Controllability of the Korteweg-de Vries equation from the right Dirichlet boundary condition

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Abstract

In this paper, we consider the controllability of the Korteweg-de Vries equation in a bounded interval when the control operates via the right Dirichlet boundary condition, while the left Dirichlet and the right Neumann boundary conditions are kept to zero. We prove that the linearized equation is controllable if and only if the length of the spatial domain does not belong to some countable critical set. When the length is not critical, we prove the local exact controllability of the nonlinear equation.

1 Introduction

In this paper, we are interested in a controllability problem concerning the Korteweg-de Vries equation in the bounded interval $[0, L]$ controlled via the right Dirichlet boundary condition

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = 0, \ y_{|x=L} = v(t), \ y_{x|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L). \end{cases} \tag{1}$$

The question that we raise is the problem of exact controllability, that is, given $y_0, y_1 \in L^2(0, L)$ and $T > 0$, does there exist $v \in L^2(0, T)$ such that the unique solution of (1) starting at y_0 reaches y_1 at time $t = T$? In fact, we will study the *local* exact controllability near zero, that is, the previous problem limited to $\|y_0\|_{L^2(0,L)}$ and $\|y_1\|_{L^2(0,L)}$ small enough.

The problem of controllability of equation (1) has been deeply studied by many authors [12, 13, 14, 15, 16]. In particular several different cases have been considered: the case where all three boundary conditions are used as controls (see in particular Russell and Zhang [15]), the case with only the right Neumann boundary control (see Rosier [12], Coron and Crépeau [8], Cerpa [3] and Cerpa and Crépeau [4]), the case with only the left Dirichlet boundary control (see Rosier [14] and the authors [9]), and the case with an additional control in the left hand side (see Chapouly [5]).

In the case when the only control is the right Dirichlet boundary condition, the question remains, up to our knowledge, open. In this context, the result that we get is comparable to the one of Rosier in the context of the right Neumann boundary control [12]. We prove that the linearized equation around 0 is controllable if and only if the length L does not belong to some countable critical set. The same pathological behaviour was raised by Rosier in [12]. Precisely, Rosier shows that equation (1) is controllable via the right Neumann condition if and only if

$$L \notin \mathcal{N}^* \text{ where } \mathcal{N}^* := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, \ k, l \in \mathbb{N}^* \right\}. \tag{2}$$

The critical set which we find in the present situation is different.

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Our first result concerns the linearized system, which is the following

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = 0,\ y_{|x=L} = v(t),\ y_{x|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L). \end{cases} \tag{3}$$

As it is classical (see e.g. [9, 10]), for any $v \in L^2(0,T)$ and any $y_0 \in H^{-1}(0,1)$, there exists a unique (transposition) solution of (3) in $C^0([0,T];H^{-1}(0,L)) \cap L^2(0,T;L^2(0,L))$.

The problem of exact controllability is the following. Given $L > 0$, $T > 0$, is it true that for any $y_0, y_1 \in L^2(0,L)$, there exists a control $v \in L^2(0,T)$ such that the solution of (3) satisfies

$$y_{|t=T} = y_1 \text{ on } (0,L)? \tag{4}$$

We answer to this question in the following statement.

Theorem 1. *There exists a countable set $\mathcal{N} \subset (0,+\infty)$ such that the following happens*

$$L \notin \mathcal{N} \Longleftrightarrow \textit{Equation } (3) \textit{ is exactly controllable.} \tag{5}$$

Remark 1. *As will be established in the proof, one can describe the set $\mathcal{N}$ as follows*

$$\mathcal{N} = \Big\{ L \in \mathbb{R}^{+*} \ /\ \exists (a,b) \in \mathbb{C}^2 \textit{ such that} \\ a\exp(a) = b\exp(b) = -(a+b)\exp(-(a+b)) \textit{ and } L^2 = -(a^2+ab+b^2) \Big\}. \tag{6}$$

Note that the critical lengths found by Rosier for the control by the right Neumann condition (2) *can be reformulated as follows [12]*

$$\mathcal{N}^* = \Big\{ L \in \mathbb{R}^{+*} \ /\ \exists (a,b) \in \mathbb{C}^2 \textit{ such that} \\ \exp(a) = \exp(b) = \exp(-(a+b)) \textit{ and } L^2 = -(a^2+ab+b^2) \Big\}. \tag{7}$$

As often, this controllability result for the linearized system yields a local controllability result for the nonlinear system.

Theorem 2. *Let us consider $L \in \mathbb{R}^{+*} \setminus \mathcal{N}$. There exists $\mu > 0$ such that for any $y_0, y_1 \in L^2(0,L)$ satisfying*

$$\|y_0\|_{L^2(0,L)} + \|y_1\|_{L^2(0,L)} < \mu, \tag{8}$$

there exists $v \in H^{1/6-\varepsilon}(0,T)$ such that the solution $y \in L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L))$ of (1) *satisfies*

$$y_{|t=T} = y_1 \textit{ in } (0,L). \tag{9}$$

2 Controllability of the linearized system

In this section, we prove Theorem 1.

