A Study on Sparse Signal Reconstruction from Interlaced Samples by l1-Norm Minimization

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Abstract:

We propose a sparse signal reconstruction algorithm from interlaced samples with unknown offset parameters based on the l_1-norm minimization principle. A typical application of the problem is superresolution from multiple low-resolution images. The algorithm first minimizes the l_1-norm of a vector that satisfies data constraint with the offset parameters fixed. Second, the minimum value is further minimized with respect to the parameters. Even though this is a heuristic approach, the computer simulations show that the proposed algorithm perfectly reconstructs sparse signals without failure when the reconstruction functions are polynomials and with more than 99% probability for large dimensional signals when the reconstruction functions are Fourier cosine basis functions.

1. Introduction

Sampling theory is at the interface of analog/digital conversion, and sampling theorems provide bridges between the continuous and the discrete-time worlds. A fundamental framework of the sampling theorems consists of data acquisition (sampling) process of a target signal and reconstruction process from the data. Classical studies assumed that both processes are fixed and known. Then, sampling theorems yield in linear formulations [9].

On the other hand, recent studies assume that sampling or reconstruction processes contain unknown factors. Then, sampling theorems become nonlinear. For example, Vetterli *et al.* discussed problems in which locations of reconstruction functions are unknown [11], [5]. They introduced the notion of rate of innovation, and provided perfect reconstruction procedures for signals with finite rate of innovation. The recent hot topic, compressive sampling, assumes that signals are sparse in the sense that signals are expressed by a small subset of reconstruction functions, but the subset is unknown [3], [1], [4]. It is interesting that the solution is obtained by the l_1-norm minimization.

In contrast to the above studies, problems with unknown factors in the sampling process have also been discussed. For example, sampling locations are assumed to be unknown and completely arbitrary in [8] and [2]. A more restricted sampling process is interlaced sampling [7], in which a signal is sampled by a sampling device several times with slightly shifted locations. If the offset parameters are unknown, the sampling theorem becomes nonlinear. A typical application is superresolution from a set of multiple low-resolution images. A replacement of a single high-rate A/D converter by multiple lower rate converters also yields within this formulation.

To this problem, Vandewalle *et al.* proposed perfect reconstruction algorithms under a condition that the total number of unknown parameters is less than or equal to the number of samples [10]. We can find, however, practical situations in which the condition is not true. The method proposed in [2] can be applied to such situations. However, it hardly provides a high quality stable result. In order to solve these difficulties, the present author proposed an algorithm that reconstructs the closest function to a mean signal under data constraint assuming that signals are generated from a probability distribution [6]. The mean signal is, however, not always available.

Hence, in this paper we propose a signal reconstruction algorithm from interlaced samples with unknown offsets using a relatively weak *a priori* knowledge, sparsity. The algorithm first minimizes the l_1-norm of a vector that satisfies data constraint with the offset parameters fixed. Then, the minimum value is further minimized with respect to the parameters. Even though this is a heuristic approach, the computer simulations show that the proposed algorithm perfectly reconstructs sparse signals without failure when the reconstruction functions are polynomials and with more than 99% probability for large dimensional signals when the reconstruction functions are Fourier cosine basis functions.

This paper is organized as follows. Section 2 formulates the fundamental framework and defines the notion of sparsity. Section 3 introduces interlaced sampling and summarizes the conventional studies. In Section 4, we propose the l_1-norm minimization algorithm. Section 5 evaluates the algorithm through simulations, and shows that the algorithm perfectly reconstruct sparse signals with high probability. Section 6 concludes the paper.

2. Sparse Signals

A signal f to be reconstructed is defined on a continuous domain $\mathcal{D}$. We assume that f belongs to a Hilbert space $H = H(\mathcal{D})$ of a finite dimension K. The inner product for f and g in H is denoted by $\langle f, g\rangle$, and the norm is induced as $\|f\| = \sqrt{\langle f, f\rangle}$. By using an arbitrarily fixed basis $\{\varphi_k\}_{k=0}^{K-1}$, any f in H is expressed as

$$f = \sum_{k=0}^{K-1} a_k \varphi_k. \quad (1)$$

A K-dimensional vector with k-th element a_k is denoted by $\boldsymbol{a}$.

Definition 1 *A signal f is J-sparse if at most J coefficients of $\{a_k\}_{k=0}^{K-1}$ in Eq. (1) are non-zero and the rest are zero.*

It should be noted that unknown factors in J-sparse signals are not only values of non-zero coefficients but also their locations. Hence, there are $2J$ unknown factors in a J-sparse signal. If $2J \geq K$, then the number of unknown factors is more than K, which is the number of the original unknown coefficients $\{a_k\}_{k=0}^{K-1}$ without sparsity. Hence, in order for sparsity to be meaningful, we assume that

$$J < K/2.$$

In real applications, J is supposed to be much smaller than $K/2$.

