Global existence vs. blowup for the one dimensional quasilinear Smoluchowski-Poisson system

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Abstract

We prove that, unlike in several space dimensions, there is no critical (nonlinear) diffusion coefficient for which solutions to the one dimensional quasilinear Smoluchowski-Poisson equation with small mass exist globally while finite time blowup could occur for solutions with large mass.

1 Introduction

In a previous paper [4] we investigate the influence of the diffusion coefficient a on the life span of solutions to the one dimensional Smoluchowski-Poisson system

$$\begin{aligned} \partial_t u &= \partial_x \left(a(u)\partial_x u - u\partial_x v\right) \quad \text{in} \quad (0,\infty)\times(0,1), && (1)\\ 0 &= \partial_x^2 v + u - M \quad \text{in} \quad (0,\infty)\times(0,1), && (2)\\ a(u)\partial_x u &= \partial_x v = 0 \quad \text{on} \quad (0,\infty)\times\{0,1\}, && (3)\\ u(0) &= u_0 \ge 0 \quad \text{in} \quad (0,1), \quad \int_0^1 v(t,x)dx = 0 \quad \text{for any} \quad t\in(0,\infty), && (4) \end{aligned}$$

where

$$M := \langle u_0\rangle = \int_0^1 u_0(x)dx$$

denotes the mean value of u_0, and uncover a fundamental difference with the quasilinear Smoluchowski-Poisson system in higher space dimensions. More precisely, when the space dimension n is greater or equal

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to two, there is a critical diffusion $a_*(r) := (1+r)^{(n-2)/2}$ which separates different behaviours for the quasilinear Smoluchowski-Poisson system. Roughly speaking,

(a) if the diffusion coefficient a is stronger than a_* (in the sense that $a(r) \geq C(1+r)^\alpha$ for some $\alpha > (n-2)/n$ and $C > 0$), then all solutions exist globally whatever the value of the mass of the initial condition u_0 [5],

(b) if the diffusion coefficient a is weaker than a_* (in the sense that $a(r) \leq C(1+r)^\alpha$ for some $\alpha < (n-2)/n$ and $C > 0$), then there exists for all $M > 0$ an initial condition u_0 with $\langle u_0 \rangle = M$ for which the corresponding solution to the quasilinear Smoluchowski-Poisson system blows up in finite time (in the sense that $\|u(t)\|_\infty \to \infty$ as $t \to T$ for some $T \in (0,\infty)$) [3, 5, 7],

(c) if the diffusion coefficient a behaves as a_* for large values of r, solutions starting from initial data u_0 with small mass $\langle u_0 \rangle$ exist globally while there are initial data with large mass for which the corresponding solution to the quasilinear Smoluchowski-Poisson system blows up in finite time [3, 7].

Observe that, in space dimension $n = 2$, the critical diffusion is constant and a more precise description of the situation **(c)** is actually available. Namely, when $a \equiv 1$, there is a threshold mass M_* such that, if $\langle u_0 \rangle < M_*$, the corresponding solution is global while, for any $M > M_*$, there are initial data with $\langle u_0 \rangle = M$ for which the corresponding solution blows up in finite time [6, 7, 8]. The threshold mass M_* is known explicitly ($M_* = 4\pi$) but it is worth mentioning that for radially symmetric solutions in a ball, the threshold mass is 8π. Similar results are also available for the quasilinear Smoluchowski-Poisson system in $\mathbb{R}^n$, $n \geq 2$ [1, 2, 9, 10].

Most surprisingly, the above description fails to be valid in one space dimension and we prove in particular in [4] that all solutions are global for the diffusion $a(r) = (1+r)^{-1}$ though it is a natural candidate to be critical. We actually identify two classes of diffusion coefficients a in [4], one for which all solutions exist globally as in **(a)** and the other for which there are solutions blowing-up in finite time starting from initial data with an arbitrary positive mass as in **(b)**, but the situation **(c)** does not seem to occur in one space dimension. The purpose of this note is to show that the dichotomy **(a)** or **(b)** can be extended to larger classes of diffusion, thereby extending the analysis performed in [4].

Theorem 1 *Let the diffusion coefficient $a \in \mathcal{C}^1((0,\infty))$ be a positive function.*

(i) Assume first that $a \in L^1(1,\infty)$ and one of the following assumptions is satisfied, either

$$\gamma := \sup_{r\in(0,1)} r \int_r^\infty a(s)ds < \infty, \tag{5}$$

or there exist $\vartheta > 0$ and $\alpha \in (\vartheta/(1+\vartheta), 2]$ such that

$$\gamma_\vartheta := \sup_{r\in(0,1)} r^{2+\vartheta} a(r) < \infty \quad \text{and} \quad C_\infty := \sup_{r\geq 1} r^\alpha a(r) < \infty. \tag{6}$$

For any $M > 0$, there exists a positive initial condition $u_0 \in \mathcal{C}([0,1])$ such that $\langle u_0 \rangle = M$ and the corresponding classical solution to (1)-(4) blows up in finite time.

(ii) Assume next that $a \notin L^1(1,\infty)$ and consider an initial condition $u_0 \in \mathcal{C}([0,1])$ such that $u_0 \geq m_0 > 0$ and $\langle u_0 \rangle = M$ for some $M > 0$ and $m_0 \in (0,M)$. Then the corresponding classical solution to (1)-(4) exists globally.

