Acyclic coloring of graphs with maximum degree five

Hervé Hocquard and Mickaël Montassier
LaBRI, Université Bordeaux I, 33405 Talence Cedex, France

February 5, 2010

Abstract

An acyclic k-coloring of a graph G is a proper vertex coloring of G which uses at most k colors such that the graph induced by the union of every two color classes is a forest. In this paper, we mainly prove that every 5-connected graph with maximum degree five is acyclically 8-colorable, improving partially [5].

1 Introduction

A *proper vertex coloring* of a graph $G = (V, E)$ is an assignment of colors to the vertices of the graph such that two adjacent vertices do not use the same color. A proper vertex coloring of a graph G is *acyclic* if G contains no bicolored cycles; in other words, the graph induced by every two color classes is a forest. The *acyclic chromatic number* of G, denoted by $\chi_a(G)$, is the smallest integer k such that G is acyclically k-colorable. Acyclic colorings were introduced by Grünbaum [6] who proved that every planar graph is acyclically 9-colorable and conjectured that 5 colors are sufficient. Mitchem [9] improved this result to 8 colors, Albertson and Berman [2] to 7 colors and Kostochka [8] to 6 colors. Finally, in 1979, Borodin [3] proved that 5 colors are sufficient. This bound is best possible since there exist 4-regular planar graphs [6] which are not acyclically colorable with four colors. Concerning graphs with bounded maximum degree, Alon *et al.* [1] proved that asymptotically every graph with maximum degree Δ is acyclically colorable with $O(\Delta^{4/3})$ colors; moreover they exhibited graphs with maximum degree Δ with acyclic chromatic number at least $\Omega(\Delta^{4/3}/(\log \Delta)^{1/3}$. For small maximum degrees, it was proved that 4 colors are sufficient to acyclically color graph with maximum degree 3 (this bound is best possible because of K_4). In 1979, Burstein [4] proved that every graph with maximum degree 4 is acyclically 5-colorable (this bound is tight because of K_5). It was proved by Fertin and Raspaud [5] that every graph of maximum degree 5 can be acyclically colored with 9 colors. In this note we improve partially this result proving that:

Lemma 1 *Every graph G with maximum degree at most 5 and minimum degree strictly less than 5 is acyclically 8-colorable.*

Then, we will refine our approach to prove :

Theorem 1 *Every 5-connected graph with maximum degree five is acyclically 8-colorable.*

We now introduce some notations. The following terminology was introduced in [5]. A partial acyclic coloring of G is a coloring φ of a subset S of V such that φ is an acyclic coloring of $G[S]$ (the subgraph induced by S). A partial acyclic coloring using at most k colors is said to be a partial acyclic k-coloring of G. Let φ be a partial acyclic 8-coloring of G and let v be an uncolored vertex of G. We say that a color c for v allows us to extend φ if the partial coloring φ' defined by $\varphi'(u) = \varphi(u)$ for all colored vertex u and by $\varphi'(v) = c$ is a partial acyclic 8-coloring of G. For a vertex $u \in V\backslash S$, we denote the set of colored neighbors of u by $N_c(u) = N(u) \cap S$ (where $N(u)$ is the set of the neighbors of u) and $\#cn(u) = |N_c(u)|$. We denote by $SC(N_c(u))$ the set

