

ON NONUNIFORM EXPONENTIAL STABILITY FOR SKEW-EVOLUTION SEMIFLOWS ON BANACH SPACES

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ABSTRACT. The paper considers some concepts of nonuniform asymptotic stability for skew-evolution semiflows on Banach spaces. The obtained results clarify differences between the uniform and nonuniform cases. Some examples are included to illustrate the results.

1. INTRODUCTION

The exponential stability plays a central role in the theory of asymptotic behaviors for dynamical systems. In this paper we consider the more general concepts of nonuniform exponential stability for skew-evolution semiflows on Banach spaces. These seem to be more appropriate for the study of evolution equations in the nonuniform case, because of the fact that they depend on three variables, contrary to a skew-product semiflow or an evolution operator, which depend only on two, and, hence, the study of asymptotic behaviors for skew-evolution semiflows in the nonuniform setting arises as natural, relative to the third variable.

Our main objectives are to establish relations between these concepts and to give some integral characterizations for them. We also remark that we use the concept of nonuniform exponential stability, given and studied in the papers of L. Barreira and C. Valls, as for example [1], [2] or [3], and which we call "*Barreira-Valls exponential stability*".

The paper presents some generalizations for the results obtained in the uniform case in our paper [4].

We remark that Theorems 4.1 and 4.3 are generalizations of Datko type for the nonuniform exponential stability in the case of skew-evolution semiflows. The uniform exponential stability was characterized by R. Datko in [5]. The particular case of evolution operators was considered by C. Buşe in [6] and by M. Megan, A.L. Sasu and B. Sasu in [7]. Theorem 4.2 is the nonuniform variant for skew-evolution semiflows of the known result of S. Rolewicz in [8]. Theorem 5.1 is a generalization of a result proved by E.A. Barbashin in [9]. A similar result was obtained Buşe, M. Megan, M. Prajea and P. Preda for the uniform exponential stability in [10].

Some illustrating examples clarify the connections between the stability concepts considered in this paper.

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2. SKEW-EVOLUTION SEMIFLOWS

Let (X, d) be a metric space, V a Banach space and V^* its topological dual. Let $\mathcal{B}(V)$ be the space of all V -valued bounded operators defined on V . The norm of vectors on V and on V^* and of operators on $\mathcal{B}(V)$ is denoted by $\|\cdot\|$. Let us consider $Y = X \times V$ and $T = \{(t, t_0) \in \mathbb{R}_+^2 : t \geq t_0\}$. I is the identity operator.

Definition 2.1. A mapping $\varphi : T \times X \rightarrow X$ is called evolution semiflow on X if the following propositionerties are satisfied:

- (es₁) $\varphi(t, t, x) = x, \forall (t, x) \in \mathbb{R}_+ \times X;$
- (es₂) $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x), \forall (t, s), (s, t_0) \in T, \forall x \in X.$

Definition 2.2. A mapping $\Phi : T \times X \rightarrow \mathcal{B}(V)$ is called evolution cocycle over an evolution semiflow φ if it satisfies following propositionerties:

- (ec₁) $\Phi(t, t, x) = I, \forall t \geq 0, \forall x \in X;$
- (ec₂) $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \forall (t, s), (s, t_0) \in T, \forall x \in X.$

If Φ is an evolution cocycle over an evolution semiflow φ , then the mapping

$$(2.1) \quad C : T \times Y \rightarrow Y, C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$$

is called *skew-evolution semiflow* on Y .

Remark 2.1. The concept of skew-evolution semiflow generalizes the notion of skew-product semiflow, considered and studied by M. Megan, A.L. Sasu and B. Sasu in [11] and [12], where the mappings φ and Φ do not depend on the variables t and x .

