_2 \ \to_{\rm law} \ c_{22}\Big(\int_0^1(B_2^0(\tau))^2{\rm d}\tau - \Big(\int_0^1 B_2^0(\tau){\rm d}\tau\Big)^2\Big), \tag{6.9}$$

$$q^{-2d}\tilde S_{11,q} \ \to_{\rm law} \ c_{11}(1-\rho^2), \tag{6.10}$$

$$q^{-2d}S_{22,q} \ \to_{\rm law} \ c_{22} \tag{6.11}$$

as $n,q,n/q\to\infty$, as well as the *joint* convergence of the four quantities in (6.8)-(6.11). Since the limits in (6.8)-(6.11) are a.s. strictly positive and $(\hat B_1(\tau))$ is independent of $(B_2(\tau))$, this proves (2.19) and part (i).

(ii) From (2.7), (2.17) we have

$$\begin{aligned} q^{-2d_1}\tilde S_{11,q} &= q^{-2d_1}S_{11,q} - \frac{\big(q^{-d_1-d_2}S_{12,q}\big)^2}{q^{-2d_2}S_{22,q}} \\ &\to_p \ c_{11} - \frac{c_{12}^2}{c_{22}} \ = \ c_{11}(1-\rho^2). \end{aligned} \tag{6.12}$$

From (2.7) and Assumption B(d_1, d_2) we obtain that

$$\begin{aligned}
&\frac{1}{n^{d_1+1/2}} \sum_{t=1}^{[n\tau]} (\tilde{X}_1(t) - \bar{\tilde{X}}_1) \\
&= \frac{1}{n^{d_1+1/2}} \sum_{t=1}^{[n\tau]} (X_1(t) - \bar{X}_1) - \frac{q^{-d_1-d_2} S_{12,q}}{q^{-2d_2} S_{22,q}} \Big(\frac{q}{n}\Big)^{d_1-d_2} \frac{1}{n^{d_2+1/2}} \sum_{t=1}^{[n\tau]} (X_2(t) - \bar{X}_2) \\
&\to_{\mathrm{fdd}} \sqrt{c_{11}} B_1(\tau). \qquad (6.13)
\end{aligned}$$

Using similar arguments as in part (i), from (6.12) and (6.13) we get

$$\left((q/n)^{2(d_1-d_2)} \frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}}, (q/n)^{2(d_2-d_1)} \frac{V_2/S_{22,q}}{\tilde{V}_1/\tilde{S}_{11,q}} \right) \to_{\mathrm{law}} \left(\frac{U_1}{(1-\rho^2)U_2}, \frac{(1-\rho^2)U_2}{U_1} \right), \qquad (6.14)$$

where $U_i, i = 1, 2$ are defined in (2.14). Clearly, (6.14) implies (2.20) and part (ii).

(iii) In this case, (6.12) is again valid but (6.13) must be changed to

$$\begin{aligned}
&\frac{q^{d_2-d_1}}{n^{d_2+1/2}} \sum_{t=1}^{[n\tau]} (\tilde{X}_1(t) - \bar{\tilde{X}}_1) = \\
&= \frac{(q/n)^{d_2-d_1}}{n^{d_1+1/2}} \sum_{t=1}^{[n\tau]} (X_1(t) - \bar{X}_1) - \frac{q^{-d_1-d_2} S_{12,q}}{q^{-2d_2} S_{22,q}} \frac{1}{n^{d_2+1/2}} \sum_{t=1}^{[n\tau]} (X_2(t) - \bar{X}_2) \\
&\to_{\mathrm{fdd}} -\frac{c_{12}}{\sqrt{c_{22}}} B_2(\tau). \qquad (6.15)
\end{aligned}$$

From (6.15) we obtain

$$\frac{q^{2(d_2-d_1)}}{n^{2d_2}} \tilde{V}_1 \to_{\mathrm{law}} c_{11}\rho^2 \left(\int_0^1 (B_2^0(\tau))^2 \mathrm{d}\tau - \Big(\int_0^1 B_2^0(\tau) \mathrm{d}\tau \Big)^2 \right).$$