The first step is similar to [12], that is, to prove via a compactness-uniqueness type argument that the non-controllability of (3) is equivalent to the existence of an eigenvector of the operator $A\varphi = \varphi_{xxx} + \varphi_x$ with zero observation. Precisely, we have the following

Proposition 1. *System* (3) *is not exactly controllable in $H^{-1}(0,L)$ if and only if there exists $\varphi \neq 0$ satisfying for some $\lambda \in \mathbb{C}$*

$$\begin{cases} \varphi_{xxx} + \varphi_x = \lambda\varphi & \textit{in } (0,L), \\ \varphi(0) = \varphi_x(0) = \varphi(L) = \varphi_{xx}(L) = 0. \end{cases} \tag{10}$$

Proof of Proposition 1.

The proof is threefold. First, we suppose that the controllability of (3) does not occur, hence the observability of the adjoint system does not either, and we manage to find a time-varying solution of the adjoint system with zero observation. Then in a second time we deduce the existence of an eigenvector as described in (10). Finally, we show conversely that if such an eigenvector exists, then (3) is not controllable.

First step. Let us recall the adjoint system for (3):

$$\begin{cases} \psi_t + \psi_{xxx} + \psi_x = 0 & \text{in } (0,T)\times(0,L), \\ \psi_{|x=0} = \psi_{x|x=0} = \psi_{|x=L} = 0 & \text{in } (0,T), \\ \psi_{|t=T} = \psi^T & \text{in } (0,L). \end{cases} \tag{11}$$

It is a standard duality argument that the controllability of system (3) in $H^{-1}(0,L)$ is equivalent to the following observability property for the adjoint system (11):

$$\exists C>0, \quad \forall \psi^T \in H^1_0(0,L), \ \|\psi^T_x\|_{L^2(0,L)} \leq C\|\psi_{xx|x=L}\|_{L^2(0,T)}. \tag{12}$$

Recall (see for instance [9]) that given $\psi^T \in H^1_0(0,L)$, the unique solution of (11) belongs to $X := L^2(0,T;H^2(0,L)) \cap C^0([0,T];H^1_0(0,L)) \cap H^1(0,T;H^{-1}(0,L))$ and satisfies the standard inequality

$$\exists C>0, \quad \forall \psi^T \in H^1_0(0,L), \ \|\psi\|_X \leq C\|\psi^T_x\|_{L^2(0,L)}, \tag{13}$$

and the following hidden regularity inequality

$$\exists C>0, \quad \forall \psi^T \in H^1_0(0,L), \ \|\psi_{xx|x=L}\|_{L^2(0,T)} \leq C\|\psi^T_x\|_{L^2(0,L)}. \tag{14}$$

Now assuming that the system is not controllable, we deduce that there exists some sequence $\psi_n \in L^2(0,T;H^2(0,L)) \cap C^0([0,T];H^1_0(0,L)) \cap H^1(0,T;H^{-1}(0,L))$ satisfying

$$\|\psi_{n,x}(T,\cdot)\|_{L^2(0,L)} = 1 \text{ and } \|\psi_{n,xx}(\cdot,L)\|_{L^2(0,T)} \to 0 \text{ as } n\to+\infty. \tag{15}$$

Let us denote $Y := L^2(0,T;H^{7/4}(0,L)) \cap C^0([0,T];H^{3/4}(0,L))$. Now we show that there exists some positive constant C such that for any solution $\psi \in X$ of (11), one has

$$\|\psi^T_x\|_{L^2(0,L)} \leq C\big(\|\psi\|_Y + \|\psi_{xx}(\cdot,L)\|_{L^2(0,T)}\big). \tag{16}$$

For that purpose, we multiply equation (11) by $-\psi_{xx}$ and integrate by parts; we get

$$\frac{1}{2}\Big(\frac{d}{dt}\int_0^L |\psi_x|^2 - |\psi_{xx|x=L}|^2 + |\psi_{xx|x=0}|^2 - |\psi_{x|x=L}|^2\Big) = 0.$$

Integrating between $T/2$ and T, we deduce

$$\int_0^L |\psi^T_x|^2 + \int_{T/2}^T |\psi_{xx|x=0}|^2 = \int_0^L |\psi_x(T/2,\cdot)|^2 + \int_{T/2}^T |\psi_{xx|x=L}|^2 + \int_{T/2}^T |\psi_{x|x=L}|^2. \tag{17}$$

Now it is clear that one can measure the last two integrals on the right hand side by the right hand side of (16). To estimate the first one, we introduce $\theta \in C^\infty_0(0,T)$ such that $\theta \geq 0$ in $[0,T]$ and $\theta = 1$ in $[T/3, 2T/3]$. Using that

$$\begin{cases} (\theta\psi)_t + (\theta\psi)_{xxx} + (\theta\psi)_x = \theta_t\psi & \text{in } (0,T)\times(0,L), \\ (\theta\psi)_{|x=0} = (\theta\psi)_{x|x=0} = (\theta\psi)_{|x=L} = 0 & \text{in } (0,T), \\ (\theta\psi)_{|t=T} = 0 & \text{in } (0,L), \end{cases}$$

and the usual estimate for the equation with L^2 right hand side, we infer

$$\|\psi(T/2,\cdot)\|_{H^1_0(0,L)} \leq C\|\theta_t\psi\|_{L^2(0,T;L^2(0,L))} \leq C\|\psi\|_Y.$$

This immediately gives (16).