Example 2.1. Let $X = \mathbb{R}_+$. The mapping $\varphi : T \times \mathbb{R}_+ \rightarrow \mathbb{R}_+, \varphi(t, s, x) = t - s + x$ is an evolution semiflow on \mathbb{R}_+ . For every evolution operator $E : T \rightarrow \mathcal{B}(V)$ (i.e. $E(t, t) = I, \forall t \in \mathbb{R}_+$ and $E(t, s)E(s, t_0) = E(t, t_0), \forall (t, s), (s, t_0) \in T$) we obtain that $\Phi_E : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(V), \Phi_E(t, s, x) = E(t - s + x, x)$ is an evolution cocycle on V over the evolution semiflow φ . Hence, an evolution operator on V is generating a skew-evolution semiflow on Y .

Example 2.2. If $C = (\varphi, \Phi)$ denotes a skew-evolution semiflow and $\alpha \in \mathbb{R}$ a parameter, then $C_\alpha = (\varphi, \Phi_\alpha)$, where

$$(2.2) \quad \Phi_\alpha : T \times X \rightarrow \mathcal{B}(V), \Phi_\alpha(t, t_0, x) = e^{\alpha(t-t_0)}\Phi(t, t_0, x),$$

is also a skew-evolution semiflow, being the α -shifted skew-evolution semiflow.

Other examples of skew-evolution semiflows are given in [4].

3. NONUNIFORM EXPONENTIAL STABILITY

In this section we define five concepts of exponential stability for skew-evolution semiflows. For each, an equivalent definition is given. Also, we will establish some connections between these concepts and we will emphasize that they are not equivalent.

We will begin by considering the notion of uniform exponential stability for skew-evolution semiflows, as given in [4] and which was characterized for evolution operators in [13].

Definition 3.1. A skew-evolution semiflow $C = (\varphi, \Phi)$ is uniformly exponentially stable (u.e.s.) if there exist some constants $N \geq 1$ and $\alpha > 0$ such that, for all $(t, s), (s, t_0) \in T$, following relation holds:

$$(3.1) \quad \|\Phi(t, t_0, x)v\| \leq Ne^{-(t-s)\alpha} \|\Phi(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

An equivalent definition is given by

Remark 3.1. The skew-evolution semiflow $C = (\varphi, \Phi)$ is uniformly exponentially stable iff there exist $N \geq 1$ and $\alpha > 0$ such that, for all $(t, s) \in T$, the relation holds:

$$(3.2) \quad \|\Phi(t, s, x)v\| \leq Ne^{-(t-s)\alpha} \|v\|, \quad \forall (x, v) \in Y.$$

The nonuniform exponential stability is defined by

Definition 3.2. A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially stable (e.s.) if there exist a mapping $N : \mathbb{R}_+ \rightarrow [1, \infty)$ and a constant $\alpha > 0$ such that, for all $(t, s) \in T$, following relation takes place:

$$(3.3) \quad \|\Phi(t, t_0, x)v\| \leq N(s)e^{-\alpha t} \|\Phi(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

Instead of the previous definition we have

Remark 3.2. The skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially stable iff there exist $N \geq 1$ and $\alpha > 0$ such that, for all $(t, s) \in T$, we have:

$$(3.4) \quad \|\Phi(t, s, x)v\| \leq N(s)e^{-\alpha t} \|v\|, \quad \forall (x, v) \in Y.$$

A concept of nonuniform exponential stability for evolution equations is given by L. Barreira and C. Valls in [1], which we will generalize for skew-evolution semiflows. In what follows, allow us to name this asymptotic propositionerty "Barreira-Valls exponential stability".

Definition 3.3. A skew-evolution semiflow $C = (\varphi, \Phi)$ is Barreira-Valls exponentially stable (BV.e.s.) if there exist some constants $N \geq 1$, $\alpha > 0$ and $\beta \geq \alpha$ such that, for all $(t, s), (s, t_0) \in T$, the relation holds:

$$(3.5) \quad \|\Phi(t, t_0, x)v\| \leq Ne^{-\alpha t} e^{\beta s} \|\Phi(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

We also have, as an equivalent definition, the next

Remark 3.3. A skew-evolution semiflow $C = (\varphi, \Phi)$ is Barreira-Valls exponentially stable iff there some constants $N \geq 1$, $\alpha > 0$ and $\beta \geq \alpha$ such that, for all $(t, s) \in T$, following relation is verified:

$$(3.6) \quad \|\Phi(t, s, x)v\| \leq Ne^{-\alpha t} e^{\beta s} \|v\|, \quad \forall (x, v) \in Y.$$