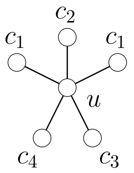

Figure 1: The list $L_u = (2,1,1,1)$

of colors used by vertices in $N_c(u)$ and $\#dcn(u) = |SC(N_c(u))|$. Given a vertex u and a color c, let $n_c(u)$ be the number of vertices in $N_c(u)$ colored with the color c. For each vertex u, we set $L_u = (n_1, n_2, \ldots, n_{\#dcn(u)})$ where each n_i denotes the number that a color appears in the neighborhood of u and $n_1 \geq n_2 \geq \ldots \geq n_{\#dcn(u)}$, and we call L_u the color list of u. For example in Figure 1 we have : all the neighbors of u are colored (thus $\#cn(u) = 5$ and $N_c(u) = N(u)$), $SC(N_c(u)) = \{c_1, c_2, c_3, c_4\}$, $\#dcn(u) = 4$ and $L_u = (2,1,1,1)$ (two neighbors colored c_1, one colored with c_2, c_3, c_4). Finally, we denote by $\Delta(G)$ and $\delta(G)$, the maximum and the minimum degree of the graph G, respectively. We use $[\![1;n]\!]$ to denote the set of integers $\{1, 2, \ldots, n\}$.

2 Proof of Lemma 1

In this section, we prove that, if G is connected with $\Delta(G) \leq 5$ and $\delta(G) < 5$, then G is acyclically 8-colorable. The proof is based on a greedy algorithm. We first define an order $\prec$ on the vertices of G, and then we color the vertices of G according to $\prec$. Let v be a vertex of degree $d(v) < 5$. Let T be a spanning tree of G rooted at v. The order $\prec$ is defined by a post order walk on T. Let $x_1, \ldots, x_n$ be the vertices of G such that for every $i,\ j,\ 1 \leq i < j \leq n$, $x_i \prec x_j$ and $x_n = v$. Observe that for all i with $1 \leq i \leq n$, x_i has at most four neighbors x_j with $j < i$. We will color the x_i's successively using Lemmas 2 and 3. The obtained coloring will be an acyclic 8-coloring of G.

We begin with an observation of [5]:

Observation 1 *[5] Let G be a graph with maximum degree 5 and let φ be a partial acyclic 8-coloring of G. Suppose that v is a uncolored vertex of G. If all colored neighbors of v have distinct colors, it suffices to proper color v to extend φ. If a color c appears $n_c(v) > 1$ times among the neighbors of v, then, to color v, we need to forbid at most $2n_c(v)$ colors to avoid the creation of possible bicolored cycles going through v and the vertices colored with c.*

Lemma 2 *Let G be a graph with $\Delta(G) \leq 5$ and let φ be a partial acyclic 8-coloring of G. Then, for any uncolored vertex u such that $\#cn(u) \leq 3$, there exists a color for u that allows us to extend φ.*

Proof
First, suppose that no color is repeated among the neighbors of u, thus $L_u = (1)$, $L_u = (1,1)$, or $L_u = (1,1,1)$. By Observation 1, u only needs to avoid the colors used by its neighbors, then it remains at least five colors to color u.

Now, suppose that a color appears (at least) twice among the neighbors of u. Then, since $\#cn(u) \leq 3$, we have exactly three cases : $L_u = (2)$, $L_u = (2,1)$, or $L_u = (3)$. When $L_u = (2)$, $L_u = (2,1)$ (resp. $L_u = (3)$), u needs to forbid four colors to avoid the creation of possible bicolored cycles by Observation 1 (resp. six colors) and at most two more colors to maintain the proper coloring (resp. one more color). In each case, at least one choice remains to color u. □

Lemma 3 *Let G be a graph with $\Delta(G) \leq 5$ and let φ be a partial acyclic 8-coloring of G. Then, for any uncolored vertex u such that $\#cn(u) = 4$, there exists a color for u that allows us to extend φ.*

Proof
Let G be a graph with $\Delta(G) \leq 5$, φ be a partial acyclic 8-coloring of G, and u be an uncolored vertex u such that $\#cn(u) = 4$. Let v_1, v_2, v_3, v_4 be its four colored neighbors, and for $1 \leq i \leq 4$ and $1 \leq j \leq 4$, let v_i^j be the four neighbors of v_i disctinct from u.

