Dichotomies for evolution equations in Banach spaces

Codruţa Stoica

Abstract

The aim of this paper is to emphasize various concepts of dichotomies for evolution equations in Banach spaces, due to the important role they play in the approach of stable, instable and central manifolds. The asymptotic properties of the solutions of the evolution equations are studied by means of the asymptotic behaviors for skew-evolution semiflows.

MSC: 34D05, 34D09, 93D20

Keywords: evolution semiflow, evolution cocycle, skew-evolution semiflow, uniform exponential dichotomy, Barreira - Valls exponential dichotomy, exponential dichotomy, uniform polynomial dichotomy, Barreira - Valls polynomial dichotomy, polynomial dichotomy

1 Preliminaries

Recently, the important progress made in the study of evolution equations had a master role in the developing of a vast literature, concerning mostly the asymptotic properties of linear operators semigroups, evolution operators or skew-product semiflows.

In this paper, the study is led throughout the notion of skew-evolution semiflow on Banach spaces, defined by means of an evolution semiflow and an evolution cocycle. As the skew-evolutions semiflows reveal themselves to be generalizations of evolution operators and skew-product semiflows, they are appropriate to study the asymptotic properties of the solutions of evolution equations having the form

$$\begin{cases} \dot{u}(t) = A(t)u(t), \ t > t_0 \geq 0 \\ u(t_0) = u_0, \end{cases}$$

where $A : \mathbf{R} \to \mathcal{B}(V)$ is an operator, $\mathrm{Dom}A(t) \subset V$, $u_0 \in \mathrm{Dom}A(t_0)$.

The fact that a skew-evolution semiflow depends on three variables t, t_0 and x, while the classic concept of cocycle depends only on t and x, justifies the study of asymptotic behaviors in a nonuniform setting (relative to the third variable t_0) for skew-evolution semiflows.

The basic concepts of asymptotic properties, such as stability, instability and dichotomy, that appear in the theory of dynamical systems, play an important role in the study of stable, instable and central manifolds. We intend to define and exemplify various concepts of dichotomies, as uniform exponential dichotomy, Barreira-Valls exponential dichotomy, exponential dichotomy, uniform polynomial dichotomy, Barreira-Valls polynomial dichotomy, polynomial dichotomy and to emphasize connections between them. We have thus considered generalizations of some asymptotic properties for evolution equations, defined by L. Barreira and C. Valls in [1]. Characterizations for the asymptotic properties in a nonuniform setting are also proved.

Some of the original results concerning the properties of stability and instability for skew-evolution semiflows were published in [13] and [7].

The exponential dichotomy for evolution equations is one of the mathematical domains with an impressive development due to its role in describing several types of differential equations. Its study led to an extended literature, which begins with the interesting results due to O. Perron in [10]. The ideas were continued by J.L. Massera and J.J. Schäffer in [5], with extensions in the infinite dimensional case accomplished by J.L. Daleckiĭ and M.G. Kreĭn in [4] and A. Pazy in [9], respectively R.J. Sacker and G.R. Sell in [12]. Diverse and important concepts of dichotomy were introduced and studied by S.N. Chow and H. Leiva in [2] and by W.A. Coppel in [3].

Some asymptotic behaviors for evolution families were given in the nonuniform case in [6] by M. Megan, A.L. Sasu and B. Sasu. The study of the nonuniform exponential dichotomy for evolution families was considered by P. Preda and M. Megan in [11].

The property of exponential dichotomy for the case of skew-evolution semiflows is treated in [8] and [14].

2 Notations. Definitions. Examples

Let us consider a metric space (X, d), a Banach space V, V^* its topological dual and $\mathcal{B}(V)$ the space of all bounded linear operators from V into itself. I is the identity operator on V. We denote $T = \{(t, t_0) \in \mathbf{R}^2,\ t \geq t_0 \geq 0\}$

and $Y = X \times V$.

Definition 1 A mapping $\varphi : T \times X \to X$ is called *evolution semiflow* on X if following relations hold:
(s_1) $\varphi(t,t,x) = x,\ \forall (t,x) \in \mathbf{R}_+ \times X$;
(s_2) $\varphi(t,s,\varphi(s,t_0,x)) = \varphi(t,t_0,x), \forall (t,s),(s,t_0) \in T, x \in X$.

Definition 2 A mapping $\Phi : T \times X \to \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow φ if:
(c_1) $\Phi(t,t,x) = I,\ \forall (t,x) \in \mathbf{R}_+ \times X$;
(c_2) $\Phi(t,s,\varphi(s,t_0,x))\Phi(s,t_0,x) = \Phi(t,t_0,x), \forall (t,s),(s,t_0) \in T, x \in X$.

Definition 3 The mapping $C : T \times Y \to Y$ defined by the relation

$$C(t,s,x,v) = (\varphi(t,s,x), \Phi(t,s,x)v),$$

where Φ is an evolution cocycle over an evolution semiflow φ, is called *skew-evolution semiflow* on Y.

Example 1 We denote by $\mathcal{C} = \mathcal{C}(\mathbf{R}_+, \mathbf{R}_+)$ the set of all continuous functions $x : \mathbf{R}_+ \to \mathbf{R}_+$, endowed with the topology of uniform convergence on compact subsets of $\mathbf{R}_+$, metrizable by means of the distance

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x,y)}{1 + d_n(x,y)}, \text{ where } d_n(x,y) = \sup_{t \in [0,n]} |x(t) - y(t)|.$$

If $x \in \mathcal{C}$, then, for all $t \in \mathbf{R}_+$, we denote $x_t(s) = x(t+s)$, $x_t \in \mathcal{C}$. Let X be the closure in $\mathcal{C}$ of the set $\{f_t, t \in \mathbf{R}_+\}$, where $f : \mathbf{R}_+ \to \mathbf{R}_+^*$ is a decreasing function. It follows that (X,d) is a metric space. The mapping $\varphi : T \times X \to X$, $\varphi(t,s,x) = x_{t-s}$ is an evolution semiflow on X.

We consider $V = \mathbf{R}^2$, with the norm $\|v\| = |v_1| + |v_2|$, $v = (v_1, v_2) \in V$. The mapping $\Phi : T \times X \to \mathcal{B}(V)$ given by

$$\Phi(t,s,x)v = \left(e^{\alpha_1 \int_s^t x(\tau - s)d\tau} v_1, e^{\alpha_2 \int_s^t x(\tau - s)d\tau} v_2 \right),\ (\alpha_1, \alpha_2) \in \mathbf{R}^2,$$

is an evolution cocycle over φ and $C = (\varphi, \Phi)$ is a skew-evolution semiflow.

Remark 1 A connection between the solutions of a differential equation

$$\dot{u}(t) = A(t)u(t),\ t \in \mathbf{R}_+ \tag{1}$$

and a skew-evolution semiflow is given by the definition of the evolution cocycle Φ, by the relation $\Phi(t,s,x)v = U(t,s)v$, where $U(t,s) = u(t)u^{-1}(s)$, $(t,s) \in T$, $(x,v) \in Y$, and where $u(t)$, $t \in \mathbf{R}_+$, is a solution of the differential equation (1).

