Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

# COSINE EFFECT IN OCEAN MODELS

Carine Lucas

MAPMO, Université d'Orléans
Bâtiment de mathématiques - Route de Chartres
B.P. 6759 - 45067 Orléans Cedex 2, FRANCE

Antoine Rousseau

INRIA
Laboratoire Jean Kuntzmann
B.P. 53, 38041 Grenoble Cedex 09, FRANCE

Abstract. This works aims at studying the impact of the cosine terms of the Coriolis force, that are usually neglected in geophysical fluid dynamics, leading to the so-called *traditional approximation*. Mathematical well-posedness arguments for simplified models, as well as numerical simulations, are presented in order to suggest the use of these terms in large scale ocean modelling.

1. **Introduction.** In this article, we aim at studying the impact of some rotation terms in (simplified) ocean models. We start with the Shallow Water Equations, obtained from the incompressible Navier Stokes Equations with free surface under the shallow water approximation. This model has been studied by numerous authors, both in the inviscid [22, 1] and viscous cases [8, 18] ; it has been widely used for theoretical studies and idealized numerical simulations: this is the framework of this article. Conversely, the operational oceanographic research community rather uses the Primitive Equations ([2, 7, 17, 21]). But it has to be mentionned that the barotropic part of the Primitive Equations corresponds to the Shallow Water Equations, and carries most of the energy (see [23]). Their study is thus particularly important.

In the sequel, we derive a new system of equations, in which anisotropic turbulent viscosities (see [14]) are taken into account. Simultaneously, the classical asymptotic analysis is modified by **new terms** that appear in the so-called viscous Shallow Water Equations (SWE):

$$\partial_t H + \mathrm{div}_x(Hu) = 0,$$

$$\begin{aligned}\partial_t(Hu) + \mathrm{div}_x(Hu\otimes u) + \frac{g}{2}\nabla_x H^2 \;&=\; -gH\,\nabla_x b - \left(1+\frac{kH}{3\mu_V}\right)^{-1} k\,u \\ &- \mu_H \nabla_x(H\mathrm{div}_x u) + 2\mu_H\,\mathrm{div}_x(HD_x u) + \boldsymbol{\Omega\cos\theta\,\nabla_x\left(u_1 H^2\right)} \\ &+ \boldsymbol{\Omega\cos\theta\,H^2 e_1 \mathrm{div}_x u} - 2\Omega\sin\theta H u^{\perp} - \boldsymbol{2\Omega\cos\theta\,He_1\nabla_x b\cdot u} + \boldsymbol{2\Omega\cos\theta\,u_1\,H\,\nabla_x b},\end{aligned}$$

2000 *Mathematics Subject Classification.* Primary: 76M45, 76U05 ; Secondary: 35B40, 35Q35, 76M20.

*Key words and phrases.* Coriolis Force, Multiscale Analysis, Traditional Approximation, Shallow Water Equation, Quasi-Geostrophic Equation.

where $u$ denotes the mean velocity and $H$ the water height, $b$ the topography and $\mu_V$, $\mu_H$ some eddy viscosities (see below). The angular speed of the Earth is $\Omega$, $\theta$ is the latitude, and $k$ represents the friction coefficient.
These new terms (bold-faced in the above equation) are of the order of the viscous terms; they are classically neglected with no rigorous justification, in the so-called *traditional approximation* (see *e.g.* [12, 24, 9]). They also arise in the derivation of the viscous Quasi-Geostrophic (QG) Equation:

$$D_t\left(\left(\partial^2_{x_1} + \left(1+\boldsymbol{\delta^2}\right)\partial^2_{x_2}\right)\psi - \frac{(2\Omega\sin\theta_0)^2}{gH_{char}}\psi + \left(1 - \frac{\boldsymbol{H_{char}}}{\boldsymbol{2\tan\theta_0}}\boldsymbol{\partial_{x_2}}\right)\frac{2\Omega\sin\theta_0}{H_{char}}b + \beta x_2\right)$$
$$= -\frac{1}{\varepsilon L_{char}}\alpha_0(H_{char})\Delta\psi + \mu_H\Delta^2\psi + \mathrm{curl} f,$$

where $\psi$ is the stream function, $D_t = \left(\partial_t + u^0\cdot\nabla_x\right)$, $H_{char} = \varepsilon L_{char}$ the characteristic height of the domain and $\delta = \Omega\sqrt{H_{char}/g}\,\cos\theta_0$. The coefficient $\alpha_0$ is related to the friction factor.

As we will see hereafter, the complete Coriolis force (when the traditional approximation is relaxed) does not modify the asymptotic development at the first order of [22]. However, and this is the main point of this work, the corresponding *viscous* equations differ from the one in [18]. Starting from the new SWE and considering the quasi-geostrophic approximation, we end up with another model ; this so-called viscous Quasi-Geostrophic Shallow Water model has been introduced in [3] and we aim at considering the effect of the traditional approximation in this model.
The objective of this work is to show that the so-called "cosine effect" (see **new terms** in equations above) has to be considered in the models.

The article is organized as follows. In Section 2, following the ideas of [8, 18, 14], we perform the asymptotic analysis that leads to the Shallow Water Equations, and recall a theorem of well-posedness that has been proved in [15]. In Section 3, we consider the Quasi-Geostrophic Equation for which derivation of the model, well-posedness result and numerical simulations are presented in the case where the traditional approximation is relaxed.

## 2. **Shallow Water Model.**

The two dimensional Shallow Water system is obtained from three dimensional Navier-Stokes Equations in a shallow domain. We look for the equations satisfied by the horizontal mean velocity field and the free surface.

We consider 3D Navier-Stokes Equations (NSE) for an homogenous fluid:

$$\partial_t U + \mathrm{div}(U\otimes U) = \mathrm{div}\,\sigma - 2\overrightarrow{\Omega}\times U + f, \tag{1a}$$
$$\mathrm{div}\,U = 0, \tag{1b}$$

for $(x,z)$ in $\mathbb{T}^2\times[b(x),h(t,x)]$, where $U=(u,w)\in\mathbb{R}^2\times\mathbb{R}$ is the fluid velocity, $\sigma$ is the stress tensor (given by $-p\mathrm{Id}+S$, $S$ to be detailed in the sequel), $2\overrightarrow{\Omega}\times U$ is the Coriolis force with $\overrightarrow{\Omega} = \Omega(0,\cos\theta,\sin\theta)$, $\theta$ represents the latitude and will be first considered as a constant. Finally $f = -g\overrightarrow{e_3}$ is the gravity force.
Figure 1 describes the computational domain, together with the bathymetry $b$, the water column height $H$ and the free surface $h$. We supplement Equation (1) with usual boundary conditions:

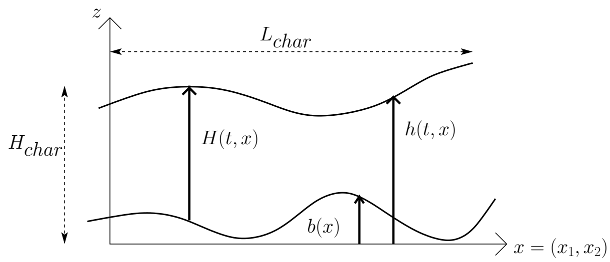


FIGURE 1. Notations used for the Shallow Water system

- at the free surface $z = h(t, x)$:

$$\partial_t h + u \cdot \nabla_x h = w, \tag{2}$$
$$\sigma\, n = 0. \tag{3}$$

- at the bottom $z = b(x)$:

$$-u \cdot \nabla_x b + w = 0, \tag{4}$$
$$(\sigma\, n) \cdot \tau_1 \tau_1 + (\sigma\, n) \cdot \tau_2 \tau_2 = kU \cdot \tau_1 \tau_1 + kU \cdot \tau_2 \tau_2, \tag{5}$$

  where $k$ is the friction coefficient, and $(\tau_1, \tau_2)$ a basis of the tangential surface. Equation (5) is called the Navier slip condition; the reader is referred to [11] (resp. [13]) for a theoretical (resp. physical) discussion on this boundary condition.

In what follows we write these equations under their non-dimensional form, make an asymptotic development of $U$ and study its first orders for a specific scaling of turbulent viscosities. Shallow Water Equations are obtained after integrating the first momentum equation over the water height. We also present the corresponding Shallow Water system when the latitude is not constant.