To extend φ to u, we will consider all possible L_u:

> **Case $\boldsymbol{L_u = (1,1,1,1)}$.** By Observation 1, it suffices to proper color u (we have four remaining colors).
>
> **Case $\boldsymbol{L_u = (2,1,1)}$.** By Observation 1, u needs to forbid four colors to avoid the possible creation of bicolored cycles and three more colors to maintain the proper coloring. Then, one choice remains to color u.
>
> **Case $\boldsymbol{L_u = (3,1)}$.** W.l.o.g. suppose that $\varphi(v_1) = \varphi(v_2) = \varphi(v_3) = 1$ and $\varphi(v_4) = 2$. If there exists a color $\alpha \in \llbracket 3;8 \rrbracket$ that it is not involved in a possible bicolored cycle with colors 1, α, then we color u with α. Otherwise, this implies that each color in $\llbracket 3;8 \rrbracket$ appears twice (and exactly twice) in the neighborhood of v_1, v_2, v_3, and for each v_i ($1 \leq i \leq 3$), we have $\#dcn(v_i) = 4$. It suffices now to recolor v_1 with a color different from 1, 2 and those in $SC(N_c(v_1))$ (we have two colors to recolor v_1). Hence, L_u becomes $(2,1,1)$, a case seen previously.
>
> **Case $\boldsymbol{L_u = (4)}$.** W.l.o.g. suppose that $\varphi(v_1) = \varphi(v_2) = \varphi(v_3) = \varphi(v_4) = 1$. Observe that if one of the v_i's has $\#dcn(v_i) = 4$, say v_1, then we can recolor v_1 with a color different from 1 and those in $SC(N_c(v_1))$. We obtain $L_u = (3,1)$, a case seen previously. So suppose that for $1 \leq i \leq 4$, $\#dcn(v_i) \leq 3$. Observe that, to avoid the possible creation of bicolored cycles, at most six colors may be forbidden for u. Hence it remains a choice to color u.
>
> **Case $\boldsymbol{L_u = (2,2)}$.** W.l.o.g. suppose that $\varphi(v_1) = \varphi(v_2) = 1$ and $\varphi(v_3) = \varphi(v_4) = 2$. Observe that if one of the v_i's has $\#dcn(v_i) = 4$, say v_j, then we can recolor v_j with a color different from 1, 2 and those in $SC(N_c(v_j))$. We obtain $L_u = (2,1,1)$, a case seen previously. So suppose that for $1 \leq i \leq 4$, $\#dcn(v_i) \leq 3$. Now, if we cannot color u with a color in $\llbracket 3;8 \rrbracket$, this implies that for $1 \leq i \leq 4$, $\#dcn(v_i) = 3$, and w.l.o.g. $SC(N_c(v_1)) = SC(N_c(v_2)) = \{3,4,5\}$, $SC(N_c(v_3)) = SC(N_c(v_4)) = \{6,7,8\}$. We focus on v_1 and its neighborhood. We will try to recolor v_1 with a color different from 1: if we succeed, then we will obtain a new L_u solved previously ; if not, then we will show that there exists a color for u that extends φ. If $\#cn(v_1) = 3$, then we recolor v_1 with a color different from $\llbracket 1;5 \rrbracket$ and we are done. So assume that $\#cn(v_1) = 4$ and w.l.o.g set $\varphi(v_1^1) = \varphi(v_1^2) = 3, \varphi(v_1^3) = 4$, and $\varphi(v_1^4) = 5$. We try to recolor v_1 with a color different from 3, 4, 5, and those of $SC(N_c(v_1^1)) \setminus \{1\}$. If there is a choice different from 1, we are done. If not, then we can color u with 3 (since 3 does not appear in $SC(N_c(v_3))$, $SC(N_c(v_4))$, and v_1^1 has a unique neighbor colored with 1 (v_1)). That completes the proof.

$\square$

3 Proof of Theorem 1

In this section we prove that every 5-connected graph with maximum degree five is acyclically 8-colorable. In fact we will prove that following stronger result. Let G be a 5-regular graph. A *good spanning tree* T^* of G is a spanning tree of G having a vertex with four leaves.

