Stochastic viscosity solution for stochastic PDIEs with nonlinear Neumann boundary condition

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Abstract

This paper is an attempt to extend the notion of viscosity solution to nonlinear stochastic partial differential integral equations with nonlinear Neumann boundary condition. Using the recently developed theory on generalized backward doubly stochastic differential equations driven by a Lévy process, we prove the existence of the stochastic viscosity solution, and further extend the nonlinear Feynman-Kac formula.

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1 Introduction

The notion of the viscosity solution for partial differential equations, first introduced by Crandall and Lions [7], has an impact on the modern theoretical and applied mathematics. Today the theory has become an indispensable tool in many applied fields, especially in optimal control theory and numerous subjects related to it. We refer to the well-known "User's Guide" by Crandall et al. [8] and the books by Bardi et al. [1] and Fleming and Soner [9] for a detailed account for the theory of (deterministic) viscosity solutions.

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Since it is well known that almost all the deterministic problems in these applied fields have their stochastic counterparts, many works have extended the notion of viscosity solution to stochastic partial differential equations (SPDEs, in short). The first among them is done by Lions and Souganidis [12, 13]. They use the so-called "stochastic characteristic" to remove the SPDEs. Next, two other ways of defining a stochastic viscosity solution of SPDEs is considered by Buckdahn and Ma respectively in [4, 5] and [6]. In the two first paper, they used the "Doss-Sussman" transformation to connect the stochastic viscosity solution of SPDEs with the solution of associated backward doubly stochastic differential equations (BDSDEs, in short). In the second one, they introduced the stochastic viscosity solution by using the notion of stochastic sub and super jets. Recently, based on both previous work, Boufoussi et al. introduced in [3], the notion of viscosity solution of SPDEs with nonlinear Neumann boundary condition. The existence result is derived via the so-called generalized BDSDEs and the "Doss-Sussman" transformation.

Inspired by the aforementioned works, especially [3] and [4, 5], this paper consider the following nonlinear stochastic partial differential integral equations (SPDIEs, in short) with nonlinear Neumann boundary condition

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + Lu(t,x) + f(t,x,u(t,x),(u_k^1(t,x))_{k=1}^{\infty}) \\ \qquad\qquad + g(t,x,u(t,x))\Diamond B_t = 0, \quad 0<t<T,\ x\in\overline{\Theta}, \\ \frac{\partial u}{\partial n}(t,x) + \phi(t,x,u(t,x)) = 0, \quad 0<t<T,\ x\in\partial\Theta, \\ u(T,x) = u_0(x), \quad x\in\overline{\Theta}. \end{cases} \tag{1.1}$$

Here $\Diamond$ denotes the Wick product and, thus, indicates that the differential is to be understood in Itô's backward integral sense with respect to Brownian motion B. Moreover f, g, ϕ and u_0 are some measurable functions with appropriate dimensions and L is the second-order differential integral operator of the form:

$$\begin{aligned} L\varphi(t,x) &= m_1\sigma(x)\frac{\partial\varphi}{\partial x}(t,x) + \frac{1}{2}\sigma(x)^2\frac{\partial^2\varphi}{\partial x^2}(t,x) \\ &\quad + \int_{\mathbb{R}}\left[\varphi(t,x+\sigma(x)y) - \varphi(t,x) - \frac{\partial\varphi}{\partial x}(t,x)\sigma(x)y\right]\nu(dy); \end{aligned} \tag{1.2}$$

in which σ is a certain function and $m_1 = \mathbb{E}(L_1)$, which will be given in Section 3. We denote

$$\varphi_k^1(t,x) = \int_{\mathbb{R}}[\varphi(t,x+\sigma(x)y) - \varphi(t,x) - \frac{\partial\varphi}{\partial x}(t,x)\sigma(x)y]p_k(y)\nu(dy),\ k\geq 1$$

and

$$\frac{\partial\varphi}{\partial n}(t,x) = \sum_{i=1}^{d}\frac{\partial\psi}{\partial_i}(x)\frac{\partial\varphi}{\partial x_i}(t,x),\ \forall\, x\in\partial\Theta,$$

where the function $\psi \in C_b^2(\mathbb{R}^n)$ is connected to the domain Θ by the following relation:

$$\Theta = \{x \in \mathbb{R}^n : \psi(x) > 0\} \text{ and } \partial\Theta = \{x \in \mathbb{R}^n : \psi(x) = 0\}.$$

The goal of this paper is to determine the definition and next naturally establish the existence of the stochastic viscosity solution to SPDIEs (1.1). More precisely, we give some direct links between this stochastic viscosity solution and the solution of the so-called generalized backward doubly stochastic differential equations driven by a Lévy process (BDSDELs, for short) initiated by Hu and Ren [10]. Such a relation in a sense could be viewed as an extension of the nonlinear Feynman-Kac formula to stochastic PDIEs, which, to our best knowledge, is new. Note also that this work could be considered as a generalization for the updated result obtained by Ren and Otmani [15], where the authors treat deterministic PDIEs with nonlinear Neumann boundary conditions.

