

February 14, 2010

ABOUT THE CHARACTERIZATION OF SOME RESIDUE CURRENTS

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This unpublished paper is a copy (completed by a development of section 5 and by minor corrections) of the article with the same title published in:

Complex Analysis and Digital Geometry, *Proceedings from the Kiselmanfest, 2006*, Acta Universitatis Upsaliensis, C. Organisation och Historia, **86**, Uppsala University Library (2009), 147-157.

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1. Introduction.

1.1. Residue current in dimension 1. Let $\omega = g(z)dz$ be a meromorphic 1-form on a small enough open set $0 \in U \subset \mathbb{C}$ having 0 as unique pole, with multiplicity k :

$$g = \sum_{l=1}^k \frac{a_{-l}}{z^l} + \text{holomorphic function}$$

Note that ω is d -closed.

Let $\psi = \psi_0 d\bar{z} \in \mathcal{D}^1(U)$ be a 1-test form. In general $g\psi$ is not integrable, but the principal value

$$Vp[\omega](\psi) = \lim_{\epsilon \rightarrow 0} \int_{|z| \geq \epsilon} \omega \wedge \psi$$

exists, and $dVp[\omega] = d''Vp[\omega] = \text{Res}[\omega]$ is the residue current of ω . For any test function φ on U ,

$$\text{Res}[\omega](\varphi) = \lim_{\epsilon \rightarrow 0} \int_{|z|=\epsilon} \omega \wedge \varphi$$

Then $\text{Res}[\omega] = 2\pi i \text{res}_0(\omega)\delta_0 + dB = \sum_{j=0}^{k-1} b_j \frac{\partial^j}{\partial z^j} \delta_0$ where $\text{res}_0(\omega) = a_{-1}$ is the Cauchy residue. We remark

that δ_0 is the integration current on the subvariety $\{0\}$ of U , that $D = \sum_{j=0}^{k-1} b_j \frac{\partial^j}{\partial z^j}$ and that $b_j = \lambda_j a_{-j}$ where the λ_j are universal constants.

Conversely, given the subvariety $\{0\}$ and the differential operator D , then the meromorphic differential form ω is equal to gdz , up to holomorphic form; hence the residue current $\text{Res}[\omega] = D\delta_0$, can be constructed.

1.2. Characterization of holomorphic chains. P. Lelong (1957) proved that a complex analytic subvariety V in a complex analytic manifold X defines an integration current $\varphi \mapsto [V](\varphi) = \int_{\text{Reg}V} \varphi$ on X . More

generally, a *holomorphic p -chain* is a current $\sum_{l \in L} n_l [V_l]$ where $n_l \in \mathbb{Z}$, $[V_l]$ is the integration current defined

by an irreducible p -dimensional complex analytic subvariety V_l , the family $(V_l)_{l \in L}$ being locally finite.

During more than twenty years, J. King [K 71], Harvey-Shiffman [HS 74], Shiffman [S 83], H. Alexander [A 97] succeeded in proving the following structure theorem: *Holomorphic p -chains on a complex manifold X are exactly the rectifiable d -closed currents of bidimension (p, p) on X .*

In the case of section 1.1, $\text{Res}[\omega]$ is the holomorphic chain with complex coefficients $2\pi i \text{res}_0(\omega)\delta_0$ if and only if 0 is a simple pole of ω .

1.3. Our aim is to characterize residue currents using rectifiable currents with coefficients that are principal values of meromorphic differential forms and holomorphic differential operators acting on them.

We present a few results in this direction.

The structure theorem of section 1.2 concerns complex analytic varieties and closed currents. So, after generalities on residue currents of semi-meromorphic differential forms, we will concentrate on residue currents of closed meromorphic forms.

2. Preliminaries: local description of a residue current ([D 93], section 6)

2.1. We will consider a finite number of holomorphic functions defined on a small enough open neighborhood U of the origin 0 of \mathbb{C}^n , with coordinates (z_1, \dots, z_n) . For convenient coordinates, any semi-meromorphic differential form, for U small enough, can be written $\frac{\alpha}{f}$, where $\alpha \in \mathcal{E}(U)$, $f \in \mathcal{O}(U)$ and

$$f = u_j \prod_k {}_j\rho_k^{r_k},$$

where the ${}_j\rho_k$ are irreducible distinct Weierstrass polynomials in z_j and the $r_k \in \mathbb{N}$ are independent of j , moreover u_j is a unit at 0, i.e., for U small enough, u_j does not vanish on U . Let B_j be the discriminant of the polynomial ${}_j\rho = \prod_k {}_j\rho_k$ and let $Y_k = Z({}_j\rho_k)$; it is clear that Y_k is independent of j . Let $Y = \cup_k Y_k$ and $Z = \text{Sing } Y$.