Combining this result with (6.12) and the convergences in (6.9), (6.11), with $d = d_2$, one obtains

$$\frac{\tilde{V}_1/\tilde{S}_{11,q}}{V_2/S_{22,q}} \to_{\mathrm{law}} \frac{\rho^2}{1-\rho^2},$$

proving (2.21). □

Proof of Proposition 2.11. We shall prove the second inequality in (2.30) only since the first one can be proved analogously. We shall use the following elementary inequalities: for any r.v. $\xi, \eta \geq 0$ and any $x > 0$

$$\begin{aligned}
&\mathrm{P}(\frac{\xi}{\eta} \leq x) \leq \mathrm{P}(\xi \leq \sqrt{x}) + \mathrm{P}(\eta > \frac{1}{\sqrt{x}}), \qquad (6.16) \\
&\mathrm{P}(\xi\eta \leq x) \leq \mathrm{P}(\xi \leq \sqrt{x}) + \mathrm{P}(\eta \leq \sqrt{x}), \quad \mathrm{P}(\xi - \eta \leq x) \leq \mathrm{P}(\xi \leq 2x) + \mathrm{P}(\eta > x).
\end{aligned}$$

Denote

$$\begin{aligned}
&\tilde{\xi}_1 = n^{-2d_1} \tilde{V}_1, \quad \xi_1 = n^{-2d_1} V_1, \quad \xi_2 = q^{-2d_2} S_{22,q}, \quad \tilde{\xi}_3 = q^{-2d_1} \tilde{S}_{11,q}, \\
&\xi_3 = q^{-2d_1} S_{11,q}, \quad \xi_4 = n^{-2d_2} V_2, \quad x = a \Big(\frac{q}{n}\Big)^{2(d_1-d_2)}.
\end{aligned}$$

Then

$$
\begin{aligned}
\mathrm{P}(\tilde{T}_n \le a) &\le \mathrm{P}\Big(\frac{\tilde{\xi}_1 \xi_2}{\tilde{\xi}_3 \xi_4} \le x\Big) \qquad (6.17)\\
&\le \mathrm{P}(\tilde{\xi}_1 \xi_2 \le x^{1/2}) + \mathrm{P}(\tilde{\xi}_3 \xi_4 > x^{-1/2})\\
&\le \mathrm{P}(\tilde{\xi}_1 \le x^{1/4}) + \mathrm{P}(\xi_2 \le x^{1/4}) + \mathrm{P}(\tilde{\xi}_3 > x^{-1/4}) + \mathrm{P}(\xi_4 > x^{-1/4}).
\end{aligned}
$$

Next, using the inequality $\tilde{V}_1 \ge (1/2)V_1 - \hat{\beta}^2 V_2 = (1/2)V_1 - \frac{\hat{\rho}^2 S_{11,q}}{S_{22,q}} V_2$ and the facts that $|\hat{\rho}| \le 1,\ V_2 \ge 0$, we get

$$
\tilde{\xi}_1 \ \ge \ (1/2)\xi_1 - \eta_1, \qquad (6.18)
$$

where

$$
\eta_1 \ = \ \frac{S_{11,q}}{S_{22,q}}(n^{-2d_1} V_2) \ = \ x\frac{\xi_3 \xi_4}{\xi_2}.
$$

Relations (6.18) and (6.16) yield

$$
\begin{aligned}
\mathrm{P}(\tilde{\xi}_1 \le x^{1/4}) &\le \mathrm{P}((1/2)\xi_1 - \eta_1 \le x^{1/4}) \qquad (6.19)\\
&\le \mathrm{P}(\xi_1 \le 4x^{1/4}) + \mathrm{P}(\eta_1 > x^{1/4})\\
&= \mathrm{P}(\xi_1 \le 4x^{1/4}) + \mathrm{P}(x\frac{\xi_3\xi_4}{\xi_2} > x^{1/4})\\
&\le \mathrm{P}(\xi_1 \le 4x^{1/4}) + \mathrm{P}(\xi_3 > x^{-1/4}) + \mathrm{P}(\xi_4 > x^{-1/4}) + \mathrm{P}(\xi_2 \le x^{1/4}).
\end{aligned}
$$

Combining (6.17) and (6.19) and using $\tilde{S}_{11,q} \ge S_{11,q}$, see (2.17), we obtain

$$
\mathrm{P}(\tilde{T}_n \le a) \ \le \ \mathrm{P}(\xi_1 \le 4x^{1/4}) + 2\mathrm{P}(\xi_2 \le x^{1/4}) + 2\mathrm{P}(\xi_3 > x^{-1/4}) + 2\mathrm{P}(\xi_4 > x^{-1/4}) \qquad (6.20)
$$

From Assumption $\mathrm{A}(d_1, d_2)$ and (6.30) we obtain

$$
\begin{aligned}
\mathrm{E}S_{11,q} &= \frac{1}{q+1}\mathrm{E}(\sum_{t=0}^{q} X_1(t))^2 + \mathrm{E}\sum_{|h|\le q}(1 - \frac{|h|}{q+1})(\hat{\gamma}_{11}(h) - \hat{\gamma}^{\circ}_{11}(h)) \qquad (6.21)\\
&\le K_2 q^{2d_1},\\
\mathrm{E}V_2 &\le K_3 n^{2d_2}
\end{aligned}
$$

for some constants K_2, K_3 independent of n, q, implying $\mathrm{E}\xi_3 \le K_2,\ \mathrm{E}\xi_4 \le K_3$. From (6.20), (6.21), the Markov inequality, and assumption (2.29), the statement of the proposition easily follows. □