The rest of this paper is organized as follows. In Section 2, we introduced notion of stochastic viscosity solutions and all details associated. In Section 3, we review the generalized backward doubly stochastic differential equations driven by a Lévy process and its connection to stochastic PDIEs, from which the existence of the stochastic viscosity solution will follow.

2 Notion of viscosity solution for SPDIE

21 Notations, assumptions and definitions

Let $(\Omega, \mathcal{F}; \mathbb{P})$ be a complete probability space on which a d-dimensional Brownian motion $B = (B_t)_{t\geq 0}$ is defined . Let $\mathbf{F}^B = \mathcal{F}^B_{t,T}$ denote the natural filtration generated by B, augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. Further, let $\mathcal{M}^B_{0,T}$ denote all the $\mathbf{F}^B$-stopping times τ such $0 \leq \tau \leq T$, a.s. and $\mathcal{M}^B_\infty$ be the set of all almost surely finite $\mathbf{F}^B$-stopping times. Let us introduce

$$\ell^2 = \Big\{ x = (x^{(i)})_{i\geq 1};\ \|x\|_{\ell^2} = (\sum_{i=1}^{\infty} |x^{(i)}|^2)^{1/2} < \infty \Big\}.$$

For generic Euclidean spaces $E, E_1 = \mathbb{R}^n$ or ℓ^2 and we introduce the following:

1. The symbol $\mathcal{C}^{k,n}([0,T] \times E; E_1)$ stands for the space of all E_1-valued functions defined on $[0,T] \times E$ which are k-times continuously differentiable in t and n-times continuously differentiable in x, and $\mathcal{C}_b^{k,n}([0,T] \times E; E_1)$ denotes the subspace of $\mathcal{C}^{k,n}([0,T] \times E; E_1)$ in which all functions have uniformly bounded partial derivatives.

2. For any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}^B_T$, $\mathcal{C}^{k,n}(\mathcal{G}, [0,T] \times E; E_1)$ (resp. $\mathcal{C}_b^{k,n}(\mathcal{G}, [0,T] \times E; E_1)$) denotes the space of all $\mathcal{C}^{k,n}([0,T] \times E; E_1)$ (resp. $\mathcal{C}_b^{k,n}([0,T] \times E; E_1)$)-valued random variable that are $\mathcal{G} \otimes \mathcal{B}([0,T] \times E)$-measurable;

3. $\mathcal{C}^{k,n}(\mathbf{F}^B, [0,T] \times E; E_1)$ (resp.$\mathcal{C}_b^{k,n}(\mathbf{F}^B, [0,T] \times E; E_1)$) is the space of all random fields $\phi \in \mathcal{C}^{k,n}(\mathcal{F}_T, [0,T] \times E; E_1$ (resp. $\mathcal{C}^{k,n}(\mathcal{F}_T, [0,T] \times E; E_1)$, such that for fixed $x \in E$ and $t \in [0,T]$, the mapping $\omega \to \alpha(t, \omega, x)$ is $\mathbf{F}^B$-progressively measurable.

4. For any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}^B$ and a real number $p \geq 0$, $L^p(\mathcal{G}; E)$ to be all E-valued $\mathcal{G}$-measurable random variable ξ such that $\mathbb{E}|\xi|^p < \infty$.

Furthermore, regardless of their dimensions we denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the inner product and norm in E and E_1, respectively. For $(t, x, y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$, we denote $D_x = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_d})$,
$D_{xx} = (\partial^2_{x_i x_j})_{i,j=1}^d$, $D_y = \frac{\partial}{\partial y}$, $D_t = \frac{\partial}{\partial t}$. The meaning of D_{xy} and D_{yy} is then self-explanatory.
Let Θ be an open connected and smooth bounded domain of $\mathbb{R}^n$ $(d \geq 1)$ such that for a function $\psi \in \mathcal{C}_b^2(\mathbb{R}^n)$, Θ and its boundary $\partial\Theta$ are characterized by $\Theta = \{\psi > 0\}$, $\partial\Theta = \{\psi = 0\}$ and, for any $x \in \partial\Theta$, $\nabla\psi(x)$ is the unit normal vector pointing towards the interior of Θ.
Throughout this paper, we shall make use of the following standing assumptions:

(**A1**) The function $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is uniformly Lipschitz continuous, with a Lipschitz constant $K > 0$.

(**A2**) The function $f : \Omega \times [0,T] \times \overline{\Theta} \times \mathbb{R} \times \ell^2 \to \mathbb{R}$ is a continuous random field such that for fixed , (x, y, q), $f(\cdot, \cdot, x, y, \sigma^* q)$ is a $\mathcal{F}_{t,T}^B$-measurable; and there exists a constant $K > 0$, for all (t, x, y, z), $(t', x', y', z') \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \ell^2$, such that for $\mathbb{P}$-a.e. ω,

$$|f(\omega, 0, 0, 0, 0)| \leq K$$

$$|f(\omega, t, x, y, z) - f(\omega, t', x', y', z')| \leq K(|t - t'| + |x - x'| + |y - y'| + |z - z'|).$$

