Adaptive and Ultra-Wideband Sampling via Signal Segmentation and Projection

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Abstract:

Adaptive frequency band (AFB) and ultra-wide-band (UWB) systems require either rapidly changing or very high sampling rates. Conventional analog-to-digital devices have non-adaptive and limited dynamic range. We investigate AFB and UWB sampling via a basis projection method. The method decomposes the signal into a basis over time segments via a continuous-time inner product operation and then samples the basis coefficients in parallel. The signal may then be reconstructed from the basis coefficients to recover the signal in the time domain. We develop the procedure of this method, analyze various methods for signal segmentation and close by creating systems designed for binary signals.

1. Introduction

Adaptive frequency band (AFB) and ultra-wide-band (UWB) systems, requiring either rapidly changing or very high sampling rates, stress classical sampling approaches. At UWB rates, conventional analog-to-digital devices have limited dynamic range and exhibit undesired nonlinear effects such as timing jitter. Increased sampling speed leads to less accurate devices that have lower precision in numerical representation. This motivates alternative sampling schemes that use mixed-signal approaches, coupling analog processing with parallel sampling, to provide improved sampling accuracy and parallel data streams amenable to lower speed (parallel) digital computation.

We investigate AFB and UWB sampling via a basis projection method. The method was introduced as a means of UWB parallel sampling by Hoyos *et. al.* [7] and applied to UWB communications systems [8, 9, 10]. The method first decomposes the signal into a basis over time segments via a continuous-time inner product operation and then samples the basis coefficients in parallel. The signal may then be reconstructed from the basis coefficients to recover time domain samples, or further processing may be carried out in the new domain [7].

We address several issues associated with the basis-expansion and sampling procedure, including choice of basis, truncation error, rate of convergence and segmentation of the signal. We develop a mathematical model of the procedure, using both standard (sine, cosine) basis elements and general basis elements, and give this representation in both the time and frequency domains. We compute exact truncation error bounds, and compare the method with traditional sampling. We close by developing the method for binary signals.

2. Sampling via Projection

Let f be a signal of finite energy whose Fourier transform $\widehat{f}$ has compact support, i.e., $f, \widehat{f} \in L^2$, with $\mathrm{supp}(\widehat{f}) \subset [-\Omega, \Omega]$. The signal is in the Paley-Wiener class $PW(\Omega)$. For a block of time T_c, let

$$f(t) = \sum_{k \in \mathbb{Z}} f(t) \chi_{[(k)T_c, (k+1)T_c]}(t) \, .$$

For this original development, we keep T_c fixed. We later let T_c be adaptive and will denote the adaptive time segmentation as $\tau_c(t)$. A given block $f_k(t) = f(t)\chi_{[(k)T_c,(k+1)T_c]}(t)$ can be T_c- periodically continued, getting

$$(f_k)^{\circ}(t) = (f(t)\chi_{[(k)T_c,(k+1)T_c]}(t))^{\circ} \, .$$

Expanding $(f_k)^{\circ}(t)$ in a Fourier series, we get

$$(f_k)^{\circ}(t) = \sum_{n \in \mathbb{Z}} \widehat{(f_k)^{\circ}}[n] e^{(2\pi i n t / T_c)} \, , \text{ where}$$

$$\widehat{(f_k)^{\circ}}[n] = \frac{1}{T_c} \int_{(k)_{T_c}}^{(k+1)T_c} f(t) e^{(-2\pi i n t / T_c)} \, dt \, .$$

Given that the original function f is Ω band-limited, we can estimate the value of n for which $f_k[n]$ is non-zero. At minimum, $f_k[n]$ is non-zero if

$$\frac{n}{T_c} \leq \Omega \, , \text{or equivalently, } n \leq T_c \cdot \Omega \, .$$

Let

$$N = \lceil T_c \cdot \Omega \rceil \, .$$

(Note that the truncated block functions f_k are not band-limited. We discuss this in section 3.) For this choice of N, we compute

$$\begin{aligned} f(t) &= \sum_{k \in \mathbb{Z}} f(t) \chi_{[(k)T_c,(k+1)T_c]}(t) \\ &= \sum_{k \in \mathbb{Z}} \left[(f_k)^{\circ}(t) \right] \chi_{[(k)T_c,(k+1)T_c]}(t) \\ &\approx \sum_{k \in \mathbb{Z}} \left[\sum_{n=-N}^{n=N} \widehat{(f_k)^{\circ}}[n] e^{(2\pi i n t / T_c)} \right] \chi_{[(k)T_c,(k+1)T_c]}(t) \, . \end{aligned}$$