After shrinkage of $(0 \in) U$, the following expressions of $\frac{1}{f}$ are valid on U : for every $j \in [1, \dots, n]$,

$$\frac{1}{f} = u_j^{-1} \sum_k \sum_{\mu=1}^{r_k} {}_j c_{\mu}^k \frac{1}{{}_j\rho_k^{\mu}}$$

where ${}_j c_{\mu}^k$ is a meromorphic function whose polar set, in Y_k , is contained in $Z(B_j)$. Notice that B_j is a holomorphic function of $(z_1, \dots, \widehat{z}_j, \dots, z_n)$. In the following, for simplicity, we omit the unit u_j^{-1} .

2.2. Let $\omega = \frac{1}{f}$, $Vp[\omega](\psi) = \lim_{\epsilon \rightarrow 0} \int_{|f| \geq \epsilon} \omega \wedge \psi$; $\psi \in \mathcal{D}^{n,n}(U)$. The residue of ω is

$$\text{Res}[\omega] = (dVp - Vpd)[\omega] = (d^n Vp - Vpd^n)[\omega]$$

For every $\varphi \in \mathcal{D}^{n,n-1}(U)$, let $\varphi = \sum_{j=1}^n \varphi_j$ with

$$\varphi_j = \psi_j dz_1 \wedge \dots \wedge d\bar{z}_1 \wedge \dots \wedge \widehat{d\bar{z}_j} \wedge \dots$$

Then, from Herrera-Lieberman [HL 71], and the next lemma about B_j , we have:

$$\text{Res}[\omega](\varphi) = \sum_{j=1}^n \sum_k \sum_{\mu=1}^{r_k} \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta |{}_j\rho_k| = \epsilon} {}_j c_{\mu}^k \frac{1}{{}_j\rho_k^{\mu}} \varphi_j.$$

The lemma we have used here is the following:

Lemma 2.1. ([D 93], Lemma 6.2.2).

$$\text{Res}[\omega](\varphi_j) = \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta |f| = \epsilon} \omega \varphi_j.$$

Outside $Z(B_j)$, for $|{}_j\rho_k|$ small enough (since $\frac{\partial_j\rho_k}{\partial z_j} \neq 0$), we take $(z_1, \dots, z_{j-1}, {}_j\rho_k, z_{j+1}, \dots, z_n)$ as local coordinates.

2.3. Notations. For the sake of simplicity, until the end of this section, we assume $j = 1$ and write ρ_k, c_μ^k instead of ${}_1\rho_k, {}_1c_\mu^k$. Outside $Z(B_1)$, we take $(\rho_k, z_2, \dots, z_n)$ as local coordinates; then, for every C^∞ function h and every $s \in \mathbb{N}$, we have

$$\frac{\partial^s h}{\partial \rho_k^s} = \frac{1}{\left(\frac{\partial \rho_k}{\partial z_1}\right)^{2s-1}} D_s h, \quad \text{for } s \geq 1,$$

where $D_s = \sum_{\alpha=1}^s \beta_\alpha^s \frac{\partial^\alpha}{\partial z_1^\alpha}$, β_α^s is a holomorphic function determined by ρ_k and $D_0 = \left(\frac{\partial \rho_k}{\partial z_1}\right)^{-1}$.

Let

$$g_l^\mu = \binom{\mu-1}{l} \frac{1}{\left(\frac{\partial \rho_k}{\partial z_1}\right)^{2\mu-4}} D_l \left(\frac{c_\mu^k}{\frac{\partial \rho_k}{\partial z_1}}\right), \quad (0 \leq l \leq \mu-2);$$

$$g_{\mu-1}^\mu = \frac{1}{\left(\frac{\partial \rho_k}{\partial z_1}\right)^{2\mu-3}} D_{\mu-1} \left(\frac{c_\mu^k}{\frac{\partial \rho_k}{\partial z_1}}\right)$$