As already mentioned, Theorem 1 extends the results obtained in [4]. More precisely, in [4, Theorem 5], the assertion (ii) of Theorem 1 is proved under the additional assumption that, for each $\varepsilon \in (0,\infty)$, there is $\kappa_\varepsilon > 0$ for which

$$a(r) \leq \varepsilon\, ra(r) + \frac{\kappa_\varepsilon}{r} \quad \text{for} \quad r \in (0,1)\,,$$

which roughly means that a cannot have a singularity stronger than $1/r$ near $r = 0$. This assumption turns out to be unnecessary for global existence but nevertheless ensures the global boundedness of the solution in L^∞. Under the sole assumption of Theorem 1 (ii), our proof does not exclude that solution to (1)-(4) becomes unbounded as $t \to \infty$. Concerning Theorem 1 (i), it is established in [4, Theorem 10] for $a \in L^1(1,\infty)$ such that there is a concave function B for which

$$0 \leq -rA(r) \quad \leq \quad B(r) \quad \text{with} \quad A(r) = -\int_r^\infty a(s)\, ds\,, \qquad r \in (0,\infty)\,, \tag{7}$$

$$\lim_{r\to\infty} \frac{B(r)}{r} \quad = \quad 0\,. \tag{8}$$

We make this criterion more explicit here by showing that the integrability of a on $(1,\infty)$ and (5) guarantee the existence of a concave function B satisfying (7) and (8), see Lemma 3 below. Let us point out here that the assumption (5) somehow means that a cannot have a singularity stronger that $1/r^2$ near $r = 0$. However, the result remains true if a has an algebraic singularity of higher order near $r = 0$ which is allowed by (6) provided a decays suitably at infinity. Observe that the second condition in (6) is compatible with the integrability of a at infinity as $\vartheta/(1+\vartheta) < 1$.

Summarizing the outcome of Theorem 1, we realize that, for a given diffusion coefficient a with a singularity weaker than $1/r^2$ near $r = 0$, the integrability or non-integrability of a at infinity completely determines whether we are in the situation **(a)** or **(b)** described above and excludes the situation **(c)**. There is thus no critical diffusion in this class. The same comment applies to the class of diffusion coefficients satisfying (6) with an algebraic singularity stronger than $1/r^2$ near $r = 0$. In particular there is no critical nonlinearity in the class of functions $\mathcal{C}([0,\infty)) \cap \mathcal{C}^1((0,\infty))$.

The paper is organized as follows: in section 2 we recall some statements from [4]. Section 3 is devoted to proving the finite time blowup of solutions to (1)-(4) when $a \in L^1(1,\infty)$. Global existence of solutions for all initial data when a is not integrable at infinity is proved in the last section.

2 Preliminaries.

In this section we summarize some results and methods introduced in [4]. Let $a \in \mathcal{C}^1((0,\infty))$ be a positive function and consider an initial condition $u_0 \in \mathcal{C}([0,1])$ such that $u_0 \geq m_0 > 0$ and $\langle u_0 \rangle = M$ for some $M > 0$ and $m_0 \in (0, M)$. By [4, Propositions 2 and 3] there is a unique maximal classical solution (u, v) to (1)-(4) defined on $[0, T_{max})$ which satisfies

$$\min_{x\in[0,1]} u(t,x) > 0\,, \quad \langle u(t)\rangle := \int_0^1 u(t,x)\ dx = M\,, \quad \text{and} \quad \langle v(t)\rangle := \int_0^1 v(t,x)\ dx = 0 \tag{9}$$

for $t \in (0, T_{max})$. In addition, $T_{max} = \infty$ or $T_{max} < \infty$ with $\|u(t)\|_\infty \to \infty$ as $t \to T_{max}$.

We next recall the approach introduced in [4] which will be used herein as well. Owing to the positivity (9) and the regularity of u, the indefinite integral

$$U(t,x) := \int_0^x u(t,z)dz\,, \quad x \in [0,1]\,,$$

is a smooth increasing function from $[0,1]$ onto $[0,M]$ for each $t \in [0, T_{max})$ and has a smooth inverse F defined by

$$U(t, F(t,y)) = y\,, \qquad (t,y) \in [0,T_{max}) \times [0,M]\,. \tag{10}$$

Introducing $f(t,y) := \partial_y F(t,y)$, we have

$$f(t,y)\ u(t, F(t,y)) = 1\,, \qquad (t,y) \in [0,T_{max}) \times [0,M]\,, \tag{11}$$

and it follows from (1)-(4) that f solves

$$\partial_t f = \partial_y^2 \Psi(f) - 1 + Mf\,, \qquad (t,y) \in (0,T_{max}) \times (0,M)\,, \tag{12}$$

$$\partial_y f(t,0) = \partial_y f(t,M) = 0\,, \qquad t \in (0, T_{max})\,, \tag{13}$$

$$f(0,y) = f_0(y) := \frac{1}{u_0(F(0,y))}\,, \qquad y \in (0,M)\,, \tag{14}$$

where

$$\Psi'(r) := \frac{1}{r^2}\ a\left(\frac{1}{r}\right) \quad \text{for any} \ \ r > 0\,, \qquad \Psi(1) := 0\,, \tag{15}$$

Moreover the conservation of mass (9) yields

$$\int_0^M f(t,y)dy = F(t,M) - F(t,0) = 1\,, \qquad t \in [0,T_{max})\,. \tag{16}$$

At this point, the crucial observation is that, thanks to (11), finite time blowup of u is equivalent to the vanishing (or touch-down) of f. In other words, u exist globally if the minimum of $f(t)$ is positive for each $t > 0$. We refer to [4, Proposition 1] for a more detailed description.