Now since X is compactly embedded in Y, one may extract some subsequence of ψ_n converging to some $\psi \in X$ for the Y (strong) topology. Let us call this subsequence (ψ_n) again. From (15) and (16), we deduce that the sequence $(\psi_n(T,\cdot))$ is a Cauchy sequence in $H^1_0(0,L)$. Call $\hat{\psi}^T$ its limit. From (13) we deduce that ψ is the solution of (11) associated to the condition $\hat{\psi}^T$ at time $t = T$. Now using (14) and (15), we deduce that

$$\psi_{xx|x=L} = 0.$$

It remains only to prove that $\psi \neq 0$. It is a consequence of (16) that $\psi_n(T,\cdot)$ is a Cauchy sequence in $H^1_0(0,L)$. Hence it follows from (15) that $\|\hat{\psi}^T\|_{H^1_0(0,L)} = 1$, which ends the first step of the proof.

Second step. This step of the proof follows [12] (which itself relies on an argument due to Bardos, Lebeau and Rauch [1]) very closely. Let A be the operator

$$A = \partial^3_{xxx} + \partial_x \text{ on the domain } D(A) := \{\psi \in H^3(0,L) \ / \ \psi(0) = \psi_x(0) = \psi(0) = 0\}.$$

Call N_τ the vector subspace of $H^1_0(0,L)$ consisting of ψ^T such that the solution of (11) in $[0,T] \times [0,L]$ satisfies $\psi_{xx|x=L} = 0$ in $[\tau, T]$. It is clear that N_τ is non decreasing as a function of τ. Now the same argument as above shows that from any bounded sequence of N_τ, one can extract a converging subsequence in $H^1_0(0,L)$. Hence by Riesz's theorem, N_τ is finite-dimensional. Since it is non decreasing, there is some non trivial subinterval in $[0,T]$, say $[a,b]$, on which it is constant, say, equal to N. Moreover this can be done for some b arbitrarily close to T but different from it.

Call M the subspace of $H^1_0(0,L)$ consisting of $\psi(b,\cdot)$ as ψ^T describes N. Now calling S the semi-group associated to A, one can see that, due to the fact that N_τ is constant on $[a,b]$, N is stable by the action of $S(-t)$ for $t > 0$ small enough. It follows that M is also stable by this action. Recalling that

$$\psi_t + A\psi = 0 \text{ in } [a,b],$$

this yields that $A\psi(b,\cdot) \in M$, since M being finite-dimensional, it is closed for the H^{-2} topology and since the solution ψ belongs to $C^1([0,T]; H^{-2}(0,L))$. Note that due to the regularizing effect of the equation, M is composed of smooth functions. Now the finite-dimensional linear mapping $\psi(b,\cdot) \in M \mapsto A\psi(b,\cdot) \in M$ on a non trivial complex vector space admits a complex eigenvalue, which establishes the existence of a non trivial solution of (10).

Third step. Now we show that if there exists an nontrivial solution φ to (10), then the system (3) is not controllable. It is enough to see that, in that case, given a solution y of (3), the integral

$$\exp(-\lambda t) \int_0^1 y(t,x)\varphi(x)\,dx,$$

is constant over time, independently of v. Using (3) and (10) performing then inetegration by parts (or using directly the definition of a transposition solution), it is not difficult to see that

$$\frac{d}{dt}\int_0^1 y(t,x)\varphi(x)\,dx = -\int_0^1 (y_{xxx} + y_x)(t,x)\varphi(x)\,dx = \lambda \int_0^1 y(t,x)\varphi(x)\,dx = 0,$$

which involves in particular that (3) is not controllable.

Proposition 2. *System* (3) *is not exactly controllable in* $H^{-1}(0,L)$ *if and only if there exist* $a, b \in \mathbb{C}$ *such that*

$$a\exp(a) = b\exp(b) = -(a+b)\exp(-(a+b)). \tag{18}$$

$$L^2 = -(a^2 + ab + b^2). \tag{19}$$

Proof of Proposition 2.

We simply solve equation (10). Let μ_0, μ_1 and μ_2 be the three roots of the characteristic equation associated to (10):

$$\xi^3 + \xi - \lambda = 0. \tag{20}$$

Now μ_0, μ_1 and μ_2 have to satisfy

$$\begin{cases} \mu_0 + \mu_1 + \mu_2 = 0, \\ \mu_0\mu_1 + \mu_1\mu_2 + \mu_0\mu_2 = 1, \\ \mu_0\mu_1\mu_2 = \lambda. \end{cases} \tag{21}$$

Note that if there were a double root, it would necessarily be (up to reindexing) $\mu_0 = \mu_1 = \pm i\sqrt{3}/3$, hence $\mu_2 = \mp i2\sqrt{3}/3$. Call $\sigma = \pm\sqrt{3}/3$. Let us explain why a corresponding function

$$\varphi(x) = (c_0 + c_1 x)\exp(i\sigma x) + c_2 \exp(-2i\sigma x),$$

cannot solve (10) unless $c_0 = c_1 = c_2 = 0$. From $\varphi(0) = \varphi_x(0) = 0$, one easily deduces that $c_1 = -3i\sigma c_0$ and $c_2 = -c_0$. Now, unless $c_0 = 0$, the condition $\varphi(L) = 0$ reads

$$(1 - 3i\sigma L)\exp(3i\sigma L) = 1.$$

Taking the modulus, this gives $L = 0$.