The fact that the skew-evolution semiflows are generalizations for skew-product semiflows is emphasized by

Example 2 Let X be the metric space defined as in Example 1. The mapping $\varphi_0 : \mathbf{R}_+ \times X \to X$, $\varphi_0(t,x) = x_t$, where $x_t(\tau) = x(t+\tau)$, $\forall \tau \geq 0$, is a semiflow on X. Let us consider for every $x \in X$ the parabolic system with Neumann's boundary conditions:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,y) = x(t)\dfrac{\partial^2 v}{\partial y^2}(t,y), & t>0, y \in (0,1) \\ v(0,y) = v_0(y), & y \in (0,1) \\ \dfrac{\partial v}{\partial y}(t,0) = \dfrac{\partial v}{\partial y}(t,1) = 0, & t > 0. \end{cases} \tag{2}$$

Let $V = \mathcal{L}^2(0,1)$ be a separable Hilbert space with the orthonormal basis $\{e_n\}_{n\in\mathbf{N}}$, $e_0 = 1$, $e_n(y) = \sqrt{2}\cos n\pi y$, where $y \in (0,1)$, $n \in \mathbf{N}$. We denote $D(A) = \{v \in \mathcal{L}^2(0,1),\ v(0) = v(1) = 0\}$ and we define the operator

$$A : D(A) \subset V \to V,\ Av = \frac{d^2 v}{dy^2},$$

which generates a $\mathcal{C}_0$-semigroup S, defined by $S(t)v = \sum\limits_{n=0}^{\infty} e^{-n^2\pi^2 t}\langle v, e_n\rangle e_n$, where $\langle \cdot,\cdot \rangle$ denotes the scalar product in V. We define for every $x \in X$, $A(x) : D(A) \subset V \to V$, $A(x) = x(0)A$, which allows us to rewrite system (2) in V as

$$\begin{cases} \dot{v}(t) = A(\varphi_0(t,x))v(t), & t > 0 \\ v(0) = v_0. \end{cases} \tag{3}$$

The mapping

$$\Phi_0 : \mathbf{R}_+ \times X \to \mathcal{B}(V),\ \Phi_0(t,x)v = S\left(\int_0^t x(s)ds\right)v$$

is a cocycle over the semiflow φ_0 and $C_0 = (\varphi_0, \Phi_0)$ is a linear skew-product semiflow strongly continuous on Y. Also, for all $v_0 \in D(A)$, we have obtained that $v(t) = \Phi(t,x)x_0,\ t \geq 0$, is a strongly solution of system (3).

As $C_0 = (\varphi_0, \Phi_0)$ is a skew-product semiflow on Y, then the mapping $C : T \times Y \to Y$, $C(t,s,x,v) = (\varphi(t,s,x), \Phi(t,s,x)v)$, where

$$\varphi(t,s,x) = \varphi_0(t-s,x) \text{ and } \Phi(t,s,x) = \Phi_0(t-s,x),\ \forall (t,s,x) \in T \times X$$

is a skew-evolution semiflow on Y. Hence, the skew-evolution semiflows generalize the notion of skew-evolution semiflows.

An interesting class of skew-evolution semiflows, useful to describe some asymptotic properties, is given by

Example 3 Let us consider a skew-evolution semiflow $C = (\varphi, \Phi)$ and a parameter $\lambda \in \mathbf{R}$. We define the mapping

$$\Phi_\lambda : T \times X \to \mathcal{B}(V),\ \Phi_\lambda(t, t_0, x) = e^{\lambda(t-t_0)}\Phi(t, t_0, x). \tag{4}$$

One can remark that $C_\lambda = (\varphi, \Phi_\lambda)$ also satisfies the conditions of Definition 3, being called λ-*shifted skew-evolution semiflow* on Y.

Let us consider on the Banach space V the Cauchy problem

$$\begin{cases} \dot{v}(t) = Av(t),\ t > 0 \\ v(0) = v_0 \end{cases}$$

with the nonlinear operator A. Let us suppose that A generates a nonlinear C_0-semigroup $\mathcal{S} = \{S(t)\}_{t\geq 0}$. Then $\Phi(t, s, x)v = S(t-s)v$, where $t \geq s \geq 0$, $(x, v) \in Y$, defines an evolution cocycle. Moreover, the mapping defined by $\Phi_\lambda : T \times X \to \mathcal{B}(V)$, $\Phi_\lambda(t, s, x)v = S_\lambda(t-s)v$, where $\mathcal{S}_\lambda = \{S_\lambda(t)\}_{t\geq 0}$ is generated by the operator $A - \lambda I$, is also an evolution cocycle.

Definition 4 A skew-evolution semiflow $C = (\varphi, \Phi)$ is said to be *strongly measurable* if, for all $(t_0, x, v) \in T \times Y$, the mapping $s \mapsto \|\Phi(s, t_0, x)v\|$ is measurable on $[t_0, \infty)$.

Definition 5 The skew-evolution semiflow $C = (\varphi, \Phi)$ is said to have *exponential growth* if there exist $M, \omega : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that:

$$\|\Phi(t, t_0, x)v\| \leq M(s)e^{\omega(t-s)} \|\Phi(s, t_0, x)v\|, \forall (t, s), (s, t_0) \in T, \forall (x, v) \in Y.$$

Remark 2 If $C = (\varphi, \Phi)$ is a skew-evolution semiflow with exponential growth, as following relations

$$\|\Phi_\lambda(t, t_0, x)v\| = e^{\lambda(t-t_0)} \|\Phi(t, t_0, x)v\| \leq M(t_0)e^{[\omega(t_0)+\lambda](t-t_0)} \|v\|,$$

hold for all $(t_0, x, v) \in \mathbf{R}_+ \times Y$, then $C_\lambda = (\varphi, \Phi_\lambda)$, $\lambda > 0$, has also exponential growth.

Remark 3 (i) If we consider in Definition 5 the constants $M \geq 1$ and $\omega > 0$, the skew-evolution semiflow C is said to have *uniform exponential growth*;

(ii) If in Definition 5 we consider $M \geq 1$ to be a constant such that the relation $\|\Phi(t, s, x)\| \leq Me^{\omega(t-s)}$ holds for all $(t, s) \in T$ and all $x \in X$, the skew-evolution semiflow C is said to have *bounded exponential growth.*

Definition 6 The skew-evolution semiflow $C=(\varphi,\Phi)$ is said to have *exponential decay* if there exist $M,\omega:\mathbf{R}_+\to\mathbf{R}_+^*$ such that:

$$\|\Phi(s,t_0,x)v\|\leq M(t)e^{\omega(t-s)}\|\Phi(t,t_0,x)v\|,\forall(t,s),(s,t_0)\in T,\forall(x,v)\in Y.$$

Remark 4 If $C=(\varphi,\Phi)$ be a skew-evolution semiflow with exponential decay, as following relations

$$\|\Phi_{-\lambda}(s,t_0,x)v\|=e^{-\lambda(s-t_0)}\|\Phi(s,t_0,x)v\|\leq M(t)e^{[\omega(t)+\lambda](t-s)}\|\Phi_{-\lambda}(t,t_0,x)v\|,$$

hold for all $(t,s),(s,t_0)\in T$ and all $(x,v)\in Y$, then $C_{-\lambda}=(\varphi,\Phi_{-\lambda})$, $\lambda>0$, has also exponential decay.

Remark 5 If in Definition 6 we consider $M\geq 1$ and $\omega>0$ to be constants, the skew-evolution semiflow C is said to have *uniform exponential decay*.

3 On various classes of dichotomy

Let $C:T\times Y\to Y$, $C(t,s,x,v)=(\varphi(t,s,x),\Phi(t,s,x)v)$ be a skew-evolution semiflow on Y.

Definition 7 A continuous mapping $P:Y\to Y$ defined by:

$$P(x,v)=(x,P(x)v),\ \forall(x,v)\in Y, \tag{5}$$

where $P(x)$ is a linear projection on Y_x, is called *projector* on Y.

