Double Dirichelet Averages and Complex B-Splines

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Abstract:

A relation between double Dirichlet averages and multivariate complex B-splines is presented. Based on this reationship, a formula for the computation of certain moments of multivariate complex B-splines is derived.

1. Introduction

Recently, a new class of B-splines with complex order z, $\operatorname{Re} z > 1$, was introduced in [4]. It was shown that complex B-splines generate a multiresolution analysis of $L^2(\mathbb{R})$. Unlike the classical cardinal B-splines, complex B-splines B_z possess an additional modulation and phase factor in the frequency domain:

$$\widehat{B}_z(\omega) = \widehat{B}_{\operatorname{Re} z}(\omega)\, e^{i \operatorname{Im} z \ln|\Omega(\omega)|}\, e^{-\operatorname{Im} z \arg \Omega(\omega)},$$

where $\Omega(\omega) := (1 - e^{-i\omega})/(i\omega)$. The existence of these two factors allows the extraction of additional information from sampled data and the manipulation of images.

In [6] and [9], some further properties of complex B-splines were investigated. In particular, connections between complex derivatives of Riemann-Liouville or Weyl type and Dirichlet averages were exhibited. Whereas in [6] the emphasis was on univariate complex B-splines and their applications to statistical processes, multivariate complex B-splines were defined in [9] using a well-known geometric formula for classical multivariate B-splines [7, 10]. It was also shown that Dirichlet averages are especially well-suited to explore the properties of multivariate complex B-splines. Using Dirichlet averages, several classical multivariate B-spline identities were generalized to the complex setting. There also exist interesting relationships between complex B-splines, Dirichlet averages and difference operators, several of which are highlighted in [5].

This short paper presents a generalization of some results found in [3, 12] to complex B-splines. For this purpose, the concept of double Dirichlet average [1] was introduced and its definition extended via projective limits to an infinite-dimensional setting suitable for complex B-splines. Moments of complex B-splines are defined and a formula for their computation in terms of a special double Dirichlet average presented.

2. Complex B-Splines

Let $n \in \mathbb{N}$ and let $\triangle^n$ denote the standard n-simplex in $\mathbb{R}^{n+1}$:

$$\triangle^n := \left\{ u := (u_0, \ldots, u_n) \in \mathbb{R}^{n+1} \;\middle|\; u_j \geq 0; \; j = 0, 1, \ldots, n; \; \sum_{j=0}^{n} u_j = 1 \right\}.$$

The extension of $\triangle^n$ to infinite dimensions is done via projective limits. The resulting infinite-dimensional standard simplex is given by

$$\triangle^\infty := \left\{ u := (u_j)_j \in (\mathbb{R}_0^+)^{\mathbb{N}_0} \;\middle|\; \sum_{j=0}^{\infty} u_j = 1 \right\},$$

and endowed with the topology of pointwise convergence, i.e., the weak$*$-topology. We denote by $\mu_b = \varprojlim \mu_b^n$ the projective limit of *Dirichlet measures* μ_b^n on the n-dimensional standard simplex $\triangle^n$ with density

$$\frac{\Gamma(b_0)\cdots\Gamma(b_n)}{\Gamma(b_0 + \cdots + b_n)}\, u_0^{b_0-1} u_1^{b_1-1} \cdots u_n^{b_n-1}. \tag{1}$$

Here, $\Gamma : \mathbb{C}\setminus\mathbb{Z}_0^- \to \mathbb{C}$ denotes the Euler Gamma function. Let $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$ and let $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

Definition 1 ([6])**.** Given a weight vector $b \in \mathbb{C}_+^{\mathbb{N}_0}$ and an increasing knot sequence $\tau := \{\tau_k\}_k \in \mathbb{R}^{\mathbb{N}_0}$ with the property that $\lim_{k\to\infty} \sqrt[k]{\tau_k} \leq \varrho$, for some $\varrho \in [0, e)$, a complex B-spline $B_z(\bullet \mid b; \tau)$ of order z, $\operatorname{Re} z > 1$, with weight vector b and knot sequence τ is a function satisfying

$$\int_{\mathbb{R}} B_z(t \mid b; \tau) g^{(z)}(t)\, dt = \int_{\triangle^\infty} g^{(z)}(\tau \cdot u)\, d\mu_b(u) \tag{2}$$

for all $g \in \mathscr{S}(\mathbb{R})$.