Let $Vp_{Y_k, B_1}^1[g_l^\mu]$ also denote the direct image, by the inclusion $Y_k \rightarrow U$, of the Cauchy principal value $Vp_{Y_k, B_1}[g_l^\mu]$ of $g_l^\mu|_{Y_k}$;

$$D_{1,k}^{\mu,l} = \sum_{\alpha=1}^{\mu-1-l} (-1)^\alpha \beta_\alpha^{\mu-1-l} \frac{\partial^\alpha}{\partial z_1^\alpha}, \quad \text{and } D_{1,k}^{\mu,\mu-1} = \text{id}.$$

2.4. Final expression of the residue. All what has been done for $j = 1$ is valid for any $j \in \{1, \dots, n\}$: the principal value $Vp^j(k, \mu, l) = Vp_{Y_k, B_j}^j[g_l^\mu]$ defined on Y_k and the holomorphic differential operator $D_{j,k}^{\mu,l}$. We also denote $Vp^j(k, \mu, l)$ the direct image of the principal value by the canonical injection $Y \hookrightarrow U$. Then, denoting L the inner product, we have:

$$(*) \quad \text{Res}[\omega](\varphi) = 2\pi i \sum_{j=1}^n \left[\sum_k \sum_{\mu=1}^{r_k} \frac{1}{(\mu-1)!} \sum_{l=0}^{\mu-1} D_{j,k}^{\mu,l} Vp^j(k, \mu, l) \right] \left(\frac{\partial}{\partial z_j} L\varphi_j \right)$$

3. The case of simple poles.

3.1. The case $\omega = \frac{1}{f}$.

Lemma 3.1. *For a simple pole and for every k , ${}^j c_1^k$ is holomorphic.*

Proof. Let $w = z_j$ and $y = (z_1, \dots, \hat{z}_j, \dots, z_n)$. At points $z \in U$ where $B_j(z) \neq 0$, for given y , let $w_{ks}, s = 1, \dots, s_k$, be the zeros of ρ_k . For given y , $\rho_k = \prod_{s=1}^{s_k} (w - w_{ks})$,

$$\frac{1}{f} = u_j \sum_k \sum_{s=1}^{s_k} {}^j C_1^{k,s} (w - w_{ks})^{-1}$$

where ${}^j C_1^{k,s} = \frac{1}{\frac{\partial}{\partial w} f(w_{ks}, y)}$; let \prod_σ^s denote the product for all $\sigma \neq s$,

$$\sum_{s=1}^{s_k} {}^j C_1^{k,s} (w - w_{ks})^{-1} = \sum_{s=1}^{s_k} {}^j C_1^{k,s} \frac{\prod_\sigma^s (w - w_{k\sigma})}{\prod_\sigma (w - w_{k\sigma})} = {}^j c_1^k(w, y) \rho_k^{-1},$$

with

$${}^j c_1^k(w, y) = \sum_{s=1}^{s_k} \frac{\prod_{\sigma}^s (w - w_{k\sigma})}{\frac{\partial}{\partial w} f(w_{ks}, y)} \quad ([D 57], \text{IV.B.3 et C.1}).$$

Here ${}^j c_1^k(w, y)$ holomorphically extends to points of U where the w_s are not all distinct because: if w_s appears m times in $\prod_{\sigma} (w - w_{k\sigma})$, it appears $(m-1)$ times in the numerator and the denominator of $\frac{\prod_{\sigma}^s (w - w_{k\sigma})}{\frac{\partial}{\partial w} f(w_{ks}, y)}$.

□

All the poles of ω are simple, i.e. for every k , $r_k = 1$; then $\mu = 1$, $l = 0$.

$$\text{Res}[\omega](\varphi) = 2\pi i \sum_{j=1}^n \left[\sum_k D_{j,k}^{1,0} V p^j(k, 1, 0) \right] \left(\frac{\partial}{\partial z_j} L\varphi_j \right)$$

$$\begin{aligned} \text{But } D_{1,k}^{1,0} &= \text{id}; \quad D_0 = \left(\frac{\partial \rho_k}{\partial z_1} \right)^{-1}; \quad g_{\mu-1}^{\mu} = \frac{1}{\left(\frac{\partial \rho_k}{\partial z_1} \right)^{2\mu-3}} D_{\mu-1} \left(\frac{c_{\mu}^k}{\frac{\partial \rho_k}{\partial z_1}} \right); \quad g_0^1 = \frac{1}{\left(\frac{\partial \rho_k}{\partial z_1} \right)^{-1}} D_0 \left(\frac{c_1^k}{\frac{\partial \rho_k}{\partial z_1}} \right) \\ &= \frac{1}{\left(\frac{\partial \rho_k}{\partial z_1} \right)^{-1}} \left(\frac{\partial \rho_k}{\partial z_1} \right)^{-1} \left(\frac{c_1^k}{\frac{\partial \rho_k}{\partial z_1}} \right) = \left(\frac{\partial \rho_k}{\partial z_1} \right)^{-1} c_1^k; \end{aligned}$$