*Non-Dimensional Navier-Stokes Equations.* We write the Navier-Stokes system and the boundary conditions in a non-dimensionalized form, using some characteristic scales specially chosen to get the Shallow Water model.

Let us start with the 3D NSE for an homogenous fluid. In the following, the subscript $x$ denotes horizontal variables, $u_1$ the first component of the vector $u$ and $e_1$ the unit vector ${}^t(1\ 0)$. The stress tensor reads $\sigma = -p\mathrm{Id} + S = -p\mathrm{Id} + \mu_H D^H + \mu_V D^V + \mu_E D^E$. In this relation, $p$ is the pressure, $\mu_H$, $\mu_V$ and $\mu_E$ represent eddy viscosities. More precisely, if $\Lambda$ is the vector:

$$\Lambda = \gamma(\xi) \begin{pmatrix} \xi \\ 1 \end{pmatrix} \quad \text{where} \quad \xi = \frac{h - z}{b - h} \nabla_x b + \frac{z - b}{b - h} \nabla_x h \quad \text{and} \quad \gamma(\xi) = \frac{1}{\sqrt{1 + |\xi|^2}},$$

then $\mu_H$ denotes the eddy viscosity related to the shear in the direction orthogonal to $\Lambda$, $\mu_V$ the viscosity linked to the shear in the direction of $\Lambda$, and $\mu_E$ can be interpreted as the compression rate in the direction of $\Lambda$ (or expansion in the orthogonal direction)[1]. Lastly, the symmetric part of the gradient of $U$ is split as:

$$2D(U) = \begin{pmatrix} \nabla_x u + {}^t\nabla_x u & \partial_z u + \nabla_x w \\ {}^t(\partial_z u + \nabla_x w) & 2\partial_z w \end{pmatrix} = D^H + D^V + D^E,$$

where the tensors are given by:

$$D^H = 2(\mathrm{Id} - \Lambda\, {}^t\Lambda)\, D(U)\, (\mathrm{Id} - \Lambda\, {}^t\Lambda) + {}^t\Lambda\, D(U)\, \Lambda(\mathrm{Id} - \Lambda\, {}^t\Lambda), \tag{6a}$$

$$D^V = 2(\mathrm{Id} - \Lambda\, {}^t\Lambda)\, D(U)\, \Lambda\, {}^t\Lambda + 2\Lambda\, {}^t\Lambda\, D(U)\, (\mathrm{Id} - \Lambda\, {}^t\Lambda), \tag{6b}$$

$$D^E = {}^t\Lambda\, D(U)\, \Lambda(3\Lambda\, {}^t\Lambda - \mathrm{Id}). \tag{6c}$$

Then, the Navier-Stokes system reads:

$$\begin{aligned} &\partial_t u + u\cdot\nabla_x u + w\partial_z u = -\nabla_x p + \mathrm{div}_x S_{xx} + \partial_z S_{xz} - 2\Omega\sin\theta\, u^\perp - 2\Omega\cos\theta w e_1, \\ &\partial_t w + u\cdot\nabla_x w + w\partial_z w = -\partial_z p + \mathrm{div}_x S_{zx} + \partial_z S_{zz} + 2\Omega\cos\theta\, u_1 - g, \\ &\mathrm{div}_x u + \partial_z w = 0, \end{aligned}$$

where the tensor $S$ is written as: $S = \begin{pmatrix} S_{xx} & S_{xz} \\ S_{zx} & S_{zz} \end{pmatrix}$.

We introduce the following dimensionless variables and numbers:

$$\begin{aligned} x &= L_{char}\,\widetilde{x}, & z &= H_{char}\,\widetilde{z}, & &\text{with } \varepsilon = \frac{H_{char}}{L_{char}} \ll 1, \\ u &= u_{char}\,\widetilde{u}, & w &= w_{char}\,\widetilde{w}, & &\text{with } w_{char} = \varepsilon u_{char}, \\ t &= \frac{L_{char}}{u_{char}}\,\widetilde{t}, & p &= p_{char}\,\widetilde{p}, & &\text{with } p_{char} = u_{char}^2, \\ \xi &= \varepsilon\widetilde{\xi}, & Ro &= \frac{u_{char}}{2L_{char}\Omega}, & Fr &= \frac{u_{char}}{\sqrt{gH_{char}}}, \\ \mu_H &= \varepsilon L_{char} u_{char}\nu_H, & \mu_V &= \varepsilon L_{char} u_{char}\nu_V, & \mu_E &= \varepsilon^3 L_{char} u_{char}\nu_E, \\ S_{xx} &= u_{char}^2\varepsilon\widetilde{S}_{xx}, & S_{xz} &= u_{char}^2\widetilde{S}_{xz}, \\ S_{zx} &= u_{char}^2\widetilde{S}_{zx}, & S_{zz} &= u_{char}^2\varepsilon\widetilde{S}_{zz}, \end{aligned}$$

where $\varepsilon$ is the aspect ratio, $Ro$ the Rossby number, $Fr$ the Froude number. We write the 3D NSE with non-dimensionnal variables:

$$\partial_{\widetilde{t}}\widetilde{u} + \widetilde{u}\cdot\nabla_{\widetilde{x}}\widetilde{u} + \widetilde{w}\partial_{\widetilde{z}}\widetilde{u} = -\nabla_{\widetilde{x}}\widetilde{p} + \varepsilon\mathrm{div}_{\widetilde{x}}\widetilde{S}_{xx} + \frac{1}{\varepsilon}\partial_{\widetilde{z}}\widetilde{S}_{xz} - \frac{\sin\theta}{Ro}\widetilde{u}^\perp - \varepsilon\frac{\cos\theta}{Ro}\widetilde{w}e_1, \tag{7a}$$

$$\partial_{\widetilde{t}}\widetilde{w} + \widetilde{u}\cdot\nabla_{\widetilde{x}}\widetilde{w} + \widetilde{w}\partial_{\widetilde{z}}\widetilde{w} = -\frac{1}{\varepsilon^2}\partial_{\widetilde{z}}\widetilde{p} + \frac{1}{\varepsilon}\mathrm{div}_{\widetilde{x}}\widetilde{S}_{zx} + \frac{1}{\varepsilon}\partial_{\widetilde{z}}\widetilde{S}_{zz} + \frac{1}{\varepsilon}\frac{\cos\theta}{Ro}\widetilde{u}_1 - \frac{1}{\varepsilon^2 Fr^2}, \tag{7b}$$

$$\mathrm{div}_{\widetilde{x}}\widetilde{u} + \partial_{\widetilde{z}}\widetilde{w} = 0. \tag{7c}$$

Naturally, all the non-dimensionnal variables (except the $\widetilde{S}_{i\,j}$) are $O(1)$ as $\varepsilon$ goes to zero; we present in Appendix A the rigorous asymptotic development of the non-dimensionnal stress tensor $\widetilde{S}$.

Let us now consider the boundary conditions. We first replace $\sigma$ with its expression and change the variables:

[1] Note that $\xi$ is a barycentric combination of $-\nabla_x b$ and $-\nabla_x h$. In particular, $\xi(z = b) = -\nabla_x b$ and $\xi(z = h) = -\nabla_x h$. That is, for a flat bottom (resp. a rigid lid), $\Lambda(z = b)$ (resp. $\Lambda(z = h)$) corresponds to the vertical direction, hence the subscripts $H$ and $V$ (vertical and horizontal shear).