Remark 6 The mapping $P(x):Y_x\to Y_x$ is linear and bounded and satisfies the relation $P(x)P(x)=P^2(x)=P(x)$ for all $x\in X$.

For all projectors $P:Y\to Y$ we define the sets

$$ImP=\{(x,v)\in Y,P(x)v=v\}\text{ and }KerP=\{(x,v)\in Y,P(x)v=0\}.$$

Remark 7 Let P be a projector on Y. Then ImP and $KerP$ are closed subsets of Y and for all $x\in X$ we have

$$ImP(x)+KerP(x)=Y_x\text{ and }ImP(x)\cap KerP(x)=\{0\}.$$

Remark 8 If P is a projector on Y, then the mapping

$$Q:Y\to Y,\ Q(x,v)=(x,v-P(x)v) \tag{6}$$

is also a projector on Y, called *the complementary projector* of P.

Definition 8 A projector P on Y is called *invariant* relative to a skew-evolution semiflow $C = (\varphi, \Phi)$ if following relation holds:

$$P(\varphi(t,s,x))\Phi(t,s,x) = \Phi(t,s,x)P(x), \tag{7}$$

for all $(t,s) \in T$ and all $x \in X$.

Remark 9 If the projector P is invariant relative to a skew-evolution semiflow C, then its complementary projector Q is also invariant relative to C.

Definition 9 A projector P_1 and its complementary projector P_2 are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if

(d_1) the projectors P_1 and P_2 are invariant on Y;

(d_2) for all $x \in X$, the projections $P_1(x)$ and $P_2(x)$ commute and the relation $P_1(x)P_2(x) = 0$ holds.

In what follows we will denote

$$\Phi_k(t,t_0,x) = \Phi(t,t_0,x)P_k(x),\ \forall (t,t_0) \in T,\ \forall x \in X,\ \forall k \in \{1,2\}.$$

We remark that Φ_k, $k \in \{1,2\}$ are evolution cocycles and

$$C_k(t,s,x,v) = (\varphi(t,s,x), \Phi_k(t,s,x)v),\ \forall (t,t_0,x,v) \in T \times Y,\ \forall k \in \{1,2\},$$

are skew-evolution semiflows, over all evolution semiflows φ on X.

Definition 10 The skew-evolution semiflow $C = (\varphi, \Phi)$ is called *uniformly exponentially dichotomic* if there exist two projectors P_1 and P_2 compatible with C, some constants $N_1 \geq 1$, $N_2 \geq 1$ and ν_1, $\nu_2 > 0$ such that:

$$e^{\nu_1(t-s)} \left\| \Phi_1(t,t_0,x)v \right\| \leq N_1 \left\| \Phi_1(s,t_0,x)v \right\|; \tag{8}$$

$$e^{\nu_2(t-s)} \left\| \Phi_2(s,t_0,x)(x)v \right\| \leq N_2 \left\| \Phi_2(t,t_0,x)(x)v \right\|, \tag{9}$$

for all $(t,s),(s,t_0) \in T$ and all $(x,v) \in Y$.

Remark 10 Without any loss of generality we can consider

$$N = \max\{N_1, N_2\} \text{ and } \nu = \min\{\nu_1, \nu_2\}.$$

We will call N_1, N_2, ν_1, ν_2, respectively N, ν *dichotomic characteristics.*

In what follows we will define generalizations for skew-evolution semiflows of some asymptotic properties given by L. Barreira and C. Valls for evolution equations in [1].

Definition 11 The skew-evolution semiflow $C=(\varphi,\Phi)$ is called *Barreira-Valls exponentially dichotomic* if there exist two projectors P_1 and P_2 compatible with C, some constants $N\geq 1$, $\alpha_1,\ \alpha_2>0$ and $\beta_1,\ \beta_2>0$ such that:

$$\|\Phi_1(t,t_0,x)v\|\leq Ne^{-\alpha_1 t}e^{\beta_1 s}\|\Phi_1(s,t_0,x)v\|; \tag{10}$$

$$\|\Phi_2(s,t_0,x)v\|\leq Ne^{-\alpha_2 t}e^{\beta_2 s}\|\Phi_2(t,t_0,x)v\|, \tag{11}$$

for all $(t,s),(s,t_0)\in T$ and all $(x,v)\in Y$.

Definition 12 The skew-evolution semiflow $C=(\varphi,\Phi)$ is called *exponentially dichotomic* if there exist two projectors P_1 and P_2 compatible with C, some mappings $N_1,\ N_2:\mathbf{R}_+\to\mathbf{R}_+^*$ and some constants $\nu_1,\ \nu_2>0$ such that:

$$\|\Phi_1(t,t_0,x)v\|\leq N_1(s)e^{-\nu_1 t}\|\Phi_1(s,t_0,x)v\|; \tag{12}$$

$$\|\Phi_2(s,t_0,x)v\|\leq N_2(s)e^{-\nu_2 t}\|\Phi_2(t,t_0,x)v\|, \tag{13}$$

for all $(t,s),(s,t_0)\in T$ and all $(x,v)\in Y$.

Some immediate connections concerning the previously defined asymptotic properties for skew-evolution semiflows are given by

Remark 11 (i) A uniformly exponentially dichotomic skew-evolution semiflow is Barreira-Valls exponentially dichotomic;

(ii) Barreira-Valls exponentially dichotomic skew-evolution semiflow is exponentially dichotomic.

The reciprocal statements are not true, as shown in what follows. Hence, the next example emphasizes a skew-evolution semiflow which is Barreira-Valls exponentially dichotomic, but is not uniformly exponentially dichotomic.

Example 4 Let $f:\mathbf{R}_+\to(0,\infty)$ be a decreasing function with the property that there exists $\lim\limits_{t\to\infty}f(t)=l>0$. We will consider $\lambda>f(0)$. Let $\mathcal{C}=\mathcal{C}(\mathbf{R},\mathbf{R})$ be the metric space of all continuous functions $x:\mathbf{R}\to\mathbf{R}$, with the topology of uniform convergence on compact subsets of $\mathbf{R}$. $\mathcal{C}$ is metrizable relative to the metric given in Example 1. We denote X the closure in $\mathcal{C}$ of the set $\{f_t,\ t\in\mathbf{R}_+\}$, where $f_t(\tau)=f(t+\tau)$, $\forall\tau\in\mathbf{R}_+$. Then (X,d) is a metric space. The mapping

$$\varphi:T\times X\to X,\ \varphi(t,s,x)(\tau)=x_{t-s}(\tau)=x(t-s+\tau)$$

is an evolution semiflow on X. Let us consider the Banach space $V = \mathbf{R}^2$ with the norm $\|v\| = |v_1| + |v_2|$, $v = (v_1, v_2) \in V$. The mapping

$$\Phi : T \times X \to \mathcal{B}(V),\ \Phi(t,s,x)v =$$

$$= \left(e^{t\sin t - s\sin s - 2(t-s) - \int_s^t x(\tau - s)d\tau} v_1,\ e^{3(t-s) - 2t\cos t + 2s\cos s + \int_s^t x(\tau-s)d\tau} v_2\right),$$

where $t \geq s \geq 0$, $(x,v) \in Y$, is an evolution cocycle over the evolution semiflow φ. We consider the projectors $P_1, P_2 : Y \to Y$, $P_1(x,v) = (v_1, 0)$, $P_2(x,v) = (0, v_2)$, for all $x \in X$ and all $v = (v_1, v_2) \in V$, compatible with the skew-evolution semiflow $C = (\varphi, \Phi)$.