The asymptotic propositionerty of nonuniform stability is considered in

Definition 3.4. A skew-evolution semiflow $C = (\varphi, \Phi)$ is stable (s.) if there exists a mapping $N : \mathbb{R}_+ \rightarrow [1, \infty)$ such that, for all $(t, s), (s, t_0) \in T$, the relation is true:

$$(3.7) \quad \|\Phi(t, t_0, x)v\| \leq N(s) \|\Phi(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

We also have

Remark 3.4. The skew-evolution semiflow $C = (\varphi, \Phi)$ is stable iff there exists a mapping $N : \mathbb{R}_+ \rightarrow [1, \infty)$ such that, for all $(t, s) \in T$, the relation is verified:

$$(3.8) \quad \|\Phi(t, s, x)v\| \leq N(s) \|v\|, \quad \forall (x, v) \in Y.$$

Let us remind the propositionerty of exponential growth for skew-evolution semiflows, given by

Definition 3.5. A skew-evolution semiflow $C = (\varphi, \Phi)$ has exponential growth (e.g.) if there exist two nondecreasing mappings $M, \omega : \mathbb{R}_+ \rightarrow [1, \infty)$ such that, for all $(t, s), (s, t_0) \in T$, we have:

$$(3.9) \quad \|\Phi(t, t_0, x)v\| \leq M(s)e^{\omega(t-s)} \|\Phi(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

Similarly, we have

Remark 3.5. The skew-evolution semiflow $C = (\varphi, \Phi)$ has exponential growth iff there exist two nondecreasing mappings $M, \omega : \mathbb{R}_+ \rightarrow [1, \infty)$ such that, for all $(t, s) \in T$, the relation holds:

$$(3.10) \quad \|\Phi(t, s, x)v\| \leq M(s)e^{\omega(t-s)} \|v\|, \quad \forall (x, v) \in Y.$$

We obtain following relations concerning the previously defined asymptotic propositionerties for skew-evolution semiflows.

Remark 3.6. From the previous definitions it follows that:

$$(3.11) \quad (u.e.s.) \implies (BV.e.s.) \implies (e.s.) \implies (s.) \implies (e.g.)$$

The reciprocal statements are not true, as shown in what follows.

The next example emphasizes a skew-evolution semiflow which is Barreira-Valls exponentially stable but is not uniformly exponentially stable.

Example 3.1. Let $X = \mathbb{R}_+$ and $V = \mathbb{R}$. The mapping $\varphi : T \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, where $\varphi(t, s, x) = t - s + x$ is an evolution semiflow on \mathbb{R}_+ .

We will consider the function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, given by $u(t) = e^{2t-t \sin t}$. We define

$$\Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v, \quad \text{with } (t, s) \in T, (x, v) \in Y.$$

As we have

$$|\Phi_u(t, s, x)v| \leq |v| \cdot e^{t \sin t - s \sin s + 2s - 2t} \leq |v|e^{3s-2t} = e^{-2t}e^{3t}|v|,$$

for all $(t, s, x, v) \in T \times Y$. It follows that $C_u = (\varphi, \Phi_u)$ is Barreira-Valls exponentially stable.

Let us suppose now that the skew-evolution semiflow $C_u = (\varphi, \Phi_u)$ is uniformly exponentially stable. According to Definition 3.1, there exist $N \geq 1$, $\alpha > 0$ and $t_1 > 0$ such that

$$e^{t \sin t - s \sin s + 2s - 2t} \leq Ne^{\alpha(s-t)}, \quad \forall t \geq s \geq t_1.$$

If we consider $t = 2n\pi + \frac{\pi}{2}$ and $s = 2n\pi$, we have that

$$\exp\left(2n\pi - \frac{3\pi}{2}\right) \leq N \exp\left(-\frac{\pi}{2}\right),$$

which, for $n \rightarrow \infty$, leads to a contradiction, which proves that C_u is not uniformly exponentially stable.