- at the free surface $z = h(t,x)$:
  the normal vector $n$ is $n = (1 + (\nabla_x h)^2)^{-\frac{1}{2}} \begin{pmatrix} -\nabla_x h \\ 1 \end{pmatrix}$. The horizontal variable $h$ is rescaled as $h = H_{char}\widetilde{h}$ to get the dimensionless conditions at the free surface:

$$\widetilde{p}\,\nabla_{\widetilde{x}}\widetilde{h} - \varepsilon \widetilde{S}_{xx}\nabla_{\widetilde{x}}\widetilde{h} + \frac{1}{\varepsilon}\widetilde{S}_{xz} = 0, \tag{8a}$$

$$\widetilde{p} - \varepsilon \widetilde{S}_{zz} + \varepsilon \widetilde{S}_{zx}\nabla_{\widetilde{x}}\widetilde{h} = 0, \tag{8b}$$

$$\partial_{\widetilde{t}}\widetilde{h} + \widetilde{u}\cdot\nabla_{\widetilde{x}}\widetilde{h} = \widetilde{w}. \tag{8c}$$

- at the bottom $z = b(x)$:
  if we write $b = H_{char}\widetilde{b}$, the non-penetration condition reads

$$-\widetilde{u}\cdot\nabla_{\widetilde{x}}\widetilde{b} + \widetilde{w} = 0. \tag{9}$$

  The Navier condition is more involved. We choose the tangential vectors

$$\tau_1 = \frac{1}{|\nabla_x b|}\begin{pmatrix} \nabla_x^{\perp} b \\ 0 \end{pmatrix} \text{ and } \tau_2 = \frac{1}{\sqrt{|\nabla_x b|^2 + |\nabla_x b|^4}}\begin{pmatrix} -\nabla_x b \\ -|\nabla_x b|^2 \end{pmatrix}.$$

  We define $K = k\,u_{char}^{-1}\varepsilon^{-1}$ and obtain a complex expression that we do not detail here. We perform some approximations in the following and give a simplified equality obtained from this expression.

*Hydrostatic Approximation.* We now use the hydrostatic approximation, that is we suppose the aspect ratio $\varepsilon$ to be small. We only keep the first two orders in Equation (7b) and we also leave out terms in boundary conditions. In order to ligthen the notations, we drop the tilde for the non-dimensionnal variables.

We are led to study the system:

$$\partial_t u + u\cdot\nabla_x u + w\partial_z u = -\nabla_x p + \varepsilon \mathrm{div}_x S_{xx} + \frac{1}{\varepsilon}\,\partial_z S_{xz} - \frac{\sin\theta}{Ro}u^{\perp} - \varepsilon\frac{\cos\theta}{Ro}we_1, \tag{10a}$$

$$\partial_z p = \varepsilon \mathrm{div}_x S_{zx} + \varepsilon\partial_z S_{zz} - \frac{1}{Fr^2} + \varepsilon\frac{\cos\theta}{Ro}u_1, \tag{10b}$$

$$\mathrm{div}_x u + \partial_z w = 0. \tag{10c}$$

We can simplify the Navier condition at the bottom

$$\varepsilon S_{xx}\nabla_x b - \frac{1}{\varepsilon}S_{xz} = -Ku + O(\varepsilon^2). \tag{11}$$

At the free surface, Equation (8c) is not modified but we can rewrite (8a) as follows:

$$\varepsilon S_{xz} = -\varepsilon^2(p\,\nabla_x h - \varepsilon S_{xx}\nabla_x h) = \varepsilon\,{}^tS_{zx} \quad \text{for } z = h(x,t),$$

and plug it into Equation (8b):

$$\begin{aligned} p - \varepsilon S_{zz} &= -\varepsilon S_{zx}\nabla_x h = -\varepsilon S_{xz}\cdot\nabla_x h \\ &= \varepsilon^2(p\,\nabla_x h - \varepsilon S_{xx}\nabla_x h)\cdot\nabla_x h, \\ p - \varepsilon S_{zz} &= O(\varepsilon^2), \quad \text{for } z = h(x,t). \end{aligned} \tag{12}$$

We integrate Equation (10b) from $h$ to $z$, with $z$ between $b$ and $h$. The value of $p$ at the free surface is given by (12), and we find the pressure at order $\varepsilon$:

$$p(t,x,z) = \frac{1}{Fr^2}\,(h(t,x) - z) + \varepsilon\int_h^z \mathrm{div}_x S_{zx} + \varepsilon S_{zz} + \varepsilon\frac{\cos\theta}{Ro}\int_h^z u_1 + O(\varepsilon^2). \tag{13}$$

As we are looking for equations on the mean velocity and on the evolution of the free surface, we first integrate the momentum equation (10a) over the water height (between $z = b(x)$ and $z = h(t, x)$). We apply Leibniz formula and get:

$$\begin{aligned}\partial_t \int_b^h u - \partial_t h\, u_{|_{z=h}} + \mathrm{div}_x \int_b^h (u \otimes u) - ((u \cdot \nabla_x h)\, u)_{|_{z=h}} + ((u \cdot \nabla_x b)\, u)_{|_{z=b}} \\ +(u\, w)_{|_{z=h}} - (u\, w)_{|_{z=b}} + \nabla_x \int_b^h p \;=\; \nabla_x h\, p_{|_{z=h}} - \nabla_x b\, p_{|_{z=b}} \\ +\varepsilon\, \mathrm{div}_x \int_b^h S_{xx} - \varepsilon S_{xx|_{z=h}} \nabla_x h + \varepsilon S_{xx|_{z=b}} \nabla_x b + \frac{1}{\varepsilon} S_{xz|_{z=h}} \\ -\frac{1}{\varepsilon} S_{xz|_{z=b}} - \frac{\sin\theta}{Ro} \int_b^h u^\perp - \varepsilon \frac{\cos\theta}{Ro} \int_b^h w e_1.\end{aligned}$$

Then, we use boundary conditions (8a), (8c), (9) and (11) to simplify the expressions at the surface and at the bottom and obtain the integrated momentum equation:

$$\begin{aligned}\partial_t \int_b^h u + \mathrm{div}_x \int_b^h (u \otimes u) + \nabla_x \int_b^h p = -\nabla_x b\, p_{|_{z=b}} - K u_{|_{z=b}} \\ + \varepsilon\, \mathrm{div}_x \int_b^h S_{xx} - \frac{\sin\theta}{Ro} \int_b^h u^\perp - \varepsilon \frac{\cos\theta}{Ro} \int_b^h w e_1, \quad (14)\end{aligned}$$

with a new Coriolis term (the last one).

We also want the evolution of the free surface: we integrate the divergence free equation (10c) from the bottom to the surface, use Leibniz formula again together with Equations (8c) and (9) to find:

$$\partial_t h(t, x) + \mathrm{div}_x \int_{b(x)}^{h(t,x)} u = 0. \quad (15)$$

In the sequel we study the integrated momentum equation (14) and the free surface equation (15) when we approximate $u$ at the first order and at the second order.

*Shallow Water System.* We have already done the main assumption to get the Shallow Water system from 3D NSE, that is the depth is small compared to the length of the domain, whereas the Froude and Rossby number are fixed. Now we develop $u$, $w$, $H$, $p$, $b$ in powers of $\varepsilon$, that is $u = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots$ (and so on) with $H(t, x) = h(t, x) - b(x)$.

We look for $u^0$, the first order of the velocity. We use the horizontal momentum equation (10a) and the boundary conditions (8a) and (11). We obtain

$$\partial_z^2 u = O(\varepsilon), \qquad (\partial_z u)_{|_{z=b}} = O(\varepsilon), \qquad (\partial_z u)_{|_{z=h}} = O(\varepsilon),$$

which means that at the first order $u$ does not depend on $z$: $u^0(t, x, z) = u^0(t, x)$. As we are looking for the dynamics of $u^0$, we study the previous equations at the first order. Let us first rewrite the evolution equation (15)

$$\partial_t H^0 + \mathrm{div}_x (H^0 u^0) = 0. \quad (16)$$

Then we have $p$ at the first order with (13):

$$p^0(t, x, z) = Fr^{-2}(h - z).$$

We replace this value in the integrated momentum equation (14) and obtain:

$$\partial_t(H^0u^0) + \mathrm{div}_x(H^0u^0 \otimes u^0) + \frac{1}{2\,Fr^2}\nabla_x(H^0)^2 = \\ -\frac{1}{Fr^2}H^0\,\nabla_x b^0 - K u^0 - \frac{\sin\theta}{Ro}H^0{u^0}^{\perp}. \quad (17)$$

Equations (16) - (17) form the Shallow Water system at the first order in non-dimensional variables.
If we go back to dimensional variables we have the Shallow Water system at the first order for a viscosity of order $\varepsilon$:

$$\partial_t H + \mathrm{div}_x(Hu) = 0, \quad (18a)$$

$$\partial_t(Hu) + \mathrm{div}_x(Hu \otimes u) + \frac{g}{2}\nabla_x H^2 = -gH\nabla_x b - ku - 2\Omega\sin\theta H u^{\perp}. \quad (18b)$$

At this point, we get the inviscid Shallow Water system; the cosine part of the Coriolis force does not modify these equations at the first order. But there are no viscosity effects in this system: we are led to study the second order to make viscous terms appear in our Shallow Water system.