Given this choice of the standard (sines, cosines) basis, we can, for a fixed value of N, adjust to a large bandwidth Ω by choosing small time blocks T_c. Also, after a given set of time blocks, we can deal with a increase or decrease in bandwidth Ω by again adjusting the time blocks, e.g., given an increase in Ω, decrease the time blocks adaptively to $\tau_c(t)$, and vice versa. There is, of course, a price to be paid. The quality of the signal, as expressed in the accuracy the representation of f, depends on N, Ω and T_c.

Theorem : [The Projection Formula] Let $f, \widehat{f} \in L^2(\mathbb{R})$ and $f \in PW_\Omega$, i.e. $\text{supp}(\widehat{f}) \subset [-\Omega, \Omega]$. Let T_c be a fixed block of time. Then, for $N = \lceil T_c \cdot \Omega \rceil$, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k \in \mathbb{Z}} \left[\sum_{n=-N}^{N} f_k[n] e^{(2\pi int/T_c)} \right] \chi_{[kT_c,(k+1)T_c]}(t). \tag{1}$$

The Projection Method can adapt to changes in the signal. Suppose that the signal $f(t)$ has a band-limit $\Omega(t)$ which changes with time. For example, let f be a signal from a cell phone which changes from voice to a highly detailed musical piece. This change effects the time blocking $\tau_c(t)$ and the number of basis elements $N(t)$. This, of course, makes the analysis more complicated, but is at the heart of the advantage the Projection Method has over conventional methods.

During a given $\tau_c(t)$, let $\overline{\Omega}(t) = \sup\{\Omega(t) : t \in \tau_c(t)\}$. For a signal f that is $\Omega(t)$ band-limited, we can estimate the value of n for which $f_k[n]$ is non-zero. At minimum, $f_k[n]$ is non-zero if

$$\frac{n}{\tau_c(t)} \leq \overline{\Omega}(t) \text{ , or equivalently, } n \leq \tau_c(t) \cdot \overline{\Omega}(t) \, .$$

Let

$$N(t) = \lceil \tau_c(t) \cdot \overline{\Omega}(t) \rceil \, .$$

For this choice of $N(t)$, we have the following.

Theorem : [The Adaptive Projection Formula] Let f, $\widehat{f} \in L^2(\mathbb{R})$ and f have a variable but bounded band-limit $\Omega(t)$. Let $\tau_c(t)$ be an adaptive block of time and given $\tau_c(t)$, let $\overline{\Omega}(t) = \sup\{\Omega(t) : t \in \tau_c(t)\}$. Then, for $N(t) = \lceil \tau_c(t) \cdot \Omega(t) \rceil$, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k \in \mathbb{Z}} \left[\sum_{n=-N(t)}^{N(t)} f_k[n] e^{(2\pi int/\tau_c)} \right] \chi_{[k\tau_c,(k+1)\tau_c]}(t). \tag{2}$$

In comparison, Shannon Sampling examines the function at specific points, then uses those individual points to recreate the signal. The Projection Method breaks the signal into segments in the time domain and then approximates their respective periodic expansions with a Fourier series. This process allows the system to individually evaluate each piece and base its calculation on the needed bandwidth. The individual Fourier series are then summed, recreating a close approximation of the original signal. It is important to note that instead of fixing T_c, the method allows us to fix any of the three while allowing the other two to fluctuate. The easiest and most practical parameter from the design factor to fix is N. For situations in which the bandwidth does not need flexibility, it is possible to fix Ω and T_c by the equation $N = \lceil T_c \cdot \Omega \rceil$. However, if greater bandwidth Ω is need, choose shorter time blocks T_c.