Proof of Proposition 3.1. With exception of (2.6), all other facts in the statement of the proposition follow similarly or using the argument developed in Davydov (1970), Bružaitė and Vaičiulis (2005), Giraitis et al. (2006) and other papers. In particular, the joint convergence of partial sums of (X_1, X_2) can be proved by using the scheme of discrete stochastic integrals in Surgailis (2003). See also Chung (2002, proof of Theorem 1).

Let us prove the convergence of empirical long-run covariances in (2.6) or, equivalently, in (2.7). Denote

$$X_{ij}(t) \;=\; \sum_{k=0}^{\infty} \psi_{ij}(k)\xi_j(t-k), \quad i,j=1,2 \tag{6.22}$$

It suffices to show the convergence of the HAC estimates of long-run covariances $c_{ij,i'j'}$ of components X_{ij} and $X_{i'j'}$ in (6.22), for any pairs $(i,j),(i',j') \in \{1,2\}\times\{1,2\}$; more precisely, to show that

$$q^{-d_{ij}-d_{i'j'}} S_{ij,i'j',q} \;\;\to_p\;\; c_{ij,i'j'} \qquad (i,j,i',j'=1,2), \tag{6.23}$$

where $S_{ij,i'j',q}$ is defined as in (2.1) with $\hat{\gamma}_{ij}(h)$ replaced by the empirical covariance $\hat{\gamma}_{ij,i'j'}(h)$ between observations $X_{ij}(t), t=1,\cdots,n$ and $X_{i'j'}(t), t=1,\cdots,n$.

Fix $i,j,i',j' \in \{1,2\}$ and denote $X(t)=X_{ij}(t),\ X'(t)=X_{i'j'}(t),\ d=d_{ij},\ d'=d_{i'j'}$,

$$S_q = S_{ij,i'j',q}, \quad \gamma(h)=\gamma_{ij,i'j'}(h), \quad \hat{\gamma}(h)=\hat{\gamma}_{ij,i'j'}(h), \quad c=c_{ij,i'j'},$$

$\psi(k)=\psi_{ij}(k),\ \psi'(k)=\psi_{i'j'}(k),\ \xi(s)=\xi_j(s),\ \xi'(s)=\xi_{j'}(s),\ \tilde{\rho}_\xi=\rho_{\xi,jj'}$ for short. Write $S_q=S'_q+S''_q$, where

$$S'_q \;=\; \sum_{h=-q}^{q}\left(1-\frac{|h|}{q+1}\right)\tilde{\gamma}(h), \qquad S''_q \;=\; \sum_{h=-q}^{q}\left(1-\frac{|h|}{q+1}\right)(\hat{\gamma}(h)-\tilde{\gamma}(h)),$$

where

$$\tilde{\gamma}(h) \;=\; n^{-1}\begin{cases}\sum_{t=1}^{n-h} X(t)X'(t+h), & h\ge 0,\\ \sum_{t=1-h}^{n} X(t)X'(t+h), & h\le 0\end{cases}.$$

is the empirical covariance from noncentered observations; c.f. (2.2). Then (6.23) follows from

$$q^{-d-d'}S'_q \;\;\to_p\;\; c, \qquad S''_q \;=\; o_p(q^{d+d'}). \tag{6.24}$$

In the subsequent proof of the first relation of (6.24), we first assume $d>0, d'>0$. Split $S'_q=\tilde{\gamma}(0)+\sum_{h=-q}^{-1}\left(1-\frac{|h|}{q+1}\right)\tilde{\gamma}(h)+\sum_{h=1}^{q}\left(1-\frac{|h|}{q+1}\right)\tilde{\gamma}(h)$. Here, the last two sums can be treated similarly and $\tilde{\gamma}(0)=O_p(1)$ is negligible. Consider

$$\sum_{h=1}^{q}\left(1-\frac{h}{q+1}\right)\tilde{\gamma}(h) \;\;=\;\; \sum_{i=1}^{3} U_i,$$

where

$$U_1 \;=\; \tilde{\rho}_\xi \sum_s \sum_{h=1}^{q}\left(1-\frac{h}{q+1}\right)\frac{1}{n}\sum_{t=1}^{n-h}\psi(t-s)\psi'(t+h-s),$$