We will denote by a bold letter the approximation of the variables at order $\varepsilon$ (for example, $\boldsymbol{u^1} = u^0 + \varepsilon u^1$) and by a bar the mean value on the water height defined by:

$$\bar{u}(t,x) := \frac{1}{H(t,x)}\int_b^h u\,dz.$$

Let us rewrite the divergence condition at the second order

$$\partial_t \boldsymbol{H^1} + \mathrm{div}_x(\boldsymbol{H^1}\boldsymbol{\bar{u}^1}) = O(\varepsilon^2). \quad (19)$$

As before, we rewrite the momentum equation (10a) but at order $\varepsilon$, and with the Shallow Water system at the first order (16) - (17) we obtain an expression for the second derivative of $u$:

$$\frac{\nu_V}{\varepsilon}\partial_z^2 u = -\frac{K}{H^0}u^0 + O(\varepsilon).$$

We integrate this equality from $b$ to $z$ (for $z$ between $b(x)$ and $h(t,x)$) with the boundary condition (11). We integrate it again from $b$ to $z$ to find an approximation of $u$ at the second order:

$$\begin{aligned} u &= \boldsymbol{u^1}_{\,|z=b} + \varepsilon\frac{K}{\nu_V}u^0\int_b^z\left(1 - \frac{s-b^0}{H^0}\right)ds + O(\varepsilon^2) \\ &= \boldsymbol{u^1}_{\,|z=b}\left(1 + \varepsilon\frac{K}{\nu_V}(z-b^0)\left(1 - \frac{z-b^0}{2H^0}\right)\right) + O(\varepsilon^2). \end{aligned}$$

With this expression, we can compute the mean value of $u$:

$$\bar{u} = \boldsymbol{u^1}_{\,|z=b}\left(1 + \varepsilon\frac{K}{\nu_V}\frac{H^0}{3}\right) + O(\varepsilon^2).$$

One can easily check that:

$$\begin{aligned} \overline{u^2} &= \overline{u}^2 + O(\varepsilon^2), \\ \overline{u \otimes u} &= \overline{u}\otimes\overline{u} + O(\varepsilon^2). \end{aligned}$$

This is used in the sequel.
We also have the value of $p$ at the second order with Equation (13). Thanks to the

asymptotic analysis provided in Appendix A (see Equation (36)), we can simplify this equation:

$$p(t,x,z) = \frac{1}{Fr^2}\left(h(t,x)-z\right) + \varepsilon\frac{\cos\theta}{Ro}\int_h^z u_1 + O(\varepsilon^2). \tag{20}$$

Then, we replace it in the integrated momentum equation (14) and, using again the divergence free condition (10c) to express $w^0$ as a function of $h^0$ and $u^0$, we get

$$\begin{aligned}
&\partial_t(\boldsymbol{H^1}\bar{u}) + \mathrm{div}_x(\boldsymbol{H^1}\bar{u}\otimes\bar{u}) + \frac{1}{2Fr^2}\nabla_x(\boldsymbol{H^1})^2 - \varepsilon\frac{\cos\theta}{2Ro}\nabla_x\left(u_1^0(H^0)^2\right)\\
&\quad = -K\boldsymbol{u^1}_{|_{z=b}} - \nabla_x b\left(\frac{\boldsymbol{H^1}}{Fr^2} - \varepsilon\frac{\cos\theta}{Ro}u_1^0 H^0\right) + \varepsilon\,\mathrm{div}_x(H^0 S_{xx})\\
&\quad -\frac{\sin\theta}{Ro}\boldsymbol{H^1}\bar{u}^\perp + \varepsilon\frac{\cos\theta}{2Ro}(H^0)^2 e_1\mathrm{div}_x u^0 - \varepsilon\frac{\cos\theta}{Ro}H^0 e_1\nabla_x b^0\cdot u^0 + O(\varepsilon^2).
\end{aligned}$$

The velocity at the bottom $u_{|_{z=b}}$ is given by

$$\boldsymbol{u^1}_{\ |_{z=b}} = \bar{u}\left(1+\frac{\varepsilon K}{\nu_V}\frac{H^0}{3}\right)^{-1} + O(\varepsilon^2),$$

which leads to:

$$\begin{aligned}
&\partial_t(\boldsymbol{H^1}\bar{u}) + \mathrm{div}_x(\boldsymbol{H^1}\bar{u}\otimes\bar{u}) + \frac{1}{2Fr^2}\nabla_x(\boldsymbol{H^1})^2 = -K\bar{u}\left(1+\frac{\varepsilon K}{\nu_V}\frac{\boldsymbol{H^1}}{3}\right)^{-1}\\
&\quad -\varepsilon\nu_H\nabla_x(\boldsymbol{H^1}\mathrm{div}_x\bar{u}) + 2\varepsilon\nu_H\,\mathrm{div}_x(\boldsymbol{H^1}D_x\bar{u}) + \varepsilon\frac{\cos\theta}{2Ro}\nabla_x\left(\bar{u}_1(\boldsymbol{H^1})^2\right)\\
&\quad +\varepsilon\frac{\cos\theta}{2Ro}(\boldsymbol{H^1})^2 e_1\mathrm{div}_x\bar{u} - \varepsilon\frac{\cos\theta}{Ro}\boldsymbol{H^1}e_1\nabla_x b\cdot\bar{u} - \frac{\sin\theta}{Ro}\boldsymbol{H^1}\bar{u}^\perp\\
&\quad -\nabla_x b\left(\frac{\boldsymbol{H^1}}{Fr^2} - \varepsilon\frac{\cos\theta}{Ro}\bar{u}_1\boldsymbol{H^1}\right) + O(\varepsilon^2).
\end{aligned} \tag{21}$$

Equations (19)-(21) form the Shallow Water system at the second order in non-dimensional variables, with new cosine terms.
Finally, let us go back to the dimensional form to get the viscous Shallow Water system at the second order:

$$\partial_t H + \mathrm{div}_x(Hu) = 0, \tag{22a}$$

$$\begin{aligned}
\partial_t(Hu) + \mathrm{div}_x(Hu\otimes u) + \frac{g}{2}\nabla_x H^2 \ &= \ -\alpha_0(H)\,u - gH\,\nabla_x b\\
&- \mu_H\nabla_x(H\mathrm{div}_x u) + 2\mu_H\,\mathrm{div}_x(HD_x u) + \Omega\,\cos\theta\,\nabla_x\left(u_1 H^2\right)\\
&+ \Omega\,\cos\theta\,H^2 e_1\mathrm{div}_x u - 2\Omega\,\sin\theta Hu^\perp - 2\Omega\,\cos\theta\,He_1\nabla_x b\cdot u + 2\Omega\,\cos\theta\,u_1\,H\,\nabla_x b,
\end{aligned} \tag{22b}$$

where $\alpha_0(H) = k/\left(1+\frac{kH}{3\mu_V}\right)$.

Conversely to the first order approximation (18), we now have viscous terms in the SWE.

**Remark 1.** Note that the coefficients of the viscous terms slightly differ from those in [18] (which is a generalization of [8] to the 2D case), due to the choice we made for the stress tensor.

**Remark 2.** If the latitude is not constant, the only difference with the previous development is that the term that reads $\Omega\cos\theta\,\nabla_x\left(u_1 H^2\right)$ in the constant case must be replaced by $\Omega\,\nabla_x\left(\cos\theta u_1 H^2\right)$ (with no additional difficulty), the other ones remaining unchanged.

**Remark 3.** Note that, if we take the capillarity into account, the term $aH\nabla_x\Delta_x H + aH\nabla_x\Delta_x b$ adds to the right hand side of Equation (22b), where $a$ is the capillarity coefficient.

