Explicit localization estimates for spline-type spaces

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Abstract:

We give some explicit decay estimates for the dual system of a basis of functions that are polynomially localized in space.

1. Introduction

A spline-type space S is a closed subspace of $L^2(\mathbb{R}^d)$ possessing a Riesz basis of functions well localized is space. That is, there exists a family of functions $\{f_k\}_k \subseteq S$ and constants $0 < A \leq B < +\infty$ such that

$$A\|c\|_{\ell^2} \leq \|\sum_k c_k f_k\|_{L^2} \leq B\|c\|_{\ell^2}, \qquad (1)$$

holds for every $c \in \ell^2$, and the functions $\{f_k\}_k$ satisfy an spatial localization condition.

In a spline-type space any function in $f \in S$ has a unique expansion $f = \sum_k c_k f_k$. Moreover the coefficients are given by $c_k = \langle f, g_k \rangle$, where $\{g_k\}_k \subseteq S$ is the *dual basis*, a set of functions characterized by the relation $\langle g_k, f_j \rangle = \delta_{k,j}$. These spaces provide a very natural framework for the sampling problem.

The general theory of localized frames (see [6], [5] and [2]) asserts that the functions forming the dual basis satisfy a similar spatial localization. This can be used to extend the expansion in (1) to other spaces, so that the family $\{f_k\}_k$ becomes a Banach frame for an associated family of Banach spaces (see [4] and [6]). In the case of a spline-type space S, this means that the decay of a function in S can be characterized by the decay of its coefficients and, in particular, that the functions $\{f_k\}_k$ form a so called p-Riesz basis for its L^p-closed linear span, for the whole range $1 \leq p \leq \infty$.

We derive, in some concrete case, explicit bounds for the localization of the dual basis. We will work with a set of functions satisfying a polynomial decay condition around a set of nodes forming a lattice. By a change of variables, we can assume that the lattice is $\mathbb{Z}^d$. So, we will consider a set of functions $\{f_k\}_k \subseteq L^2(\mathbb{R}^d)$ satisfying the condition,

$$|f_k(x)| \leq C\,(1 + |x-k|)^{-s}, \quad x \in \mathbb{R}^d \text{ and } k \in \mathbb{Z}^d,$$

for some constant C. This type of spatial localization is specifically covered by the results in [5], but the constants given there are not explicit. We will derive a polynomial decay condition for the dual basis $\{g_k\}_k$, giving explicit information on the resulting constants. This yields some qualitative information, like the dependence of theses constants on A, C and s and the corresponding p-Riesz basis bounds for the original basis.

2. Main result

Theorem 1 *Let $C \geq 1$, and let $t > d$ be integers. Let $s > d + t$ be a real number. For $k \in \mathbb{Z}^d$ let $f_k : \mathbb{R}^d \to \mathbb{C}$ be a measurable function such that*

$$|f_k(x)| \leq C\,(1 + |x-k|)^{-s}, \qquad (x \in \mathbb{R}^d).$$

Suppose that $\{f_k\}_k$ is a Riesz basis for its L^2 closed linear span S, with bounds $0 < A \leq B < \infty$. Let $\{g_k\}_k \subseteq S$ be its dual basis.
Then, the dual functions satisfy,

$$|g_k(x)| \leq D\,(1 + |x-k|)^{-t}, \qquad (x \in \mathbb{R}^d).$$

where D is given by,

$$D = \frac{E^{st} C^{2t+1}}{(s-t-d)^t} \frac{1 + A^{t-1}}{A^{t+1}},$$

for some constant $E > 0$ that only depends on the dimension d.

Remark 1 *The constant E can be explicitly determined from the proof.*

The results in [6] prescribe polynomial decay estimates for the dual basis similar to those possessed by the original basis. As a trade-off for the explicit constants we will not obtain the full preservation of these decay conditions. Nevertheless, any degree of polynomial decay on the dual system can be granted, provided that the original basis has sufficiently good decay.

Finally observe that, although the basis $\{f_k\}_k$ is assumed to be concentrated around a lattice of nodes, the functions f_k are not assumed to be shifts of a single function. In particular, Theorem 1 below allows for a basis of functions whose 'optimal' concentration nodes do not form a lattice but are comparable to one. The 'eccentricity' of the configuration of concentration nodes is, however, penalized by the constants modelling the decay.