Now let us consider the case with simple roots of (20). A combination

$$\varphi(x) := c_0 \exp(\mu_0 x) + c_1 \exp(\mu_1 x) + c_2 \exp(\mu_2 x),$$

solves the boundary conditions in (10) if and only if

$$c_0 + c_1 + c_2 = 0, \tag{22}$$
$$\mu_0 c_0 + \mu_1 c_1 + \mu_2 c_2 = 0, \tag{23}$$
$$c_0 \exp(\mu_0 L) + c_1 \exp(\mu_1 L) + c_2 \exp(\mu_2 L) = 0, \tag{24}$$
$$c_0\mu_0^2 \exp(\mu_0 L) + c_1\mu_1^2 \exp(\mu_1 L) + c_2\mu_2^2 \exp(\mu_2 L) = 0. \tag{25}$$

By elementary combinations of these equations we get:

$$c_0 = -c_1 - c_2, \tag{26}$$
$$(\mu_1 - \mu_0)c_1 + (\mu_2 - \mu_0)c_2 = 0, \tag{27}$$
$$(\exp(\mu_1 L) - \exp(\mu_0 L))c_1 + (\exp(\mu_2 L) - \exp(\mu_0 L))c_2 = 0, \tag{28}$$
$$(\mu_1^2 - \mu_0^2)\exp(\mu_1 L)c_1 + (\mu_2^2 - \mu_0^2)\exp(\mu_2 L)c_2 = 0. \tag{29}$$

Now we use $\mu_0 + \mu_1 + \mu_2 = 0$ as well as (27) to simplify (29). Hence, unless $c_0 = c_1 = c_2 = 0$, we arrive at

$$-\mu_2 \exp(\mu_1 L) + \mu_1 \exp(\mu_2 L) = 0.$$

Using the symmetry between μ_0, μ_1, and μ_2 we deduce

$$\mu_0 \exp(-\mu_0 L) = \mu_1 \exp(-\mu_1 L) = \mu_2 \exp(-\mu_2 L). \tag{30}$$

Reciprocally, if (30) is satisfied, using (20), it is not difficult to see that (24) and (25) are combinations of (22) and (23). Hence there exist non trivial solutions of (22)-(25) if and only if (30) holds.

Denoting $a = -L\mu_0$ and $b = -L\mu_1$, we arrive at the claim.

Proposition 3. *Equation* (18) *has an at most countable set of solutions* $(a, b) \in \mathbb{C}^2$.

Proof of Proposition 3.

Let us show that each solution $(a, b) \in \mathbb{C}^2$ of (18) is isolated in $\mathbb{C}^2$, which will prove the claim. Let us consider such a solution (a, b). Since one cannot have $a = b = -1$, and due to the symmetry between a and b, we can assume $b \neq -1$.

Let us suppose by contradiction that (a, b) is not isolated; hence there exists a sequence (a_n, b_n) of solutions of (18) converging toward (a, b).

Since $b \neq -1$, by the holomorphic inverse mapping theorem, one can find $\mathcal{V}_1$ and $\mathcal{V}_2$ two complex neighborhoods of b and $b\exp(b)$ respectively, and a holomorphic diffeomorphism $h : \mathcal{V}_2 \to \mathcal{V}_1$ such that

$$h(c)\exp(h(c)) = c \text{ for any } c \in \mathcal{V}_2. \tag{31}$$

(See [17] and references therein for some studies on this peculiar transcendental equation.) Of course, we have $b_n \exp(b_n) \in \mathcal{V}_2$ for n large enough. For such n, one clearly has $b_n = h(a_n \exp(a_n))$. We deduce that the following equation has an infinite number of solutions z converging toward a:

$$z\exp(z) = h[z\exp(z)]\exp(h[z\exp(z)]) = -(z + h[z\exp(z)])\exp(-(z + h[z\exp(z)])). \tag{32}$$

By the unique continuation principle, this equation is hence valid on a whole ball $B(a; r)$. Now from (32), we easily infer that for any $z \in B(a; r)$

$$h^2[z\exp(z)]\exp(2h[z\exp(z)]) = -h[z\exp(z)](z + h[z\exp(z)])\exp(-z),$$

hence

$$h^2[z\exp(z)] + zh[z\exp(z)] = -z^2 e^{3z}.$$

Solving this second degree equation in terms of $h[z\exp(z)]$, we obtain that for any $z \in B(a; r)$

$$h(z\exp(z)) = \frac{-z \pm z\sqrt{1 + 4\exp(3z)}}{2}.$$

This equation is valid with a fixed sign before the square root in some non trivial ball inside $B(a; r)$. Define f in $D := \mathbb{C} \setminus [-2\ln(2)/3; +\infty) + i\frac{\pi}{3}(1 + 2\mathbb{Z})$ by

$$f(z) := \frac{-z \pm z\sqrt{1 + 4\exp(3z)}}{2},$$

where we chose the usual branch of the square root in $\mathbb{C} \setminus \mathbb{R}^-$ and the same sign before the square root as above. Again by the unique continuation principle we have for any $z \in D$ that

$$f(z)\exp(f(z)) = z\exp(z).$$

Letting $\mathrm{Im}\,(z) = 0$ and $\mathrm{Re}\,(z) \to +\infty$, we get a contradiction.