An salient property of (1)-(4) is the existence of a Liapunov function [4, Lemma 8] which we recall now:

Lemma 2 *The function*

$$L_1(t) := \frac{1}{2}\int_0^M |\partial_y \Psi(f(t,y))|^2 \ dy + \int_0^M (\Psi(f(t,y)) - M \ \Psi_1(f(t,y))) \ dy$$

is a non-increasing function of time on $[0, T_{max})$*, the function* Ψ_1 *being defined by*

$$\Psi_1(1) := 0 \quad \text{and} \quad \Psi_1'(r) := r\Psi'(r) = \frac{1}{r} \ a\left(\frac{1}{r}\right), \qquad r \in (0,\infty). \tag{17}$$

3 Finite time blowup.

In this section we prove the blowup assertion of Theorem 1. To this end we first prove that the condition (5) allows us to construct a concave function B satisfying (7) and (8) so that [4, Theorem 10] can be applied.

Lemma 3 *Let* $a \in \mathcal{C}^1((0,\infty))$ *be a positive function such that* $a \in L^1(1,\infty)$ *and (5) holds. Then there exists a concave function* $B \in \mathcal{C}([0,\infty))$ *such that for all* $r \geq 0$

$$B(r) \geq r \int_r^\infty a(s) ds \tag{18}$$

and

$$\lim_{r\to\infty} \frac{B(r)}{r} = 0. \tag{19}$$

Proof of Lemma 3. We construct $B : [0,\infty) \to [0,\infty)$ in the following way: we put

$$b_i := \int_{2^i}^\infty a(s) ds\,, \qquad i \geq 0\,,$$

and notice that $\{b_i\}_{i\geq 0}$ is a decreasing sequence converging to zero as $i \to \infty$. We next define

$$B(r) = \begin{cases} b_0 r + \gamma & \text{if} \quad r \in [0,2], \\ \\ b_i r + \displaystyle\sum_{j=0}^{i-1} (b_j - b_{j+1}) 2^{j+1} + \gamma & \text{if} \quad r \in (2^i, 2^{i+1}] \ \text{ and } \ i \geq 1, \end{cases} \tag{20}$$

Clearly, $B \in \mathcal{C}([0,\infty))$ and

$$B'(r) = \begin{cases} b_0 & \text{if} \quad r \in (0,2), \\ b_i & \text{if} \quad r \in (2^i, 2^{i+1}) \ \text{ and } \ i \geq 1. \end{cases} \tag{21}$$

Hence B is concave as a consequence of the fact that the sequence $\{b_i\}_{i\geq 0}$ is decreasing. Furthermore, for $r \in [0,1]$, we have

$$B(r) \geq \gamma \geq r \int_r^\infty a(s) ds,$$

and for $r \in [2^i, 2^{i+1}]$, $i \geq 0$,

$$B(r) \geq b_i r = r \int_{2^i}^{\infty} a(s) ds \geq r \int_r^{\infty} a(s) ds.$$

Therefore, B satisfies (18).

Finally, let $k \geq 1$. If $i \geq k+1$ and $r \in (2^i, 2^{i+1}]$, then

$$\begin{aligned} \frac{B(r)}{r} &= b_i + \frac{\gamma}{r} + \sum_{j=0}^{i-1} (b_j - b_{j+1}) \frac{2^{j+1}}{r} \leq b_i + \frac{\gamma}{r} + \sum_{j=k}^{i-1} (b_j - b_{j+1}) + \sum_{j=0}^{k-1} (b_j - b_{j+1}) \frac{2^{j+1}}{r} \\ &\leq b_i + \frac{1}{r} \left(\gamma + 2^k \sum_{j=0}^{k-1} (b_j - b_{j+1}) \right) + (b_k - b_i) \leq b_k + \frac{1}{r} \left(\gamma + 2^k b_0 \right). \end{aligned}$$

Consequently,

$$\limsup_{r \to \infty} \frac{B(r)}{r} \leq b_k \text{ for all } k \geq 1 \, .$$

Letting $k \to \infty$, we obtain (19) since $b_k \to 0$ as $k \to \infty$ and Lemma 3 is proved. □

Proof of Theorem 1 (i), Part 1. When a belongs to $L^1(1, \infty)$ and satisfies (5), it follows from Lemma 3 that the conditions (7) and (8) are satisfied so that Theorem 1 (i) follows from [4, Theorem 10]. □

To handle the other case, we proceed in a different way by showing an upper bound for the function f defined in section 2. We first observe that the function Ψ defined in (15) satisfies

$$\Psi(r) = \int_1^{\infty} \frac{1}{s^2} a\left(\frac{1}{s}\right) \, ds = \int_{1/r}^{1} a(s) \, ds \, , \quad r \in (0, \infty) \, ,$$

so that, if $a \in L^1(1, \infty)$, $\Psi(r)$ has a finite limit $\Psi(0) := -\|a\|_{L^1(1,\infty)}$ as $r \to 0$. We then define

$$\tilde{\Psi}(r) := \Psi(r) - \Psi(0) = \int_0^r \frac{1}{s^2} a\left(\frac{1}{s}\right) \, ds = \int_{1/r}^{\infty} a(s) \, ds \, , \quad r \in (0, \infty) \, . \tag{22}$$