$$U_2 \;=\; \sum_s \eta_s \sum_{h=1}^{q}\left(1-\frac{h}{q+1}\right)\frac{1}{n}\sum_{t=1}^{n-h}\psi(t-s)\psi'(t+h-s), \tag{6.25}$$

$$U_3 \;=\; \sum_{s\ne s'}\sum_{h=1}^{q}\left(1-\frac{h}{q+1}\right)\frac{1}{n}\sum_{t=1}^{n-h}\psi(t-s)\psi'(t+h-s')\xi(s)\xi'(s'), \tag{6.26}$$

where $\eta_s = \xi(s)\xi'(s) - \mathrm{E}\xi(s)\xi'(s) = \xi(s)\xi'(s) - \tilde\rho_\xi$, the sums $\sum_s$ and $\sum_{s\ne s'}$ are taken over all $s \in \mathbb{Z}$ and $s, s' \in \mathbb{Z}, s \ne s'$, respectively, and where we put $\psi(t) = \psi'(t) = 0$ $(t < 0)$.

First, consider the (nonrandom) term U_1. Using the asymptotics of ψ and ψ' and the dominated convergence theorem, we easily obtain that, as $q \to \infty,\ n \to \infty,\ q/n \to 0$,

$$\begin{aligned} U_1 &= \tilde\rho_\xi \sum_{h=1}^{q} \Big(1 - \frac{h}{q+1}\Big)\Big(\frac{1}{n}\sum_{t=1}^{n-h} 1\Big) \sum_{k=0}^{\infty} \psi(k)\psi'(h+k) \\ &\sim \tilde\rho_\xi\, \alpha\,\alpha' \sum_{h=1}^{q} \int_0^\infty k^{d-1}(h+k)^{d'-1}\mathrm{d}k \\ &\sim \tilde\rho_\xi\, \alpha\,\alpha' \mathrm{B}(d, 1-d-d') \sum_{h=1}^{q} h^{d+d'-1} \\ &\sim \frac{1}{2} c q^{d+d'}, \end{aligned}$$

where $c = 2\tilde\rho_\xi\, \alpha\,\alpha'\, \mathrm{B}(d, 1-d-d')/(d+d')$. Then, the first relation in (6.24) follows from

$$q^{-d-d'} U_i \ = \ o_p(1) \qquad (i = 2, 3). \tag{6.27}$$

To estimate U_2, we use the fact that $(\eta(s), s \in \mathbb{Z})$ are i.i.d.r.v.'s, the well-known inequality $\mathrm{E}|\sum_i M_i|^p \le 2\sum_i \mathrm{E}|M_i|^p$ for independent zero mean random variables M_i with $\mathrm{E}|M_i|^p < \infty$ and $1 \le p \le 2$ (see Bahr and Esséen (1965)), the fact $\mathrm{E}|\eta_s|^p = C_p < \infty$ for some $p \in (1, 2)$ and the Minkowski inequality. Using these facts, we obtain

$$\begin{aligned} \mathrm{E}|U_2|^p &\le 2C_p \sum_s \Big(\sum_{h=1}^{q} \frac{1}{n} \sum_{t=1}^{n} |\psi(t-s)\psi'(t+h-s)|\Big)^p \\ &\le C\Big(\sum_{h=1}^{q} \frac{1}{n} \sum_{t=1}^{n} \Big(\sum_s |\psi(t-s)\psi'(t+h-s)|^p\Big)^{1/p}\Big)^p \\ &\le C\Big(\sum_{h=1}^{q} \Big(\sum_{s=0}^{\infty} s_+^{p(d-1)} (h+s)^{p(d'-1)}\Big)^{1/p}\Big)^p \\ &\le C\Big(\sum_{h=1}^{q} \Big(h^{p(d+d'-2)+1}\Big)^{1/p}\Big)^p \\ &\le C\Big(\sum_{h=1}^{q} h^{(d+d'-2)+(1/p)}\Big)^p \ \le \ C q^{p((1/p)+d+d'-1)} \end{aligned}$$

and therefore $\mathrm{E}^{1/p}|U_2|^p = O(q^{d+d'+(1/p)-1}) = o(q^{d+d'})$, as $p > 1$, proving (6.27) for $i = 2$.