The following example presents a skew-evolution semiflow which is exponentially stable but not Barreira-Valls exponentially stable.

Example 3.2. Let $X = \mathbb{R}_+$. The mapping $\varphi : T \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\varphi(t, s, x) = x$ is an evolution semiflow on \mathbb{R}_+ .

Let us consider a continuous function $u : \mathbb{R}_+ \rightarrow [1, \infty)$ with

$$u(n) = n \cdot 2^{2n} \text{ and } u\left(n + \frac{1}{2^{2n}}\right) = 1.$$

We define

$$\Phi_u(t, s, x)v = \frac{u(s)e^s}{u(t)e^t}v, \text{ where } (t, s) \in T, (x, v) \in Y.$$

As following relation

$$\|\Phi_u(t, s, x)v\| \leq u(s)e^s e^{-t} \|v\|$$

holds for all $(t, s, x, v) \in T \times Y$, it results that the skew-evolution semiflow $C_u = (\varphi, \Phi_u)$ is exponentially stable.

Let us now suppose that the skew-evolution semiflow $C_u = (\varphi, \Phi_u)$ is Barreira-Valls exponentially stable. Then, according to Definition 3.3, there exist $N \geq 1$, $\alpha > 0$, $\beta > 0$ and $t_1 > 0$ such that

$$\frac{u(s)e^s}{u(t)e^t} \leq N e^{-\alpha t} e^{\beta s}, \quad \forall t \geq s \geq t_1.$$

For $t = n + \frac{1}{2^{2n}}$ and $s = n$ it follows that

$$e^{n(2^{2n}+1)} \leq N e^{n+\frac{1}{2^{2n}}} e^{-\alpha\left(n+\frac{1}{2^{2n}}\right)} e^{\beta n},$$

which is equivalent with

$$e^{n(2^{2n}-\beta)} \leq N e^{\frac{1}{2^{2n}}-\alpha\left(n+\frac{1}{2^{2n}}\right)}.$$

For $n \rightarrow \infty$, a contradiction is obtained, which proves that C_u is not Barreira-Valls exponentially stable.

There exist skew-evolution semiflows that are stable but not exponentially stable, as results from the following

Example 3.3. Let us consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and

$$u : \mathbb{R}_+ \rightarrow [1, \infty) \text{ with the property } \lim_{t \rightarrow \infty} \frac{u(t)}{e^t} = 0.$$

The mapping

$$\Phi_u : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}), \quad \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v$$

is an evolution cocycle. As $|\Phi(t, s, x)v| \leq u(s)|v|$, $\forall (t, s, x, v) \in T \times Y$, it follows that $C_u = (\varphi, \Phi_u)$ is a stable skew-evolution semiflow, for every evolution semiflow φ on \mathbb{R}_+ .

On the other hand, if we suppose that C_u is exponentially stable, according to Definition 3.2, there exist a mapping $N : \mathbb{R}_+ \rightarrow [1, \infty)$ and a constant $\alpha > 0$ such that, for all $(t, s), (s, t_0) \in T$, we have

$$\|\Phi(t, t_0, x)v\| \leq N(s)e^{-\alpha t} \|\Phi(s, t_0, x)v\|, \quad \forall (x, v) \in Y.$$

It follows that

$$\frac{u(s)}{N(s)} \leq \frac{u(t)}{e^{\alpha t}}.$$

For $t \rightarrow \infty$ we obtain a contradiction, and, hence, C_u is not exponentially stable.

Following example gives a skew-evolution semiflow that has exponential growth but is not stable.

Example 3.4. We consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and

$$u : \mathbb{R}_+ \rightarrow [1, \infty) \text{ with the property } \lim_{t \rightarrow \infty} \frac{e^t}{u(t)} = \infty.$$

The mapping

$$\Phi_u : T \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}), \quad \Phi_u(t, s, x)v = \frac{u(s)e^t}{u(t)e^s}v$$

is an evolution cocycle. We have $|\Phi(t, s, x)v| \leq u(s)e^{t-s}|v|$, $\forall (t, s, x, v) \in T \times Y$. Hence, $C_u = (\varphi, \Phi_u)$ is a skew-evolution semiflow, over every evolution semiflow φ , and has exponential growth.