We now state a well-posedness theorem for the Shallow Water Equations: consider the following system in a bounded domain $\mathcal{D}$ with periodic boundary conditions:

$$\begin{aligned}
&\partial_t H + \operatorname{div}(Hu) = 0,\\
&\partial_t(Hu) + \operatorname{div}(Hu\otimes u) + \frac{1}{2Fr^2}\nabla H^2 = -\tilde\alpha_0(H)u - \tilde\alpha_1(H)Hu|u|\\
&\quad + AH\nabla\Delta H + AH\nabla\Delta b + \nu\operatorname{div}(HD(u)) + \varepsilon\frac{\cos\theta}{2Ro}\nabla\left(u_1H^2\right)\\
&\quad + \varepsilon\frac{\cos\theta}{2Ro}H^2 e_1 \operatorname{div} u - \varepsilon\frac{\cos\theta}{Ro}He_1\nabla b\cdot u - \frac{\sin\theta}{Ro}Hu^\perp\\
&\quad - \nabla b\left(\frac{H}{Fr^2} - \varepsilon\frac{\cos\theta}{Ro}u_1 H\right).
\end{aligned}\tag{23}$$

We suppose that the bottom is regular enough ($b\in H^3(\mathcal{D})$) and that the initial condition on the water height satisfies: $H(0) = H_0 \geq 0$, $\sqrt{H_0}\in H^1_{per}(\mathcal{D})$, $H_0\in L^2_{per}(\mathcal{D})$, $H_0 u_0^2 \in L^1_{per}(\mathcal{D})$, $\log\left(K_0^{-1}H_0\tilde\alpha_0(H_0)\right)\in L^1_{per}(\mathcal{D})$. In [15], we proved the following theorem:

**Theorem 2.1.** *If $A > 0$, $\tilde\alpha_0(H) > 0$ and $\tilde\alpha_1(H)\geq 0$, with $H_0$ satisfying the conditions above, there exists a global weak solution of (23).*

## 3. **Quasi-Geostrophic Model.**

The Quasi-Geostrophic model is used for the abstract modeling of the ocean at mid-latitudes (see [4]). It is obtained from the Shallow Water system assuming that the Rossby and Froude numbers are very small (the aspect ratio $\varepsilon$ is now considered as fixed).

### 3.1. **Derivation of the Quasi-Geostrophic Equation with Cosine Effect.**

We consider the dimensionless Shallow Water system (19) - (21), that depends on the aspect ratio $\varepsilon$. We add in these equations a source term $\tilde f$ that could have been introduced in the previous derivation with no additional difficulty. In order to lighten the notations, we set $\mu = \mu_H$ and $\nu = \nu_H$ in the sequel.

We write an asymptotic development $u = u^0 + \eta\,u^1 + \ldots$, $H = 1 + \eta\,FH^1 + \ldots$ where we suppose $Ro = \eta$, $Fr^2 = F\eta^2$, $b = \eta\,\tilde b$, with $\eta \ll 1$, $F = (2\Omega L_{char})^2/(gH_{char})$ and $\varepsilon$ fixed. Let us underline that, from now on, $\varepsilon$ is a fixed parameter, and the Rossby number $\eta$ is *the asymptotic parameter*, meant to go to zero. Doing this, we are not considering a crossed asymptotic $(\varepsilon,\eta)\to 0$ as for the direct derivation of the QG model (see [6]), but we let $\eta$ go to zero as $\varepsilon$ remains fixed. This crossed asymptotic is beyond the scope of this paper and is left to further studies. We also use the beta-plane approximation around the latitude $\theta_0$ (see [19]).

We then have to study the following equations:

$$\partial_t H + \mathrm{div}_x(Hu) = 0, \tag{24a}$$

$$\begin{aligned}
&\partial_t(Hu) + \mathrm{div}_x(Hu\otimes u) + \frac{1}{F\eta^2}H\nabla_x H = -\frac{\sin\theta_0}{\eta}H\,u^\perp - \beta\,x_2\,H\,u^\perp \\
&\quad + \varepsilon\frac{\cos\theta_0}{2\eta}e_1\,H^2\mathrm{div}_x u - \frac{\varepsilon}{2}\tan\theta_0\,\beta\,x_2 e_1\,H^2\mathrm{div}_x u + \varepsilon\frac{\cos\theta_0}{2\eta}\nabla_x(H^2u_1) \\
&\quad - \frac{\varepsilon}{2}\tan\theta_0\,\beta\,\nabla_x(H^2u_1\,x_2) - \tilde\alpha_0(H)\,u + 2\nu\mathrm{div}_x(HD(u)) - \nu\nabla_x(H\,\mathrm{div}u) \qquad (24b)\\
&\quad - \varepsilon\frac{\cos\theta_0}{\eta}\,He_1\nabla_x b\cdot u + \varepsilon\tan\theta_0\,\beta\,x_2\,He_1\nabla_x b\cdot u + \varepsilon\frac{\cos\theta_0}{\eta}\,u_1\,H\,\nabla_x b \\
&\quad - \varepsilon\tan\theta_0\,\beta\,x_2\,u_1\,H\,\nabla_x b - \frac{1}{F\eta^2}H\,\nabla_x b + H\tilde f + O(\eta).
\end{aligned}$$

At the first and second orders, Equation (24a) gives:

$$\mathrm{div}_x u^0 = 0, \qquad \text{and} \qquad F\partial_t H^1 + \mathrm{div}_x u^1 + F\nabla_x H^1\cdot u^0 = 0.$$

We also look at the first and second orders of the momentum equation (24b). The first order gives

$$\nabla_x H^1 + \left(\sin\theta_0 - \varepsilon\frac{\cos\theta_0}{2}\partial_{x_2}\right)u^{0\perp} + \frac{\nabla_x\tilde b}{F} = 0.$$

Then we take the *curl* (i.e. $-\partial_{x_2}$ of the first component + $\partial_{x_1}$ of the second one) of the second order and get

$$\begin{aligned}
(\partial_t + u^0\cdot\nabla_x)(\mathrm{curl}\,u^0) &= -\tilde\alpha_0(1)\mathrm{curl}\,u^0 + \nu\Delta(\mathrm{curl}\,u^0) + \sin\theta_0\,F(\partial_t H^1 + u^0\cdot\nabla_x H^1) \\
&\quad - \sin\theta_0\,Fu^0\cdot\nabla_x H^1 - \beta u_2^0 + \varepsilon F\frac{\cos\theta_0}{2}\partial_{x_2}(\partial_t H^1 + u^0\cdot\nabla_x H^1) \\
&\quad - \nabla_x^\perp H^1\cdot\nabla_x\tilde b + \varepsilon\cos\theta_0\,\partial_{x_2}(u^0\cdot\nabla_x\tilde b) + \varepsilon\cos\theta_0\,\nabla_x^\perp u_1^0\cdot\nabla_x\tilde b + \mathrm{curl}\tilde f.
\end{aligned}$$

We note that

$$\sin\theta_0\,u^0\cdot\nabla_x H^1 - \varepsilon\frac{\cos\theta_0}{2}(\partial_{x_2}u^0)\cdot\nabla_x H^1 + \nabla_x^\perp H^1\cdot\frac{\nabla_x\tilde b}{F} = 0,$$

and

$$(\partial_{x_2}u^0)\cdot\nabla_x\tilde b + \nabla_x^\perp u_1^0\cdot\nabla_x\tilde b = 0.$$

We define $\psi$ by $u^0 = \nabla_x^\perp\psi$, and consequently $H^1 = (\sin\theta_0 - \varepsilon\cos\theta_0\partial_{x_2}/2)\,\psi - \tilde b/F$, and find

$$\begin{aligned}
&(\partial_t + u^0\cdot\nabla_x)\left(\left(\partial_{x_1}^2 + \left(1+\varepsilon^2F\frac{\cos^2\theta_0}{4}\right)\partial_{x_2}^2\right)\psi - \sin^2\theta_0\,F\psi\right. \\
&\qquad\left. + \left(\sin\theta_0 - \varepsilon\frac{\cos\theta_0}{2}\partial_{x_2}\right)\tilde b + \beta x_2\right) = -\tilde\alpha_0(1)\,\Delta\psi + \nu\Delta^2\psi + \mathrm{curl}\tilde f. \qquad (25)
\end{aligned}$$

Equation (25) is the non-dimensional Quasi-Geostrophic Equation obtained from the viscous Shallow Water system at the second order with variable topography. The new terms (due to the Coriolis force) are $\varepsilon^2F\cos^2\theta_0\partial_{x_2}^2\psi/4$ and $\varepsilon\cos\theta_0\partial_{x_2}\tilde b/2$. The "unusual" $\sin\theta_0$ coefficient is linked to an "unusual" Rossby number expression that arises in $F$ and $\tilde b$; in the "usual" case, the term $\sin\theta_0$ is replaced by 1, and

$\cos\theta_0$ by $1/\tan\theta_0$.