Next, consider U_3. Using the fact that $\sum_{s=1}^{\infty} s^{d-1}(t+s)^{d'-1} \le C t^{d+d'-1}$ $(t \ge 0)$, we obtain

$$\mathrm{E}U_3^2 \ \le \ 2\sum_{s\ne s'} \Big(\sum_{h=1}^{q} \Big(1 - \frac{h}{q+1}\Big) \frac{1}{n} \sum_{t=1}^{n-h} \psi(t-s)\psi'(t+h-s')\Big)^2$$

$$\leq \frac{C}{n^2} \sum_{t,t'=1}^{n} \sum_{h,h'=1}^{q} \sum_{s,s'} |\psi(t-s)\psi'(t+h-s')\psi(t'-s)\psi'(t'+h'-s')| \quad (6.28)$$

$$\leq \frac{C}{n^2} \sum_{t,t'=1}^{n} \sum_{h,h'=1}^{q} |t-t'|_+^{2d-1} |t-t'+h-h'|_+^{2d'-1}$$

$$\leq \frac{Cq}{n} \sum_{|t|\leq n} \sum_{|h|\leq q} |t|_+^{2d-1} |t+h|_+^{2d'-1} \ \leq \ C(J_1+J_2),$$

where

$$J_1 \ = \ (q/n) \sum_{|t|\leq 2q} |t|_+^{2d-1} \sum_{|h|\leq 3q} |h|_+^{2d'-1} \ \leq \ C(q/n) q^{2(d+d')} \ = \ o(q^{2(d+d')}),$$

$$J_2 \ = \ (q^2/n) \sum_{2q<|t|\leq n} |t|^{2d+2d'-2} \ \leq \ C(q^2/n) \begin{cases} n^{2d+2d'-1}, & d+d'>1/2 \\ q^{2d+2d'-1}, & d+d'<1/2 \\ \log(n/q), & d+d'=1/2 \end{cases}$$

and so $J_2 = o(q^{2(d+d')})$ as $q, n, n/q \to \infty$ in all three cases (in the last case $d+d'=1/2$ this follows from the fact that $x \to 0$ entails $x \log(1/x) \to 0$).

This proves the first relation of (6.24) for $d>0, d'>0$.

Consider now the case $d=d'=0$ we want to prove that

$$S'_q \ \to_p \ c,$$

where

$$\begin{aligned} c \ &= \ \lim_{n\to\infty} n^{-1} \mathrm{E} \Big(\sum_{t=1}^{n} X(t) \Big) \Big(\sum_{s=1}^{n} X'(s) \Big) \\ &= \ \tilde{\rho}_\xi \lim_{n\to\infty} n^{-1} \sum_u \sum_{t=1}^{n} \psi(t-u) \sum_{s=1}^{n} \psi'(s-u) \\ &= \ \tilde{\rho}_\xi \, \alpha \, \alpha'. \end{aligned} \quad (6.29)$$

By writing $S'_q = \mathrm{E}S'_q + (S'_q - \mathrm{E}S'_q)$, the convergence $\mathrm{E}S'_q \to c$ follows similarly as in (6.29) above. Relation $S'_q - \mathrm{E}S'_q = o_p(1)$ can be shown similarly to (6.27), i.e., by splitting $S'_q - \mathrm{E}S'_q$ into "diagonal" and "off-diagonal" parts in the quadratic form in noise variables. Consider the "diagonal" part U_2 in (6.25). Then

$$\mathrm{E}|U_2|^p \ \leq \ 2C_p \sum_s \left(\sum_{h=1}^{q} \frac{1}{n} \sum_{t=1}^{n} |\psi(t-s)\psi'(t+h-s)| \right)^p \ \leq \ C(W_1+W_2),$$

where

$$\begin{aligned} W_1 \ &= \ \sum_{s=1}^{n} \left(\sum_{h=1}^{q} \frac{1}{n} \sum_{t=1}^{n} |\psi(t-s)\psi'(t+h-s)| \right)^p \\ &\leq \ Cn^{-p} \sum_{s=1}^{n} \left(\sum_{t=1}^{\infty} |\psi(t-s)| \right)^p \ = \ O(n^{1-p}) \ = o(1) \end{aligned}$$

since $p > 1$, while

$$\begin{aligned} W_2 &= \sum_{s=0}^{\infty}\Big(\sum_{h=1}^{q}\frac{1}{n}\sum_{t=1}^{n}|\psi(t+s)\psi'(t+h+s)|\Big)^p \\ &\le Cn^{-p}\sum_{s=0}^{\infty}\Big(\sum_{t=1}^{n}|\psi(t+s)|\Big)^p \\ &\le C\Big(n^{-1}\sum_{t=1}^{n}\Big(\sum_{s=0}^{\infty}|\psi(t+s)|^p\Big)^{1/p}\Big)^p \;=\; o(1) \end{aligned}$$

where we used the Minkowski inequality and the dominated convergence theorem to get $o(1)$, in view of the fact that $\sum_{s=0}^{\infty}|\psi(t+s)|^p$ is bounded in t and tends to zero as $t\to\infty$.