Lemma 4 *Let $a \in \mathcal{C}^1((0, \infty))$ be a positive function such that $a \in L^1(1, \infty)$. There exists a positive constant $\mu_M > 0$ depending only on M and a such that, for any non-negative function $g \in H^1(0, M)$ satisfying $\|g\|_{L^1(0,M)} = 1$, we have*

$$\|\tilde{\Psi}(g)\|^2_{L^\infty(0,M)} \leq 32 M \mathcal{L}_1(g) + \mu_M, \tag{23}$$

with

$$\mathcal{L}_1(g) := \frac{1}{2} \|\partial_y \Psi(g)\|^2_{L^2(0,M)} + \int_0^M \left(\Psi(g) - M \Psi_1(g) \right)(y) \, dy \, , \tag{24}$$

the functions Ψ and Ψ_1 being defined in (15) and (17), respectively.

Proof of Lemma 4. We set $G := \|g\|_{L^\infty(0,M)}$ which is finite owing to the continuous embedding of $H^1(0,M)$ in $L^\infty(0,M)$. Assume first that $G > 1$. Then, for $y \in (0,M)$ and $z \in (0,M)$, we have

$$\tilde{\Psi}(g(y)) = \tilde{\Psi}(g(z)) + \int_z^y \partial_x \tilde{\Psi}(g(x))\ dx \le \tilde{\Psi}(g(z)) + M^{1/2} \|\partial_y \Psi(g)\|_{L^2(0,M)}.$$

Integrating the above inequality over $(0,M)$ with respect to z gives

$$\begin{aligned} M\tilde{\Psi}(g(y)) &\le \int_0^M \tilde{\Psi}(g(z))\ dz + M^{3/2} \|\partial_y \Psi(g)\|_{L^2(0,M)} \\ &\le \int_0^M \mathbf{1}_{[0,2/M]}(g(z)) \tilde{\Psi}(g(z))\ dz + \int_0^M \mathbf{1}_{(2/M,\infty)}(g(z)) \tilde{\Psi}(g(z))\ dz + M^{3/2} \|\partial_y \Psi(g)\|_{L^2(0,M)} \\ &\le M\tilde{\Psi}\left(\frac{2}{M}\right) + \frac{M\tilde{\Psi}(G)}{2} \int_0^M \mathbf{1}_{(2/M,\infty)}(g(z)) g(z)\ dz + M^{3/2} \|\partial_y \Psi(g)\|_{L^2(0,M)} \\ &\le M\tilde{\Psi}\left(\frac{2}{M}\right) + \frac{M\tilde{\Psi}(G)}{2} + M^{3/2} \|\partial_y \Psi(g)\|_{L^2(0,M)}\,, \end{aligned}$$

where we have used the property $\|g\|_{L^1(0,M)} = 1$ to obtain the last inequality. Taking the supremum over $y \in (0,M)$ and using the monotonicity and non-negativity of $\tilde{\Psi}$, we deduce that

$$\tilde{\Psi}(G) \le 2\tilde{\Psi}\left(\frac{2}{M}\right) + 2M^{1/2} \|\partial_y \Psi(g)\|_{L^2(0,M)}. \tag{25}$$

We next observe that the integrability of a at infinity also ensures that $\Psi_1(0) > -\infty$, so that $\tilde{\Psi}_1 := \Psi_1 - \Psi_1(0)$ is well-defined and satisfies

$$\tilde{\Psi}_1(r) = \int_0^r s\Psi'(s)\ ds \le r\tilde{\Psi}(r)\,, \quad r \in (0,\infty)\,. \tag{26}$$

Since $\|g\|_{L^1(0,M)} = 1$, it follows from (25) and (26) that

$$\int_0^M \tilde{\Psi}_1(g)\ dy \le \int_0^M g\tilde{\Psi}(g)\ dy \le \tilde{\Psi}(G) \int_0^M g dy \le 2\tilde{\Psi}\left(\frac{2}{M}\right) + 2M^{1/2} \|\partial_y \Psi(g)\|_{L^2(0,M)}. \tag{27}$$

We next infer from (27) and the non-negativity of $\tilde{\Psi}$ that

$$\begin{aligned} \mathcal{L}_1(g) &\ge \frac{1}{2} \|\partial_y \Psi(g)\|_{L^2(0,M)}^2 + \int_0^M \tilde{\Psi}(g)\ dy + M\Psi(0) - M \int_0^M \tilde{\Psi}_1(g)\ dy \\ &\ge \frac{1}{2} \|\partial_y \Psi(g)\|_{L^2(0,M)}^2 + M\Psi(0) - 2M\tilde{\Psi}\left(\frac{2}{M}\right) - 2M^{3/2} \|\partial_y \Psi(g)\|_{L^2(0,M)} \\ &\ge \frac{1}{4} \|\partial_y \Psi(g)\|_{L^2(0,M)}^2 + \left(\frac{1}{2} \|\partial_y \Psi(g)\|_{L^2(0,M)} - 2M^{3/2}\right)^2 - 4M^3 + M\Psi(0) - 2M\tilde{\Psi}\left(\frac{2}{M}\right) \\ &\ge \frac{1}{4} \|\partial_y \Psi(g)\|_{L^2(0,M)}^2 - 4M^3 + M\Psi(0) - 2M\tilde{\Psi}\left(\frac{2}{M}\right), \end{aligned}$$

whence

$$\|\partial_y \Psi(g)\|_{L^2(0,M)}^2 \leq 4\mathcal{L}_1(g) + 16M^3 - 4M\Psi(0) + 8M\tilde{\Psi}\left(\frac{2}{M}\right).$$