3. Interlaced Sampling

Interlaced sampling means that a signal f is sampled M times by an identical observation device with offsets $\{\delta^{(m)}\}_{m=0}^{M-1}$, where $\delta^{(0)} = 0$. An M-dimensional vector with m-th element $\delta^{(m)}$ is denoted by $\boldsymbol{\delta}$. The observation device is characterized by sampling functions $\{\psi_n\}_{n=0}^{N-1}$, which are given *a priori*. Then, the sampling function for the n-th sample in the m-th sequence is given by

$$\psi_n^{(m)}(x) = \psi_n(x - \delta^{(m)}),$$

and the sample is expressed as

$$d_n^{(m)} = \langle f, \psi_n^{(m)} \rangle. \quad (2)$$

Let $\boldsymbol{d}$ be an MN-dimensional vector in which $d_n^{(m)}$ is the $n+mN$-th element. An $MN \times K$ matrix with the $n+mN$, k-th element $\langle \varphi_k, \psi_n^{(m)} \rangle$ is denoted by B_δ. Substituting Eq. (1) into Eq. (2) yields

$$B_\delta \boldsymbol{a} = \boldsymbol{d}. \quad (3)$$

For simplicity, we assume that the column vectors of B_δ are linearly independent. Figure 1 illustrates the formulation of interlaced sampling.

In order to reconstruct the signal f from interlaced samples with unknown offsets, we have to determine both $\{a_k\}_{k=0}^{K-1}$ and $\{\delta^{(m)}\}_{m=1}^{M-1}$. To this problem, Vandewalle *et al.* proposed perfect reconstruction algorithms under a condition that the number of unknown parameters is less than or equal to the number of samples $\{\{d_n^{(m)}\}_{n=0}^{N-1}\}_{m=0}^{M-1}$, or

$$K + M - 1 \leq MN. \quad (4)$$

We can find, however, practical situations in which the condition is not true. The method in [2] can be applied

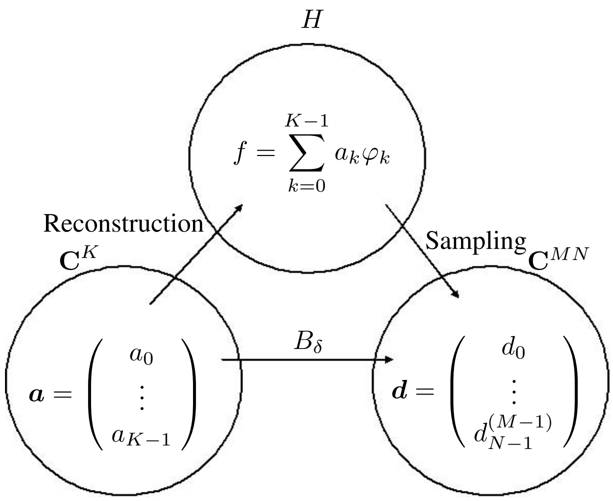


Figure 1: Formulation of sampling and reconstruction. The vector $\boldsymbol{a}$ is to be estimated from the vector $\boldsymbol{d}$. Note that there are unknown offset parameters $\boldsymbol{\delta}$ in B_δ.

to the situation without Eq. (4). However, the results obtained by the method tend to be unstable. The present author proposed an algorithm which uses a mean signal as a prior [6]. However, the mean signal is not always available. Hence, in this paper, we propose perfect reconstruction algorithms using a relatively weak prior, sparsity.

4. l_1-Norm Minimization Algorithm

The problem which we are going to solve in this paper is stated as follows.

Problem 1 *Determine J-sparse vector $\boldsymbol{a}$ and $\boldsymbol{\delta}$ which satisfy Eq. (3) under the condition that the column vectors of B_δ are linearly independent.*

Because of the linear independentness, a vector $\boldsymbol{a}$ that satisfies $B_\delta \boldsymbol{a} = \boldsymbol{d}$ is uniquely determined as

$$\boldsymbol{a} = B_\delta^\dagger \boldsymbol{d},$$

where $B_\delta^\dagger$ is the Moore-Penrose generalized inverse of B_δ. Let us define a matrix B_ε by setting an arbitrarily fixed parameter $\boldsymbol{\varepsilon}$ instead of $\boldsymbol{\delta}$. By using this matrix, a vector $\boldsymbol{c}_\varepsilon$ is defined as

$$\boldsymbol{c}_\varepsilon = B_\varepsilon^\dagger \boldsymbol{d}. \quad (5)$$

Then, our problem becomes a problem of finding a parameter $\boldsymbol{\varepsilon}$ such that the vector $\boldsymbol{c}_\varepsilon$ is J-sparse.