Proposition 4. *Equation* (18) *has at least a countable quantity of solutions* $(a, b) \in \mathbb{C}^2$ *such that* $a^2 + ab + b^2 \in \mathbb{R}^{-*}$.

Proof of Proposition 4.

We use the ansatz $a = \bar{b}$. Then we denote

$$a = \rho\exp(i\theta) \text{ and } b = \rho\exp(-i\theta), \tag{33}$$

with $\rho > 0$ and $\theta \in (-\pi, \pi) \setminus \{0\}$. Now $a\exp(a) = b\exp(b)$ reads

$$\exp(2i\theta) = \exp(-2i\rho\sin(\theta)),$$

i.e.

$$\rho = -\frac{\theta}{\sin(\theta)} + k\frac{\pi}{\sin(\theta)},\ \ k \in \mathbb{Z}. \tag{34}$$

Now we write $a\exp(a) = -(a + b)\exp(-(a + b))$ in the form

$$\exp(i\theta) + \exp(-i\theta) = -\exp(i\theta)\exp(\rho\exp(i\theta))\exp(\rho[\exp(i\theta) + \exp(-i\theta)]).$$

Using (34) we easily arrive at

$$2\cos(\theta) = -(-1)^k \exp(-3(\theta - k\pi)\cot(\theta)), \tag{35}$$

so that under the ansatz (33), (18) is equivalent to (34)-(35).

Now it is easy to see that for k even and sufficiently negative, (35) has a solution $\theta_k \in (-\frac{7\pi}{12}, -\frac{\pi}{2})$. Indeed, denoting

$$h_k(\theta) := 2\cos(\theta) + (-1)^k \exp(-3(\theta - k\pi)\cot(\theta)),$$

one checks that

$$h_k(\theta) = 1 \text{ for } \theta = -\frac{\pi}{2} \text{ and } h_k(\theta) < 0 \text{ for } \theta = -\frac{7\pi}{12} \text{ and } -k \text{ large enough.}$$

Now fixing ρ_k by (34) and then a_k and b_k by (33), it is then elementary to check that

$$a_k^2 + a_k b_k + b_k^2 = \rho_k^2(1 + 2\cos(2\theta_k)) \in \mathbb{R}^{-*},$$

so that there exists a corresponding critical length L. This gives a countable family of critical lengths, since it is elementary to check that all the θ_k that we have constructed are different.

3 Controllability of the nonlinear system

The proof of the controllability of the nonlinear system will rely on a fixed point scheme. But in that order, we will need more regular controls than the ones provided by Theorem 1. To do so, we first reduce the exact controllability problem to a null controllability one, using the reversibility of some ad hoc system. Next we prove that one can regularize the controls for this null controllability problem. Finally, we give the fixed point argument.

3.1 Reduction to a null controllability problem

Proposition 5. *Fix $L > 0$. Given $z_1 \in L^2(0, L)$, there exists a unique solution $z \in L^2(0, T; H^1(0, L)) \cap C^0([0, T]; L^2(0, L))$ of the following boundary-value problem*

$$\begin{cases} z_t + z_{xxx} + z_x = 0 & \text{in } (0,T)\times(0,L), \\ z_{|x=0} = z_{xx|x=0} = z_{x|x=L} = 0 & \text{in } (0,T), \\ z_{|t=T} = z_1 & \text{in } (0,L). \end{cases} \tag{36}$$

Proof of Proposition 5.

Step 1. We consider the simplified system:

$$\begin{cases} z_t + z_{xxx} = f & \text{in } (0,T)\times(0,L), \\ z_{|x=0} = z_{xx|x=0} = z_{x|x=L} = 0 & \text{in } (0,T), \\ z_{|t=T} = z_1 & \text{in } (0,L), \end{cases} \tag{37}$$

and the following one, satisfied by $w = z_x$ when z satisfies (37) and $f = 0$ at $x = 0$:

$$\begin{cases} w_t + w_{xxx} = f_x & \text{in } (0,T)\times(0,L), \\ w_{xx|x=0} = w_{x|x=0} = w_{|x=L} = 0 & \text{in } (0,T), \\ w_{|t=T} = w_1 := z_{1x} & \text{in } (0,L). \end{cases} \tag{38}$$

System (38) is the linearized version of the one considered in [6]. One easily checks that $P := -\partial_{xxx}^3$ defined on $D(P) := \{u \in H^3(0,L)/u(0) = u_x(L) = u_{xx}(L) = 0\}$ is dissipative, as well as its adjoint

$P^* := \partial^3_{xxx}$ defined on $D(P^*) := \{u \in H^3(0,L)/u(0) = u_x(0) = u_{xx}(L) = 0\}$. One deduces by the Lumer-Phillips Theorem that the system

$$\begin{cases} w_t + w_{xxx} = g & \text{in } (0,T)\times(0,L),\\ w_{|x=0} = w_{x|x=L} = w_{xx|x=L} = 0 & \text{in } (0,T),\\ w_{|t=0} = w_0 & \text{in } (0,L), \end{cases}$$

is well posed for $w_0 \in D(P)$ and $g \in C^1([0,T];L^2(0,L))$, and yields a solution $w \in C^0([0,T];D(P)) \cap C^1([0,T];L^2(0,L))$. A simple change of variable $(t,x) \to (T-t, L-x)$ yields the equivalent statement (in $D(P^*)$) for system (38).