Here, $\mathscr{S}(\mathbb{R})$ denotes the space of Schwartz functions on $\mathbb{R}$, and $\tau \cdot u = \sum_{k\in\mathbb{N}_0} \tau_k u_k$ for $u = \{u_k\}_{k\in\mathbb{N}_0} \in \triangle^\infty$. In addition, we used the Weyl or Riemann-Liouville fractional derivative [8, 11, 13] of complex order z, $\operatorname{Re} z > 0$, $W^z : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$, defined by

$$(W^z f)(x) := \frac{(-1)^n}{\Gamma(\nu)} \frac{d^n}{dx^n} \int_x^\infty (t - x)^{\nu-1} f(t)\, dt,$$

with $n = \lceil \operatorname{Re} z \rceil$, and $\nu = n - z$. Here $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$, $x \mapsto \min\{n \in \mathbb{Z} \mid n \geq x\}$, denotes the *ceiling function*. To simplify notation, we write $f^{(z)}$ for $W^z f$

It is easy to show that the univariate complex B-spline $B_z(t \mid b; \tau)$ is an element of $L^2(\mathbb{R})$ [5].

Remark 2. For finite $\tau = \tau(n)$ and $b = b(n)$ and $z := n \in \mathbb{N}$, (2) defines also *Dirichlet splines* if g is chosen in $C^n(\mathbb{R})$. For, Dirichlet splines $D_n(\cdot \mid b; \tau)$ of order n are defined as those functions for which

$$\int_{\mathbb{R}} g^{(n)}(t) D_n(t| \, b; \tau) \, dt = \int_{\Delta^n} g^{(n)}(\tau \cdot u) \, d\mu_b(u),$$

holds true for $\tau \in \mathbb{R}^{n+1}$ and for all $g \in C^n(\mathbb{R})$, and thus for $g \in \mathscr{S}(\mathbb{R})$.

To define a multivariate analogue of the univariate complex B-splines, we proceed as follows. Let $\lambda \in \mathbb{R}^s \setminus \{0\}$ be a direction, and let $g : \mathbb{R} \to \mathbb{C}$ be a function. The *ridge function* corresponding to g is defined as $g_\lambda : \mathbb{R}^s \to \mathbb{C}$,

$$g_\lambda(x) = g(\langle \lambda, x \rangle) \quad \text{for all } x \in \mathbb{R}^s.$$

We denote the canonical inner product in $\mathbb{R}^s$ by $\langle \bullet, \bullet \rangle$ and the norm induced by it by $\| \bullet \|$.

Definition 3 ([9])**.** Let $\tau = \{\tau^n\}_{n \in \mathbb{N}_0} \in (\mathbb{R}^s)^{\mathbb{N}_0}$ be a sequence of knots in $\mathbb{R}^s$ with the property that

$$\exists \, \varrho \in [0, e) : \limsup_{n \to \infty} \sqrt[n]{\|\tau^n\|} \leq \varrho. \tag{3}$$

The multivariate complex B-spline $\boldsymbol{B}_z(\bullet \mid b, \tau) : \mathbb{R}^s \to \mathbb{C}$ of order z, $\operatorname{Re} z > 1$, with weight vector $b \in \mathbb{C}_+^{\mathbb{N}_0}$ and knot sequence τ is defined by means of the identity

$$\int_{\mathbb{R}^s} g(\langle \lambda, x \rangle) \boldsymbol{B}_z(x \mid b, \tau) \, dx = \int_{\mathbb{R}} g(t) B_z(t \mid b, \lambda\tau) \, dt, \tag{4}$$

where $g \in \mathscr{S}(\mathbb{R})$, and where $\lambda \in \mathbb{R}^s \setminus \{0\}$ such that $\lambda\tau := \{\langle \lambda, \tau^n \rangle\}_{n \in \mathbb{N}_0}$ is separated.

