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Stochastic viability and dynamic programming

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Abstract

This paper deals with the stochastic control of nonlinear systems in the presence of state and control constraints, for uncertain discrete-time dynamics in finite dimensional spaces. In the deterministic case, the viability kernel is known to play a basic role for the analysis of such problems and the design of viable control feedbacks. In the present paper, we show how a stochastic viability kernel and viable feedbacks relying on probability (or chance) constraints can be defined and computed by a dynamic programming equation. An example illustrates most of the assertions.

Key words: stochastic control, state constraints, viability, discrete time, dynamic programming.

1 Introduction

Risk, vulnerability, safety or precaution constitute major issues in the management and control of dynamical systems. Regarding these motivations, the role played by the acceptability constraints or targets is central, and it has to be articulated with uncertainty and, in particular, with stochasticity when a probability distribution is given. The present paper addresses the issue of state and control constraints in the stochastic context. For the sake of simplicity, we consider noisy control dynamics systems. This is a natural extension of deterministic control systems, which covers a large class of situations. Thus we consider the following state equation as the uncertain dynamic model

$$x(t+1) = f\big(t, x(t), u(t), w(t)\big) , \quad t = t_0, \dots, T-1 , \quad \text{with} \quad x(t_0) = x_0 \quad (1)$$

where $x(t) \in \mathbb{X} = \mathbb{R}^n$ represents the system *state* vector at time t, $x_0 \in \mathbb{X}$ is the *initial condition* at *initial time* t_0, $u(t) \in \mathbb{U} = \mathbb{R}^p$ represents *decision* or *control*

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vector while $w(t) \in \mathbb{W} = \mathbb{R}^q$ stands for the *uncertain variable*, or *disturbance*, or *noise*.

The admissibility of decisions and states is first restricted by a non empty subset $\mathbb{B}(t,x)$ of admissible controls in $\mathbb{U}$ for all (t,x):

$$u(t) \in \mathbb{B}\big(t, x(t)\big) \subset \mathbb{U} \,. \tag{2}$$

Similarly, the relevant states of the system are limited by a non empty subset $\mathbb{A}\big(t, w(t)\big)$ of the state space $\mathbb{X}$ possibly uncertain for all t,

$$x(t) \in \mathbb{A}\big(t, w(t)\big) \subset \mathbb{X} \,, \tag{3}$$

and a target

$$x(T) \in \mathbb{A}\big(T, w(T)\big) \subset \mathbb{X} \,. \tag{4}$$

We assume that

$$w(t) \in \mathbb{S}(t) \subset \mathbb{W} \,, \tag{5}$$

so that the sequences

$$w(\cdot) := \big(w(t_0), w(t_0+1), \ldots, w(T-1), w(T)\big) \tag{6}$$

belonging to

$$\Omega := \mathbb{S}(t_0) \times \cdots \times \mathbb{S}(T) \subset \mathbb{W}^{T+1-t_0} \tag{7}$$

capture the idea of possible *scenarios* for the problem. A scenario is an *uncertainty trajectory*.

These control, state or target constraints may reduce the relevant paths of the system (1). Such a feasibility issue can be addressed in a robust or stochastic framework. Here we focus on the stochastic case assuming that the domain of scenarios Ω is equipped with some probability $\mathbb{P}$. In this probabilistic setting, one can relax the constraint requirements (2)-(3)-(4) by satisfying the state constraints along time with a given confidence level β

$$\mathbb{P}\Big(w(\cdot) \in \Omega \mid x(t) \in \mathbb{A}\big(t, w(t)\big) \ \text{ for } \ t = t_0, \ldots, T\Big) \geq \beta \tag{8}$$

by appropriate controls satisfying (2). Such probabilistic constraints are often called chance constraints in the stochastic literature as in [14, 16]. We shall give proper mathematical content to the above formula in the following section. Concentrating now on motivation, the idea of stochastic viability is basically to require the respect of the constraints at a given confidence level β (say 90%, 99%). It implicitly assumes that some extreme events makes irrelevant the robust approach [12] that is closely related to stochasticity with a confidence level 100%.

The problems of dynamic control under constraints usually refers to viability [1] or invariance [9, 17] framework. Basically, such an approach focuses on intertemporal feasible paths. From the mathematical viewpoint, most of viability and weak invariance results are addressed in the continuous time case. However,

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some mathematical works deal with the discrete-time case. This includes the study of numerical schemes for the approximation of the viability problems of the continuous dynamics as in [1, 15]. Important contributions for discrete-time case are also captured by the study of the positivity for linear systems as in [4], or by the hybrid control as in [2, 17] or [11]. Other references may be found in the control theory literature, such as [5, 13] and the survey paper [6]. A large study focusing on the discrete-time case is also provided in [10].