It follows that (38) is well posed for $z_{1x} \in D(P)$ and $f \in C^1([0,T];H^1_0(0,L))$.

Step 2. Now we are going to prove that for $f \in L^2(0,T;H^{-1/3}(0,L))$, we have the following estimate on the solution z of (37):

$$\|z\|_{L^2(0,T;H^1(0,L))\cap C^0([0,T];L^2(0,L))} \leq C\big(\|f\|_{L^2(0,T;H^{-1/3}(0,L))} + \|z_1\|_{L^2(0,L)}\big), \tag{39}$$

where the constant $C > 0$ is independent of T as long as it is bounded.

a. Let us first prove that

$$\|z\|_{L^2(0,T;H^2(0,L))\cap C^0([0,T];H^1(0,L))} \leq C\big(\|f\|_{L^2((0,T)\times(0,L))} + \|z_1\|_{H^1_0(0,L)}\big). \tag{40}$$

To prove this, we consider a regular solution of (38), and operate in three steps. First we multiply by z_{xx} and integrate by parts to get

$$-\frac{1}{2}\partial_t \int_0^L |z_x|^2\,dx + \frac{|z_{xx}(L)|^2}{2} = \int_0^L z_{xx} f\,dx.$$

Next we multiply by $(L-x)z_{xx}$ to deduce

$$-\frac{1}{2}\partial_t \int_0^L (L-x)|z_x|^2\,dx + \int_0^L z_t z_x\,dx + \frac{1}{2}\int_0^L |z_{xx}|^2\,dx = \int_0^L (L-x) z_{xx} f\,dx.$$

Finally we multiply by $-z_x$ to find

$$-\int_0^L z_t z_x\,dx + \int_0^L |z_{xx}|^2\,dx = -\int_0^L z_x f\,dx.$$

We sum up the three previous equalities to get

$$-\frac{1}{2}\partial_t \int_0^L (L+1-x)|z_x|^2\,dx + \frac{|z_{xx}(L)|^2}{2} + \frac{3}{2}\int_0^L |z_{xx}|^2\,dx = \int_0^L (L-x+1) z_{xx} f\,dx - \int_0^L z_x f\,dx. \tag{41}$$

We use Young's inequality on the first term in the right hand side to deduce

$$-\frac{1}{2}\partial_t \int_0^L (L+1-x)|z_x|^2\,dx + \frac{|z_{xx}(L)|^2}{2} + \int_0^L |z_{xx}|^2\,dx \leq -\int_0^L z_x f\,dx + C\int_0^L f^2\,dx. \tag{42}$$

We finally deduce estimate (40) by using Gronwall's lemma.

b. Let us now prove that

$$\|z\|_{L^2((0,T)\times(0,L))\cap C^0([0,T];H^{-1}(0,L))} \leq C\big(\|f\|_{L^2(0,T;H^{-2/3}(0,L))} + \|z_1\|_{H^{-1}(0,L)}\big). \tag{43}$$

In order to prove this, we use that z satisfies for every $h \in L^2((0,T)\times(0,L)) + L^1(0,T;H^1_0(0,L))$ that

$$\int_0^T\int_0^L hz = \int_0^T\int_0^L f\varphi + \int_0^L z_1\varphi(T,\cdot),$$

where φ is the solution of the adjoint problem

$$\begin{cases} \varphi_t + \varphi_{xxx} = h & \text{in } (0,T)\times(0,L), \\ \varphi_{x|x=0} = \varphi_{|x=L} = \varphi_{xx|x=L} = 0 & \text{in } (0,T), \\ \varphi_{|t=0} = 0 & \text{in } (0,L). \end{cases} \tag{44}$$

From the previous computations (see (41)), we have that there exists $C > 0$ such that

$$\|\varphi\|_{L^2(0,T;H^2(0,L))\cap C^0([0,T];H^1(0,L))} \leq C\|h\|_{L^2((0,T)\times(0,L))}.$$

Following the same computation, one can also prove that the same is true with $\|h\|_{L^1(0,T;H^1_0(0,L))}$ in the right hand side. This readily establishes (43).

c. Interpolating the inequalities obtained in the two previous paragraphs, we obtain inequality (39).

Step 3. Finally, we use a Schauder fixed point argument to the operator $\mathcal{F} : z \mapsto \tilde{z}$ where $\tilde{z}$ is the solution of (37) with $f = -z_x$. The domain on which we define the operator is for some $K > 0$ the following convex set:

$$Z_K := \Big\{ z \in L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L)) \ / \\ \forall t \in [0,T],\ \sup_{s\in[0,t]} \|u(s)\|^2_{L^2(0,L)} + \|u\|^2_{L^2(0,t;H^1(0,L))} \leq Ke^{Kt}\|z_1\|^2_{L^2} \Big\},$$

endowed with the $L^2(0,T;H^{2/3}(0,L))$ topology. First, from (39), $\mathcal{F}$ is continuous on Z_K. Using (39) and that $\|\cdot\|^2_{H^{-1/3}} \leq \varepsilon\|\cdot\|^2_{L^2} + C_\varepsilon\|\cdot\|^2_{H^{-1}}$, we see that

$$\|\tilde{z}\|^2_{L^2(0,t;H^1(0,L))} + \|\tilde{z}\|^2_{C^0([0,t];L^2(0,L))} \leq C\|z_1\|^2 + \frac{1}{2}\|z_x\|^2_{L^2(0,t;L^2(0,L))} + C_{\frac{1}{2}}\|z_x\|^2_{L^2(0,t;H^{-1}(0,L))}.$$