Let us remove this problem coming back to dimensional variables. We get:

$$D_t\left(\left(\partial^2_{x_1}+\left(1+\delta^2\right)\partial^2_{x_2}\right)\psi-\frac{(2\Omega\sin\theta_0)^2}{gH_{char}}\psi+\left(1-\frac{H_{char}}{2\tan\theta_0}\partial_{x_2}\right)\frac{2\Omega\sin\theta_0}{H_{char}}b+\beta x_2\right)$$
$$=-\frac{1}{\varepsilon L_{char}}\alpha_0(H_{char})\Delta\psi+\mu\Delta^2\psi+\mathrm{curl}f, \quad (26)$$

where $D_t=\left(\partial_t+u^0\cdot\nabla_x\right)$, $H_{char}=\varepsilon L_{char}$ and $\delta=\Omega\sqrt{H_{char}/g}\,\cos\theta_0$.

We add to this equation the following boundary conditions:

$$\psi=0 \text{ and } \Delta\psi=0 \text{ on } \partial\mathcal{D},$$

which respectively express the non-penetration condition and the slip condition.

**Remark 4.** We can already notice that the cosine term has two different contributions. First, the Laplacian is modified in the second direction by the small coefficient $\delta$. The other change is on the topography coefficient: we see the derivative of the topography in the second variable.

3.2. **Mathematical Properties of the Quasi-Geostrophic Equation.** The following theorem has been proved in [16], we just recall it hereafter.

**Theorem 3.1.** *If $\mathcal{D}$ is a rectangular domain, for all $f$ in $L^2\left(0,T;L^2(\mathcal{D})\right)$, Equation (26) with the initial condition $\psi_0$ in $H^3(\mathcal{D})\cap H^1_0(\mathcal{D})$ and the boundary conditions $\psi=0$ and $\Delta\psi=0$ on $\partial\mathcal{D}$ has a unique solution $\psi$ in $\mathcal{C}\left([0,T];H^3(\mathcal{D})\cap H^1_0(\mathcal{D})\right)\cap L^2\left(0,T;H^4(\mathcal{D})\cap H^1_0(\mathcal{D})\right)$.*

3.3. **Numerical Results: Cosine Effects on Large Scale Computations.** We now present numerical results for the QG Equation and show that the cosine effect cannot be neglected. The details of the scheme are presented in [16], we do not repeat them here, but recall the choice of physical parameters.

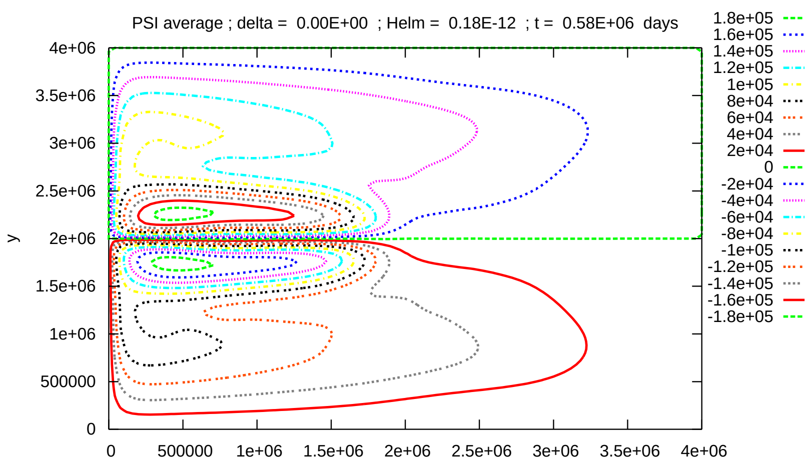


Figure 2. Time-average of the stream function over 1600 years. The upper part is negative so the velocity is in the counterclockwise direction.

*Choice of the Parameters.* We consider that $\mathcal{D}$ is a square basin of length $L = 4.000$km, which is 5.000m deep with a $100 \times 100$ points grid, and thus the aspect ratio is $\varepsilon = 1.25 \times 10^{-3}$. We borrow the other physical parameters from [10]. We check that these parameters match the case studied at the beginning: in particular, the non-dimensional horizontal viscosity is of order of the aspect ratio. We choose the forcing term (wind) as $-10^{-2}\sin(2\pi x_2/L)$ and let the model run over nearly 1600 years (note that we have $T = O(\varepsilon^{-1})$) to ensure that the convergence error is small compared to the cosine effect. We then compare the results obtained with and without the cosine terms, with a flat or a varying bottom. A first look at the evolution of the energy behaviour indicates that the system is chaotic; we thus present the time-average of $\psi$ instead of $\psi$ in the following.

3.3.1. *Flat Bottom.* As mentionned above, the cosine effect has two correlated contributions. In order to disconnect these terms, we first consider a flat bottom so that the second term (depending on the topography variations) vanishes. Figure 2 shows the stream function without the cosine effect, and we plot in Figure 3 the difference between the two models: this corresponds to the numerical contribution of the cosine effet.

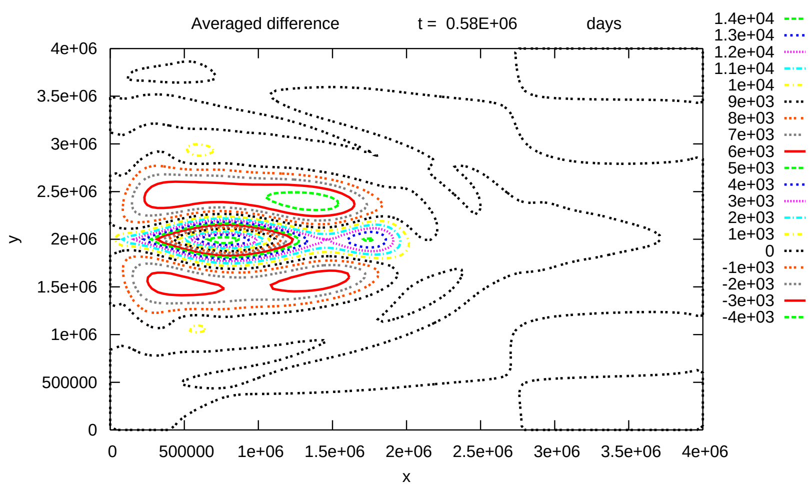


FIGURE 3. Time-average of the difference between the stream function with the cosine terms and the stream function without these terms, over 1600 years.

The difference is mainly located in the jet, with a maximum value of $1.4 * 10^4$. Comparing this to Figure 2 where the maximum value is $1.8 * 10^5$, we show that the contribution of the new model is about ten percents.

3.4. **Varying Bottom.** Let us now consider the cosine effect, with a bottom depending on the second variable $x_2$. We use a topography roughly resembling to the Mid-Atlantic ridge, as can be seen in Figure 4.
As for the flat bottom, we present in Figure 5 the time-average of the stream function, solution of the QG model without any cosine effect. Figure 6 plots the difference between the two models, with and without the cosine terms.

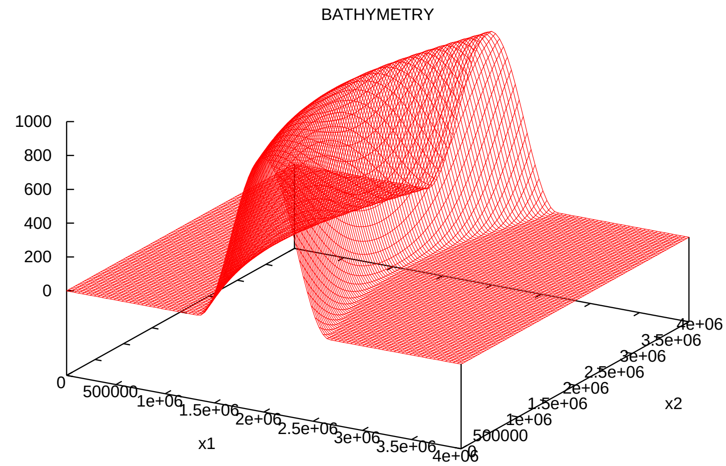


FIGURE 4. An example of bottom.