Viability is defined as the ability to choose, at each time step, a control such that the system configuration remains admissible. The *viability kernel* associated with the dynamics and the constraints play a major role regarding such issues. It is the set of initial states x_0 from which starts an acceptable solution. For a decision maker or control designer, knowing the viability kernel has practical interest since it describes the states from which controls can be found that maintain the system in a desirable configuration forever. However, computing this kernel is not an easy task in general. Of major interest is the fact that a dynamic programming equation underlies the computation or approximation of viability kernels as pointed out in [1, 10].

The present paper aims at expanding viability concepts and results in the stochastic case for discrete-time systems. In particular, we adapt the notions of viability kernel and viable controls in the probabilistic or chance constraint framework. Mathematical materials of stochastic viability can be found in [3, 8, 7] but they rather focus on the continuous time case and cope with constraints satisfied almost surely. We here provide a dynamic programming and Bellman perspective for the probabilistic framework.

The paper is organized as follows. Section 2 is devoted to the statement of the probabilistic viability problem. Then, Section 3 exhibits the dynamic programming structure underlying such stochastic viability. An example is exposed in Section 4 to illustrate some of the main findings.

2 The stochastic viability problem

Here we address the issue of state constraints in the probabilistic sense. This is basically related to risk assessment and management. This requires some specific tools inspired from the viability and invariance approach known for the certain case. In particular, within the probabilistic framework, we adapt the notions of viability kernel and viable controls.

2.1 Probabilistic assumptions and expected value

Probabilistic assumptions on the uncertainty $w(\cdot) \in \Omega$ are now added, providing a stochastic nature to the problem. Mathematically speaking, we suppose that the domain of scenarios $\Omega \subset \mathbb{W}^{T+1} = \mathbb{R}^q \times \cdots \times \mathbb{R}^q$ is equipped with a σ-field[1] $\mathcal{F}$ and a *probability* $\mathbb{P}$: thus, $(\Omega, \mathcal{F}, \mathbb{P})$ constitutes a *probability space*. The

[1]For instance, $\mathcal{F}$ is the trace of Ω on the usual borelian σ-field $\mathcal{F} = \bigotimes_{t=t_0}^{T} \mathcal{B}(\mathbb{R}^q)$.

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sequences

$$w(\cdot) = \big(w(0), w(1), \dots, w(T-1), w(T)\big) \in \Omega$$

now become the *primitive random variables*.

Hereafter, we shall assume that the random process $w(\cdot)$ is independent and identically distributed (i.i.d.) under probability $\mathbb{P}$. In other words, we suppose that the probability is the product $\mathbb{P} = \bigotimes_{t=t_0}^{T} \mu$ of a common marginal distribution μ. The *expectation operator* $\mathbb{E}$ is defined on the set of measurable and integrable functions by

$$\mathbb{E}[g] = \mathbb{E}_{\mathbb{P}}\left[g\left(w(\cdot)\right)\right] = \int_{\Omega} g\big(w(t_0), \dots, w(T)\big) d\mu(w(t_0)) \cdots d\mu(w(T)) \ ,$$

and we have that

$$\mathbb{E}_{\mathbb{P}}\left[g\left(w(t)\right)\right] = \mathbb{E}_{\mu}\left[g\left(w(t)\right)\right] \ .$$

2.2 Controls and feedback strategies

It is well-known that control issues in the uncertain case are much more complicated than in the deterministic case. In the uncertain context, we must drop the idea that the knowledge of open-loop decisions $u(\cdot) = \big(u(t_0), \dots, u(T-1)\big)$ induces one single path of sequential states $x(\cdot) = \big(x(t_0), \dots, x(T)\big)$. Open loop controls $u(t)$ depending only upon time t are no longer relevant, contrarily to closed loop or feedback controls $\mathfrak{u}(t, x(t))$ which display more adaptive properties by taking into account the uncertain state evolution $x(t)$. In the stochastic setting, all the objects considered will be implicitly equipped with appropriate measurability properties. Thus we define a *feedback* as an element of the set of all measurable functions from the couples time-state towards the controls:

$$\mathfrak{U} := \{\mathfrak{u} : (t,x) \in \{t_0, \dots, T-1\} \times \mathbb{X} \mapsto \mathfrak{u}(t,x) \in \mathbb{U},\ \mathfrak{u}\,\text{measurable}\,\} \ . \tag{9}$$

The control constraints case restricts feedbacks to admissible feedbacks accounting for control constraints (2) as follows

$$\mathfrak{U}^{ad} = \{\mathfrak{u} \in \mathfrak{U} \mid \mathfrak{u}(t,x) \in \mathbb{B}(t,x) \ , \quad \forall (t,x) \in \{t_0, \dots, T-1\} \times \mathbb{X}\} \ . \tag{10}$$

Let us mention that, in the stochastic context, a feedback decision is also termed a *pure Markovian strategy*. Markovian means that the current state contains all the sufficient information of past system evolution to determine the statistical distribution of future states. Thus, only current state $x(t)$ is needed in the feedback loop among the whole sequence of past states $x(t_0)$,..., $x(t)$.