Let us suppose that C_u is stable. According to Definition 3.4, there exists a mapping $N : \mathbb{R}_+ \rightarrow [1, \infty)$ such that $u(s)e^t \leq N(s)u(t)e^s$, for all $(t, s) \in T$. If $t \rightarrow \infty$, a contradiction is obtained. Hence, C_u is not stable.

4. DATKO TYPE THEOREMS FOR THE NONUNIFORM EXPONENTIAL STABILITY

A different type of stability for skew-evolution semiflows in the nonuniform setting is presented in this section, as well a particular class of skew-evolution semiflows, which allows connections between various stability types.

Definition 4.1. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called integrally stable (i.s.) if there exists a mapping $D : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that:

$$(4.1) \quad \int_s^\infty \|\Phi(t, t_0, x)v\| dt \leq D(s) \|\Phi(s, t_0, x)v\|,$$

for all $(s, t_0) \in T$ and all $(x, v) \in Y$.

An equivalent definition can be considered the next

Remark 4.1. A skew-evolution semiflow $C = (\varphi, \Phi)$ is integrally stable iff there exists a mapping $D : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that:

$$(4.2) \quad \int_s^\infty \|\Phi(t, s, x)v\| dt \leq D(s) \|v\|,$$

for all $s \in \mathbb{R}_+$ and all $(x, v) \in Y$.

Definition 4.2. A skew-evolution semiflow $C = (\varphi, \Phi)$ has bounded exponential growth if C has exponential growth and function M from Definition 3.5 is bounded.

Proposition 4.1. An integrally stable skew-evolution semiflow $C = (\varphi, \Phi)$ with bounded exponential growth is stable.

Proof. Let us denote $M = \sup_{t \geq 0} M(t)$ and $c = \int_0^1 e^{-\omega(t)}$, where functions M and ω are given by Definition 3.5.

We observe that for $t \geq s + 1$ we have

$$c \leq \int_0^{t-s} e^{-\omega(r)} dr = \int_s^t e^{-\omega(t-\tau)} d\tau$$

and, further,

$$\begin{aligned} |c| \langle v^*, \Phi(t, s, x)v \rangle &\leq \int_s^t e^{-\omega(t-\tau)} \|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\| \|\Phi(\tau, s, x)v\| d\tau \leq \\ &\leq M \int_s^t \|\Phi(\tau, s, x)v\| d\tau \leq MD(s) \|v\| \|v\|^*, \end{aligned}$$

for all $(t, t_0) \in T$, all $(x, v) \in Y$ and all $v^* \in V^*$, function D being given by Remark 4.1.

By taking supremum relative to $\|v\|^* \leq 1$, we obtain

$$\|\Phi(t, s, x)v\| \leq \frac{MD(s)}{c}, \quad \forall t \geq s + 1, (x, v) \in Y.$$

Finally, it follows that

$$\|\Phi(t, s, x)v\| \leq N(s) \|v\|, \quad \forall (t, s) \in T, \forall (x, v) \in Y,$$

where we have denoted

$$N(s) = M \left[\frac{D(s)}{c} + e^{\omega(s)} \right],$$

and which proves that C is stable. \square

Definition 4.3. A skew-evolution semiflow $C = (\varphi, \Phi)$ is said to be exponentially integrally stable (e.i.s.) if there exist a mapping $D : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ and a constant $d > 0$ such that following relation:

$$(4.3) \quad \int_s^\infty e^{(t-s)d} \|\Phi(t, s, x)v\| dt \leq D(s) \|\Phi(s, t_0, x)v\|,$$

holds for all $(s, t_0) \in T$ and all $(x, v) \in Y$.