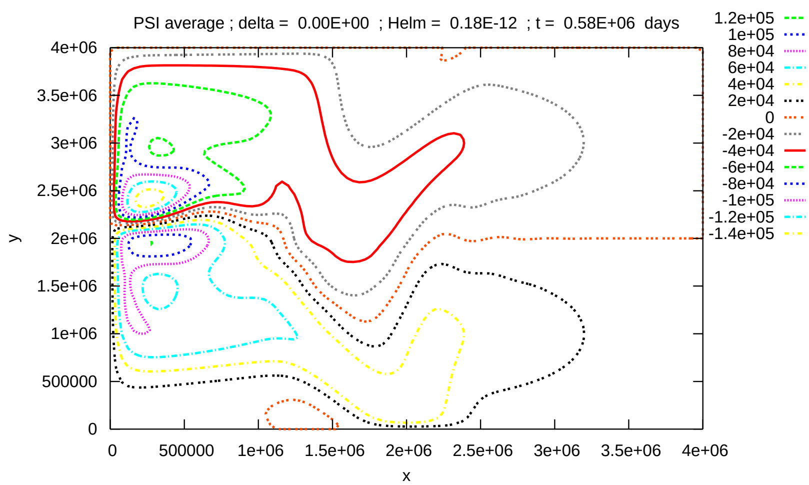


FIGURE 5. Time-average of the stream function over 1600 years with a varying bottom. The upper part is negative so the velocity is in the counterclockwise direction.

In this configuration, the contribution of the cosine effect is weaker: only a few percents.

4. **Conclusion.** We derived new SW and QG models from NSE with anisotropic turbulent viscosities. We proved that, when the horizontal viscosity is of the order of the aspect ratio (which is physically reasonnable), some new terms appear in the derivation of the viscous models. The theoretical studies have been reinforced by numerical simulations for the QG Equation.

The authors now aim to study the numerical behavior of viscous Shallow Water Equations, as well as well-posedness study of the Hydrostatic Primitive Equations

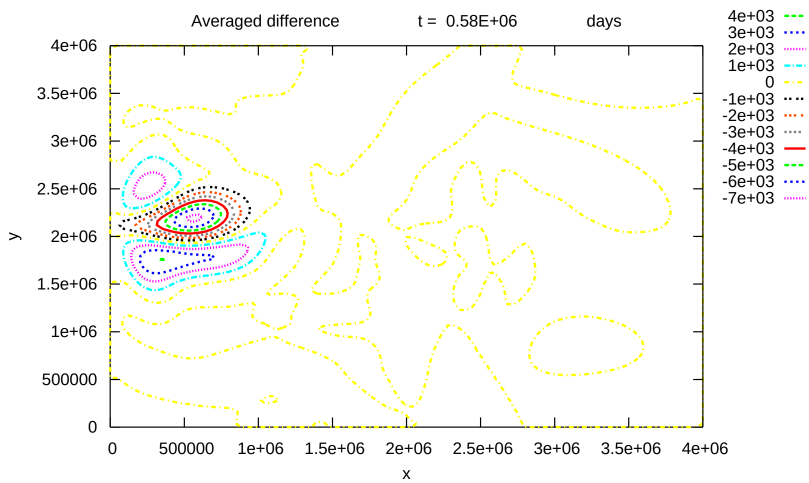


FIGURE 6. Time-average of the difference between the stream function with the cosine terms and the stream function without these terms, over 1600 years, with a varying bottom.

for which the traditional approximation is relaxed, leading to the so-called quasi-hydrostatic model. We believe that we can extend recent results of [5, 20] to the Quasi-Hydrostatic Primitive Equations.

**Acknowledgements.** We would like to thank Didier Bresch, Christine Kazantsev and the audience of the conference for raising questions related to this work. We also thank the anonymous referees for the very interesting remarks they pointed out during the review process.

**Appendix** A. **Asymptotic Development of the Viscous tensor.** In this appendix, we present the asymptotic development of the viscous tensor that is needed in the derivation of the models (see Section 2). For the sake of readability, we start this study with the hypotheses of flat bottom and rigid lid. That is, the vector $\Lambda$ is exactly the vertical unit vector ${}^t(0,0,1)$.

The expression of the tensor $S$ reads:

$$S = \begin{pmatrix} S_{xx} & S_{xz} \\ S_{zx} & S_{zz} \end{pmatrix}. \tag{27}$$

On the one hand, using the definitions of the dimensionless variables in Section 2, we have:

$$S = u_{char}^2 \begin{pmatrix} \varepsilon \widetilde{S}_{xx} & \widetilde{S}_{xz} \\ \widetilde{S}_{zx} & \varepsilon \widetilde{S}_{zz} \end{pmatrix}. \tag{28}$$

On the other hand, we recall the definition of $S$ with the viscosities:

$$S = \mu_H D^H + \mu_V D^V + \mu_E D^E = \varepsilon L_{char} u_{char} \left( \nu_H D^H + \nu_V D^V + \varepsilon^2 \nu_E D^E \right). \tag{29}$$

We are led to develop the expressions of the tensor $D^H$, $D^V$ and $D^E$ (with $\Lambda$ the vertical unit vector, see above) in order to write them under their non-dimensional form:

$$D^H = \begin{pmatrix} \nabla_x u + {}^t\nabla_x u + \partial_z w \,\mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} = \frac{u_{char}}{L_{char}} \begin{pmatrix} \nabla_{\widetilde{x}} \widetilde{u} + {}^t\nabla_{\widetilde{x}} \widetilde{u} + \partial_{\widetilde{z}} \widetilde{w} \,\mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}, \tag{30a}$$

$$D^V = \begin{pmatrix} 0 & \partial_z u + \nabla_x w \\ {}^t(\partial_z u + \nabla_x w) & 0 \end{pmatrix} = \frac{u_{char}}{L_{char}} \begin{pmatrix} 0 & \frac{1}{\varepsilon}\partial_{\widetilde{z}} \widetilde{u} + \varepsilon \nabla_{\widetilde{x}} \widetilde{w} \\ {}^t\left(\frac{1}{\varepsilon}\partial_{\widetilde{z}} \widetilde{u} + \varepsilon \nabla_{\widetilde{x}} \widetilde{w}\right) & 0 \end{pmatrix}, \tag{30b}$$

$$D^E = \partial_z w \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & 2 \end{pmatrix} = \frac{u_{char}}{L_{char}} \partial_{\widetilde{z}} \widetilde{w} \begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & 2 \end{pmatrix}. \tag{30c}$$

Using (30) together with (29), we have:

$$S = \varepsilon u_{char}^2 \begin{pmatrix} \nu_H\left(\nabla_{\widetilde{x}} \widetilde{u} + {}^t\nabla_{\widetilde{x}} \widetilde{u}\right) + \left(\nu_H - \varepsilon^2 \nu_E\right)\partial_{\widetilde{z}} \widetilde{w} \,\mathrm{Id} & \nu_V \left(\frac{1}{\varepsilon}\partial_{\widetilde{z}} \widetilde{u} + \varepsilon \nabla_{\widetilde{x}} \widetilde{w}\right) \\ \nu_V \;{}^t\left(\frac{1}{\varepsilon}\partial_{\widetilde{z}} \widetilde{u} + \varepsilon \nabla_{\widetilde{x}} \widetilde{w}\right) & 2\varepsilon^2 \nu_E \partial_{\widetilde{z}} \widetilde{w} \end{pmatrix}. \tag{31}$$