At this stage, we need to introduce some notations which will appear quite useful in the sequel: the *state map* and the *control map*. Given a feedback $\mathfrak{u} \in \mathfrak{U}$, a scenario $w(\cdot) \in \Omega$ and an initial state x_0 at time $t_0 \in \{t_0, \dots, T-1\}$, the solution state $x_f[t_0, x_0, \mathfrak{u}, w(\cdot)]$ is the state path $x(\cdot) = (x(t_0), x(t_0+1), \dots, x(T))$ solution of dynamics

$$x(t+1) = f\big(t, x(t), \mathfrak{u}(t, x(t)), w(t)\big) \ , \quad t = t_0, \dots, T-1$$

starting from the initial condition $x(t_0) = x_0$ at time t_0 and associated with feedback control $\mathfrak{u}$ and scenario $w(\cdot)$. The solution control $u_f[t_0, x_0, \mathfrak{u}, w(\cdot)]$ is the associated decision path $u(\cdot) = (u(t_0), u(t_0+1), \ldots, u(T-1))$ where $u(t) = \mathfrak{u}(t, x(t))$.

2.3 The stochastic viability kernel and viable feedbacks

The viability kernel plays a major role in the viability analysis. In the determinsitic case, it is the set of initial states x_0 such that the state constraints hold true for at least one control stategy. In the probabilistic setting, one relaxes the constraints requirement by satisfying the state constraints along time with a given confidence level as in (8). We give proper mathematical content to this latter formula (8) inspired by chance constraints [14] in the following Definition.

Definition 1 *The* stochastic viability kernel *at time t_0 and at confidence level $\beta \in]0, 1]$ is*

$$\mathbb{V}\mathrm{iab}_\beta(t_0) := \left\{ x_0 \in \mathbb{X} \;\middle|\; \begin{array}{l} \textit{there exists } \mathfrak{u} \in \mathfrak{U}^{ad} \textit{ such that} \\ \mathbb{P}\Big(w(\cdot) \in \Omega \mid x(t) \in \mathbb{A}\big(t, w(t)\big) \textit{ for } t = t_0, \ldots, T\Big) \geq \beta \end{array} \right\} \tag{11}$$

where $x(t)$ is a shorthand for the solution map $x(t) = x_f[t_0, x_0, \mathfrak{u}, w(\cdot)](t)$.

Stochastic viable feedbacks are measurable feedback controls that allow the stochastic viability property to hold true.

Definition 2 Stochastic viable feedbacks *are those for which the above relations occur:*

$$\mathfrak{U}^{viab}_\beta(t_0, x_0) := \left\{ \mathfrak{u} \in \mathfrak{U}^{ad} \;\middle|\; \mathbb{P}\Big(w(\cdot) \in \Omega \mid x(t) \in \mathbb{A}\big(t, w(t)\big) \textit{ for } t = t_0, \ldots, T\Big) \geq \beta \right\} . \tag{12}$$

We have the following strong link between stochastic viable feedbacks and the viability kernel:

$$x_0 \in \mathbb{V}\mathrm{iab}_\beta(t_0) \iff \mathfrak{U}^{\mathrm{viab}}_\beta(t_0, x_0) \neq \emptyset \, .$$

Of particular interest is the case where the confident rate is $\beta = 1$ which is very close to robust viability and control. Indeed, when the scenario domain Ω is countable and that every scenario $w(\cdot)$ has strictly positive probability under $\mathbb{P}$, $\mathbb{V}\mathrm{iab}_1(t_0)$ is the *robust viability kernel* (the set of initial states x_0 such that the state constraints hold true for at least one control stategy, *whatever the scenario*). When the uncertainty domain $\mathbb{S}(t)$ in (5) is reduced to a single element, so is also the scenario domain Ω in (7): this is the deterministic case for which $\mathbb{V}\mathrm{iab}_1(t_0)$ coincides with the classical viability kernel [1, 10].

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3 Stochastic dynamic programming equation

We shall now exhibit a a characterization of stochastic viability in terms of dynamic programming. It relies on the the *maximal viability probability* defined recursively as follows.

Definition 3 *Assume that the random process $w(\cdot)$ is i.i.d. under probability $\mathbb{P}$, with marginal distribution μ. The* stochastic viability value function $V(t,x)$, *associated with dynamics (1), control constraints (2), state constraints (3) and target constraints (4) is defined by the following backward induction:*

$$\begin{cases} V(T,x) & := \mathbb{E}_\mu\Big[\mathbf{1}_{\mathbb{A}\big(T,w\big)}(x)\Big], \\ V(t,x) & := \max_{u\in\mathbb{B}(t,x)} \mathbb{E}_\mu\Big[\mathbf{1}_{\mathbb{A}\big(t,w\big)}(x)\, V\big(t+1, f(t,x,u,w)\big)\Big] . \end{cases} \quad (13)$$

Here, $\mathbf{1}_A$ stands for the indicator function of a set A. It is defined by $\mathbf{1}_A(x)=1$ if $x\in A$, and $\mathbf{1}_A(x)=0$ if $x\notin A$.