$$V p^j(k, 1, 0) = V p_{Y_k, B_j}^j [g_0^1] = V p_{Y_k, B_j}^j \left[\left(\frac{\partial \rho_k}{\partial z_j} \right)^{-1} {}^j c_1^k \right],$$

hence

$$\text{Res}[\omega](\varphi) = 2\pi i \sum_{j=1}^n \left[\sum_k V p_{Y_k, B_j}^j \left[\left(\frac{\partial \rho_k}{\partial z_j} \right)^{-1} {}^j c_1^k \right] \right] \left(\frac{\partial}{\partial z_j} L\varphi_j \right)$$

where ${}^j c_1^k$ is holomorphic.

3.2. The case of any degree. Let $\omega = \frac{\alpha}{f}$. Then $\text{Res}[\omega] = \alpha \wedge \text{Res}\left(\frac{1}{f}\right)$. Moreover, $d \text{Res}[\omega] = \pm \text{Res}[d\omega]$, then $\text{Res}[\omega]$ is d -closed if ω is d -closed.

4. Expression of the residue current of a closed meromorphic differential form.

In this section and a part of the following one, we give statements on residue currents according to the general hypotheses and proofs of sections 2 and 3. Proofs in a particular case where the polar set is equisingular and the singularity of the polar set is a 2-codimensional smooth submanifold are given in ([D 57], IV.D).

4.1. Closed meromorphic differential forms.

4.1.1. Let $\omega = \frac{\alpha}{f}$ be a d -closed meromorphic differential p -form on a small enough open neighborhood U of the origin 0 of \mathbb{C}^n . From section 2.1, we get $\omega = \sum \omega_k$ with $\omega_k = \sum_{\mu=1}^{r_k} {}^j c_{\mu}^k \frac{\alpha}{j \rho_{\mu}^k}$ for every $j = 1, \dots, n$. We have

$${}^j c_{\mu}^k = \frac{{}^j a_{\mu}^k(z_1, \dots, z_n)}{{}^j b_{\mu}^k(z_1, \dots, \widehat{z}_j, \dots, z_n)},$$

where a and b are holomorphic. Then $d\omega = \sum d\omega_k$ and $d\omega_k$ is the quotient of a holomorphic form by a product of ${}^j b_{\mu}^k(z_1, \dots, \widehat{z}_j, \dots, z_n)$ and ${}^j \rho_k^{r_k+1}$ (see [D 57], IV.D.1).

As at the end of section 2.2, using the local coordinates

$$(z_1, \dots, z_{j-1}, \rho_k, z_{j+1}, \dots, z_n),$$

we have

$$(4.1) \quad \omega_k = \sum_{\mu=1}^{r_k} [{}^j A_{\mu}^k \wedge {}^j \rho_k^{-\mu} d_j \rho_k + {}^j \rho_k^{-\mu} B_{\mu}^k],$$

where the coefficients are meromorphic.

Let \mathcal{R}_j be the ring of meromorphic forms on U whose coefficients are quotients of holomorphic forms on U by products of powers of $\frac{\partial_j \rho_k}{\partial z_j}$ and $j b_\mu^k$.

Lemma 4.1 ([D 57], Lemme 4.10). *Assume that $d\omega_k \in \mathcal{R}_j$. Then*

$$\omega_k =_j \rho_k^{-1} d_j \rho_k \wedge a_j^k + \beta_j^k + dR_j^k$$

with

$$R_j^k = \sum_{\nu=1}^{r_k-1} j e_{\nu,j}^k \rho_k^{-\nu} \quad \text{and} \quad da_j^k = d_j \rho_k \wedge {}^k a_j' + C_j^k \rho_k,$$

where $a_j^k, \beta_j^k, j e_{\nu,j}^k, {}^k a_j', C_j \in \mathcal{R}_j$ and are independent of dz_j .