As consequence of the fact that $B_z(\bullet \mid b; \tau) \in L^2(\mathbb{R})$, one obtains from the above definition that $\boldsymbol{B}_z(\bullet \mid b, \tau) \in L^2(\mathbb{R}^s)$ [5]. Moreover, it follows from the Hermite-Genocchi formula for the univariate complex B-splines $B_z(\bullet \mid b, \lambda\tau)$ and (4), that $\boldsymbol{B}_z(x \mid b, \tau) = 0$, when $x \notin [\tau]$, the convex hull of τ.

3. Dirichlet Averages

Let Ω to be a nonempty open convex set in $\mathbb{C}^s$, $s \in \mathbb{N}$, and let $b \in \mathbb{C}_+^{\mathbb{N}_0}$. Let $f \in \mathscr{S}(\Omega) := \mathscr{S}(\Omega, \mathbb{C})$ be a measurable function. For $\tau \in \Omega^{\mathbb{N}_0} \subset (\mathbb{C}^s)^{\mathbb{N}_0}$ and $u \in \triangle^\infty$, define $\tau \cdot u$ to be the bilinear mapping $(\tau, u) \mapsto \sum_{i=1}^\infty u_i \tau^i$. The infinite sum exists if there exists a $\varrho \in [0, e)$ so that

$$\limsup_{n \to \infty} \sqrt[n]{\|\tau^n\|} \leq \varrho. \tag{5}$$

Here, $\| \cdot \|$ now denotes the canonical Euclidean norm on $\mathbb{C}^s$. (See also [6].)

Definition 4. Let $f : \Omega \subset \mathbb{C}^s \to \mathbb{C}$ be a measurable function. The Dirichlet average $F : \mathbb{C}_+^{\mathbb{N}_0} \times \Omega^{\mathbb{N}_0} \to \mathbb{C}$ over $\triangle^\infty$ is defined by

$$F(b; \tau) := \int_{\triangle^\infty} f(\tau \cdot u) \, d\mu_b(u),$$

where $\mu_b = \varprojlim \mu_b^n$ is the projective limit of Dirichlet measures on the n-dimensional standard simplex $\triangle^n$.

We remark that the Dirichlet average is holomorphic in $b \in (\mathbb{C}_+)^{\mathbb{N}_0}$ when $f \in C(\Omega, \mathbb{C})$ for every fixed $\tau \in \Omega^{\mathbb{N}_0}$. (See [2] for the finite-dimensional case and [9] for the infinite-dimensional setting.)

Definition 5. [1] Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be continuous. Let $b \in \mathbb{C}_+^{k+1}$ and $\beta \in \mathbb{C}_+^{\varkappa+1}$. Suppose that for fixed $k, \varkappa \in \mathbb{N}$, $X \in \mathbb{C}^{(k+1)\times(\varkappa+1)}$ and that the convex hull $[X]$ of X is contained in Ω. Then the double Dirichlet average of f is defined by

$$\mathscr{F}(b; X; \beta) := \int_{\triangle^k} \int_{\triangle^\varkappa} f(u \cdot Xv) d\mu_b^k(u) d\nu_\beta^\varkappa(v),$$

where $u \cdot Xv := \sum_{i=0}^k \sum_{j=0}^\varkappa u_i X_{ij} v_j$.

Note that $\mathscr{F}(b; X; \beta)$ is holomorphic on Ω in the elements of b, β, and X.