It then follows from (25) and the above inequality that

$$\begin{aligned}\tilde{\Psi}(G)^2 &\leq 8\tilde{\Psi}\left(\frac{2}{M}\right)^2 + 8M\|\partial_y \Psi(g)\|_{L^2(0,M)}^2 \\ &\leq 8\tilde{\Psi}\left(\frac{2}{M}\right)^2 + 32M\mathcal{L}_1(g) + 128M^4 - 32M^2\Psi(0) + 64M^2\tilde{\Psi}\left(\frac{2}{M}\right) \\ &\leq 32M\mathcal{L}_1(g) + \mu_M,\end{aligned}$$

with

$$\mu_M := 1 + 128M^4 - 32M^2\Psi(0) + 64M^2\tilde{\Psi}\left(\frac{2}{M}\right) + 8\tilde{\Psi}\left(\frac{2}{M}\right)^2 + \Psi(0)^2 - 32M\Psi(0).$$

We have thus shown Lemma 4 when $G = \|g\|_{L^\infty(0,M)} > 1$. To complete the proof, we finally consider the case $G \in [0,1]$ and notice that, in that case,

$$0 \leq \tilde{\Psi}(G) \leq -\Psi(0) \quad \text{and} \quad \mathcal{L}_1(g) \geq \int_0^M \tilde{\Psi}(g)\ dy + M\Psi(0) \geq M\Psi(0),$$

since $\Psi_1 \leq 0$ in $(0,1)$ and $\tilde{\Psi} \geq 0$. Consequently,

$$\tilde{\Psi}(G)^2 \leq \Psi(0)^2 = 32M\Psi(0) + \Psi(0)^2 - 32M\Psi(0) \leq 32M\mathcal{L}_1(g) + \mu_M,$$

and the proof of Lemma 4 is complete. □

As an obvious consequence of Lemmas 2 and 4 we have the following result:

Corollary 5 *Let $a \in \mathcal{C}^1((0,\infty))$ be a positive function such that $a \in L^1(1,\infty)$. For $t \in [0,T_{max})$ and $y \in [0,M]$, we have*

$$0 \leq \tilde{\Psi}(f(t,y)) \leq \left(32M \max\left\{\mathcal{L}_1(f_0),0\right\} + \mu_M\right)^{1/2}.$$

Proof of Corollary 5. Clearly $\mathcal{L}_1(f(t)) = L_1(t) \leq L_1(0) = \mathcal{L}_1(f_0) \leq \max\left\{\mathcal{L}_1(f_0),0\right\}$ for $t \in [0,T_{max})$ by Lemma 2 and Corollary 5 readily follows from Lemma 4. □

Remark 6 *Corollary 5 provides an L^∞-bound on f only if $\Psi(r) \to \infty$ as $r \to \infty$, that is, if $a \notin L^1(0,1)$. In that case, it gives a positive lower bound for u by (11).*

We next turn to the proof of the second part of Theorem 1 for which we develop further the arguments from [4, Theorem 10].

Proof of Theorem 1 (i), Part 2. Assume now that $a \in L^1(1,\infty)$ and satisfies (6). We fix $M > 0$, $q > 2$, and $\varepsilon_M \in (0,1)$ such that

$$q > \max\left\{3+\vartheta, \frac{5+3\vartheta}{\alpha(\vartheta+1)-\vartheta}\right\} \quad \text{and} \quad \frac{q(q+1)}{M^2}\int_{1/\varepsilon_M}^\infty a(s)\ ds \le \frac{1}{2}\,, \tag{28}$$

the existence of ε_M being guaranteed by the integrability of a at infinity.

For

$$\delta \in \left(0, \min\left\{1, 2M, (2M)^{-1/q}\right\}\right)\,, \tag{29}$$

we put

$$f_0(y) := \frac{2(1-M\delta^q)}{\delta^2}\ (\delta-y)_+ + \delta^q \ge \delta^q > 0\,, \quad y \in [0,M]\,. \tag{30}$$

Then

$$\int_0^M f_0(y)\ dy = 1, \quad \|f_0\|_{L^\infty(0,M)} = \frac{2(1-M\delta^q)}{\delta} + \delta^q \le \frac{2}{\delta}\,. \tag{31}$$

Introducing next

$$m_q(t) := \int_0^M y^q f(t,y)\ dy\,, \quad t \in [0,T_{max})\,,$$

we have

$$m_q(0) = \left(\frac{2(1-M\delta^q)}{(q+1)(q+2)} + \frac{M^{q+1}}{q+1}\right)\ \delta^q \le C_1\delta^q \quad \text{with} \quad C_1 := \left(\frac{2+(q+2)M^{q+1}}{(q+1)(q+2)}\right)\,. \tag{32}$$