The backward dynamic programming equation (13) allows us to define the value function $V(t,x)$. By writting a max instead of a sup, we implicitly assume the existence of an optimal solution for each time t and state x. It turns out that the stochastic viability function $V(t_0,x)$ at time t_0 is related to the stochastic viability kernels $\{\mathbb{V}\mathrm{iab}_\beta(t_0), \beta\in[0,1]\}$, and that dynamic programming induction reveals relevant stochastic feedback controls. To achieve this, we first claim that the value function V is the solution of a (stochastic) optimal control problem involving the viability criterion π defined as follows:

$$\pi\big(t_0, x(\cdot), u(\cdot), w(\cdot)\big) = \prod_{t=t_0}^{T} \mathbf{1}_{\mathbb{A}\big(t,w(t)\big)}(x(t)) . \quad (14)$$

Proposition 1 *Assume that the random process $w(\cdot)$ is i.i.d. under probability $\mathbb{P}$, with marginal distribution μ. For any initial conditions (t_0,x_0), we have*

$$V(t_0,x_0) = \max_{\mathfrak{u}\in\mathfrak{U}^{ad}} \mathbb{E}_{\mathbb{P}}\left[\pi\big(t_0, x(\cdot), u(\cdot), w(\cdot)\big)\right] ,$$

where the stochastic viability value function $V(t_0,x_0)$ is given by the backward induction (13), where the criterion π is defined in (14), and where $x(\cdot) = x_f[t_0,x_0,\mathfrak{u},w(\cdot)](\cdot)$ and $u(\cdot) = u_f[t_0,x_0,\mathfrak{u},w(\cdot)]$ are shorthand expressions for the solution maps.

The proof of this previous Proposition is exposed in Appendix A. We also derive the following assertion regarding the stochastic viability kernel.

Proposition 2 *Assume that the random process $w(\cdot)$ is i.i.d. under probability $\mathbb{P}$, with marginal distribution μ. The stochastic viability kernel at confidence level β is the section of level β of the stochastic value function:*

$$V(t_0,x_0)\geq\beta \iff x_0\in\mathbb{V}\mathrm{iab}_\beta(t_0) . \quad (15)$$

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The proof of this previous Proposition is also exposed in Appendix A. As regard the viable feedbacks, we obtain the following assertion.

Proposition 3 *Assume that the random process $w(\cdot)$ is i.i.d. under probability $\mathbb{P}$, with marginal distribution μ. For any time $t = t_0, \ldots, T-1$ and state x, let us assume that*

$$\mathbb{B}^{viab}(t,x) := \arg\max_{u \in \mathbb{B}(t,x)} \mathbb{E}_\mu \left[\mathbf{1}_{\mathbb{A}\left(t,w\right)}(x)\, V\Big(t+1, f\big(t,x,u,w(t)\big)\Big) \right] \qquad (16)$$

is not empty. Then, for any $x_0 \in \mathbb{V}\mathrm{iab}_\beta(t_0)$, any measurable selection[2] $\mathfrak{u}^\star \in \mathbb{B}^{viab}$ belongs to the set of stochastic viable feedbacks $\mathfrak{U}_\beta^{viab}(t_0, x_0)$.

4 A simple academic example

To illustrate the general statements, we consider a simple academic model and perform a probabilistic viability analysis.

4.1 Example statement

The evolution of a scalar $x(t)$ is governed by the discrete-time dynamics

$$x(t+1) = x(t) + u(t) + w(t) \ ,$$

where control is constrained by

$$u(t) \in \{-1, 1\} = \mathbb{B}(t,x) = \mathbb{B}$$

and uncertainty scenarii are induced by

$$w(t) \in \{-1, 0, 1\} = \mathbb{S}(t) = \mathbb{S} \ .$$

We assume that $w(\cdot)$ is an i.i.d. sequence, with probability

$$\mu(w(t) = 1) = \mu(w(t) = -1) = p; \ \mu(w(t) = 0) = 1 - 2p \ .$$

The state constraint is

$$x(t) \in \{-1, 0, 1\} = \mathbb{A}\big(t, w(t)\big) = \mathbb{A} \ .$$

The decision maker intends to exhibit controls such that this constraint is satisfied with a high enough probability

$$\mathbb{P}\Big(x(t) \in \{-1, 0, 1\} \ , \quad t = t_0, \ldots, T\Big) \geq \beta \ .$$

[2] Any $\mathfrak{u}^\star \in \mathfrak{U}$ such that $\mathfrak{u}^\star(t,x) \in \mathbb{B}^{\mathrm{viab}}(t,x)$ for any t and x.