We again use projective limits to extend the notion of double Dirichlet average to an infinite-dimenional setting. To this end, let $u, v \in \triangle^\infty$ and let $\mu_b = \varprojlim \mu_b^n$ and $\nu_\beta = \varprojlim \nu_\beta^n$ be the projective limits of Dirichlet measures μ_b^n and ν_β^n of the form (1) on the n-dimensional standard simplex, where $b, \beta \in \mathbb{C}_+^{\mathbb{N}_0}$. Now suppose that $X \in \mathbb{C}^{\mathbb{N}_0 \times \mathbb{N}_0}$ is a infinite matrix with the property that $\sum_{i=0}^\infty \sum_{j=0}^\infty |X_{ij}|$ converges. Let

$$u \cdot Xv := \sum_{i=0}^\infty \sum_{j=0}^\infty u_i X_{ij} v_j.$$

Suppose that $\Omega \subset \mathbb{C}$ contains the convex hull $[X]$ of X and that $f : \Omega \to \mathbb{C}$ is continuous. The double Dirichlet average of f over $\triangle^\infty$ is then given by

$$\mathscr{F}(b; X; \beta) := \int_{\triangle^\infty} \int_{\triangle^\infty} f(u \cdot Xv) d\mu_b(u) d\nu_\beta(v). \tag{6}$$

(We use the same symbol for the (double) Dirichlet average over $\triangle^\infty$ and its finite-dimensional projections $\triangle^n$.)

It is easy to show that

$$\mathscr{F}(b; X; \beta) = \int_{\triangle^\infty} F(\beta; uX) d\mu_b(u), \tag{7}$$

where $uX := \{\langle u, X_j \rangle\}_{j \in \mathbb{N}_0}$, with X_j denoting the j-column of X.

We note that $\mathscr{F}(b; X; \beta)$ is holomorphic in the elements of b, β, and X over $\triangle^\infty$.

For $z \in \mathbb{C}_+$, we define

$$\mathscr{F}^{(z)}(b; X; \beta) := \int_{\triangle^\infty} \int_{\triangle^\infty} f^{(z)}(u \cdot Xv) d\mu_b(u) d\nu_\beta(v).$$

(See also [9] for the case of a single Dirichlet average.)

4. Double Dirichlet Averages and Complex B-Splines

Assume now that the matrix X is real-valued and of the form $X_{ij} = 0$, for $i \geq s$ and all $j \in \mathbb{N}_0$, some $s \in \mathbb{N}$. In other words, $X \in \mathbb{R}^{s\times\mathbb{N}_0}$.

Theorem 6. *Suppose that $\beta \in \mathbb{R}_+^\infty$ and that $\operatorname{Re} z > 1$. Let $b := (b_0, b_1, \ldots, b_{s-1}) \in \mathbb{R}^s$ be such that $\sum_{i=0}^{s-1} b_i \notin -\mathbb{N}_0$. Assume that $f \in \mathscr{S}(\mathbb{R}^+)$. Further assume that uX is separated for all $u \in \triangle^{s-1}$. Then*

$$\mathscr{F}^{(z)}(b; X; \beta) = \int_{\mathbb{R}^s} \boldsymbol{B}_z(x \mid \beta, X)\, F^{(z)}(b; x) dx.$$

Proof. We prove the formula first for $b \in \mathbb{R}_+^s$. To this end, we identify $u = (u_0, u_1, \ldots, u_{s-1}, 0, 0, \ldots) \in \triangle^\infty$ with $(u_0, u_1, \ldots, u_{s-1}) \in \triangle^{s-1}$. By the Hermite-Genocchi formula for complex B-splines (see [6] and to some extend [9]), we have that

$$\begin{aligned} F^{(z)}(\beta; uX) &= \int_{\triangle^\infty} f^{(z)}(u' \cdot uX)\, d\mu_\beta(u') \\ &= \int_{\mathbb{R}} f^{(z)}(t) B_z(t \mid \beta, uX) dt \end{aligned}$$

Substituting this expression into (7) and using (4) gives

$$\mathscr{F}^{(z)}(b; X; \beta) = \int_{\triangle^\infty} \int_{\mathbb{R}^s} f^{(z)}(\langle u, x\rangle) \boldsymbol{B}_z(x \mid \beta, uX)\, dx\, d\mu_b(u).$$

Interchanging the order of integration yields the statement for $b \in \mathbb{R}_+^s$. To obtain the general case $b \in \mathbb{R}^s$, we note that by Theorem 6.3-7 in [2], the Dirichlet average F can be holomorphically continued in the b-parameters provided $\sum_{i=0}^{s-1} b_i \notin -\mathbb{N}_0$. □

Remark 7. Theorem 6 extends Theorem 6.1 in [12] to complex B-splines and the $\triangle^\infty$-setting.