It follows from (12), (13), and the non-negativity of $\tilde\Psi$ that

$$\begin{aligned} \frac{dm_q}{dt} &= -q\int_0^M y^{q-1}\partial_y\tilde\Psi(f)\ dy + Mm_q - \frac{M^{q+1}}{q+1}, \\ \frac{dm_q}{dt} &\le q(q-1)\int_0^M y^{q-2}\tilde\Psi(f)\ dy + Mm_q - \frac{M^{q+1}}{q+1}\,. \end{aligned} \tag{33}$$

We shall now estimate the integral on the right-hand side of (33): to this end, we split the domain of integration into three parts which we handle differently. As a preliminary step, we notice that, by (6),

$$\Psi'(r) \le \gamma_\vartheta r^\vartheta \quad \text{and} \quad \Psi(r) \le \frac{\gamma_\vartheta}{\vartheta+1}\ r^{\vartheta+1} \le \gamma_\vartheta\ r^{\vartheta+1}\,, \quad r \ge 1\,. \tag{34}$$

We next define

$$K_0 := \left(32M\max\{\mathcal{L}_1(f_0),0\} + \mu_M\right)^{1/2(2+\vartheta)} > 1\,,$$

and consider $(t,y) \in [0,T_{max}) \times [0,M]$.

- If $f(t,y)\in(0,\varepsilon_M]$, it follows from (28) and the monotonicity of $\tilde{\Psi}$ that

$$\tilde{\Psi}(f(t,y)) \le \tilde{\Psi}(\varepsilon_M) = \int_{1/\varepsilon_M}^\infty a(s)\ ds \le \frac{M^2}{2q(q+1)}\,. \tag{35}$$

- If $f(t,y)\in(\varepsilon_M,K_0)$, then (34) and the monotonicity of $\tilde{\Psi}$ yield

$$\tilde{\Psi}(f(t,y)) = \frac{\tilde{\Psi}(f(t,y))}{f(t,y)}\ f(t,y) \le \frac{\Psi(K_0)-\Psi(0)}{\varepsilon_M}\ f(t,y) \le \frac{\gamma_\vartheta K_0^{\vartheta+1}-\Psi(0)}{\varepsilon_M}\ f(t,y)\,. \tag{36}$$

- If $f(t,y)\ge K_0$, Corollary 5 ensures that

$$\tilde{\Psi}(f(t,y)) = \frac{\tilde{\Psi}(f(t,y))}{f(t,y)}\ f(t,y) \le \frac{K_0^{\vartheta+2}}{K_0}\ f(t,y) \le K_0^{\vartheta+1}\ f(t,y)\,. \tag{37}$$

Consequently, recalling that $K_0>1$ and $\Psi(0)<0$, we deduce from (33) and (35)-(37) that

$$\begin{aligned}
\frac{dm_q}{dt} &\le q(q-1)\ \int_0^M y^{q-2}\tilde{\Psi}(f)\ \mathbf{1}_{(0,\varepsilon_M]}(f)\ dy + q(q-1)\ \int_0^M y^{q-2}\tilde{\Psi}(f)\ \mathbf{1}_{(\varepsilon_M,K_0)}(f)\ dy \\
&+ q(q-1)\ \int_0^M y^{q-2}\tilde{\Psi}(f)\ \mathbf{1}_{[K_0,\infty)}(f)\ dy + Mm_q - \frac{M^{q+1}}{q+1} \\
&\le \frac{(q-1)M^2}{2(q+1)}\ \int_0^M y^{q-2}\ dy + q(q-1)\ \frac{\gamma_\vartheta K_0^{\vartheta+1}-\Psi(0)}{\varepsilon_M}\ \int_0^M y^{q-2} f\ dy \\
&+ q(q-1)K_0^{\vartheta+1}\ \int_0^M y^{q-2} f\ dy + Mm_q - \frac{M^{q+1}}{q+1} \\
&\le C_2\ K_0^{\vartheta+1}\ \int_0^M y^{q-2} f\ dy + Mm_q - \frac{M^{q+1}}{2(q+1)}\,,
\end{aligned}$$

with $C_2 := q(q-1)(\gamma_\vartheta-\Psi(0)+\varepsilon_M)/\varepsilon_M$. We next use Hölder's inequality and (16) to conclude that

$$\frac{dm_q}{dt} \le C_2\ K_0^{\vartheta+1}\ m_q^{(q-2)/q} + Mm_q - \frac{M^{q+1}}{2(q+1)}\,. \tag{38}$$

It remains to estimate K_0 and in fact $\mathcal{L}_1(f_0)$. Since Ψ is negative on $(0,1)$ and Ψ_1 is bounded from below by $\Psi_1(0)$, it follows from (29) and (30) that

$$\begin{aligned}
\mathcal{L}_1(f_0) &\le \frac{2}{\delta^4}\ (1-M\delta^q)^2\ \int_0^\delta |\Psi'(f_0)|^2\ dy + \int_0^\delta \Psi(f_0)\ dy - M^2\Psi_1(0) \\
&\le \frac{2}{\delta^4}\ \int_0^\delta |\Psi'(f_0)|^2\ dy + \int_0^\delta \Psi(f_0)\ dy - M^2\Psi_1(0).
\end{aligned}$$

On the one hand, we infer from (31), (34), and the monotonicity of Ψ that

$$\int_0^\delta \Psi(f_0)\ dy \le \delta\ \Psi\left(\frac{2}{\delta}\right) \le \gamma_\vartheta 2^{\vartheta+1}\ \delta^{-\vartheta}\,.$$