The intuition to satisfy the above probability constraint is as follows. When $x(t)$ belongs to the border $\{-1, 1\}$ of the domain $\mathbb{A} = \{-1, 0, 1\}$, there is an obvious decision to make: if $x(t) = -1$, take $u(t) = 1$ so that $x(t)+u(t) = 0$ and thus $x(t+1) = w(t) \in \{-1, 0, 1\}$ (the same with $x(t) = 1$ and $u(t) = -1$). But when $x(t) = 0$, then $x(t+1) = u(t) + w(t)$ and, whatever $u(t) \in \{-1, 1\}$, there is a chance that $w(t)$ takes the same value, sending $x(t)$ outside $\mathbb{A} = \{-1, 0, 1\}$.

4.2 Results

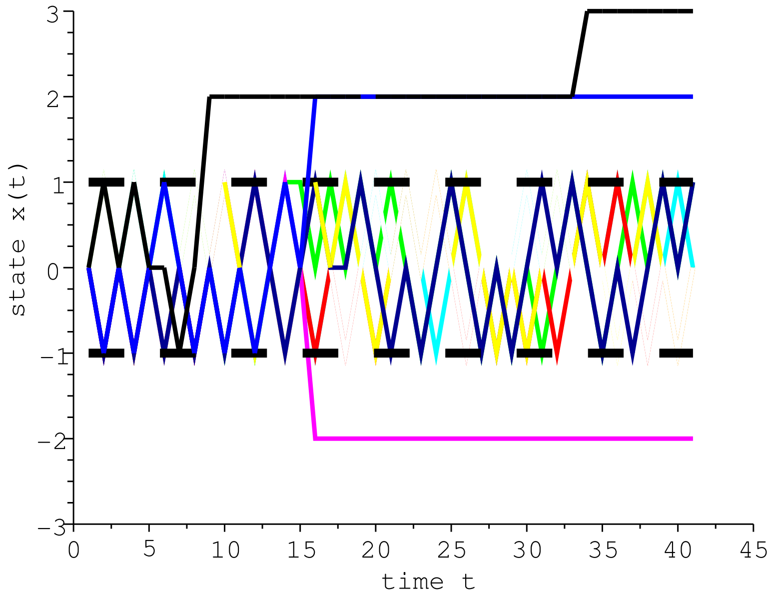

Figure 1: 9 simulations of state trajectories $x(t)$ over time horizon $[0, 40]$ for dynamics $x(t+1) = x(t) + u(t) + w(t)$ starting from $x_0 = 0$ with stochastic viable feedback controls $\mathfrak{u}^\star(t, x) \in \mathbb{B}^{\text{viab}}(t, x)$ as defined in (17). Probability of facing high disturbances $w \in \{-1, +1\}$ is low with $p = 1\%$. Viability probability value function $V(0, 0) \approx 67\%$ and 3 trajectories over 9 violate the constraint.

By dynamic programming equation (13), we compute the maximal viability probability $V(t, x)$ and associated viable feedback controls $\mathbb{B}^{\text{viab}}(t, x)$.

Result 1 *Introduce matrix M, vectors $\vec{\mathbf{1}}$ and $\vec{\mathbf{1}}_i(x)$ by*

$$M = \begin{pmatrix} p & 1-2p & p \\ p & 1-2p & 0 \\ p & 1-2p & p \end{pmatrix}, \ \vec{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ (\vec{\mathbf{1}}_i(x))_j = \mathbf{1}_{\{i\}}(x) = \mathbf{1}_{\{x=i\}} \ .$$

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The stochastic viability value function is given by

$$V(t,x) = \sum_{i=-1,0,1} \left\langle \vec{\mathbf{1}}_i(x), M^{T-t}\vec{\mathbf{1}} \right\rangle ,$$

or, in other words, $V(t,x) = 0$ *for all* $x \notin \{-1,0,1\}$ *and*

$$V(t,x) = (M^{T-t}\vec{\mathbf{1}})_{x+2} , \quad \forall x \in \{-1,0,1\} .$$

The associated viable feedback controls are given by

$$\mathbb{B}^{viab}(t,x) = \begin{cases} 1 & \text{if} \quad x = -1 \\ \{-1,1\} & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x = 1 , \end{cases} \tag{17}$$

Consequently the viability kernel reads:

Result 2

$$\mathbb{V}\text{iab}_\beta(t) = \begin{cases} \mathbb{A} & \text{if} \quad \beta \le (M^{T-t}\vec{\mathbf{1}})_2 \\ \{-1,1\} & \text{if} \quad (M^{T-t}\vec{\mathbf{1}})_2 < \beta \le (M^{T-t}\vec{\mathbf{1}})_1 \\ \emptyset & \text{if} \quad (M^{T-t}\vec{\mathbf{1}})_1 < \beta. \end{cases}$$