We have, according to the properties of function x,

$$|\Phi(t,s,x)P_1(x)v| = e^{t\sin t - s\sin s + 2s - 2t} e^{-\int_s^t x(\tau-s)d\tau} |v_1| \leq$$

$$\leq e^{-t+3s} e^{-l(t-s)} |v_1| = e^{-(1+l)t} e^{(3+l)s} |v_1|,$$

for all $(t,s,x,v) \in T \times Y$.

Also, following relations

$$|\Phi(t,s,x)P_2(x)v| = e^{3t - 3s - 2t\cos t + 2s\cos s + \int_s^t x(\tau-s)d\tau} |v_2| \geq$$

$$\geq e^{t-s} e^{l(t-s)} |v_2| = e^{(1+l)t} e^{-(1+l)s} |v_2|,$$

hold for all $(t,s,x,v) \in T \times Y$.

Hence, the skew-evolution semiflow $C = (\varphi, \Phi)$ is Barreira-Valls exponentially dichotomic with $N = 1$, $\alpha_1 = \alpha_2 = \beta_2 = 1 + l$, $\beta_1 = 3 + l$.

Let us suppose now that $C = (\varphi, \Phi)$ is uniformly exponentially dichotomic. According to Definition 10, there exist $N \geq 1$ and $\nu_1 > 0$, $\nu_2 > 0$ such that

$$e^{t\sin t - s\sin s + 2s - 2t} e^{-\int_s^t x(\tau-s)d\tau} |v_1| \leq N e^{-\nu_1(t-s)} |v_1|,\ \forall t \geq s \geq 0$$

and

$$N e^{3t - 3s - 2t\cos t + 2s\cos s} e^{\int_s^t x(\tau-s)d\tau} |v_2| \geq e^{\nu_2(t-s)} |v_2|,\ \forall t \geq s \geq 0.$$

If we consider $t = 2n\pi + \frac{\pi}{2}$ and $s = 2n\pi$, we have in the first inequality

$$e^{2n\pi - \frac{\pi}{2}} \leq N e^{-\nu\frac{\pi}{2}} e^{\int\limits_{2n\pi}^{2n\pi+\frac{\pi}{2}} x(\tau - 2n\pi)d\tau} \leq N e^{(-\nu_1+\lambda)\frac{\pi}{2}},$$

which, for $n \to \infty$, leads to a contradiction. In the second inequality, if we consider $t = 2n\pi$ and $s = 2n\pi - \pi$, we obtain

$$Ne^{-4n\pi+3\pi} \geq e^{\nu_2\pi} e^{\int\limits_{2n\pi-\pi}^{2n\pi} x(\tau-2n\pi+\pi)d\tau} \geq e^{(\nu_2-\lambda)\pi}.$$

For $n \to \infty$, a contradiction is obtained.

We obtain that C is not uniformly exponentially dichotomic.

There exist exponentially dichotomic skew-evolution semiflows that are not Barreira-Valls exponentially dichotomic, as in the next

Example 5 We consider the metric space (X, d), the Banach space V, the evolution semiflow φ and the projectors P_1 and P_2 defined as in Example 4. Let us consider a continuous function

$$g : \mathbf{R}_+ \to [1, \infty) \text{ with } g(n) = e^{n \cdot 2^{2n}} \text{ and } g\left(n + \frac{1}{2^{2n}}\right) = 1.$$

The mapping $\Phi : T \times X \to \mathcal{B}(V)$, defined by

$$\Phi(t, s, x)v = \left(\frac{g(s)}{g(t)} e^{-(t-s)-\int_s^t x(\tau-s)d\tau} v_1, \frac{g(t)}{g(s)} e^{t-s+\int_s^t x(\tau-s)d\tau} v_2\right)$$

is an evolution cocycle over the evolution semiflow φ. As

$$|\Phi(t, s, x)P_1(x)v| \leq g(s)e^{-(1+l)(t-s)}|v_1|, \; \forall(t, s, x, v) \in T \times Y$$

and

$$g(s)\,|\Phi(t, s, x)P_2(x)v| \geq e^{(1+l)(t-s)}|v_2|, \; \forall(t, s, x, v) \in T \times Y,$$

the skew-evolution semiflow $C = (\varphi, \Phi)$ is exponentially dichotomic, with $N_1(u) = N_2(u) = g(u) \cdot e^{(1+l)u}$, $u \geq 0$, and $\nu_1 = \nu_2 = 1 + l$.

Let us suppose that C is Barreira-Valls exponentially dichotomic. There exist $N \geq 1$ and α_1, α_2, β_1, $\beta_2 > 0$ such that

$$\frac{g(s)}{g(t)} e^s \leq Ne^t e^{-\alpha_1 t} e^{\beta_1 s} e^{\int_s^t x(\tau-s)d\tau}$$

and

$$e^{\alpha_2 t} e^{-t} \leq N \frac{g(t)}{g(s)} e^{\beta_2 s} e^{-s} e^{\int_s^t x(\tau-s)d\tau}.$$

Further, if we consider $t = n + \dfrac{1}{2^{2n}}$ and $s = n$, it follows that

$$e^{n(2^{2n}+1+\alpha_1-\beta_1)} \leq Ne^{\frac{\lambda-\alpha_1}{2^{2n}}} \text{ and } e^{n(2^n+\alpha_2-\beta_2)} \leq Ne^{\frac{1+\lambda-\alpha_2}{2^{2n}}}.$$

As, for $n \to \infty$, two contradictions are obtained, it follows that C is not Barreira-Valls exponentially dichotomic.

Let us present some particular classes of dichotomy, given by

Definition 13 A skew-evolution semiflow $C = (\varphi, \Phi)$ is *uniformly polynomially dichotomic* if there exist two projectors P_1 and P_2 compatible with C and some constants $N \geq 1$ and $\alpha_1 > 0$, $\alpha_2 > 0$ such that:

$$\|\Phi_1(t,s,x)v\| \leq Nt^{-\alpha_1}s^{\alpha_1}\|P_1(x)v\|; \tag{14}$$

$$\|P_2(x)v\| \leq Nt^{-\alpha_2}s^{\alpha_2}\|\Phi_2(t,s,x)v\|; \tag{15}$$

for all $(t,s) \in T$ and all $(x,v) \in Y$.

Definition 14 A skew-evolution semiflow $C = (\varphi, \Phi)$ is *Barreira-Valls polynomially dichotomic* if there exist some constants $N \geq 1$, $\alpha_1 > 0$, $\alpha_2 > 0$ and $\beta_1 > 0$, $\beta_2 > 0$ such that:

$$\|\Phi_1(t,s,x)v\| \leq Nt^{-\alpha_1}s^{\beta_1}\|P_1(x)v\|; \tag{16}$$

$$\|P_2(x)v\| \leq Nt^{-\alpha_2}s^{\beta_2}\|\Phi_2(t,s,x)v\|, \tag{17}$$

for all $(t,s) \in T$ and all $(x,v) \in Y$.

Definition 15 A skew-evolution semiflow $C = (\varphi, \Phi)$ is *polynomially dichotomic* if there exist a function $N : \mathbf{R}_+ \to [1,\infty)$, some constants $\alpha_1 > 0$ and $\alpha_2 > 0$ such that:

$$\|\Phi_1(t,s,x)v\| \leq N(s)t^{-\alpha_1}\|P_1(x)v\|; \tag{18}$$

$$\|P_2(x)v\| \leq N(s)t^{-\alpha_2}\|\Phi_2(t,s,x)v\|, \tag{19}$$

for all $(t,s) \in T$ and all $(x,v) \in Y$.