Consider the “off-diagonal” term U_3 in (6.26). Noting that, for fixed h, h', the sum in (6.28) over all $t, t', s, s' \in \mathbb{Z}$ is bounded by a constant independent of h, h', we get $\mathrm{E}U_3^2 \le C(q/n)^2 = o(1)$.

This proves the first relation of (6.24) for $d > 0, d' > 0$ and $d = d' = 0$.

Let us prove the second relation in (6.24). It follows from

$$\sum_{|h|\le q}\mathrm{E}|\hat\gamma(h)-\tilde\gamma(h)| = o(q^{d+d'}). \tag{6.30}$$

Using definitions of $\hat\gamma(h), \tilde\gamma(h)$, the Cauchy-Schwartz inequality and (2.5), for $h \ge 0$ one obtains

$$\begin{aligned} &(\mathrm{E}|\hat\gamma(h)-\tilde\gamma(h)|)^2 \\ &\le \mathrm{E}\left(\bar X\right)^2\mathrm{E}\Big(\frac{1}{n}\sum_{t=1}^{n-h}X'(t+h)\Big)^2 + \mathrm{E}\left(\bar X'\right)^2\mathrm{E}\Big(\frac{1}{n}\sum_{t=1}^{n-h}X(t)\Big)^2 + \mathrm{E}\left(\bar X\right)^2\mathrm{E}\left(\bar X'\right)^2 \\ &\le Cn^{2d+2d'-2} \end{aligned} \tag{6.31}$$

and so (6.30) reduces to $Cqn^{d+d'-1} = o(q^{d+d'})$ which is a consequence of $d + d' < 1$ and $q/n \to 0$.

This concludes the proof of (6.24) for $d > 0, d' > 0$ and $d = d' = 0$. The cases $d > 0 = d'$ and $d = 0 < d'$ can be treated in a similar way. Proposition 3.1 is proved. □

6.3 Derivation of the adaptive bandwidth

The aim of this section is to derive the adaptive bandwidth formula used in our simulations, viz.

$$\hat q \;=\; 0.3|\hat I|^{1/2}\begin{cases} n^{1/(3+4\hat d)}, & \text{if } \hat d < 1/4,\\ n^{1/2-\hat d}, & \text{if } \hat d > 1/4, \end{cases} \tag{6.32}$$

where $\hat d = (\hat d_1 + \hat d_2)/2$ is an estimator of the (common) long memory parameter d,

$$\hat I \;=\; \int_0^{\pi}\Big(\frac{\hat g_1(x)}{\hat g_1(0)} - \frac{\hat g_2(x)}{\hat g_2(0)}\Big)\frac{\mathrm{d}x}{x^{2\hat d}\sin^2(x/2)}, \tag{6.33}$$

and where $\hat{g}_i$ is an estimator of the short memory part $g_i(x) = f_i(x)/|x|^{2d_i}$ of the spectral density f_i of $X_i,\ i = 1, 2$. In this paper, $\hat{g}_i$ is the spectral density of the best AR approximation of g_i which is computed following the two step procedure in Ray and Crato (1996). Namely, we first estimate d_i and then we fit an AR process to $(1-L)^{\hat{d}_i} X_i$ using the BIC criterion.

From Abadir et al. (2009, Theorem 2.1) under similar assumptions on X_i as in Section 2 we have the following expansion of $S_{ii,q}$: for $0 < d_i < 1/4$,

$$q^{-2d_i} S_{ii,q} \quad = \quad c_{ii} + (q/n)^{1/2} g_i(0)(Z_{ni} + o_p(1)) + q^{-1-2d_i} g_i(0)(B_i + o_p(1)), \qquad (6.34)$$

where $Z_{ni} \to_{\text{law}} Z_i \sim N(0, v(d_i))$,

$$v(d_i) \quad = \quad 8\pi \int_0^\infty \Big(\frac{\sin(x/2)}{x/2}\Big)^4 x^{-4d_i} \mathrm{d}x,$$

$$B_i \quad = \quad \int_0^\infty \Big(\frac{g_i(x)}{g_i(0)\sin^2(x/2)} \mathbf{1}_{\{0<x<\pi\}} - \frac{1}{(x/2)^2}\Big) \frac{\mathrm{d}x}{x^{2d_i}}$$

and where $g_i(x) = f_i(x)|x|^{2d_i}$ is the short memory component of the spectral density f_i of X_i, which is assumed to be continuous at $x = 0$ and $g_i(x) = g_i(0) + O(x^2),\ x \to 0,\ g_i(0) > 0$. Note that the long-run variance c_{ii} is related to $g_i(0)$ by

$$c_{ii} = g_i(0) p(d_i), \qquad (6.35)$$

where $p(d) = 2\Gamma(1-2d)\sin(\pi d)/d(1+2d)$ depends only on d.