The difficulty of the control is captured by the second row of the matrix M where the sum is not equal to 1 which suggests that the state $x = 0$ can escape from $\mathbb{A}$. The results are illustrated by Figure 1 where 9 simulations of state trajectories $x(t)$ starting from $x_0 = 0$ are displayed over time horizon $[0, 40]$ with stochastic viable feedback controls $\mathfrak{u}^\star(t,x) \in \mathbb{B}^{\text{viab}}(t,x)$ as defined in (17). Probability of facing high disturbances $w \in \{-1,+1\}$ is low with $p = 1\%$. However viability probability value function turns out to be $V(0,0) \approx 67\%$ which points out a significant risk of leaving viability set $\mathbb{A} = \{-1,0,1\}$ due the accumulation of risks over 40 periods; Therefore it is intuitive that 3 paths over 9 leave the state constraint set $\mathbb{A} = \{-1,0,1\}$ along time.

Proof. We shall check that $V(t,x) = \sum_{i=-1,0,1} \left\langle \vec{\mathbf{1}}_i(x), M^{T-t}\vec{\mathbf{1}} \right\rangle$ is solution to the dynamic programming equation (13).

This is true for final time $t = T$ because

$$\sum_{i=-1,0,1} \left\langle \vec{\mathbf{1}}_i(x), M^{T-T}\vec{\mathbf{1}} \right\rangle = \sum_{i=-1,0,1} \left\langle \vec{\mathbf{1}}_i(x), \vec{\mathbf{1}} \right\rangle = \sum_{i=-1,0,1} \mathbf{1}_i(x) = \mathbf{1}_{\{-1,0,1\}}(x) = \mathbf{1}_{\mathbb{A}}(x) .$$

Proceeding by backward induction, let us suppose that

$$V(t+1,x) = \sum_{i=-1,0,1} \left\langle \vec{\mathbf{1}}_i(x), M^{T-(t+1)}\vec{\mathbf{1}} \right\rangle .$$

The dynamic programming equation (13) gives

$$V(t,x) = \mathbf{1}_{\{-1,0,1\}}(x) \max_{u \in \{-1,1\}} \mathbb{E}_\mu \Big[V\big(t+1, x+u+w\big) \Big] .$$

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Whenever $x \notin \mathbb{A} = \{-1, 0, 1\}$, we clearly have that $V(t, x) = 0$. Whenever $x = -1$, we deduce that

$$\begin{aligned} V(t,-1) &= \max\big\{pV(t+1,-3) + (1-2p)V(t+1,-2) + pV(t+1,-1), \\ &\quad pV(t+1,-1) + (1-2p)V(t+1,0) + pV(t+1,1)\big\} \\ &= \max\big\{pV(t+1,-1) + (1-2p)V(t+1,0) + pV(t+1,1), pV(t+1,-1)\big\} \\ &= pV(t+1,-1) + (1-2p)V(t+1,0) + pV(t+1,1) \end{aligned}$$

and the viable control is provided by $\mathfrak{u}^\star(t,-1) = 1$. By induction, we deduce that

$$\begin{aligned} V(t,-1) &= pV(t+1,-1) + (1-2p)V(t+1,0) + pV(t+1,1) \\ &= \sum_{i=-1,0,1} M_{1,i+2}(M^{T-(t+1)}\vec{\mathbf{1}})_{i+2} \\ &= (MM^{T-(t+1)}\vec{\mathbf{1}})_1 \\ &= (M^{T-t}\vec{\mathbf{1}})_1 \\ &= \Big\langle \vec{\mathbf{1}}_{-1}(-1), M^{T-t}\vec{\mathbf{1}} \Big\rangle \\ &= \sum_{i=-1,0,1} \Big\langle \vec{\mathbf{1}}_i(-1), M^{T-t}\vec{\mathbf{1}} \Big\rangle \ . \end{aligned}$$

In the same way, we check the expression for the stochastic viability value function $V(t,1)$ when $x = 1$, and obtain the viable control $\mathfrak{u}^\star(t,1) = -1$. The case $x = 0$ is treated in the same vein, with the difference that viable control is not unique since $\mathfrak{u}^\star(t,0) \in \{-1,+1\}$ and

$$V(t,0) = pV(t+1,-1) + (1-2p)V(t+1,0).$$

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A Proofs

A.1 Proof of Proposition 1

We use the following notations for any strategy $\mathfrak{u} \in \mathfrak{U}$:

- $\pi^{\mathfrak{u}}$ is the evaluation of the criterion π defined in (14)

$$\pi^{\mathfrak{u}}\big(t_0, x_0, w(\cdot)\big) := \pi\big(t_0, x_f[t_0, x_0, \mathfrak{u}, w(\cdot)](\cdot), u_f[t_0, x_0, \mathfrak{u}, w(\cdot)](\cdot), w(\cdot)\big) \tag{18}$$

  where $w(\cdot) \in \Omega$ and x_f, u_f are the solution maps;