On the other hand, we have

$$\begin{array}{rcl} f_0(y) & \ge & 1 \quad \text{for} \quad y\in[0,y_\delta] \quad \text{with} \quad y_\delta := \delta - \dfrac{1-\delta^q}{2(1-M\delta^q)}\delta^2 > 0\,, \\ f_0(y) & \in & [\delta^q,1] \quad \text{for} \quad y\in[y_\delta,\delta]\,, \end{array}$$

so that, if $y\in[0,y_\delta]$,

$$\Psi'(f_0(y))^2 \le \gamma_\vartheta^2 f_0(y)^{2\vartheta} \le \gamma_\vartheta 4^\vartheta\ \delta^{-2\vartheta}$$

by (31) and (34), while, if $y\in(y_\delta,\delta]$,

$$\Psi'(f_0(y))^2 \le \frac{1}{f_0(y)^4} a\left(\frac{1}{f_0(y)}\right)^2 \le C_\infty^2\ f_0(y)^{2(\alpha-2)} \le C_\infty^2\ \delta^{-2q(2-\alpha)}$$

by (6) since $\alpha\le 2$. Therefore,

$$\begin{array}{rcl} \mathcal{L}_1(f_0) & \le & \dfrac{2}{\delta^4}\left[\displaystyle\int_0^{y_\delta} \gamma_\vartheta 4^\vartheta\ \delta^{-2\vartheta}\ dy + \int_{y_\delta}^\delta C_\infty^2\ \delta^{-2q(2-\alpha)}\ dy\right] + \gamma_\vartheta 2^{\vartheta+1}\ \delta^{-\vartheta} - M^2\Psi_1(0) \\ & \le & \gamma_\vartheta 4^{\vartheta+1}\ \delta^{-3-2\vartheta} + C_\infty^2\ \dfrac{1-\delta^q}{2(1-M\delta^q)}\ \delta^{-2-2q(2-\alpha)} + \gamma_\vartheta 2^{\vartheta+1}\ \delta^{-\vartheta} - M^2\Psi_1(0) \\ & \le & \gamma_\vartheta 4^{\vartheta+1}\ \delta^{-2(2+\vartheta)} + C_\infty^2\ \delta^{-2-2q(2-\alpha)} + \gamma_\vartheta 2^{\vartheta+1}\ \delta^{-\vartheta} - M^2\Psi_1(0) \\ & \le & C_3\ \left(\delta^{-2(2+\vartheta)} + \delta^{-2-2q(2-\alpha)}\right) \end{array}$$

with $C_3 := \gamma_\vartheta 4^{\vartheta+2} + C_\infty^2 - M^2\Psi_1(0)$. Therefore,

$$K_0^{\vartheta+1} \le C_4\ \left(\delta^{-(\vartheta+1)} + \delta^{-(\vartheta+1)(1+q(2-\alpha))/(\vartheta+2)}\right) \tag{39}$$

for some constant $C_4>0$ depending only on M and a.

Combining (38) and (39) yields

$$\frac{dm_q}{dt} \le \Lambda_\delta(m_q) := C_5\ \left(\delta^{-(\vartheta+1)} + \delta^{-(\vartheta+1)(1+q(2-\alpha))/(\vartheta+2)}\right)\ m_q^{(q-2)/q}\ + Mm_q - \frac{M^{q+1}}{2(q+1)} \tag{40}$$

for $t\in[0,T_{max})$ and some constant $C_5>0$ depending only on M and a. At this point, we note that the monotonicity of Λ_δ and (40) imply that $\Lambda_\delta(m_q(t)) \le \Lambda_\delta(m_q(0))$ for $t\in[0,T_{max})$ if $\Lambda_\delta(m_q(0))<0$, the latter condition being satisfied for δ small enough as

$$\Lambda_\delta(m_q(0)) \le C_1^{(q-2)/q} C_5\ \left(\delta^{q-3-\vartheta} + \delta^{(q(\alpha(\vartheta+1)-\vartheta)-3\vartheta-5)/(\vartheta+2)}\right) + MC_1\ \delta^q - \frac{M^{q+1}}{2(q+1)}$$

by (28) and (32).

Summarizing, we have shown that, if δ satisfies (29) and

$$C_1^{(q-2)/q} C_5 \ \left(\delta^{q-3-\vartheta} + \delta^{(q(\alpha(\vartheta+1)-\vartheta)-3\vartheta-5)/(\vartheta+2)} \right) + M C_1 \ \delta^q < \frac{M^{q+1}}{2(q+1)} \,, \tag{41}$$

we have

$$\frac{dm_q}{dt}(t) \le \Lambda_\delta(m_q(t)) \le \Lambda_\delta(m_q(0)) < 0 \,, \quad t \in [0, T_{max}) \,,$$

an inequality which can only be true on a finite time interval owing to the non-negativity of m_q. Therefore, $T_{max} < \infty$ in that case and, for any $M > 0$, we have found an initial condition u_0 given by (10), (11), and (30) (for δ small enough according to the above analysis) such that $\langle u_0 \rangle = M$ and the first component u of the corresponding solution to (1)-(4) blows up in finite time. □

4 Global existence.

The proof of Theorem 1 (ii) also relies on the study of the function L_1 defined in Lemma 2. For that purpose, we first recall another property from [4]. We define the function E_1 by

$$E_1(h) := \frac{1}{2} \|\partial_y h\|_2^2 + \int_0^M \mathbf{1}_{(-\infty,0)}(h(y)) \ h(y) \ dy \,, \quad h \in H^1(0, M) \,, \tag{42}$$

for which we have the following lower bound.