From the form of statistic T_n it is clear that q must be chosen so that the ratio c_{11}/c_{22} is well estimated by $S_{11,q}/S_{22,q}$. From (6.34), assuming $d_1 = d_2 = d$ as under the null hypothesis, we obtain

$$\frac{S_{11,q}/c_{11}}{S_{22,q}/c_{22}} - 1 \quad = \quad (q/n)^{1/2} \frac{Z_{n1} - Z_{n2}}{p(d)}(1 + o_p(1)) + q^{-1-2d} \frac{B_1 - B_2}{p(d)}(1 + o_p(1)). \qquad (6.36)$$

Therefore as $n, q, q/n \to \infty$,

$$E\Big(\frac{S_{11,q}/c_{11}}{S_{22,q}/c_{22}} - 1\Big)^2 \quad \sim \quad \frac{1}{p(d)^2} \Big((q/n)\mathrm{E}(Z_1 - Z_2)^2 + q^{-2(1+2d)} I^2\Big), \qquad (6.37)$$

since for $d_1 = d_2 = d$, we have $B_1 - B_2 = I$, where

$$I \quad = \quad \int_0^\pi \Big(\frac{g_1(x)}{g_1(0)} - \frac{g_2(x)}{g_2(0)}\Big) \frac{\mathrm{d}x}{x^{2d}\sin^2(x/2)},$$

c.f. (6.33). Minimizing the right-hand side of (6.37) with respect to q, we obtain

$$q \quad = \quad K_1(d)|I|^{2/(3+4d)} n^{1/(3+4d)}, \qquad (6.38)$$

where $K_1(d)$ depends on d and $\mathrm{E}(Z_1 - Z_2)^2$. Numerical computation of the function $K_1(d)$ reveals that it is well approximated by the constant value 0.3 on the interval $(0, 1/4)$ except for the case when d is close to $1/4$ and then $K_1(d)$ diverges but then also the approximations in (6.34) and (6.37) are less accurate. Therefore we choose to replace $K_1(d)$ in (6.38) by 0.3 on the whole interval $d \in (0, 1/4)$ in order that the test is not too conservative. For similar reasons, we replace the exponent of $|I|$ in (6.38) by $1/2$, since otherwise the test turns out to be too conservative for small values of d when (X_1) and (X_2) have very different short memory parts (or high values of $|I|$). The result of these replacements is $q = 0.3|I|^{1/2}n^{1/(3+4d)}$, c.f. (6.32).

Next, let us turn to the case $1/4 < d < 1/2$. From Abadir et al. (2009, Theorem 2.1) we obtain that for $1/4 < d_i < 1/2$

$$q^{-2d_i}S^{\circ}_{ii,q} \;=\; c_{ii} + (q/n)^{1-2d_i}g_i(0)(\tilde{Z}_{ni} + o_p(1)) + q^{-1-2d_i}g_i(0)(B_i + o_p(1)), \quad (6.39)$$

where $\tilde{Z}_{ni} \to_{\text{law}} \tilde{Z}_i$ and $\tilde{Z}_i$ is a (non-Gaussian) r.v. whose distribution depends only on d_i. From Proposition 2.4 (2.8) we have that

$$q^{-2d_i}(S_{ii,q} - S^{\circ}_{ii,q}) \;=\; -2(q/n)^{1-2d_i}g_i(0)(Y^2_{ni} + o_p(1)), \quad (6.40)$$

where

$$Y_{ni} = g_i(0)^{-1/2}n^{1/2-d_i}\bar{X}_i \;\to_{\text{law}}\; Y_i \sim N\Big(0, \frac{c_{ii}}{g_i(0)}\Big) = N(0, p(d_i)), \quad (6.41)$$

see (6.35). Combining (6.39)-(6.41) and using the facts that $\mathrm{E}\tilde{Z}_i = 0$, $i = 1, 2$ and $\mathrm{E}Y_1^2 = \mathrm{E}Y_2^2$, similarly as in (6.36) and (6.37) we obtain

$$\begin{aligned}\frac{S_{11,q}/c_{11}}{S_{22,q}/c_{22}} - 1 \;&=\; (q/n)^{1-2d}\frac{\tilde{Z}_{n1} - 2Y^2_{n1} - \tilde{Z}_{n2} + 2Y^2_{n2}}{p(d)})(1 + o_p(1)) \\ &+\; q^{-1-2d}\frac{B_1 - B_2}{p(d)}(1 + o_p(1))\end{aligned}$$