4.1.2. Let φ be of type $(n-p, n-1)$. Then

$$\varphi = \sum \varphi_j, \quad \text{with} \quad \varphi_j = \sum \psi_{l_1, \dots, l_{n-p}} dz_{l_1} \wedge \dots \wedge dz_{l_{n-p}} \wedge \dots \wedge \widehat{dz_j} \wedge \dots$$

Proposition 4.2. *Let $\omega = \frac{\alpha}{f}$ be a d -closed meromorphic p -form on U . Given a coordinate system on U , and with notations of section 2.1, there exists a current $S_j^{p-1,1}$ such that $d'' S_j|_{U \setminus Z} = 0$, $\text{supp} S_j = Y$ and, for every k, j , a d -closed meromorphic $(p-1)$ -form A_j^k on Y_k with polar set Z such that*

$$\text{Res}[\omega](\varphi) = \sum_{j=1}^n \left(2\pi i \sum_k V p_{Y_k, B_j} A_j^k + d' S_j \right) \left(\frac{\partial}{\partial z_j} \mathbf{L} \varphi_j \right).$$

When the coordinate system is changed, the first term of the parenthesis is modified by addition of $2\pi i \sum_k d' V p_{Y_k, B_j} [F_j^k]$ where F_j^k is a meromorphic $(p-2)$ -form on Y_k with polar set Z .

Here $2\pi i \sum_{j=1}^n \sum_k V p_{Y_k, B_j} A_j^k(\cdot_j)$ will be called the *reduced residue* of ω .

Proof. Apply the proof of (*) (section 2) to the meromorphic form of Lemma 4.1.

We shall use the expression of $\text{Res}[\omega](\varphi)$ of section 2.2, for ω closed.

For k and j fixed, we consider

$$J_{kj} = \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |j \rho_k| = \epsilon} \omega_k(\varphi_j).$$

Then $\text{Res}[\omega](\varphi) = \sum_{k,j} J_{kj}$.

$$\lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |j \rho_k| = \epsilon} dR_j^k \wedge \varphi_j = (-1)^p \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |j \rho_k| = \epsilon} R_j^k \wedge d\varphi_j.$$

Let S_j^k be the current defined by

$$S_j^k(\psi_j) = - \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |j \rho_k| = \epsilon} R_j^k \wedge \psi_j.$$

By Lemme 4.1. R_j^k is independent of dz_j .

Let $\psi_j = dz_j \wedge \eta^j + \xi^j$, where ξ^j is independent of dz_j , then $\eta^j = \frac{\partial}{\partial z_j} \mathbf{L} \psi_j$.

After change of coordinates:

$$(4.2) \quad S_j^k(\psi_j) = - \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |j \rho_k| = \epsilon} \left(\frac{\partial_j \rho_k}{\partial z_j} \right)^{-1} R_j^k \wedge d_j \rho_k \wedge \eta^j$$

$$= (-1)^p 2\pi i \lim_{\delta \rightarrow 0} \sum_{\nu} \int_{Y_k |_{B_j} \geq \delta} (\nu - 1)!^{-1} \left(\frac{\partial^{\nu-1} (j e_{\nu}^k \wedge \eta^j \left(\frac{\partial_j \rho_k}{\partial z_j} \right)^{-1})}{\partial_j \rho_k^{\nu-1}} \right)_{j \rho_k = 0}$$

We have $S_j(\psi_j) = \sum_k S_j^k$.

$$\lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |j \rho_k| = \epsilon} j \rho_k^{-1} d_j \rho_k \wedge a_j^k + \beta_j^k = 2\pi i \lim_{\delta \rightarrow 0} \int_{|B_j| \geq \delta} a_j^k |_{Y_k} = 2\pi i V p_{Y_k, B_j} A_j^k, \text{ with } A_j^k = a_j^k |_{Y_k}$$

The last ainea is proved as in ([D 57], IV.D.4). \square

Corollary 4.3. *The current S_j is obtained by application of holomorphic differential operators to currents principal values of meromorphic forms supported by the irreducible components of Y .*

Proof. The corollary follows from the above expression for S_j and the computations in section 2. \square

We remark that d' itself is a holomorphic differential operator.

4.2. Particular cases.

4.2.1. The case $p = 1$. With the notations of Proposition 4.2, the forms A^k are of degree 0 and are d -closed, hence constant and unique: the reduced residue is a *divisor with complex coefficients*.

4.2.2. With the hypotheses and the notations of section 2.1, if all the multiplicities r_k are equal to 1, the reduced residue is uniquely determined and the current $S = 0$.