Identifying (31) with (28), we finally obtain the non-dimensional values of the coefficients $\widetilde{S}_{i\,j}$ as functions of $\widetilde{u}$ and $\widetilde{w}$:

$$\widetilde{S}_{xx} = \nu_H\left(\nabla_{\widetilde{x}} \widetilde{u} + {}^t\nabla_{\widetilde{x}} \widetilde{u}\right) + \left(\nu_H - \varepsilon^2 \nu_E\right)\partial_{\widetilde{z}} \widetilde{w} \,\mathrm{Id}, \tag{32a}$$

$$\widetilde{S}_{xz} = \nu_V \left(\partial_{\widetilde{z}} \widetilde{u} + \varepsilon^2 \nabla_{\widetilde{x}} \widetilde{w}\right), \tag{32b}$$

$$\widetilde{S}_{zx} = \nu_V \;{}^t\left(\partial_{\widetilde{z}} \widetilde{u} + \varepsilon^2 \nabla_{\widetilde{x}} \widetilde{w}\right), \tag{32c}$$

$$\widetilde{S}_{zz} = 2\varepsilon^2 \nu_E \partial_{\widetilde{z}} \widetilde{w}. \tag{32d}$$

Finally, we recall (see the derivation of the Shallow Water system at the first order) that $\partial_{\widetilde{z}} \widetilde{u}^0 = 0$, which provides, together with (32), the following informations for the $\widetilde{S}_{i\,j}$:

$$\widetilde{S}_{xx} = \nu_H\left(\nabla_{\widetilde{x}} \widetilde{u} + {}^t\nabla_{\widetilde{x}} \widetilde{u}\right) + \nu_H \partial_{\widetilde{z}} \widetilde{w} \,\mathrm{Id} + O(\varepsilon^2), \tag{33a}$$

$$\widetilde{S}_{xz} = O(\varepsilon^2), \tag{33b}$$

$$\widetilde{S}_{zx} = O(\varepsilon^2), \tag{33c}$$

$$\widetilde{S}_{zz} = O(\varepsilon^2). \tag{33d}$$

The general case with a varying free surface and/or bottom can be reduced to a small perturbation of the vertical unit vector $\Lambda = {}^t(0\ 0\ 1)$. Indeed, we know that:

$$\Lambda = \gamma(\xi) \begin{pmatrix} \xi \\ 1 \end{pmatrix} \quad \text{where} \quad \xi = \frac{h-z}{b-h}\nabla_x b + \frac{z-b}{b-h}\nabla_x h \quad \text{and} \quad \gamma(\xi) = \frac{1}{\sqrt{1+|\xi|^2}}. \tag{34}$$

If we write (34) with non-dimensional variables, we obtain:

$$\Lambda = \gamma(\varepsilon\widetilde{\xi}) \begin{pmatrix} \varepsilon\widetilde{\xi} \\ 1 \end{pmatrix} \quad \text{where} \quad \widetilde{\xi} = \frac{h-z}{b-h}\nabla_{\widetilde{x}} \widetilde{b} + \frac{z-b}{b-h}\nabla_{\widetilde{x}} \widetilde{h}$$
$$\text{and} \quad \gamma(\varepsilon\widetilde{\xi}) = \frac{1}{\sqrt{1+|\varepsilon\widetilde{\xi}|^2}} = 1 + O(\varepsilon),$$

or, in other words,

$$\Lambda = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + O(\varepsilon). \tag{35}$$

Using Equation (35) together with the definition of the tensors (30) and the first order of the Shallow Water system, we easily obtain the following estimates:

$$\widetilde{S}_{xx} = \nu_H \left(\nabla_{\widetilde{x}}\widetilde{u} + {}^t\nabla_{\widetilde{x}}\widetilde{u}\right) + \nu_H \partial_{\widetilde{z}}\widetilde{w}\,\mathrm{Id} + O(\varepsilon), \tag{36a}$$

$$\widetilde{S}_{xz} = O(\varepsilon), \tag{36b}$$

$$\widetilde{S}_{zx} = O(\varepsilon), \tag{36c}$$

$$\widetilde{S}_{zz} = O(\varepsilon). \tag{36d}$$

## REFERENCES


[1] E. Audusse. A multilayer Saint-Venant model: derivation and numerical validation. *Discrete Contin. Dyn. Syst. Ser. B*, 5(2):189–214, 2005.

[2] P. Azérad and F. Guillén. Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics. *SIAM J. Math. Anal.*, 33(4):847–859 (electronic), 2001.

[3] P. Bougeault and R. Sadourny. *Dynamique de l'atmosphère et de l'océan*, chapter 5, pages 115–118. Éditions de l'École Polytechnique, 2001.

[4] A. Bourgeois and J. T. Beale. Validity of the quasigeostrophic model for large-scale flow in the atmosphere and ocean. *SIAM J. Math. Anal.*, 25(4):1023–1068, 1994.

[5] C. Cao and E.S. Titi. Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. *Ann. of Math.*, 166(2):245–267, 2007.

[6] B. Desjardins and E. Grenier. On the homogeneous model of wind driven ocean circulation. *SIAM J. Appl. Math.*, 60(1):43–60, 2000.

[7] F. Dumas and P. Lazure. An external-internal mode coupling for a 3d hydrodynamical model for applications at regional scale (mars). *Advances in Water Resources*, 31(2):233–250, 2008.

[8] J.-F. Gerbeau and B. Perthame. Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. *Discrete and Continuous Dynamical Systems-series B.*, 1(1):89–102, 2001.

[9] T. Gerkema, J.T.F. Zimmerman, L.R.M. Maas, and H. van Haren. Geophysical and astrophysical fluid dynamics beyond the traditional approximation. *Rev. Geophys.*, 46:1–33, 2008.

[10] A. E. Gill. *Athmosphere-Ocean Dynamics*. New York: Academic Press, 1982.

[11] D. Iftimie and F. Sueur. Viscous Boundary Layers for the Navier-Stokes Equations with the Navier Slip Conditions. Technical report, EPFL, 2006.

[12] L. Perelman J. Marshall, C. Hill and A. Adcroft. Hydrostatic, quasi-hydrostatic, and nonhydrostatic ocean modeling. *J. Geophys. Research*, 102:5733–5752, 1997.

[13] E. Lauga, M. P. Brenner, and H. A. Stone. *Handbook of Experimental Fluid Dynamics*, chapter 15. Springer, New-York, 2006.

[14] C. D. Levermore and M. Sammartino. A shallow water model with eddy viscosity for basins with varying bottom topography. *Nonlinearity*, 14(6):1493–1515, 2001.

[15] C. Lucas. Cosine effect on shallow water equations and mathematical properties. *Quarterly of Applied Mathematics*, 67(2):283–310, 2009.

[16] C. Lucas and A. Rousseau. New developments and cosine effect in the viscous shallow water and quasi-geostrophic equations. *Multiscale Model. Simul.*, 7(2):796–813, 2008.

[17] G. Madec. *NEMO reference manual, ocean dynamics component : NEMO-OPA*, volume 27. Note du Pole de modélisation, Institut Pierre-Simon Laplace (IPSL), 2008.

[18] F. Marche. *Theoretical and numerical study of shallow water models ; applications to nearshore hydrodynamics*. Phd, Université Bordeaux 1, 2005.

[19] J. Pedlosky. *Geophysical fluid dynamics, 2nd edition*. Springer, 1987.

[20] Madalina Petcu, Roger Temam, and Mohammed Ziane. Some mathematical problems in geophysical fluid dynamics. In P. G. Ciarlet, editor, *Handbook of Numerical Analysis. Special volume on Computational Methods for the Oceans and the Atmosphere*, volume XIV. Elsevier, Amsterdam, 2008.

[21] A. Rousseau, R. Temam, and J. Tribbia. Boundary value problems for the inviscid primitive equations in limited domains. In P. G. Ciarlet, editor, *Handbook of Numerical Analysis, Special volume on Computational Methods for the Oceans and the Atmosphere*, volume XIV. Elsevier, Amsterdam, 2008.

[22] A. J.-C. de Saint-Venant. Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit. *Comptes Rendus de l'Académie des Sciences*, 73:147–154, 1871.

[23] E. Simonnet, T. Tachim Medjo, and R. Temam. On the order of magnitude of the baroclinic flow in the primitive equations of the ocean. *Ann. Mat. Pura Appl. (4)*, 185(suppl.):S293–S313, 2006.

[24] A. A. White and R. A. Bromley. Dynamically consistent, quasi-hydrostatic equations for global models with a complete representation of the coriolis force. *Q. J. R. Meteorol. Soc.*, 121:399–418, 1995.

Received xxxx 20xx; revised xxxx 20xx.

*E-mail address*: Antoine.Rousseau@inria.fr
*E-mail address*: Carine.Lucas@univ-orleans.fr