5. Moments of Complex B-Splines

Following [2], we define the R-hypergeometric function $R_a(b; \tau) : \mathbb{R}_+^s \times \Omega^s \to \mathbb{C}$ by

$$R_a(b; \tau) := \int_{\triangle^{s-1}} (\tau \cdot u)^a d\mu_b^{s-1}(u), \tag{8}$$

where $\Omega := H$, H a half-plane in $\mathbb{C} \setminus \{0\}$, if $a \in \mathbb{C} \setminus \mathbb{N}$, and $\Omega := \mathbb{C}$, if $a \in \mathbb{N}$. It can be shown (see [2]) that R_{-a}, $a \in \mathbb{C}_+$, has a holomorphic continuation in τ to $\mathbb{C}_0$, where $\mathbb{C}_0 := \{\zeta \in \mathbb{C} \mid -\pi < \arg \zeta < \pi\}$.
Taking in the definition of the double Dirichlet average (6) for f the real-valued function $t \mapsto t^{-c}$, where $c := \sum_{i=0}^{s-1} b_i$, the resulting double Dirichlet average is denoted by $\mathscr{R}_{-c}(b; X; \beta)$ and generalizes power functions. The corresponding single Dirichlet average $R_{-c}(b; x)$, where $x = (x_0, \ldots, x_{s-1})$, is given by

$$R_{-c}(b; x) = \prod_{i=0}^{s-1} x_i^{-b_i}, \quad x \notin [X]. \tag{9}$$

(See, [2], (6.6-5).)
Now, let $p = (p_0, p_1, \ldots, p_{s-1}) \in \mathbb{R}^s$, $s \in \mathbb{N}$, be a multi-index all of whose components satisfy $p_i < -\frac{1}{2}$. The moment $\mathsf{M}_{|p|}^{(z)}(b; X)$ of order $|p| := \sum_{i=1}^s p_i$ of the complex B-spline $\boldsymbol{B}_z(\bullet \mid \beta, X)$ is defined by

$$\mathsf{M}_{|p|}^{(z)}(b; X) := \int_{\mathbb{R}^s} x^p\, \boldsymbol{B}_z(x \mid \beta, X)\, dx.$$

Note that since $\boldsymbol{B}_z(\bullet \mid \beta, X) \in L^2(\mathbb{R}^s)$ and $\boldsymbol{B}_z(\bullet \mid \beta, X) = 0$, for $x \notin [X]$, an easy application of the Cauchy-Schwartz inequality shows that the above integral exists provided the multi-index p satisfies the aforementioned condition on its components.
Using a result from [8], namely Property 2.5 (b), and requiring that $\operatorname{Re} z < \operatorname{Re} c$, we substitute the function $f := \frac{\Gamma(c-z)}{\Gamma(c)} (\bullet)^{-(c-z)}$ into (8) to obtain

$$R_{-(c-z)}^{(z)}(b; x) = R_{-c}(b; x) = \prod_{i=0}^{s-1} x_i^{b_i}.$$

The above considerations together with Theorem 6 immediately yield the next result.

Corollary 8. *Suppose that $\beta \in \mathbb{R}_+^\infty$ and that $\operatorname{Re} z > 1$. Let $b := (b_0, b_1, \ldots, b_{s-1}) \in (-\infty, -\frac{1}{2})^s$ be such that $c := \sum_{i=0}^{s-1} b_i \notin -\mathbb{N}_0$. Moreover, suppose that $\operatorname{Re} z < \operatorname{Re} c$. Then*

$$\mathsf{M}_{-c}^{(z)}(b; X) = \mathscr{R}_{-(c-z)}^{(z)}(b; X; \beta). \tag{10}$$

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