Since $z \in Z_K$, we deduce

$$\|\tilde{z}\|^2_{L^2(0,t;H^1(0,L))} + \|\tilde{z}\|^2_{C^0([0,t];L^2(0,L))} \leq (C + \frac{K}{2} + C_{\frac{1}{2}}) \exp(Kt)\|z_1\|^2.$$

From this we deduce that for K large enough, $\mathcal{F}(Z_K) \subset Z_K$. Now we use the compactness of the injection $L^2(0,T;H^1(0,L))\cap H^1(0,T;H^{-2}(0,L))$ into $L^2(0,T;H^{2/3}(0,L))$, to deduce via Schauder fixed point that system (36) admits a solution as claimed. Its uniqueness is proved by considering two such solutions z_1 and z_2, applying (42) to the difference $z_1 - z_2$ (which has null initial data and which corresponds to $f = z_{2x} - z_{1x}$), and finally concluding by Gronwall's lemma.

3.2 Regularizing the control

We prove the following improvement of Theorem 1.

Proposition 6. *Let* $L \in \mathbb{R}^{+*} \setminus \mathcal{N}$. *For any* $y_0, y_1 \in L^2(0,L)$ *one can find* $v \in H^{1/6-\varepsilon}(0,T)$ *such that the solution of* (3) *satisfies* (4).

Proof of Proposition 6.

Due to Proposition 5, we can restrict ourselves to the case when $y_1 = 0$. Observe that the solution given by Proposition 5 has a right Dirichlet boundary condition in $H^{1/6-\varepsilon}(0,T)$ since it belongs to $H^1(0,T;H^{-2}(0,L)) \cap L^2(0,T;H^1(0,L))$.

Next, we remark that by the regularizing effect of the system when $v = 0$ (see e.g. [9]), we can restrict ourselves to the case when $y_0 \in C^\infty([0,L])$ and satisfies $y_0(0) = y_0(L) = y_0'(L) = 0$. It suffices to choose 0 as a control during a first time interval. For such y_0, we will find a control $v \in H^1(0,T)$ driving y_0 to 0. Once this is done, joining the control 0 which we apply at the beginning and the H^1 control gives a $H^{1/2-\varepsilon}$ control and hence proves Proposition 6.

Hence, let us assume that $y_0 \in C^\infty([0,L])$. We find by Theorem 1 that there exists $h \in L^2(0,T)$ such that the solution $z \in L^2((0,T)\times(0,L)) \cap C^0([0,T];H^{-1}(0,L))$ of

$$\begin{cases} z_t + z_{xxx} + z_x = 0 & \text{in } (0,T)\times(0,L), \\ z_{|x=0} = 0,\ z_{|x=L} = h(t),\ z_{x|x=L} = 0 & \text{in } (0,T), \\ z_{|t=0} = -y_{0xxx} - y_{0x} & \text{in } (0,L), \end{cases} \tag{45}$$

satisfies

$$z_{|t=T} = 0. \tag{46}$$

Now we consider the solution $y \in L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L))$ of the following problem:

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = 0,\ y_{|x=L} = v(t) := -\int_t^T h(s)\,ds,\ y_{x|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L). \end{cases} \tag{47}$$

Here we have used that $v \in H^{1/3}(0,T)$ to get that regularity on y (see e.g. [9]). Now we claim that

$$y_t = z.$$

It suffices indeed to see that y_t satisfies system (45). Now by (46) and (47), we infer that

$$y_x(T) + y_{xxx}(T) = 0 \text{ in } (0,L) \text{ and } y(T,0) = y(T,L) = y_x(T,L) = 0. \tag{48}$$

Now there are two possibilities. If $L \notin \{k2\pi,\ k \in \mathbb{N}\setminus\{0\}\}$, then there is no non trivial solution of (48), hence we have found a control for (47) satisfying the required properties. But if on the contrary $L = k2\pi$ for some $k \in \mathbb{N}\setminus\{0\}$, then it is not difficult to see that the set of solutions of (48) consists of the vector space generated by

$$e(x) = 1 - \cos(x).$$

Hence we are led to construct a regular control steering 0 to e. By Theorem 1, there exists $\overline{h} \in L^2(0,T/2)$ such that the solution $\mathcal{E} \in L^2((0,T/2)\times(0,L)) \cap C^0([0,T/2];H^{-1}(0,L))$ of

$$\begin{cases} \mathcal{E}_t + \mathcal{E}_{xxx} + \mathcal{E}_x = 0 & \text{in } (0,T/2)\times(0,L), \\ \mathcal{E}_{|x=0} = 0,\ \mathcal{E}_{|x=L} = \overline{h}(t),\ \mathcal{E}_{x|x=L} = 0 & \text{in } (0,T/2), \\ \mathcal{E}_{|t=0} = 0 & \text{in } (0,L), \end{cases} \tag{49}$$