It is well-known that l_1-norm minimization is effective to promote sparsity as is used in the compressed sensing [3], [1], [4]. Hence, we also employ this principle to find J-sparse vector $\boldsymbol{c}_\varepsilon$. Now, our problem becomes the following problem.

Problem 2 *Determine $\boldsymbol{\varepsilon}$ that makes column vectors of the matrix B_ε linearly independent, and minimizes l_1-norm of $\boldsymbol{c}_\varepsilon$ in Eq. (5):*

$$\hat{\boldsymbol{\varepsilon}} = \mathrm{argmin}_{\boldsymbol{\varepsilon}} \|\boldsymbol{c}_\varepsilon\|_{l_1} = \mathrm{argmin}_{\boldsymbol{\varepsilon}} \|B_\varepsilon^\dagger \boldsymbol{d}\|_{l_1}. \quad (6)$$

Table 1: Parameters K, J, N and M used in simulations.

K	4	6	8	10	12
J	1	2	3	4	5
N	2	3	4	5	6
M	2	2	2	2	2

The solution to Problem 2 is different from that to Problem 1 in general. Similar to the compressed sensing, the former agrees with the latter in some cases. Theoretical analyses for the agreement are still under consideration. Instead, we show simulation results in this paper.

5. Simulations

We show computer simulations which demonstrate that the proposed algorithm perfectly reconstructs sparse signals under certain conditions. We consider two reconstruction functions, polynomial and Fourier cosine basis.

5.1 Polynomial reconstruction

Let H be a space spanned by functions

$$\varphi_k(x) = x^k \quad (0 \leq k < K)$$

for $[0, l]$ where l is a positive real number. The inner product is defined by $\langle f, g\rangle = \frac{1}{l}\int_0^l f(x)\overline{g(x)}dx$. Sampling is assumed to be ideal, i.e., $d_n^{(m)} = f(x_n + \delta^{(m)})$. The sample point x_n is given by

$$x_n = \frac{(2n+1)l}{2N} \quad (n = 0, 1, \ldots, N-1),$$

which we call the base sequence. Let $l = N$ so that the sampling interval becomes one.

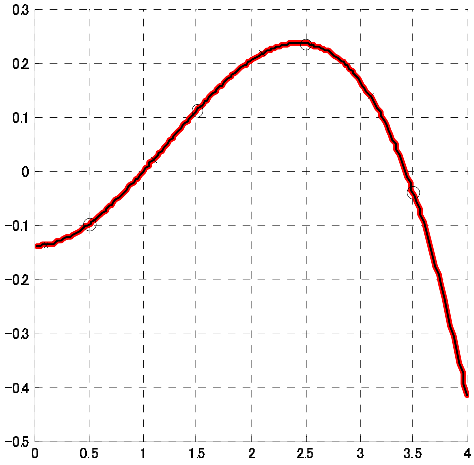
Figure 2 (a) shows a simulation result, in which the dimension of H is $K = 8$, sparsity parameter is $J = 3$, the number of samples in each sequence is $N = 4$, and the sequence was used $M = 2$ times. The offset parameter is $\delta^{(1)} = -0.4$. The black line shows the target signal f, and 'o' and 'x' respectively show the base and the first sequences. The red line shows the reconstructed signal, from which we can see the target signal is perfectly recovered. Figure 2 (b) shows that the l_1-norm of $\boldsymbol{c}_\varepsilon$ is indeed minimized at $\varepsilon = -0.4$. We repeated the simulation for one thousand target signals with the values shown in Table 1. Then, all of the signals are perfectly recovered as well as the offset parameters.

5.2 Fourier cosine basis reconstruction

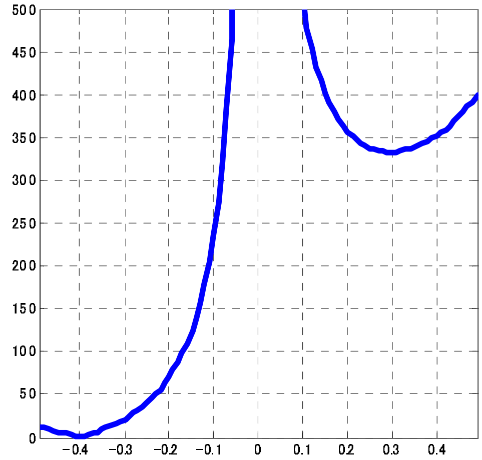
We used the same setup except that the reconstruction functions are

$$\varphi_k(x) = \begin{cases} 1 & (k = 0), \\ \sqrt{2}\cos\dfrac{k\pi x}{l} & (0 < k < K). \end{cases}$$

Under the above defined inner product, $\{\varphi_k\}_{k=0}^{K-1}$ is an orthonormal basis.