The Projection Method adapts to general orthonormal systems, much as Kramer-Weiss extends sampling to general orthonormal bases. Given a function f such that $f \in PW_\Omega$, let T_c be a fixed time block. Define $f(t)$, $f_k(t)$ and $f_k{}^\circ(t)$ as in the beginning of the computation above. Now, let $\{\varphi_n\}$ be a general orthonormal system for $L^2[0, T_c]$. Then,

$$f_k{}^\circ(t) = \sum_{n=-\infty}^{\infty} f_k[n]\varphi_n(t), \text{ where } f_k[n] = \langle f_k{}^\circ, \varphi_n \rangle.$$

Since $f \in PW_\Omega$, there exists $N = N(T_c, \Omega)$ such that $f_k[n] = 0$ for all $n > N$. Therefore, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k=-\infty}^{\infty} \left[\sum_{n=-N}^{N} f_k[n]\varphi_n(t) \right] \chi_{[kT_c,(k+1)T_c]}(t). \tag{3}$$

Given characteristics of the input class signals, the choice of basis functions used in the the Projection Method can be tailored to optimal representation of the signal or a desired characteristic in the signal. We develop a Walsh system for binary signals in section 4.

We close this section with a different system of segmentation for the time domain. This was created because it is relatively easy to implement, cuts down on frequency error and has no loss of data in time. It was developed by studying the de la Vallée-Poussin kernel used in Fourier series. Let $0 < r < T_c/2$ and let

$$\text{Tri}_L(t) = \max\{[((T_c/(4r)) + r) - |t|/(2r)], 0\} \, ,$$

$$\text{Tri}_S(t) = \max\{[((T_c/(4r)) + r - 1) - |t|/(2r)], 0\}$$

and

$$\text{Trap}(t) = \text{Tri}_L(t) - \text{Tri}_S(t) \, .$$

The Trap function has perfect overlay in the time domain and $1/\omega^2$ decay in frequency space. When one time block is ramping down, the adjacent block is ramping up at exactly the same rate. This leads to the Projection formula

$$\sum_{k \in \mathbb{Z}} \left[\sum_{n=-N}^{N} ((f \cdot \text{Trap})_k[n] e^{(2\pi int/(T_c + r))} \right] \text{Trap}(t - k(T_c/2)) \, .$$

3. Error Analysis

To compute truncation error, we first calculate the Fourier transform of both sides of the equation. Let $f \in PW(\Omega)$, so $f \in L^2$ and Ω band-limited. For $N = \lceil T_c \cdot \Omega \rceil$,

$$f_{\mathcal{P}}(t) = \sum_{k \in \mathbb{Z}} \left[\sum_{n=-N}^{N} f_k[n] e^{(2\pi int/T_c)} \right] \chi_{[kT_c,(k+1)T_c]}(t)$$

Taking the transform of both sides and evoking the relationship between the transform and convolution gives

$$\widehat{f_{\mathcal{P}}}(\omega) = \sum_{k\in\mathbb{Z}}\Bigg[\sum_{n=-N}^{N}\Big[f_k[n]\left(e^{(2\pi int/T_c)}\right)\widehat{(\omega)}\Big] * \Big[\chi_{[kT_c,(k+1)T_c]}(t)\Big]\widehat{(\omega)}\Bigg]$$

Performing the indicated transforms using the definition results in

$$\widehat{f_{\mathcal{P}}}(\omega) = \sum_{k\in\mathbb{Z}}\Bigg[\sum_{n=-N}^{N} f_k[n]\left(\delta(\omega - \frac{n}{T_c})\right) * e^{(2\pi i(k-\frac{1}{2})T_c\omega)}\frac{\sin(\pi T_c\omega)}{\pi\omega}\Bigg]$$

It is important to note that $f \cdot \chi_{[kT_c,(k+1)T_c]}$ is no longer band-limited, but it does decay at a rate less than or equal to $\frac{1}{\omega}$ in frequency. Using the relationship between translation and modulation, we get the following.

Theorem : [The Fourier Transform of the Projection Formula] Let f, $\widehat{f} \in L^2(\mathbb{R})$ and $f \in PW_\Omega$, i.e. $\mathrm{supp}(\widehat{f}) \subset [-\Omega, \Omega]$. Let T_c be a fixed block of time. Then, for $N = \lceil T_c \cdot \Omega \rceil$,

$$\widehat{f_{\mathcal{P}}}(\omega) = \sum_{k=-\infty}^{\infty}\Bigg[\sum_{n=-N}^{N} f_k[n] e^{(2\pi i(k-\frac{1}{2})T_c(\omega-\frac{n}{T_c})}\left(\frac{\sin(\pi(\frac{\omega T_c}{2}-\frac{n}{2}))}{\pi(\omega-\frac{n}{T_c})}\right)\Bigg] \quad (4)$$