We also have

Remark 4.2. A skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially integrally stable iff there exist a mapping $D : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ and a constant $d > 0$ such that:

$$(4.4) \quad \int_s^\infty e^{(t-s)d} \|\Phi(t, s, x)v\| dt \leq D(s) \|v\|,$$

for all $(t, s) \in T$ and all $(x, v) \in Y$.

Remark 4.3. As a connection between the presented asymptotic propositionerties, we have:

$$(4.5) \quad (e.i.s.) \implies (i.s.)$$

In what follows, we will emphasize some characterizations of the various types of nonuniform stability considered in Section 3. We will begin this section by considering a particular class of skew-evolution semiflows, given in

Definition 4.4. A skew-evolution semiflow $C = (\varphi, \Phi)$ is said to be strongly measurable (s.m.) if for all $(t_0, x, v) \in \mathbb{R}_+ \times Y$ the mapping $s \mapsto \|\Phi(s, t_0, x)v\|$ is measurable on $[t_0, \infty)$.

Theorem 4.1. A strongly measurable skew-evolution semiflow $C = (\varphi, \Phi)$ with bounded exponential growth is exponentially stable if and only if it is exponentially integrally stable.

Proof. Necessity. It is a simple verification for

$$d = \frac{\alpha}{2} \text{ and } D(t) = \frac{N(t)}{\alpha}, \quad t \geq 0.$$

Sufficiency. If $C = (\varphi, \Phi)$ is exponentially integrally stable, then there exists a constant $d > 0$ such that the d -shifted skew-evolution semiflow C_d , given as in Example 2.2, is integrally stable with bounded exponential growth.

According to Proposition 4.1, it follows that C_d is stable, which assures the existence of a mapping $N : \mathbb{R}_+ \rightarrow [1, \infty)$ with

$$\|\Phi(t, s, x)v\| \leq N(s)e^{-(t-s)d} \|v\|, \quad \forall (t, s) \in T, \quad \forall (x, v) \in Y,$$

which proves that C is exponentially stable. \square

Remark 4.4. Theorem 4.1 can be viewed as a Datko type theorem for the propositionerty of nonuniform exponential stability for skew-evolution semiflows. The case of uniform stability was considered in [4]. For the particular case of evolution operators, this result was proved by R. Datko in [5] in the uniform setting and by C. Buşe in [6] for the nonuniform case.

Let us denote by \mathcal{F} the set of all nondecreasing functions $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with the propositionerties $F(0) = 0$ and $F(t) > 0, \forall t > 0$.

Remark 4.5. Analogously to the uniform case studied in [4], the proof of Theorem 4.1 can be easily adapted to prove a variant of Rolewicz type for the propositionerty of exponential stability of skew-evolution semiflows in the nonuniform setting, as given by

Theorem 4.2. Let $C = (\varphi, \Phi)$ be a strongly measurable skew-evolution semiflow with exponential growth. Then C is exponentially stable if and only if there exist two mappings $F, R : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a constant $d > 0$ with $F \in \mathcal{F}$ and:

$$(4.6) \quad \int_s^\infty F\left(e^{(t-s)d} \|\Phi(t, s, x)v\| dt\right) \leq R(s)F(\|v\|),$$

for all $(s, x, v) \in \mathbb{R}_+ \times Y$.

Remark 4.6. For the particular case of evolution operators, Theorem 4.2 was proved by S. Rolewicz in [8] for the propositionerty of uniform exponential stability.

Remark 4.7. By means of the methods used in the proofs of Proposition 4.1 and of Theorem 4.1, one can obtain a Datko type theorem for the exponential stability of Barreira-Valls type, in the case of skew-evolution semiflows in the nonuniform setting, as shown by

Theorem 4.3. *Let $C = (\varphi, \Phi)$ be a strongly measurable skew-evolution semiflow with exponential growth. Then C is Barreira-Valls exponentially stable if and only if there exist some constants $N \geq 1$, $a > 0$ and $b \geq a$ such that:*

$$(4.7) \quad \int_s^\infty e^{at} \|\Phi(t, s, x)v\| dt \leq Ne^{bs} \|v\|,$$

for all $(s, x, v) \in \mathbb{R}_+ \times Y$.