3. Sketch of a proof and comments

Now we sketch the proof of the main result, for a complete proof see [11].
Consider the gram matrix of the basis $\{f_k\}_k$ given by,

$$M \equiv (m_{k,j})_{k,j\in\mathbb{Z}^d}, \quad m_{k,j} := \langle f_k, f_j \rangle .$$

Since $\{f_k\}_k$ is a Riesz sequence, M, as an operator on ℓ^2, has an inverse $N \equiv (n_{k,j})_{k,j\in\mathbb{Z}^d}$. Moreover, $\|N\|_{\ell^2\to\ell^2} \leq A^{-1}$ and $n_{k,j} = \langle g_k, g_j \rangle$, where $\{g_k\}_k \subseteq S$ is the dual basis of $\{f_k\}_k$.
The localization assumptions on the basis $\{f_k\}_k$ yield a polynomial decay estimate on the entries of M,

$$|m_{k,j}| \lesssim (1 + |k-j|)^{-s}.$$

If we can establish a similar estimate for the entries of N,

$$|n_{k,j}| \lesssim (1 + |k-j|)^{-t}.$$

with all the constants given explicitly, then, using calculations similar to those in [5], we obtain the desired polynomial concentration conditions for the dual functions.
Let us first consider the case where the basis $\{f_k\}_k$ consists of integer shifts of a single generator f (that is, $f_k = f(\cdot - k), k \in \mathbb{Z}^d$). In this case, the matrix M is constant on its diagonals. That is,

$$m_{k,j} = a_{k-j},$$

for some sequence a. Similarly, N is given by

$$n_{k,j} = b_{k-j},$$

where the sequence b satisfies $a * b = \delta$.
Therefore, in this special case, M and N are convolution operators. The off-diagonal decay of their entries is equivalent to the decay of their kernels a and b. Since the decay of a sequence x can be characterized by the smoothness of its Fourier transform $\hat{x}$, the problem can be reformulated as the preservation of the smoothness of the function $\hat{a}$ under pointwise inversion. This reasoning is present, for example, in [1].
We can measure the smoothness of $\hat{a}$ by considering weak-derivatives and use repeatedly a chain-rule argument for Sobolev spaces to obtain similar smoothness conditions for $\hat{b}$.
In the general case, where M and N need not be convolution operators, we try to imitate this reasoning, but we avoid using the Fourier transform.
Given a matrix $L \equiv (l_{k,j})_{k,j\in\mathbb{Z}^d}$ and $1 \leq h \leq d$, we consider the new matrix,

$$D_h(L)_{k,j} := (k_h - j_h) l_{k,j}.$$

Observe that, up to some multiplicative constant, the map D_h acts on a convolution operator by taking a partial derivative of its symbol (that is, the Fourier transform of its kernel.) The domain of D_h consists of those matrices L such that $D_h(L)$ defines a bounded operator on ℓ^2. We call $D_h(L)$ the *derivative* of L (with respect to x_h.)
D_h is a derivation in the sense that it satisfies the equation $D_h(AB) = D_h(A)B + AD_h(B)$, provided that $D_h(A)$ and $D_h(B)$ are both defined. Derivations are a well-known tool in operator-algebras theory (see [3], [9] and [10].)
Since $MN = I$ and $D_h(I) = 0$, we can formally express the high-order derivatives of N in terms of its lower-order ones and all the derivatives of M,

$$D_h^u(N) = -\sum_{l=0}^{u-1} \binom{u}{l} D_h^l(N) D_h^{u-l}(M) N. \qquad (2)$$