4.3. Comparison with the expression of $Res[\omega]$ in section 2, when ω is d -closed.

The reduced residue is equal to

$$2\pi i \sum_{j=1}^n \left[\sum_k V p_{Y_k, B_j}^j \left(\frac{\partial \rho_k}{\partial z_j} \right)^{-1} j c_1^k \left(\frac{\partial}{\partial z_j} L(\alpha \wedge \cdot)_j \right) \right].$$

It is well defined if all the poles of ω are simple.

5. Generalization of a theorem of Picard. Structure of residue currents of closed meromorphic forms.

5.1. The theorem of Picard [P 01] characterizes the divisor with complex coefficients associated to a d -closed differential form, of degree 1 of the third kind, on a complex projective algebraic surface; this result has been generalized by S. Lefschetz (1924): "the divisor has to be homologous to 0", then by A. Weil (1947). Locally, one of its assertions is a particular case of the theorem of Dickenstein-Sessa ([DS 85], Theorem 7.1): *Analytic cycles are locally residual currents* (see section 5.5), with a variant by D. Boudiaf ([B 92], Ch.1, sect.3).

5.2. Main results.

Theorem 5.1. *Let X be a complex manifold which is compact Kähler or Stein, and Y be a complex hypersurface of X , then $Y = \cup_{\nu} Y_{\nu}$ is a locally finite union of irreducible hypersurfaces. Let $Z = \text{Sing } Y$, and let A_{ν} be a d -closed meromorphic $(p-1)$ -form on Y_{ν} with polar set $Y_{\nu} \cap Z$ such that the current $t = 2\pi i \sum_{\nu} V p_{Y_{\nu}} A_{\nu}$ is d -closed.*

Then the following two conditions are equivalent:

(i) t is the residue current of a d -closed meromorphic p -form on X having Y as polar set with multiplicity one.

(ii) $t = dv$ on X , where v is a current, i.e., is cohomologous to 0 on X .

Proof. From section 4 locally, and a sheaf cohomology machinery globally; detailed proof will be given later for the more general theorem 5.5. \square

For $p = 1$, the A_{ν} are complex constants, then t is the divisor with complex coefficients $2\pi i \sum_{\nu} A_{\nu} Y_{\nu}$.

Corollary 5.1.1. *Under the hypotheses of Theorem 5.1, every residue current of a closed meromorphic p -form appears as a divisor, homologous to 0, whose coefficients are principal values of meromorphic $(p-1)$ -forms on the irreducible components of the support of the divisor and conversely.*

Let $\mathcal{R}_{q,q}^{loc}(X)$ be the vector space of locally rectifiable currents of bidimension (q, q) on the complex manifold X and

$$\mathcal{R}_{q,q}^{loc\mathbb{C}}(X) = \mathcal{R}_{q,q}^{loc} \otimes_{\mathbb{Z}} \mathbb{C}(X)$$

Theorem 5.2. *Let $T \in \mathcal{R}_{q,q}^{loc\mathbb{C}}(X)$, $dT = 0$. Then T is a holomorphic q -chain with complex coefficients.*

This is the structure theorem of holomorphic chains of Harvey-Shiffman-Alexander for complex coefficients; thanks to it, divisors will be translated into rectifiable currents.

Theorem 5.3. *Let X be a Stein manifold or a compact Kähler manifold. Then the following conditions are equivalent:*

(i) *T is the residue current of a d -closed meromorphic 1-form on X having $\text{supp } T$ as polar set with multiplicity 1;*

(ii) *$T \in \mathcal{R}_{n-1,n-1}^{loc\mathbb{C}}(X)$, $T = dV$.*

In the same way, we can reformulate the Theorem 5.1 with rectifiable currents:

Theorem 5.4. *Let X be a Stein manifold or a compact Kähler manifold. Then the following conditions are equivalent:*

(i) *$T = \sum_{\nu} a_{\nu} T_{\nu}$, with $T_{\nu} \in \mathcal{R}_{n-1,n-1}^{loc\mathbb{C}}(X)$, d -closed, and a_{ν} the principal value of a d -closed meromorphic $(p-1)$ -form on $\text{supp } T_{\nu}$, such that $T = dV$;*

(ii) *T is the residue current of a d -closed meromorphic p -form on X having $\cup_l T_l$ as polar set with multiplicity 1.*

5.3. Remark. The global Theorem 5.1 gives also local results since any open ball centered at 0 in \mathbb{C}^n is a Stein manifold.