satisfies

$$\mathcal{E}_{|t=T/2} = e. \tag{50}$$

Now we introduce $\rho \in C_0^\infty((0,T/2);\mathbb{R})$ with $\rho \geq 0$ and $\int_{\mathbb{R}} \rho = 1$. Due to the fact that e is a solution of (48), it is not difficult to see that $\tilde{\mathcal{E}} := \rho * \mathcal{E} \in L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L))$ (where we extend $\mathcal{E}$ by 0 outside $[0,T/2]$, and where the convolution is operated with respect to the variable t) is a solution of

$$\begin{cases} \tilde{\mathcal{E}}_t + \tilde{\mathcal{E}}_{xxx} + \tilde{\mathcal{E}}_x = 0 & \text{in } (0,T)\times(0,L), \\ \tilde{\mathcal{E}}_{|x=0} = 0,\ \tilde{\mathcal{E}}_{|x=L} = \rho * \overline{h}(t),\ \tilde{\mathcal{E}}_{x|x=L} = 0 & \text{in } (0,T), \\ \tilde{\mathcal{E}}_{|t=0} = 0 & \text{in } (0,L), \\ \tilde{\mathcal{E}}_{|t=T} = e & \text{in } (0,L). \end{cases} \tag{51}$$

From what precedes, we deduce the existence of some control $v \in H^1(0,T)$ driving y_0 to 0 and satisfying

$$\|v\|_{H^1(0,T)} \lesssim \|y_0\|_{H^2(0,L)},$$

which ends the proof.

3.3 Fixed point argument

In this paragraph we prove Theorem 2. We follow a standard fixed point scheme (see e.g. [12, 9]). Introduce the operator $L : L^2(0,L) \to L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L))$ which associates to any $z_1 \in L^2(0,L)$ the solution $z \in L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L))$ of (36) constructed in Proposition 5. Define L_0 as the operator $L : L^2(0,L) \to L^2(0,L)$ which associates $L(z_1)_{|t=0}$ to any $z_1 \in L^2(0,L)$.

Next, we introduce the operator $L_1 : L^2(0,L) \to L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L))$ which associates to any $\widehat{y}_0 \in L^2(0,L)$ the solution of the control problem provided by Proposition (6) with target 0, that is

$$\begin{cases} w_t + w_{xxx} + w_x = 0 & \text{in } (0,T) \times (0,L), \\ w_{|x=0} = w_{x|x=L} = 0;\ w_{|x=L} = v & \text{in } (0,T), \\ w_{|t=0} = \widehat{y}_0 & \text{in } (0,L), \\ w_{|t=T} = 0 & \text{in } (0,L). \end{cases} \tag{52}$$

Finally, we define the operator $L_2 : L^1(0,T;L^2(0,L)) \to L^2(0,T;H_0^1(0,L)) \cap C^0([0,T];L^2(0,L))$ which associates to f the (unique) solution u of

$$\begin{cases} u_t + u_{xxx} + u_x = f & \text{in } (0,T) \times (0,L), \\ u_{|x=0} = u_{|x=L} = u_{x|x=L} = 0 & \text{in } (0,T), \\ u_{|t=0} = 0 & \text{in } (0,L). \end{cases} \tag{53}$$

Let us now define the following fixed-point mapping $\Lambda : B(0;R) \subset L^2(0,T;H^1(0,L)) \to L^2(0,T;H^1(0,L))$, where $R > 0$ is to be determined; it is defined as:

$$\Lambda(u) = L_2(-uu_x) + L[y_1 + L_2(uu_x)(T)] + L_1[y_0 - L_0\{y_1 + L_2(uu_x)(T)\}]. \tag{54}$$

Note that of course, $uu_x = (u^2/2)_x$ belongs to $L^1(0,T;L^2(0,L))$ when $u \in L^2(0,T;H^1(0,L))$, hence Λ is well-defined. Let us prove that it maps $B(0;R)$ into itself and that it is contractive.

- Λ *is contractive.* Let $u_1, u_2 \in B(0;R)$. Using the continuity of the various operators and the bilinear estimate on uu_x, we easily deduce that for some constant C independent of u_1 and u_2,

$$\begin{aligned} \|\Lambda(u_1) - \Lambda(u_2)\|_{L^2(0,T;H^1(0,L))} &\leq C\|u_1^2 - u_2^2\|_{L^1(0,T;H^1(0,L))} \\ &\leq 2RC\|u_1 - u_2\|_{L^2(0,T;H^1(0,L))}. \end{aligned} \tag{55}$$

  Hence Λ is contractive provided that we take R small enough, namely,

$$R < \frac{1}{4C}, \tag{56}$$

  where C is the constant in the last inequality of (55).

- Λ *maps* $B(0;R)$ *into itself.* Let $u \in B(0;R)$, and observe in the same way as previously that

$$\|\Lambda(u)\|_{L^2(0,T;H^1(0,L))} \leq C(\|y_0\|_{L^2(0,L)} + \|y_1\|_{L^2(0,L)} + R^2).$$

  Hence with the choice (56) and if $\|y_0\|_{L^2(0,L)}$ and $\|y_1\|_{L^2(0,L)}$ are small enough, the operator Λ maps $B(0;R)$ into itself.

In that case, the operator Λ admits a fixed point, by the Banach-Picard Theorem. It is an elementary matter to see that this fixed point is as required. This ends the proof of Theorem 2.

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