The system using overlapping Trap functions has the advantage of $1/\omega^2$ decay in frequency. Let $\beta_L = \sqrt{T_c/(4r)+r}$, $\alpha_L = T_c/(4r) + r/2$, $\beta_S = \sqrt{T_c/(4r)+r-1}$, $\alpha_S = T_c/(4r) - r/2$. The Fourier transform of Trap is

$$\left[(\beta_L)\frac{\sin(2\pi\alpha_L(\omega))}{\pi(\omega)}\right]^2 - \left[(\beta_S)\frac{\sin(2\pi\alpha_S(\omega))}{\pi(\omega)}\right]^2.$$

This replaces the sinc term in the equation above. The Fourier coefficients are also different, and are computed in the same method as the de la Vallée-Poussin kernel used in Fourier series.

In the formula for the Projection Method, there is a reliance on a number N, representative of the number of Fourier series components. In order to ensure maximum utility from the formula, the difference between the infinitely summed series and the truncated must be made a minimum. To do this, the mean square error must be calculated. We compute this as a truncation error on the number of Fourier coefficients used to represent a given block f_k. For a fixed N, the mean square error is

$$e_N^2 = \|f_k - f_{k,N}\|_2^2 = \|\widehat{f_k} - \widehat{f_{k,N}}\|_2^2.$$

Computing and then simplifying gives

$$e_N^2 = \frac{1}{T_c}\int_{kT_C}^{(k+1)T_c} |f_k{}^\circ(t) - \sum_{|n|\le N} f_k[n]e^{(2\pi itn/T_c)}|^2 dt$$

$$= \frac{1}{T_c}\int_{kT_C}^{(k+1)T_c} |\sum_{|n|>N} f_k[n]e^{(2\pi itn/T_c)}|^2 dt.$$

Applying the triangle inequality to the right side and then exploiting the fact that $e^{(2\pi itn/T_c)}$ is an orthonormal system, thus $|e^{(2\pi itn/T_c)}| = 1$, we arrive at the following:

$$e_N^2 = \frac{1}{T_c}\int_{kT_C}^{(k+1)T_c} |\sum_{|n|>N} f_k[n]e^{(2\pi itn/T_c)}|^2 dt \quad (5)$$

$$\le \sum_{|n|>N}|f_k[n]|^2 \cdot \frac{1}{T_c}\int_{kT_c}^{(k+1)T_c} 1^2 dt = \sum_{|n|>N}|f_k[n]|^2$$

This demonstrates that the value of N has to be chosen carefully. This truncation error perpetuates over all the blocks.

The Projection Method experiences error due to truncation in two separate categories: time and frequency. The error in frequency is a function of the errors on each block due to the choice of N. By duality, this gives us errors in time. We can also get an error in time by loss of a given block or blocks of information. This is easier to compute. Given any lost or partially transmitted block $f_{k,L}$, error is simply

$$\|f_k - f_{k,L}\|_2 .$$

Error over the entire signal is computed by simply adding up the blocks. Cell phone users are used to lost information blocks, which gives rise to the following frequently used phrase – “Can you hear me now?”

4. Binary Signals

The Walsh functions $\{\omega_n\}$ form an orthonormal basis for $L^2[0,1]$. The basis functions have the range $\{1,-1\}$, with values determined by a dyadic decomposition of the interval. The Walsh functions are of modulus 1 everywhere. The functions are give by the rows of the unnormalized Hadamard matrices, which are generated recursively by

$$H(2) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H(2^{(k+1)}) = H(2) \otimes H(2^k) = \begin{bmatrix} H(2^k) & H(2^k) \\ H(2^k) & -H(2^k) \end{bmatrix}.$$

We point out that although the rows of the Hadamard matrices give the Walsh functions, the elements have to be reordered into *sequency* order. Walsh arranged the components in ascending order of zero crossings (see [1]). The Walsh functions can also be interpreted as the characters of the group G of sequences over $\mathbb{Z}_2$, i.e., $G = (\mathbb{Z}_2)^{\mathbb{N}}$. The Walsh basis is a well-developed system for the study of a wide variety of signals, including binary. The Projection Method works with the Walsh system to create a wavelet-like system to do signal analysis.