(a) Reconstruction result



(b) l_1-norm of $\boldsymbol{c}_\varepsilon$

Figure 2: Simulation result. The black line shows the target signal f, and 'o', 'x', and '+' respectively show the base, the first, and the second sequences. The red line shows the reconstructed signal which perfectly matches to the target signal.

Figure 3 (a) shows a simulation result, in which the dimension of H is $K = 60$, sparsity parameter is $J = 15$, the number of samples in each sequence is $N = 20$, and the sequence was used $M = 3$ times. The offset parameters are $\delta^{(1)} = -0.2$ and $\delta^{(2)} = 0.3$. The black line shows the target signal f, and 'o', 'x', and '+' respectively show the base, the first, and the second sequences. The red line shows the reconstructed signal, from which we can see the target signal is perfectly recovered.

Unfortunately, perfect reconstruction is not always achieved. Figure 4 shows failure rates $[\%]$ of perfect reconstruction with respect to K. The dotted red and the solid blue lines show the rates when $J = K/4$ and $J = K/6$, respectively. The failure rate for $J = K/4$ arrives at less than or equal to 1% when $K > 32$, while that for $J = K/6$ does so when $K > 30$.

Even though these results are only verified through simu-

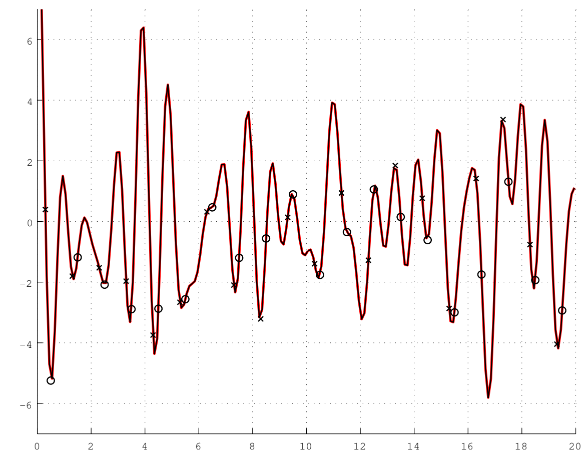


Figure 3: Simulation result for Fourier cosine basis functions. The black line shows the target signal f, and 'o', 'x', and '+' respectively show the base, the first, and the second sequences. The red line shows the reconstructed signal which perfectly matches to the target signal.

lations, the proposed approach is attractive because of its computational efficiency. It takes less than 0.4 second to find the solution for the case of $K = 60$, $N = 20$, and $M = 3$.

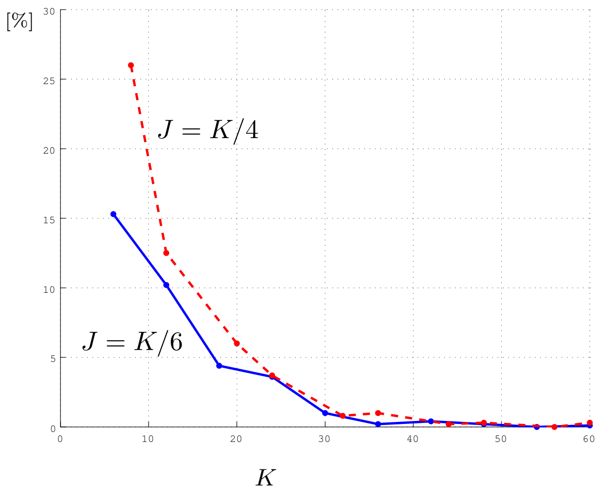


Figure 4: Failure rates of signal recovery when reconstruction functions are Fourier cosine basis functions.

6. Conclusion

We proposed a sparse signal reconstruction algorithm from interlaced samples with unknown offset parameters. The algorithm is based on the l_1-norm minimization principle: First, it minimizes the l_1-norm with the offset parameters fixed. Second, the minimum value is further minimized with respect to the parameters. Even though this is a heuristic approach, the computer simulations showed that the proposed algorithm perfectly reconstructs sparse signals without failure when the reconstruction functions are polynomials and with more than 99% probability for large dimensional signals when the reconstruction functions are Fourier cosine basis functions. Because of the computational efficiency, the proposed algorithm is very attractive. Theoretical analyses of these results are our most important future task.

Acknowledgment

This work was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Young Scientists (B), 20700164, 2008.

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