Relations between the defined classes of dichotomy are described by

Remark 12 (i) A uniformly polynomially dichotomic skew-evolution semiflow is Barreira-Valls polynomially dichotomic;

(ii) A Barreira-Valls polynomially dichotomic is polynomially dichotomic.

The next example shows a skew-evolution semiflow which is Barreira-Valls polynomially dichotomic but is not uniformly polynomially dichotomic.

Example 6 We consider the metric space (X, d), the Banach space V, the evolution semiflow φ and the projectors P_1 and P_2 defined as in Example 4. We will consider the mapping

$$g : \mathbf{R}_+ \to \mathbf{R},\ \ g(t) = (t+1)^{3-\sin \ln(t+1)}.$$

We define

$$\Phi(t,s,x)v = \left(\frac{g(s)}{g(t)} e^{-\int_s^t x(\tau-s)d\tau} v_1, \frac{g(t)}{g(s)} e^{\int_s^t x(\tau-s)d\tau} v_2\right), (t,s) \in T,\ (x,v) \in Y.$$

Φ is an evolution cocycle over φ. Due to the properties of function x and of function $f : (0,\infty) \to (0,\infty),\ f(u) = \dfrac{e^u}{u}$, we have

$$|\Phi_1(t,s,x)v| \le \frac{(s+1)^4}{(t+1)^2} e^{-l(t-s)} |v_1| \le (s+1)^2 \left(\frac{s+1}{t+1}\right)^2 e^{-lt} e^{ls} |v_1| \le$$

$$\le \frac{s(s+1)^2}{t} t^{-l} s^l |v_1| \le 4 t^{-(1+l)} s^{3+l} |v_1|,$$

for all $t \ge s \ge t_0 = 1$ and all $(x,v) \in Y$. Also, following relations

$$|\Phi_2(t,s,x)v| \ge \frac{(s+1)^4}{(t+1)^2} e^{-l(t-s)} |v_2| \ge \frac{(t+1)^2}{(s+1)^4} e^{lt} e^{-ls} |v_2| \ge t^{2+l} s^{-8-l} |v_2|,$$

hold for all $t \ge s \ge t_0 = 1$ and all $(x,v) \in Y$.

Hence, by Definition 14, the skew-evolution semiflow $C = (\varphi, \Phi)$ is Barreira-Valls polynomially dichotomic.

We suppose now that C is uniformly polynomially dichotomic. According to Definition 13, there exist $N \ge 1$ and $\alpha_1 > 0$ such that

$$\frac{(s+1)^3}{(t+1)^3} \frac{(t+1)^{\sin \ln(t+1)}}{(s+1)^{\sin \ln(s+1)}} \le N t^{-\alpha_1} s^{\alpha_1} e^{\int_s^t x(\tau-s)d\tau}$$

for all $t \ge s \ge t_0$. Let us consider

$$t = e^{2n\pi + \frac{\pi}{2}} - 1 \text{ and } s = e^{2n\pi - \frac{\pi}{2}} - 1.$$

We have, if we consider the properties of function x, that

$$e^{(2n-\lambda-1)\pi} \le N e^{2\alpha_1},$$

which, if $n \to \infty$, leads to a contradiction.

Also, as in Definition 13, there exist $N \geq 1$ and $\alpha_1 > 0$ such that

$$N \frac{(t+1)^3}{(s+1)^3} \frac{(s+1)^{\sin \ln(s+1)}}{(t+1)^{\sin \ln(t+1)}} \geq t^{\alpha_2} s^{-\alpha_2} e^{-\int_s^t x(\tau-s)d\tau}$$

for all $t \geq s \geq t_0$, which implies, for $t = e^{2n\pi+\frac{\pi}{2}} - 1$ and $s = e^{2n\pi-\frac{\pi}{2}} - 1$,

$$Ne^{(-2n+\lambda-1)\pi} \geq e^{-2\alpha_2},$$

which, for $n \to \infty$, is a contradiction.

We obtain thus that C is not uniformly polynomially dichotomic.

There exist skew-evolution semiflows that are polynomially dichotomic but are not Barreira-Valls polynomially dichotomic.

Example 7 Let us consider the data given in Example 5. We obtain

$$|\Phi(t,s,x)P_1(x)v| \leq g(s)e^{-(1+l)(t-s)}|v_1| \leq g(s)e^{(1+l)s}t^{-(1+l)}$$

and

$$g(s)e^{(1+l)s}\,|\Phi(t,s,x)P_2(x)v| \geq e^{(1+l)t}|v_2| \geq t^{(1+l)}|v_2|,$$

for all $(t,s,x,v) \in T \times Y$, which proves that the skew-evolution semiflow $C = (\varphi, \Phi)$ is polynomially dichotomic.

If we suppose that C is Barreira-Valls polynomially dichotomic, there exist $N \geq 1$, $\alpha_1 > 0$, $\alpha_2 > 0$ and $\beta_1 > 0$, $\beta_2 > 0$ such that

$$\frac{g(s)}{g(t)} \leq Nt^{-\alpha_1} s^{\beta_1} e^{t-s+\int_s^t x(\tau-s)d\tau} \text{ and } t^{\alpha_2} \leq N \frac{g(t)}{g(s)} s^{\beta_2} e^{t-s+\int_s^t x(\tau-s)d\tau}.$$

If we consider $t = n + \dfrac{1}{2^{2n}}$ and $s = n$, we obtain

$$e^{n\cdot 2^{2n}} \leq N \cdot n^{-\alpha_1} \cdot n^{\beta_1} \cdot e^{\frac{1+\lambda}{2^{2n}}} \text{ and } e^{n\cdot 2^{2n}} \leq N \left(n + \frac{1}{2^{2n}}\right)^{-\alpha_2} \cdot n^{\beta_2} \cdot e^{\frac{1+\lambda}{2^{2n}}}.$$

For $n \to \infty$, two contradictions are obtained, which proves that C is not Barreira-Valls polynomially dichotomic.

4 Main results

The first results will prove some relations between all the classes of dichotomies.

Proposition 1 *A uniformly exponentially dichotomic skew-evolution semiflow $C=(\varphi,\Phi)$ is uniformly polynomially dichotomic.*

Proof. Let us consider in Definition 10, without any loss of generality, $t_0=1$. It also assures the existence of constants $N\geq 1$ and $\nu_1>0$ such that $\|\Phi_1(t,s,x)v\|\leq Ne^{-\nu_1(t-s)}\|P_1(x)v\|$. As

$$e^{-u}\leq\frac{1}{u+1},\ \forall u\geq 0 \text{ and } \frac{t}{s}\leq t-s+1,\ \forall t\geq s\geq 1,$$

it follows that

$$\|\Phi_1(t,s,x)v\|\leq N(t-s+1)^{-\nu_1}\|P_1(x)v\|\leq Nt^{-\nu_1}s^{\nu_1}\|P_1(x)v\|,$$

for all $t\geq s\geq 1$ and all $(x,v)\in Y$.