First assume that the time domain is covered by a uniform block tiling $\chi_{[kT_c,(k+1)T_c]}(t)$. Translate and scale the function on this kth interval back to $[0,1]$ by a linear mapping. Denote the resultant mapping as f_k, which is an element of $L^2[0,1]$. Given that $f \in PW(\Omega)$, there exists an $N > 0$ ($N = N(\Omega)$) such that $\langle f_k, \omega_n\rangle = 0$ for all $n > N$. The decomposition of f_k into Walsh basis elements is

$$\sum_{n=0}^{N} \langle f_k, \omega_n\rangle\, \omega_n .$$

Translating and summing up gives the Projection representation $f_{\mathcal{P}}$

$$f_{\mathcal{P}}(t) = \sum_{k\in\mathbb{Z}} \left[\sum_{n=0}^{N} \langle f_k, \omega_n \rangle \, \omega_n \right] \chi_{[kT_c,(k+1)T_c]}(t). \quad (6)$$

Next assume that the time domain is covered by a uniform overlapping trapezoidal tiling $\text{Trap}(t - k(T_c/2))$. Note that the construction of the trapezoidal system results in the loss of no signal data, for just as a given block is ramping down, the subsequent block is ramping up at exactly the same rate. Again translate and scale the function on this kth interval back to $[0, 1]$ by a linear mapping. Denote the resultant mapping as f_{k_T}. The resultant function is an element of $L^2[0, 1]$. Given that $f \in PW(\Omega)$, there exists an $M > 0$ ($M = M(\Omega)$) such that $\langle f_{k_T}, \omega_n \rangle = 0$ for all $n > M$. The decomposition of f_{k_T} into Walsh basis elements is

$$\sum_{n=0}^{M} \langle f_k, \omega_n \rangle \, \omega_n \, .$$

Translating and summing up gives the Projection representation $f_{\mathcal{P}_T}$

$$f_{\mathcal{P}_T}(t) = \sum_{k\in\mathbb{Z}} \left[\sum_{n=0}^{N} \langle f_{k_T}, \omega_n \rangle \, \omega_n \right] \text{Trap}(t - k(T_c/2)). \quad (7)$$

5. Conclusions

The Projection Method gives a method for analog-to-digital encoding which is an alternative to Shannon Sampling. Projection gives a procedure for the sampling of a signal of variable or ultra-wide bandwidth Ω by varying the time blocks T_c. If f is Ω band-limited, we can estimate the value of n for which the Fourier coefficients $f_k[n]$ of a given time block are non-zero. At minimum, $f_k[n]$ is non-zero if $\frac{n}{T_c} \leq \Omega$, or equivalently, $n \leq T_c \cdot \Omega$. If $N = \lceil T_c \cdot \Omega \rceil$, then, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k\in\mathbb{Z}} \left[\sum_{n=-N}^{N} f_k[n] e^{(2\pi int/T_c)} \right] \chi_{[kT_c,(k+1)T_c]}(t).$$

For fixed N, if greater bandwidth Ω is need, choose shorter time blocks T_c. The price paid for this flexibility is in signal error, which has been computed above. The Projection Method can also adapt to changes in the signal, e.g., $f(t)$ has a band-limit $\Omega(t)$ which changes with time. This change effects the time blocking $\tau_c(t)$ and the number of basis elements $N(t)$. During a given $\tau_c(t)$, let $\overline{\Omega}(t) = \sup \{\Omega(t) : t \in \tau_c(t)\}$. For a signal f that is $\Omega(t)$ band-limited, we can estimate the value of n for which $f_k[n]$ is non-zero. At minimum, $f_k[n]$ is non-zero if

$$\frac{n}{\tau_c(t)} \leq \overline{\Omega}(t) \text{ , or equivalently, } n \leq \tau_c(t) \cdot \overline{\Omega}(t) \, .$$

We let

$$N(t) = \lceil \tau_c(t) \cdot \overline{\Omega}(t) \rceil \, ,$$

and have

$$f_{\mathcal{P}}(t) = \sum_{k\in\mathbb{Z}} \left[\sum_{n=-N(t)}^{N(t)} f_k[n] e^{(2\pi int/\tau_c)} \right] \chi_{[k\tau_c,(k+1)\tau_c]}(t).$$

This adaptable time segmentation makes the analysis more complicated, but is at the heart of the advantage the Projection Method has over conventional methods. Subsequent work on this method will focus on minimizing error, creating systems based on the Projection Method tailored to different types of signals and optimizing signal reconstruction in a noisy environment.

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