- the expected value

$$\pi^{\mathfrak{u}}_{\mathbb{E}}(t_0, x_0) := \mathbb{E}_{\mathbb{P}}\left[\pi^{\mathfrak{u}}\big(t_0, x_0, w(\cdot)\big)\right] . \tag{19}$$

We consider the maximization problem:

$$\pi^{\star}_{\mathbb{E}}(t_0, x_0) := \max_{\mathfrak{u} \in \mathfrak{U}^{ad}} \pi^{\mathfrak{u}}_{\mathbb{E}}(t_0, x_0) . \tag{20}$$

We aim at proving that

$$V(t, x) = \pi^{\mathfrak{u}^\star}_{\mathbb{E}}(t, x) = \max_{\mathfrak{u} \in \mathfrak{U}^{ad}} \pi^{\mathfrak{u}}_{\mathbb{E}}(t, x) .$$

Let $\mathfrak{u}^\star \in \mathfrak{U}^{ad}$ denote one of the measurable viable feedback strategies given by the dynamic programming equation (13). We perform a backward induction to prove (20).

First, the equality at $t = T$ holds true since

$$\begin{aligned}
\pi^{\mathfrak{u}^\star}_{\mathbb{E}}(T, x) &= \mathbb{E}_{\mathbb{P}}\left[\pi^{\mathfrak{u}^\star}(T, x, w(\cdot))\right] && \text{by definition (19)} \\
&= \mathbb{E}_{\mu}\left[\mathbf{1}_{\mathbb{A}\left(T, w\right)}(x)\right] && \text{by definition (14)} \\
&= V(T, x) && \text{by definition (13).}
\end{aligned}$$

Now, suppose that

$$\pi^{\mathfrak{u}^\star}_{\mathbb{E}}(t+1, x) = \max_{\mathfrak{u} \in \mathfrak{U}^{ad}} \pi^{\mathfrak{u}}_{\mathbb{E}}(t+1, x) = V(t+1, x) . \tag{21}$$

The very definition (13) of the value function V by dynamic programming combined with (22) in Lemma 1 (proved below) imply that

$$\begin{aligned}
\pi^{\mathfrak{u}^\star}_{\mathbb{E}}(t, x) &= \mathbb{E}_{\mu}\left[\mathbf{1}_{\mathbb{A}\left(t, w(t)\right)}(x)\ \pi^{\mathfrak{u}^\star}_{\mathbb{E}}\left(t+1, f\big(t, x, \mathfrak{u}^\star(t, x), w(t)\big)\right)\right] && \text{by (22)} \\
&= \mathbb{E}_{\mu}\left[\mathbf{1}_{\mathbb{A}\left(t, w\right)}(x)\ V\left(t+1, f\big(t, x, \mathfrak{u}^\star(t, x), w\big)\right)\right] && \text{by (21)} \\
&= \max_{u \in \mathbb{B}(t,x)} \mathbb{E}_{\mu}\left[\mathbf{1}_{\mathbb{A}\left(t, w\right)}(x)\ V\left(t+1, f\big(t, x, u, w\big)\right)\right] && \text{by (13)} \\
&= V(t, x) && \text{by (13).}
\end{aligned}$$

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Similarly, for any $\mathfrak{u} \in \mathfrak{U}^{ad}$, we obtain

$$\begin{array}{rcll} \pi^{\mathfrak{u}}_{\mathbb{E}}(t,x) & = & \mathbb{E}_\mu \Big[\mathbf{1}_{\mathbb{A}\big(t,w(t)\big)}(x)\; \pi^{\mathfrak{u}}_{\mathbb{E}}\left(t+1, f\big(t,x,\mathfrak{u}(t,x),w(t)\big)\right) \Big] & \text{by (22)} \\ & \leq & \mathbb{E}_\mu \Big[\mathbf{1}_{\mathbb{A}\big(t,w\big)}(x)\; V\left(t+1, f\big(t,x,\mathfrak{u}(t,x),w\big)\right) \Big] & \text{by (21)} \\ & \leq & \max_{u \in \mathbb{B}(t,x)} \mathbb{E}_\mu \Big[\mathbf{1}_{\mathbb{A}\big(t,w\big)}(x)\; V\left(t+1, f\big(t,x,u,w\big)\right) \Big] & \text{since } \mathfrak{u}(t,x) \in \mathbb{B}(t,x) \\ & = & V(t,x) & \text{by (13).} \end{array}$$

Consequently, the desired statement is obtained since

$$\max_{\mathfrak{u} \in \mathfrak{U}^{ad}} \pi^{\mathfrak{u}}_{\mathbb{E}}(t,x) \leq V(t,x) = \pi^{\mathfrak{u}^\star}_{\mathbb{E}}(t,x)$$

yields the equality

$$V(t,x) = \pi^{\mathfrak{u}^\star}_{\mathbb{E}}(t,x) = \max_{\mathfrak{u} \in \mathfrak{U}^{ad}} \pi^{\mathfrak{u}}_{\mathbb{E}}(t,x) \; .$$