We also have the property of function

$$f:(0,\infty)\to(0,\infty),\ f(u)=\frac{e^u}{u}$$

of being nondecreasing, which assures the inequality

$$\frac{e^s}{e^t}\leq\frac{s}{t},\ \forall t\geq s>0$$

and, further, for all $t\geq s\geq 1$ and all $(x,v)\in Y$, we have

$$\|P_2(x)v\|\leq Ne^{-\nu_2 t}e^{\nu_2 s}\|\Phi_2(t,s,x)v\|\leq Nt^{-\nu_2}s^{\nu_2}\|\Phi_2(t,s,x)v\|,$$

where constants $N\geq 1$ and $\nu_2>0$ are also given by Definition 10.

Thus, according to Definition 13, C is uniformly polynomially dichotomic.

We give an example of a skew-evolution semiflow which is uniformly polynomially dichotomic, but is not uniformly exponentially dichotomic.

Example 8 Let (X,d) be the metric space, V the Banach space, φ the evolution semiflow, P_1 and P_2 the projectors given as in Example 4.

Let us consider the function $g:\mathbf{R}_+\to\mathbf{R}$, given by $g(t)=t^2+1$ and let us define

$$\Phi(t,s,x)v=\left(\frac{g(s)}{g(t)}e^{-\int_s^t x(\tau-s)d\tau}v_1,\frac{g(t)}{g(s)}e^{\int_s^t x(\tau-s)d\tau}v_2\right),\ (t,s)\in T,\ (x,v)\in Y.$$

We can consider $t_0 = 1$ in Definition 13. As, $\frac{s^2+1}{t^2+1} \leq \frac{s}{t}$, for $t \geq s \geq 1$ and according to the properties of function x, we have

$$\frac{s^2+1}{t^2+1} e^{-\int_s^t x(\tau - s)d\tau} |v_1| \leq t^{-(1+l)} s^{1+l} |v_1|$$

and

$$\frac{t^2+1}{s^2+1} e^{\int_s^t x(\tau - s)d\tau} |v_2| \geq t^{(2+l)} s^{-(4+l)} |v_2|,$$

for all $t \geq s \geq 1$ and all $v \in V$. It follows that $C = (\varphi, \Phi)$ is uniformly polynomially dichotomic.

If the skew-evolution semiflow $C = (\varphi, \Phi)$ is also uniformly exponentially dichotomic, according to Definition 10, there exist $N \geq 1$ $\nu_1 > 0$ and $\nu_2 > 0$ such that

$$\frac{s^2+1}{t^2+1} |v_1| \leq N e^{-\nu_1 (t-s)} e^{-l(t-s)} |v_1| \text{ and } N \frac{t^2+1}{s^2+1} |v_2| \geq e^{\nu_2 (t-s)} e^{l(t-s)} |v_2|,$$

for all $t \geq s \geq t_0$ and all $v \in V$. If we consider $s = t_0$ and $t \to \infty$, two contradictions are obtained, which proves that C is not uniformly exponentially dichotomic.

Proposition 2 *A Barreira-Valls exponentially dichotomic skew-evolution semiflow $C = (\varphi, \Phi)$ with $\alpha_i \geq \beta_i > 0$, $i \in \{1, 2\}$, is Barreira-Valls polynomially dichotomic.*

Proof. According to Definition 11, there exist some constants $N \geq 1$, $\alpha_1 > 0$ and $\beta_1 > 0$ such that

$$\|\Phi_1(t, s, x)v\| \leq N e^{-\alpha_1 t} e^{\beta_1 s} \|P_1(x)v\|, \ \forall (t, s) \in T, \ \forall (x, v) \in Y.$$

As the mapping $f : (0, \infty) \to (0, \infty)$, defined by $f(u) = \frac{e^u}{u}$ is nondecreasing, and as, by hypothesis, we can chose $\alpha_1 \geq \beta_1$, we obtain that

$$\|\Phi_1(t, s, x)v\| \leq N t^{-\alpha_1} e^{-\beta_1 s} s^{\beta_1} e^{\beta_1 s} \|P_1(x)v\| = N t^{-\alpha_1} s^{\beta_1} \|P_1(x)v\|,$$

for all $t \geq s > 0$ and all $(x, v) \in Y$.

Analogously, we obtain

$$\|P_2(x)v\| \leq N e^{-\alpha_2 t} e^{\beta_2 s} \|\Phi_2(t, s, x)v\| \leq N t^{-\alpha_2} s^{\beta_2} \|\Phi_2(t, s, x)v\|,$$

for all $t \geq s > 0$ and all $(x, v) \in Y$, where the constants $N \geq 1$, $\alpha_2 > 0$ and $\beta_2 > 0$ are also assured by Definition 10, with the property $\alpha_2 \geq \beta_2$.

Hence, according to Definition 13, C is Barreira-Valls polynomially dichotomic.

There exist skew-evolution semiflows that are Barreira-Valls polynomially dichotomic, but are not Barreira-Valls exponentially dichotomic.

Example 9 We consider the metric space (X,d), the Banach space V, the evolution semiflow φ and the projectors P_1 and P_2 defined as in Example 4.

Let us consider the function $g:\mathbf{R}_+\to\mathbf{R}$, given by $g(t)=t+1$ and let us define an evolution cocycle Φ as in Example 8. We obtain

$$\frac{s+1}{t+1}e^{-\int_s^t x(\tau-s)d\tau}|v_1|\leq\frac{s^2}{t}e^{-l(t-s)}|v_1|\leq t^{-1-l}s^{2+l}|v_1|$$

and

$$\frac{t+1}{s+1}e^{\int_s^t x(\tau-s)d\tau}|v_2|\geq t^{1+l}s^{-2-l}|v_2|,$$

for all $t\geq s\geq 1$ and all $v\in V$. It follows that the skew-evolution semiflow $C=(\varphi,\Phi)$ is Barreira-Valls polynomially dichotomic.

Let us suppose that C is also Barreira-Valls exponentially dichotomic. According to Definition 11, there exist some constants $N\geq 1$, $\alpha_1,\beta_1>0$ and $\alpha_2,\beta_2>0$ such that

$$\frac{s+1}{t+1}e^{-\int_s^t x(\tau-s)d\tau}|v_1|\leq Ne^{-\alpha_1 t}e^{\beta_1 s}|v_1|$$

and

$$N\frac{t+1}{s+1}e^{\int_s^t x(\tau-s)d\tau}|v_2|\geq e^{\alpha_2 t}e^{-\beta_2 s}|v_2|,$$

for all $(t,s),(s,t_0)\in T$ and all $(x,v)\in Y$. We consider $s=t_0$. We have

$$\frac{e^{\alpha_1 t}}{t+1}\leq\frac{\overline{N}}{t_0+1}\text{ and }\frac{e^{\alpha_2 t}}{t+1}\leq\frac{\widetilde{N}}{t_0+1},\ \forall t\geq t_0.$$

For $t\to\infty$, we obtain two contradictions, and, hence, C is not Barreira-Valls exponentially dichotomic.

Proposition 3 *An exponentially dichotomic skew-evolution semiflow* $C=(\varphi,\Phi)$ *is polynomially dichotomic.*

Proof. Definition 12 assures the existence of a function $N_1:\mathbf{R}_+\to[1,\infty)$ and a constant $\nu_1>0$ such that

$$\|\Phi_1(t,s,x)v\|\leq N_1(s)e^{-\nu_1 t}\left\|P_1(x)v\right\|,\ \forall(t,s)\in T,\ \forall(x,v)\in Y.$$

As following inequalities $e^t \geq t+1 > t$ hold for all $t \geq 0$, we obtain

$$\left\|\Phi_1(t,s,x)v\right\| \leq N_1(s)t^{-\nu_1}\left\|P_1(x)v\right\|,$$

for all $t \geq s > 0$ and all $(x,v) \in Y$.