5.4. Generalization.

5.4.1. With the notations of section 4.1, what has been done with the current $2\pi i \sum_{\nu} V p_{Y_{\nu}} A_{\nu}$ is also possible in the general case. The current S is defined as follows: let $\psi = \sum_j \psi_j$, then $S(\psi) = \sum_j \sum_k S_j^k(\psi_j)$.

From (4.2), we have:

$$(5.3) \quad S_j^k(\psi_j) = 2\pi i \sum_{\mu=1}^{r_k} \sum_{l=0}^{\mu-1} \Delta_{j,k}^{\mu,l} V p_{Y_k, B_j}^j [\gamma_{k,l}^{\mu,j}] \left(\frac{\partial}{\partial z_j} L \psi_j \right)$$

where $\gamma_{k,l}^{\mu,j}$ is a meromorphic form on Y_k , with polar set contained in $Y_k \cap \{B_j = 0\}$, and where $\Delta_{j,k}^{\mu,l}$ is a holomorphic differential operator in the neighborhood of Y_k . In the global case, for $Y = \cup_{\nu} Y_{\nu}$ locally finite, we take $k = \nu$, the sum $\sum_{\nu} S_{\nu}^{\nu}$ being locally finite.

Then we will get generalizations of the results in sections 5.2 and 5.3 completing the programme of section 1.3.

Lemma 5.1. *Let m^p be the sheaf of closed meromorphic differential forms. Let \overline{m}^p be the image by V_p of m^p in the sheaf of germs of currents on X . Then, for X Stein or compact Kähler manifold, we have the commutative diagram*

$$\begin{array}{ccccccc} H^0(X, m^p) & \rightarrow & H^0(X, \overline{m}^p) & \rightarrow & H^0(X, \overline{m}^p/E^p) & \rightarrow & H^1(X, E^p) \\ & & \text{Res} \downarrow & & & & \downarrow \\ & & H^0(X, d''\overline{m}^p) & \rightarrow & H^{p+1}(X, \mathbb{C}) & & \end{array}$$

(from [D 57], IV.D.7)

5.4.2. The residue current of a d -closed meromorphic p -form is globally written $t = 2\pi i \sum_{\nu} V p_{Y_{\nu}} A_{\nu} + d'S$, where $S = \sum_{\nu} \sum_j S_j^{\nu}$, with $dt = 0$, from the local Proposition 4.2.

Theorem 5.5. *If X is a complex manifold which is compact Kähler, or Stein, and Y is a complex hypersurface of X , then $Y = \cup_{\nu} Y_{\nu}$ is a locally finite union of irreducible hypersurfaces. Let $Z = \text{Sing} Y$; for every ν , let A_{ν} be a d -closed meromorphic $(p-1)$ -form on Y_{ν} , and, in the notations of (5.3) with $k = \nu$, $\gamma_{\nu,l}^{\mu,j}$ be meromorphic $(p-2)$ -forms on Y_{ν} , with polar set $Y_{\nu} \cap Z$ such that the current $t = 2\pi i \sum_{\nu} V p_{Y_{\nu}} A_{\nu} + d'S$, with $S = \sum_{\nu} \sum_j S_j^{\nu}$, be d -closed.*

Then the following two conditions are equivalent:

- (i) t is the residue current of a d -closed meromorphic p -form on X having Y as polar set.
- (ii) $t = dv$ on X , where v is a current, i.e. t is cohomologous to 0 on X .

Proof.

(i) \Rightarrow (ii): From Lemma 5.1, the cohomology class of a residue current is 0; it is the case of t .

(ii) \Rightarrow (i): $t = dv$ on X ; t of type $(p, 1)$ implies: $t = dv = d^{\#}v$; v of type $(p, 0)$; the current v is closed on $X \setminus Y$, therefore it is a holomorphic p -form on $X \setminus Y$. Let m_Y^p be the sheaf of closed meromorphic p -forms with polar set Y ; the Lemma 5.1 is valid for m_Y^p instead of m^p . At a point $O \in Y$, Y is defined by $\Pi_k \rho_k = 0$ (omitting the index j); the r_k being the integers in (5.3), then $d(\Pi_k \rho_k^{r_k} v) = \Pi_k \rho_k^{r_k} d^{\#}v = \Pi_k \rho_k^{r_k} t = 0$ from Lemma 4.1; therefore $\Pi_k \rho_k^{r_k} v$ is a germ of holomorphic form at O and v extends a closed meromorphic form $G \in H^0(X, m_Y^p)$ on X .

We will show that t is the residue current of G .