Lemma 1 *We have, for $t = t_0, \ldots, T-1$ and $\mathfrak{u} \in \mathfrak{U}$,*

$$\left\{ \begin{array}{rl} \pi^{\mathfrak{u}}_{\mathbb{E}}(T,x) = & \mathbb{E}_\mu \Big[\mathbf{1}_{\mathbb{A}\big(T,w(T)\big)}(x) \Big] \\ \pi^{\mathfrak{u}}_{\mathbb{E}}(t,x) = & \mathbb{E}_\mu \Big[\mathbf{1}_{\mathbb{A}\big(t,w\big)}(x)\; \pi^{\mathfrak{u}}_{\mathbb{E}}\left(t+1, f\big(t,x,\mathfrak{u}(t,x),w\big)\right) \Big] \; . \end{array} \right. \tag{22}$$

Proof. By (14) and (18), we have

$$\left\{ \begin{array}{rcl} \pi^{\mathfrak{u}}(T,x,w(\cdot)) & = & \mathbf{1}_{\mathbb{A}\big(T,w(T)\big)}(x) \\ \pi^{\mathfrak{u}}(t,x,w(\cdot)) & = & \mathbf{1}_{\mathbb{A}\big(t,w(t)\big)}(x) \pi^{\mathfrak{u}}(t+1, f\big(t,x,\mathfrak{u}(t,x),w(t)\big), w(\cdot))) \; . \end{array} \right. \tag{23}$$

Notice that $\pi^{\mathfrak{u}}(t,x,w(\cdot))$ depends only upon the end $\big(w(t),\ldots,w(T-1)\big)$, and not upon the beginning $\big(w(t_0),\ldots,w(t-1)\big)$. We shall write this property abusively by

$$\pi^{\mathfrak{u}}(t,x,w(\cdot)) = \pi^{\mathfrak{u}}\left(t,x,(w(t),\ldots,w(T-1))\right) \; . \tag{24}$$

We have

$$
\begin{aligned}
\pi^{\mathfrak{u}}_{\mathbb{E}}(t,x) &= \mathbb{E}_{\mathbb{P}}\left[\pi^{\mathfrak{u}}(t,x,w(\cdot))\right] && \text{by (19)}\\
&= \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\mathbb{A}\left(t,w(t)\right)}(x)\ \pi^{\mathfrak{u}}\left(t+1, f\big(t,x,\mathfrak{u}(t,x),w(t)\big),w(\cdot)\right)\Big] && \text{by (23)}\\
&= \mathbb{E}_{\mu}\Big[\mathbb{E}_{\mu^{T-t-1}}\big[\\
&\quad \mathbf{1}_{\mathbb{A}\left(t,w(t)\right)}(x)\pi^{\mathfrak{u}}\left(t+1, f\big(t,x,\mathfrak{u}(t,x),w(t)\big),w(t+1),\dots,w(T-1)\right)\big]\Big]\\
&\quad \text{by Fubini theorem}\\
&= \mathbb{E}_{\mu}\Big[\mathbf{1}_{\mathbb{A}\left(t,w(t)\right)}(x)\\
&\quad \mathbb{E}_{\mu^{T-t-1}}\left[\pi^{\mathfrak{u}}\left(t+1, f\big(t,x,\mathfrak{u}(t,x),w(t+1),\dots,w(T-1)\big)\right)\right]\Big]\\
&= \mathbb{E}_{\mu}\Big[\mathbf{1}_{\mathbb{A}\left(t,w(t)\right)}(x)\mathbb{E}_{w(\cdot)\in\Omega}\left[\pi^{\mathfrak{u}}\left(t+1, F\left(t,x,\mathfrak{u}(t,x),w(\cdot)\right)\right)\right]\Big] && \text{by (24)}\\
&= \mathbb{E}_{\mu}\left[\mathbf{1}_{\mathbb{A}\left(t,w\right)}(x)\ \pi^{\mathfrak{u}}_{\mathbb{E}}\big(t+1, f\big(t,x,\mathfrak{u}(t,x),w\big)\big)\right] && \text{by (19).}
\end{aligned}
$$

Proof of Proposition 2

It is enough to remark that

$$
\mathbb{V}\mathrm{iab}_{\beta}(t) = \left\{ x_0 \in \mathbb{X} \ \middle|\ \max_{u(\cdot)} \mathbb{E}_{\mathbb{P}}\left[\pi\big(t_0, x(\cdot), u(\cdot), w(\cdot)\big)\right] \geq \beta \right\} . \tag{25}
$$

Proof of Proposition 3

Simply follow step by step the proof of Proposition 1.

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