Remark 4.8. Analogously, a Rolewicz type theorem can be given for the propositionerty of Barreira-Valls exponential stability, in the case of skew-evolution semiflows.

5. A BARBASHIN TYPE THEOREM FOR THE NONUNIFORM EXPONENTIAL STABILITY

In this section let us consider a particular class of skew-evolution semiflows, given by

Definition 5.1. *A skew-evolution semiflow $C = (\varphi, \Phi)$ is said to be $*$ -strongly measurable ($*$ -s.m.) if for every $(t, t_0, x, v^*) \in T \times X \times V^*$ the mapping defined by $s \mapsto \|\Phi(t, s, \varphi(s, t_0, x))^*v^*\|$ is measurable on $[t_0, t]$.*

The main result of this section is

Theorem 5.1. *Let $C = (\varphi, \Phi)$ be a $*$ -strongly measurable skew-evolution semiflow with exponential growth. If there exist a constant $b > 0$ and a mapping $B : \mathbb{R}_+ \rightarrow [1, \infty)$ such that:*

$$(5.1) \quad \int_s^t e^{(t-\tau)b} \|\Phi(t, \tau, \varphi(\tau, s, x))^*v^*\| d\tau \leq B(t) \|v^*\|,$$

for all $(t, s) \in T$ and all $(x, v^*) \in X \times V^*$, then C is exponentially stable.

Proof. For $t \geq s \geq 0$ we will denote

$$f_s(t) = M(s)B(t)e^{tb}e^{\omega(t)} \text{ and } K(s) = \int_0^1 \frac{du}{f_s(u)},$$

where the functions M and ω are given by Definition 3.5.

We remark that, if $t \geq s + 1$, then

$$K(s) \leq \int_0^{t-s} \frac{du}{f_s(u)} = \int_s^t \frac{d\tau}{f_s(\tau-s)}.$$

It follows that

$$\begin{aligned} & |B(t)e^{(t-s)b}K(s) \langle v^*, \Phi(t, s, x)v \rangle| \leq \\ & \leq \int_s^t \frac{e^{(t-s)b} |\langle \Phi(t, \tau, \varphi(\tau, s, x))^*v^*, \Phi(\tau, s, x)v \rangle|}{M(s)e^{(\tau-s)b}e^{\omega(\tau-s)}} d\tau \leq \\ & \leq \int_s^t e^{(t-\tau)b} \|\Phi(t, \tau, \varphi(\tau, s, x))^*v^*\| \|v\| d\tau \leq B(t) \|v\| \|v^*\|, \end{aligned}$$

which implies

$$\|\Phi(t, s, x)v\| \leq \frac{e^{-(t-s)b}}{K(s)} \|v\|$$

for all $t \geq s + 1$ and all $(x, v) \in Y$.

Now, if we consider $t \in [s, s + 1)$, we have

$$\|\Phi(t, s, x)v\| \leq M(s)e^{\omega(t-s)} \|v\| \leq M(s)e^{\omega(1)} \|v\| \leq M(s)e^{b+\omega(1)} e^{-b(t-s)} \|v\|.$$

Finally, we obtain,

$$\|\Phi(t, s, x)v\| \leq N(s)e^{-(t-s)b} \|v\|,$$

for all $(t, s) \in T$ and all $(x, v) \in X \times V$, where we have denoted

$$N(s) = M(s)e^{b+\omega(1)} + \frac{1}{K(s)},$$

and which proves the exponential stability of the skew-evolution semiflow C . \square

Remark 5.1. Theorem 5.1 is a generalization of a known result of E.A. Barbashin emphasized in [9]. A similar result was obtained by C. Bușe, M. Megan, M. Prajea and P. Preda for the uniform exponential stability of evolution operators in [10].

Remark 5.2. Analogously as in the proof of Theorem 5.1, one can prove a Barbashin type theorem for the propositionerty of Barreira-Valls exponential stability, in the case of skew-evolution semiflows.

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