Using the polynomial off-diagonal decay bounds on M and the bound $\|N\|_{\ell^2\to\ell^2} \leq A^{-1}$ we can obtain bounds for the $\ell^2 \to \ell^2$ norms of some derivatives of N. These imply polynomial off-diagonal decay estimates for N, and hence yield the desired spatial localization bounds for the dual basis.
In the argument above we related the off-diagonal decay of a matrix with the $\ell^2 \to \ell^2$ norm of its derivatives. The $\ell^2 \to \ell^2$ norm of a matrix is not determined by the size of its entries. However, there are some necessary and (other) sufficient conditions on the size of the entries of a matrix for it to be bounded on ℓ^2. This "gap" in the conditions accounts for the loss of some decay information in Theorem 1, when passing from the original basis to its dual system.
Finally we point out that the formal computations in the above argument are not sufficient to prove the theorem. Consider again the simple case of a basis of integer shifts. With the notation of the discussion above, we have the relation

$$a * b = \delta, \qquad (3)$$

we have some decay estimate on a (that can be reformulated as a smoothness condition on $\hat{a}$) and we want to prove a similar decay condition for b. There may be various sequences x satisfying the relation $a * x = \delta$; b can be singled out as the only one of them having a bounded Fourier transform. For example, when a is finitely supported, equation 3 is a linear difference equation which has other solutions besides b (that grow exponentially). The decay of the sequence b can be rigorously proved by resorting to some Sobolev-space smoothing argument.
In the general case, to derive equation (2), one needs to use the associativity of the product of matrices. This is justified only if all the matrices involved represent bounded operators. In other words, we need to know a priori that the derivatives of N that are involved in equation (2) define bounded operators. This can be proved using the general results on derivations on Banach algebras (see [3], [9]) or Jaffard's Theorem [8].
The use of derivations is somehow implicit in Jaffard's paper [8]. Recently, Gröchenig and Klotz [7] have systematically studied the use of derivations in connection to various problems including the preservation under inversion of various kinds of off-diagonal decay conditions.

4. Application

From Theorem 1 we can derive the following qualitative statement.

Theorem 2 *Let $\{F^i\}_{i\in I}$ be a family of Riesz sequences,*

$$F^i \equiv \{f_k^i\}_{k\in\mathbb{Z}^d} \subseteq \boldsymbol{L}^2(\mathbb{R}^d), \quad (i \in I).$$

sharing a uniform lower basis bound. Suppose that the family $\{F^i\}_i$ *satisfies a uniform concentration condition,*

$$\left|f_k^i(x)\right| \leq C\left(1+|x-k|\right)^{-s}, (x \in \mathbb{R}^d, k \in \mathbb{Z}^d, i \in I),$$

for some constants $C \geq 1$, $s > d+t$ *and* $t > d$*, with* t *an integer.*
Then the following holds.

(a) The respective family of dual systems $\{G^i\}_i$ *- where* $G^i \equiv \{g_k^i\}_{k\in\mathbb{Z}^d}$ *- satisfies a uniform concentration condition,*

$$\left|g_k^i(x)\right| \leq D\left(1+|x-k|\right)^{-t}, (x \in \mathbb{R}^d, k \in \mathbb{Z}^d, i \in I),$$

for some constant $D \geq 1$.

(b) A uniform p*-Riesz basis condition holds, for all* $1 \leq p \leq \infty$*. More precisely, there exist constants* $q, Q > 0$ *such that for any* $p \in [1,\infty]$ *and any* $i \in I$*, the relation*

$$q\|c\|_{\ell^p} \leq \|\sum_k c_k f_k^i\|_{L^p} \leq Q\|c\|_{\ell^p}$$

holds for all finitely supported sequences $(c_k)_{k\in\mathbb{Z}^d}$.

Statement (a) follows directly from Theorem 1. Examining the proofs in [5] we see that the uniformity of the constants given in (a) yields statement (b).
This qualitative conclusion on Theorem 2 was the original motivation for Theorem 1.
Finally, observe that the arguments given above are applicable to a generel *intrinsically localized* basis in the sense of [5].

5. Acknowledgements

The author wishes to thank Karlheinz Gröchenig and Andreas Klotz for their comments and for sharing an early draft of [7], and is indebted to Hans Feichtinger and Ursula Molter for some insightful discussions.
The author holds a fellowship from the CONICET and thanks this institution for its support. His research is also partially supported by grants: PICT06-00177, CONICET PIP N 5650, UBACyT X149.
This note was partially written during a long-term visit to NuHAG in which the author was supported by the EUCETIFA Marie Curie Excellence Grant (FP6-517154, 2005-2009).

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