As, by Definition 12 there exist a function $N_2 : \mathbf{R}_+ \to [1,\infty)$ and a constant $\nu_2 > 0$ such that

$$\left\|P_2(x)v\right\| \leq N_2(s)e^{-\nu_2 t}\left\|\Phi_2(t,s,x)v\right\|, \ \forall (t,s) \in T, \ \forall (x,v) \in Y.$$

Analogously, as previously, we have

$$\left\|P_2(x)v\right\| \leq N_2(s)t^{-\nu_2}\left\|\Phi_2(t,s,x)v\right\|,$$

for all $t \geq s > 0$ and all $(x,v) \in Y$.

Hence, according to Definition 15, C is polynomially dichotomic.

We present an example of a skew-evolution semiflow which is polynomially dichotomic, but is not exponentially dichotomic.

Example 10 We consider the metric space (X,d), the Banach space V, the evolution semiflow φ, the projectors P_1, P_2 and function g as in Example 9. Let

$$\Phi(t,s,x)v = \left(\frac{g(s)}{g(t)}e^{\int_s^t x(\tau-s)d\tau}|v_1|, \frac{g(t)}{g(s)}e^{-\int_s^t x(\tau-s)d\tau}|v_2|\right)$$

be an evolution cocycle. Analogously as in the mentioned Example, the skew-evolution semiflow C is Barreira-Valls polynomially dichotomic, and, according to Remark 12 (ii), it is also polynomially dichotomic. On the other hand, if we suppose that C is exponentially dichotomic, there exist N_1, $N_2 : \mathbf{R}_+ \to \mathbf{R}_+^*$ and ν_1, $\nu_2 > 0$ such that

$$\frac{s+1}{t+1}|v_1| \leq N_1(s)e^{-(\nu_1+l)t}e^{ls}|v_1|$$

and

$$|v_2| \leq N_2(s)e^{-(\nu_2+l)t}\frac{t+1}{s+1}|v_2|,$$

for all $(t,s) \in T$ and all $(x,v) \in Y$. If we consider $s = t_0$ and $t \to \infty$, we obtain two contradictions, which shows that C is not exponentially dichotomic.

A characterization for the classic and mostly encountered property of exponential dichotomy is given by the next

Theorem 1 *Let $C = (\varphi, \Phi)$ be a strongly measurable skew-evolution semiflow. C is exponentially dichotomic if and only if there exist two projectors P_1 and P_2 compatible with C with the properties that C_1 has bounded exponential growth and C_2 has exponential decay such that*

(i) there exist a constant $\gamma > 0$ and a mapping $D : \mathbf{R}_+ \to [1, \infty)$ with the property:

$$\int_s^\infty e^{(\tau-s)\gamma} \|\Phi_1(\tau, s, x)v\| \, d\tau \leq D(s) \|P_1(x)v\|,$$

for all $s \geq 0$ and all $(x, v) \in Y$;

(ii) there exist a constant $\rho > 0$ and a nondecreasing mapping $\widetilde{D} : \mathbf{R}_+ \to [1, \infty)$ with the property:

$$\int_{t_0}^t e^{(t-\tau)\rho} \|\Phi_2(\tau, t_0, x)v\| \, d\tau \leq \widetilde{D}(t_0) \|\Phi_2(t, t_0, x)v\|,$$

for all $t \geq t_0 \geq 0$ and all $(x, v) \in Y$.

Proof. *Necessity.* As C is exponentially dichotomic, according to Definition 12, there exist N_1, $N_2 : \mathbf{R}_+ \to \mathbf{R}_+^*$ and ν_1, $\nu_2 > 0$ such that

$$\|\Phi_1(t, t_0, x)v\| \leq N_1(s) e^{-\nu_1 t} \|\Phi_1(s, t_0, x)v\|$$

and

$$\|\Phi_2(s, t_0, x)v\| \leq N_2(s) e^{-\nu_2 t} \|\Phi_2(t, t_0, x)v\|,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

In order to prove (i), let us define $\gamma = -\dfrac{\nu_1}{2}$. We obtain successively

$$\int_s^\infty e^{(\tau-s)\gamma} \|\Phi_1(\tau, s, x)v\| \, d\tau \leq N_1(s) \|\Phi_1(s, s, x)v\| \int_s^\infty e^{-\frac{\nu_1}{2}(\tau-s)} e^{-\nu_1(s-\tau)} d\tau =$$

$$= N(s) \|P_1(x)v\| \int_s^\infty e^{-\frac{\nu_1}{2}(s-\tau)} d\tau = D(s) \|P_1(x)v\|,$$

for all $s \geq 0$ and all $(x, v) \in Y$, where we have denoted

$$D(u) = \frac{N_1(u)}{\gamma}, \ u \geq 0.$$

To prove (ii), we define $\rho = \dfrac{\nu_2}{2}$. Following relations

$$\int_{t_0}^t e^{(t-\tau)\rho} \|\Phi_2(\tau, t_0, x)v\| \, d\tau \leq N_2(t_0) \|\Phi_2(t, t_0, x)v\| \int_{t_0}^t e^{\frac{\nu_2}{2}(t-\tau)} e^{-\nu_2(t-\tau)} d\tau \leq$$

$$\leq \widetilde{D}(t_0)\, \|\Phi_2(t,t_0,x)v\|$$

hold for all $s \geq 0$ and all $(x,v) \in Y$, where we have denoted

$$\widetilde{D}(u) = \frac{2N_2(u)}{\rho},\ u \geq 0.$$

Sufficiency. According to relation (i), the γ-shifted skew-evolution semiflow $C_\gamma^1 = (\varphi, \Phi_\gamma^1)$, defined as in Example 3, has bounded exponential growth and there exists $D : \mathbf{R}_+ \to [1,\infty)$ such that

$$\int_s^\infty \left\|\Phi_\gamma^1(\tau,s,x)v\right\| d\tau \leq D(s)\, \|P_1(x)v\|\,,$$

for all $s \geq 0$ and all $(x,v) \in Y$.

First of all, we will prove that there exists $D_1 : \mathbf{R}_+ \to [1,\infty)$ such that $\left\|\Phi_\gamma^1(t,s,x)v\right\| \leq D_1(s)\, \|P_1(x)v\|$, for all $t \geq s \geq 0$ and all $(x,v) \in Y$. Let us consider, for $t \geq s+1$,

$$c = \int_0^1 e^{-\omega(u)} du \leq \int_0^{t-s} e^{-\omega(u)} du = \int_s^t e^{-\omega(t-\tau)} d\tau.$$

Hence, for $t \geq s+1$, we obtain

$$c| < v^*, \Phi_\gamma^1(t,s,x)v > | \leq \int_s^t e^{-\omega(t-\tau)} | < v^*, \Phi_\gamma^1(t,s,x)v > |d\tau =$$

$$= \int_s^t e^{-\omega(t-\tau)} \left\|\Phi_\gamma^1(t,\tau,\varphi(\tau,s,x))^* v^*\right\| \left\|\Phi_\gamma^1(\tau,s,x)v\right\| d\tau \leq$$

$$\leq M\, \|P_1(x)v^*\| \int_s^t \left\|\Phi_\gamma^1(\tau,s,x)v\right\| d\tau \leq MD(s)\, \|P_1(x)v\|\, \|P_1(x)v^*\|\,,$$

where $v \in V$, $v^* \in V^*$ and M, ω are given by Definition 5 and Remark 3. Hence,

$$\|\Phi_1(t,s,x)v\| \leq \frac{MD(s)}{c},\ \forall t \geq s+1,\ \forall (x,v) \in Y.$$