Lemma 7 *[4, Lemma 9] For $M > 0$, we have*

$$E_1(h) \ge \frac{1}{4} \ \|\partial_y h\|_2^2 - M^3 - M \left| \Psi\left(\frac{1}{M}\right) \right| \,, \tag{43}$$

and

$$\|h\|_1 \le M^{3/2} \|\partial_y h\|_2 + M \left| \Psi\left(\frac{1}{M}\right) \right| \tag{44}$$

for every $h \in H^1(0, M)$ satisfying

$$\int_0^M \Psi^{-1}(h)(y) \ dy = 1 \,. \tag{45}$$

We now show that the non-integrability of a at infinity allows us to show that $T_{max} = \infty$. To this end, we use the alternative formulation (12)-(14) as in [4] and prove that f cannot vanish in finite time.

Proof of Theorem 1 (ii). Owing to (14) and the assumptions made on u_0, we have

$$0 < f_0(y) \le \frac{1}{m_0} \,, \quad y \in [0, M].$$

Introducing $\Sigma(t) := M^{-1} + e^{Mt}\left(m_0^{-1} - M^{-1}\right)$ for $t \geq 0$ we have

$$\partial_t \Sigma - \partial_y^2 \Psi(\Sigma) - M\Sigma + 1 = M\left(\Sigma - \frac{1}{M}\right) - M\Sigma + 1 = 0,$$

$$\Sigma(0) = \frac{1}{m_0} \geq f_0(y), \quad y \in (0, M),$$

and the comparison principle warrants that

$$f(t, y) \leq \Sigma(t), \quad (t, y) \in [0, T_{max}) \times [0, M]. \tag{46}$$

We now follow the strategy of the proof of [4, Theorem 5] and first use the properties of Ψ, Ψ_1, and (46) to estimate the function L_1 defined in Lemma 2 from below. Indeed, since $\Psi \geq 0$ on $(1, \infty)$ and $\Psi_1 \leq 0$ on $(0, 1)$ we arrive at

$$\begin{aligned} L_1(0) \geq L_1(t) &= \frac{1}{2}\|\partial_y \Psi(f(t))\|_{L^2(0,M)}^2 + \int_0^M \mathbf{1}_{(0,1)}(f(t,y))(\Psi - M\Psi_1)(f(t,y))\, dy \\ &+ \int_0^M \mathbf{1}_{(1,\infty)}(f(t,y))(\Psi - M\Psi_1)(f(t,y))\, dy \\ &\geq \frac{1}{2}\|\partial_y \Psi(f(t))\|_{L^2(0,M)}^2 + \int_0^M \mathbf{1}_{(-\infty,0)}(\Psi(f(t,y)))\Psi(f(t,y))\, dy \\ &- M\int_0^M \mathbf{1}_{(1,\infty)}(f(t,y))\Psi_1(f(t,y))\, dy \\ &\geq E_1(\Psi(t)) - M^2\Psi_1(\Sigma(t)), \end{aligned}$$

where E_1 is defined in (42) and we have used (46) to obtain the last inequality. Next, by Lemma 7 and (16), we have

$$L_1(0) \geq \frac{1}{4}\|\partial_y \Psi(f(t))\|_{L^2(0,M)}^2 - M^3 - M\left|\Psi\left(\frac{1}{M}\right)\right| - M^2\Psi_1(\Sigma(t)),$$

whence

$$\frac{1}{4}\|\partial_y \Psi(f(t))\|_{L^2(0,M)}^2 \leq L_1(0) + M^3 + M\left|\Psi\left(\frac{1}{M}\right)\right| + M^2\Psi_1(\Sigma(t)). \tag{47}$$

Using again Lemma 7, we have

$$\begin{aligned} \|\Psi(f(t))\|_{L^1(0,M)} &\leq M^{3/2}\|\partial_y \Psi(f(t))\|_{L^2(0,M)} + M\left|\Psi\left(\frac{1}{M}\right)\right| \\ &\leq 2M^{3/2}\left(L_1(0) + M^3 + M\left|\Psi\left(\frac{1}{M}\right)\right| + M^2\Psi_1(\Sigma(t))\right)^{1/2} + M\left|\Psi\left(\frac{1}{M}\right)\right|. \end{aligned}$$

Combining the previous inequality with (47) and the Poincaré inequality leads us to the bound

$$\|\Psi(f(t))\|_{H^1(0,M)} \leq C_6(T)\,, \quad t \in [0, T] \cap [0, T_{max})\,, \tag{48}$$

for all $T > 0$. Together with the continuous embedding of $H^1(0,M)$ in $L^\infty(0,M)$, (48) gives

$$-C_7(T) \le \Psi(f(t,y)) \le C_7(T)\,, \quad (t,y) \in ([0,T] \cap [0,T_{max})) \times [0,M]\,.$$

Since

$$\lim_{r\to 0} \Psi(r) = -\infty$$

due to $a \notin L^1(1,\infty)$, the above lower bound on $\Psi(f)$ ensures that $f(t)$ cannot vanish in finite time, from which Theorem 1 (ii) follows as already discussed in section 2. □

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