Theorem 2 *Let G be a 5-regular graph. If G admits a good spanning tree, then G is acyclically 8-colorable.*

Theorem 1 follows from Theorem 2. Let G be a 5-connected 5-regular graph and u be a vertex adjacent to v_1, v_2, v_3, v_4, v_5. A spanning tree of $G \setminus v_1, v_2, v_3, v_4$ plus the edges uv_1, uv_2, uv_3, uv_4 is a good spanning tree of G.

Proof of Theorem 2
Let G be a 5-regular graph admitting a good spanning tree T^*. We order the vertices of G from x_1 to x_n according to a post order walk of T^* rooted at r where $x_1,\ x_2,\ x_3,\ x_4$ are four leaves of r and $r = x_n$. First, we color $x_1,\ x_2,\ x_3,\ x_4$ with distinct colors and then we will successively color $x_5, x_6, \ldots, x_n$. To color x_i with $5 \leq i \leq n-1$, we use Lemmas 2 and 3 but without never recolor the vertices x_1, x_2, x_3, x_4. In Lemma 2, no recoloring is used. In the case $L_u = (3,1)$ of Lemma 3, v_1 cannot be x_1, x_2, x_3 or x_4 since v_1 has four colored neighbors and u is not x_n. Similarly in the case $L_u = (4)$ of Lemma 3, v_1 cannot be x_1, x_2, x_3 or x_4. In the case $L_u = (2,2)$ of Lemma 3, we focus on v_1. If v_1 is, say x_1, then we just focus on v_2 instead of v_1 (since v_1 and v_2 have the same color, we are sure that v_2 is not x_2, x_3, x_4). At this point, we have an acyclic coloring of $G \backslash \{x_n\}$ such that $x_1,\ x_2,\ x_3,\ x_4$ use four distinct colors. Finally, to color x_n we have two cases:

> **Case $\boldsymbol{L_{x_n} = (1,1,1,1,1)}$.** By Observation 1, it suffices to proper color x_n (we have three remaining colors).
>
> **Case $\boldsymbol{L_{x_n} = (2,1,1,1)}$.** W.l.o.g. $\varphi(x_1) = \varphi(x_{n-1}) = 1, \varphi(x_2) = 2, \varphi(x_3) = 3, \varphi(x_4) = 4$. We choose for x_n a color different from 1, 2, 3, 4 and those in $SC(N_c(x_1))$. If there is a choice, then we are done. Otherwise, we color x_n with 1 and recolor properly x_1 and x_{n-1}.

That completes the proof of Theorem 2. □

We conclude with the following question:

Question 1 *Is it true that every 5-regular graph admits a good spanning tree?*

References

[1] N. Alon, C. McDiarmid, B. Reed, Acyclic coloring of graphs, Random structures and algorithms, 2(3):277-288, 1991.

[2] M.O Albertson and D.M. Berman. Every planar graph has an acyclic 7-coloring, Israel J. Math.,(28):169-174, 1977.

[3] O.V. Borodin, On acyclic colorings of planar graphs, Discrete Math., (25):211-236, 1979.

[4] M.I. Burstein, Every 4-valent graph has an acyclic 5-coloring, Soobšč. Akad. Nauk Gruzin SSR, (93):21-24, 1979 (in Russian).

[5] Guillaume Fertin and André Raspaud, Acyclic coloring of graphs of maximum degree five : nine colors are enough, Inform. Process. Lett., (105):65-72, 2008.

[6] B. Grünbaum, Acyclic colorings of planar graphs, Israel J. Math., 14(3):390-408, 1973.

[7] A.V. Kostochka and L.S. Mel'nikov, Note to the paper of Grünbaum on acyclic colorings, Discrete Math., (14):403-406, 1976.

[8] A.V. Kostochka, Acyclic 6-colorings of planar graphs, Metody Diskret. Anal., (28):40-56, 1976 (in Russian).

[9] J. Mitchem, Every planar graph has an acyclic 8-coloring, Duke Math. J., (41):177-181, 1974.