Now, for $t \in [s,s+1)$, we have

$$\|\Phi_1(t,s,x)v\| \leq Me^{\omega(1)}\, \|P_1(x)v\|\,,\ \forall (x,v) \in Y.$$

Thus, we obtain

$$\left\|\Phi_\gamma^1(t,s,x)v\right\| \leq D_1(s)\, \|P_1(x)v\|\,,$$

for all $t \geq s \geq 0$ and all $(x, v) \in Y$, where we have denoted

$$D_1(u) = M\left[e^{\omega(1)} + \frac{D(u)}{c}\right],\ u \geq 0.$$

Further, it follows that

$$\|\Phi_1(t, s, x)v\| \leq D_1(s)e^{-(t-s)\gamma}\|v\|,\ \forall t \geq s \geq 0.$$

According to (ii), there exist a constant $\rho > 0$ and a nondecreasing mapping $\widetilde{D} : \mathbf{R}_+ \to [1, \infty)$ such that

$$\int_{t_0}^{t} e^{-(\tau - t_0)\rho}\|\Phi_2(\tau, t_0, x)v\|\, d\tau \leq \widetilde{D}(t_0)e^{-(t-t_0)\rho}\|\Phi_2(t, t_0, x)v\|,$$

for all $t \geq t_0 \geq 0$ and all $(x, v) \in Y$. Thus,

$$\int_{t_0}^{t}\left\|\Phi^2_{-\rho}(\tau, t_0, x)v\right\| d\tau \leq \widetilde{D}(t_0)\left\|\Phi^2_{-\rho}(t, t_0, x)v\right\|,$$

for all $t \geq t_0 \geq 0$ and all $(x, v) \in Y$, where $\Phi^2_{-\rho}$ is defined as in Example 3. Let functions M and ω be given by Definition 6. Let us denote

$$c = \int_0^1 e^{-\omega(\tau)}d\tau = \int_s^{s+1} e^{-\omega(u-s)}du.$$

Further, for $t \geq s + 1$ and $s \geq t_0 \geq 0$, we obtain

$$c\left\|\Phi^2_{-\rho}(s, t_0, x)v\right\| = \int_s^{s+1} e^{-\omega(u-s)}\left\|\Phi^2_{-\rho}(s, t_0, x)v\right\| du \leq$$

$$\leq \int_s^{s+1} M(t_0)e^{-\omega(u-s)}e^{\omega(u-s)}\left\|\Phi^2_{-\rho}(u, t_0, x)v\right\| du \leq$$

$$\leq M(t_0)\int_{t_0}^{t}\left\|\Phi^2_{-\rho}(u, t_0, x)v\right\| du \leq M(t_0)\widetilde{D}(t_0)\left\|\Phi^2_{-\rho}(t, t_0, x)v\right\|.$$

We obtain

$$\left\|\Phi^2_{-\rho}(s, t_0, x)v\right\| \leq \frac{M(t_0)\widetilde{D}(t_0)}{c}\left\|\Phi^2_{-\rho}(t, t_0, x)v\right\|,$$

for all $t \geq s \geq t_0 \geq 0$ with $t \geq s + 1$ and all $(x, v) \in Y$. Now, for $t \in [s, s+1)$ and $s \geq t_0 \geq 0$, we have

$$\left\|\Phi^2_{-\rho}(s, t_0, x)v\right\| \leq M(t_0)e^{\omega(1)}\left\|\Phi^2_{-\rho}(t, t_0, x)v\right\|,$$

for all $(x, v) \in Y$. Finally, we obtain

$$\left\|\Phi^2_{-\rho}(s, t_0, x)v\right\| \leq D_2(t_0) \left\|\Phi^2_{-\rho}(t, t_0, x)v\right\|,$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in Y$, where we have denoted

$$D_2(u) = M(u) \left[\frac{\widetilde{D}(u)}{c} + e^{\omega(1)}\right],\ u \geq 0.$$

Thus, it follows that

$$e^{-(s-t_0)\rho} \left\|\Phi_2(s, t_0, x)v\right\| \leq D_2(t_0) e^{-(t-t_0)\rho} \left\|\Phi_2(t, t_0, x)v\right\|,$$

which implies

$$\left\|\Phi_2(s, t_0, x)v\right\| \leq D_2(t_0) e^{-(t-s)\rho} \left\|\Phi_2(t, t_0, x)v\right\|,$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in Y$, or

$$\left\|P_2(x)v\right\| \leq D_2(s) e^{-(t-s)\rho} \left\|\Phi_2(t, s, x)v\right\|,$$

for all $t \geq s \geq 0$ and all $(x, v) \in Y$.

Hence, the skew-evolution semiflow is exponentially dichotomic, which ends the proof.

Acknowledgments. This work is financially supported from the Exploratory Research Grant CNCSIS PN II ID 1080 No. 508/2009 of the Romanian Ministry of Education, Research and Innovation.

References

[1] L. Barreira, C. Valls. *Stability of Nonautonomous Differential Equations.* Lect. Notes Math. **1926**, 2008.

[2] S.N. Chow, H. Leiva. Existence and roughness of the exponential dichotomy for linear skew-product semiflows in Banach spaces. *J. Differential Equations* 120:429–477, 1995.

[3] W.A. Coppel. *Dichotomies in stability theory.* Lect. Notes Math. **629**, 1978.

[4] J.L. Daleckiĭ, M.G. Kreĭn. *Stability of solutions of differential equations in Banach space.* Translations of Mathematical Monographs, Amer. Math. Soc., Providence, Rhode Island, **43**, 1974.

[5] J.L. Massera, J.J. Schäffer. *Linear Differential Equations and Function Spaces.* Pure Appl. Math. **21** Academic Press, New York-London, 1966.

[6] M. Megan, A.L. Sasu, B. Sasu. Asymptotic Behaviour of Evolution Families. Ed. Mirton, 2003.

[7] M. Megan, C. Stoica. Exponential instability of skew-evolution semiflows in Banach spaces. *Studia Univ. Babeş-Bolyai Math.* LIII, No. 1:17–24, 2008.

[8] M. Megan, C. Stoica. Exponential dichotomy for skew-evolution semiflows on Banach spaces. *Ann. of Tiberiu Popoviciu Sem. on Funct. Eqs, Approx. and Conv.* 7:85–92, 2009.

[9] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer Verlag, New York, 1983.

[10] O. Perron. Die Stabitätsfrage bei Differentialgleichungen. *Math. Z.* 32:703–728, 1930.

[11] P. Preda, M. Megan. Nonuniform dichotomy of evolutionary processes in Banach spaces. *Bull. Austral. Math. Soc.* 27:31–52, 1983.

[12] R. J. Sacker, G. R. Sell. Existence of dichotomies and invariant splittings for linear differential systems III. *J. Differential Equations.* 22:497–522, 1976.

[13] C. Stoica, M. Megan. On uniform exponential stability for skew-evolution semiflows on Banach spaces. *Nonlinear Analysis.* 72, Issues 3–4:1305–1313, 2010.

[14] C. Stoica, M. Megan. Nonuniform behaviors for skew-evolution semiflows in Banach spaces. *arXiv*: 0806.1409:1–16, 2008. The 22^{nd} International Conference on Operator Theory, Timişoara, 2008.

Department of Mathematics and Computer Science,
"Aurel Vlaicu" University of Arad, Romania
E-mail address: cstoicad@yahoo.com