From Proposition 4.2,

$$\text{Res}[G] = d^{\#} \text{Vp } G = 2\pi i \sum_{\nu} \text{Vp}_{Y_{\nu}} B_{\nu} + d^{\#} T$$

where B_{ν} and T are of the same nature as A_{ν} and S .

Lemma 5.2. $M = v - \text{Vp } G$ satisfies $d^{\#} M = 0$.

Proof. We have:

$$(5.4) \quad d^{\#} M = 2\pi i \sum_{\nu} \text{Vp}_{Y_{\nu}} (A_{\nu} - B_{\nu}) + d^{\#} (S - T)$$

Let O_1 be a non singular point of Y ; there exists k such that: $O_1 \in \{j \rho_k = 0\}$, ($j = 1, \dots, n$); in the neighborhood of O_1 , $j \rho_k$ can be used as local coordinate. We have: $M = M_j$ where M_j is written with the local coordinates $(\dots, z_{j-1}, j \rho_k, z_{j+1}, \dots)$; $d^{\#} M = d^{\#} M_j$; the support of $d^{\#} M$ is Y , then, in the neighborhood of O_1 , $d^{\#} M_j$ vanishes on the differential forms containing $d_j \rho_k$ or $d_j \bar{\rho}_k$. Then

$$(5.5) \quad d^{\#} M_j = d_j \rho_k \wedge d_j \bar{\rho}_k \wedge N_j$$

M_j is of type $(p, 0)$, therefore without term in $d_j \bar{\rho}_k$ and in $d \bar{z}_l$, $l \neq j$.

From (5.5), $\frac{\partial M_j}{\partial \bar{z}_l} = 0$, then

$$(5.6) \quad d^{\#} M_j = d_j \bar{\rho}_k \wedge \frac{\partial M_j}{\partial_j \bar{\rho}_k}$$

$d^{\#} M_j$ is a differential form with distribution coefficients supported by Y_k , therefore, outside Z , from the structure theorem of distributions supported by a submanifold ([Sc 50], ch. III, théorème XXXVII), and from (5.6), the coefficients of $d^{\#} M_j$ being those of $\frac{\partial M_j}{\partial_j \bar{\rho}_k}$, then $d^{\#} M_j$ contains transversal derivatives with respect $j \rho_k$ or $j \bar{\rho}_k$ of order at least equal to $r_k + 1$, what is incompatible with the initial expression (5.4) of $d^{\#} M_j$, except if $d^{\#} M_j = 0$ outside Z . From (5.4) the $\text{Vp}_{Y_k} (A_{\nu} - B_{\nu})$ and $(S - T)$ being defined as limits of integrals of forms vanishing on $Y \setminus Z$, we have: $d^{\#} M = 0$ on X . \square

From Lemma 5.2, $\text{Res}[G] = d^{\#} v = t$. \square

Corollary 5.5.1. Under the hypotheses of Theorem 5.5, the current S is a sum of currents obtained by application of holomorphic differential operators to principal values of meromorphic forms on the irreducible components Y_{ν} of Y .

Corollary 5.5.2. Under the hypotheses of Theorem 5.5, the residue current of a d -closed meromorphic differential p -form is the sum, cohomologous to 0, of currents obtained by application of holomorphic differential operators to currents $\sum_{\nu} a_{\nu} T_{\nu}$, with $T_{\nu} \in \mathcal{R}_{n-1, n-1}^{\text{loc}}(X)$, d -closed, and a_{ν} the principal value of a meromorphic $(p - 1)$ -form on $\text{supp } T_{\nu}$.

5.5. Remarks. The Theorems of the sections 5.2 and 5.4 and their Corollaries are valid for locally residue currents in the terminology of [DS 85]. Results are also valid for any complex analytic manifold, using less natural cohomology (cf [D 57], IV.D.7).

6. Remarks about residual currents [CH 78], [DS 85].

In the classical definition and notations, we consider residual currents $R^p[\mu] = R^p P^0[\mu]$, where μ is a semi-meromorphic form $\frac{\alpha}{f_1 \dots f_p}$, and α a differential $(p, 0)$ -form. Then, $R^p[\mu]$ satisfies a formula analogous to (*) of section 2.4. ([D 93], section 8).

Locally, one of the assertions of the theorem of Picard is valid for any p , from the result of Dickenstein-Sessa quoted in section 5.1. So generalizations of theorems in sections 5.2 to 5.4, for residual currents, seem valid.

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