and

$$E\Big(\frac{S_{11,q}/c_{11}}{S_{22,q}/c_{22}} - 1\Big)^2 \;\sim\; \frac{1}{p(d)^2}\Big((q/n)^{2(1-2d)}\mathrm{E}J^2(d) + q^{-2(1+2d)}I^2\Big), \quad (6.42)$$

where $J(d) = \tilde{Z}_1 - 2Y_1^2 - \tilde{Z}_2 + 2Y_2^2$ has a distribution depending on d alone. Minimization of the right-hand side of (6.42) with respect to q leads to

$$q \;=\; K_2(d)|I|^{1/2}n^{1/2-d}, \quad (6.43)$$

where $K_2(d)$ is a function of d. In this case, we also choose $K_2(d) = 0.3$ for similar reasons as in the case $d < 1/4$ above. This completes our derivation of the bandwidth formula (6.32).

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References

Abadir, K., Distaso, W., Giraitis, L., 2009. Two estimators of the long-run variance: beyond short memory. Journal of Econometrics 150, 56–70.

Bahr, von B., Esséen, C.-G., 1965. Inequalities for the rth absolute moment of a sum of random variables, $1 \leq r \leq 2$. Annals of Mathematical Statistics 36, 299–303.

Beran, J., Terrin, N., 1996. Testing for a change of the long-memory parameter. Biometrica 83, 627–638.

Bružaitė, K., Vaičiulis, M., 2005. Asymptotic independence of distant partial sums of linear process. Lithuanian Mathematical Journal 45, 387–404.

Casas, I., Gao, J., 2008. Econometric estimation in long-range dependent volatility models: Theory and practice. Journal of Econometrics 147, 72–83.

Chung, C.-F., 2002. Sample means, sample autocovariances, and linear regression of stationary multivariate long memory processes. Econometric Theory 18, 51–78.

Crato, N., Ray, B.K., 1996. Model selection and forecasting for long-range dependent processes. Journal of Forecasting 15, 107–125.

Davydov, Yu., 1970. The invariance principle for stationary processes. Theory Probability and Its Applications 15, 487–498.

Deo, R., Hurvich, C.M., Soulier, Ph., Yi Wang, 2009. Propagation of memory parameter from durations to counts. Econometric Theory 25 (to appear).

Didier, G., Pipiras, V., 2010. Integral representations of operator fractional Brownian motion. To appear in Bernoulli.

Giraitis, L., Kokoszka, P., Leipus, R., Teyssière, G., 2003. Rescaled variance and related tests for long memory in volatility and levels. Journal of Econometrics 112, 265–294.

Giraitis, L., Leipus, R., Philippe, A., 2006. A test for stationarity versus trends and unit roots for a wide class of dependent errors. Econometric Theory 22, 989–1029.

Horváth, L., 2001. Change-point detection in long-memory processes. Journal of Multivariate Analysis 78, 218–234.

Iouditsky, A., Moulines, E., Soulier, Ph., 2001. Adaptive estimation of the fractional differencing coefficient. Bernoulli 7, 699–731.

Kwiatkowski, D., Phillips, P.C.B., Schmidt, P., Shin, Y., 1992. Testing the null hypothesis of stationarity against the alternative of a unit root: how sure are we that economic time series have a unit root? Journal of Econometrics 54, 159–178.

Lavancier, F., Philippe, A., Surgailis, D., 2009. Covariance function of vector self-similar process. Statist. Probab. Letters 79, 2415–2421.

Lo, A.W., 1991. Long-term memory in stock market prices. Econometrica 59, 1279–1313.

Lobato, I., Robinson, P.M., 1998. A nonparametric test for I(0). Revue of Economic Studies 65, 475–495.

Robinson, P.M., 1994. Efficient tests of nonstationary hypotheses. Journal of American Statistical Association 89, 1420–1437.

Samorodnitsky, G., Taqqu, M.S., 1994. Stable Non-Gaussian Random Processes. Chapman and Hall, New York.

Soofi, A.S., Wang, S., Wang, Y., 2006. Testing for long memory in the Asian foreign exchange rates. Journal of System Science and Complexity 19, 182–190.

Surgailis, D., 2003. Non CLTs: U-statistics, multinomial formula and approximations of multiple Itô-Wiener integrals. In: Doukhan, P., Oppenheim, G., Taqqu, M.S. (Eds.), Theory and Applications of Long-Range Dependence: Theory and Applications, pp. 129–142. Birkhäuser, Boston.

Surgailis, D., Teyssière, G., Vaičiulis, M., 2008. The increment ratio statistic. Journal of Multivariate Analysis 99, 510–541.

Teyssière, G., Kirman, A.P. (Eds.), 2007. Long memory in economics. Springer, Berlin - Heidelberg.