(**A3**) The function $\phi : \Omega \times [0,T] \times \overline{\Theta} \times \mathbb{R} \to \mathbb{R}$ is a continuous random field such that, for fixed (x, y), $\phi(\cdot, \cdot, x, y)$ is a $\mathcal{F}_{t,T}^B$-measurable; and there exists a constant $K > 0$, for all (t, x, y), $(t', x', y') \in [0,T] \times \mathbb{R}^n \times \mathbb{R}$, such that for $\mathbb{P}$-a.e. ω,

$$|\phi(\omega, 0, 0, 0)| \leq K$$

$$|\phi(\omega, t, x, y) - \phi(\omega, t', x', y')| \leq K(|t - t'| + |x - x'| + |y - y'|).$$

(**A4**) The function $u_0 : \mathbb{R}^n \to \mathbb{R}$ is continuous, for all $x \in \mathbb{R}^n$, such that for some positive constants K, $p > 0$,

$$|u_0(x)| \leq K(1 + |x|^p).$$

(**A5**) The function $g \in C_b^{0,2,3}([0,T] \times \overline{\Theta} \times \mathbb{R}; \mathbb{R}^d)$.

As shown by the work of Buckdahn and Ma [4, 5], our definition of stochastic viscosity solution will depend heavily on the following stochastic flow $\eta \in C(\mathbf{F}^B, [0,T]\times\mathbb{R}^n\times\mathbb{R})$, defined as the unique solution of the following stochastic differential equation in the Stratonovich sense:

$$\eta(t,x,y) \;=\; y + \int_t^T \langle g(s,x,\eta(s,x,y)), \circ dB_s\rangle. \tag{2.1}$$

We refer the reader to [4] for a lucid discussion on this topic. Under the assumption (**A5**), the mapping $y \mapsto \eta(t,x,y)$ defines a diffeomorphism for all (t,x), $\mathbb{P}$-a.s. (see Protter [16]). Let us denote its y-inverse by $\varepsilon(t,x,y)$. Then, one can show that $\varepsilon(t,x,y)$ is the solution to the following first-order SPDE:

$$\varepsilon(t,x,y) = y - \int_t^T \langle D_y\varepsilon(s,x,y),\ g(s,x,\eta(s,x,y)) \circ dB_s\rangle.$$

We now define the notion of stochastic viscosity solution for SPDIEs (1.1). In order to simply the notation, we denote:

$$A_{f,g}(\varphi(t,x)) = L\varphi(t,x) + f(t,x,\varphi(t,x),(\varphi_k^1(t,x))_{k=1}^{\infty}) - \frac{1}{2}\langle g, D_y g\rangle(t,x,\varphi(t,x)).$$

and $\Psi(t,x) = \eta(t,x,\varphi(t,x))$

Definition 2.1 *(1) A random field $u \in C(\mathbf{F}^B,[0,T]\times\overline{\Theta})$ is called a stochastic viscosity subsolution of the SPDIEs* (1.1) *if $u(T,x) \le u_0(x)$, for all $x \in \overline{\Theta}$ and if for any stopping time $\tau \in \mathcal{M}_{0,T}^B$, any state variable $\xi \in L^0(\mathcal{F}_\tau^B,[0,T]\times\Theta)$, and any random field $\varphi \in C^{1,2}(\mathcal{F}^B\tau,[0,T]\times\mathbb{R}^n)$ satisfying that*

$$u\left(t,x\right) - \Psi\left(t,x\right) \le 0 = u\left(\tau(\omega),\xi(\omega)\right) - \Psi\left(\tau(\omega),\xi(\omega)\right)$$

for all (t,x) in a neighborhood of (ξ,τ), $\mathbb{P}$-a.e. on the set $\{0<\tau<T\}$, it holds that

(a) *on the event $\{0<\tau<T\}$,*

$$\mathrm{A}_{f,g}\left(\Psi\left(\tau,\xi\right)\right) - D_y\Psi\left(\tau,\xi\right)D_t\varphi\left(\tau,\xi\right) \le 0,\ \mathbb{P}\text{-}a.e.;$$

(b) *on the event $\{0<\tau<T\}\cap\{\xi\in\partial\Theta\}$,*

$$\min\left\{\mathrm{A}_{f,g}\left(\Psi\left(\tau,\xi\right)\right) - D_y\Psi\left(\tau,\xi\right)D_t\varphi\left(\tau,\xi\right),\ -\frac{\partial\Psi}{\partial n}\left(\tau,\xi\right) - \phi\left(\tau,\xi,\Psi\left(\tau,\xi\right)\right)\right\} \le 0,\ \mathbb{P}\text{-}a.e. \tag{2.2}$$

(2) A random field $u \in C(\mathbf{F}^B,[0,T]\times\overline{\Theta})$ is called a stochastic viscosity subsolution of the SPDIE (f,g) (1.1) *if $u(T,x) \ge u_0(x)$, for all $x\in\overline{\Theta}$ and if for any stopping time $\tau \in \mathcal{M}_{0,T}^B$, any state variable $\xi \in L^0(\mathcal{F}_\tau^B,[0,T]\times\Theta)$, and any random field $\varphi \in C^{1,2}(\mathcal{F}^B\tau,[0,T]\times\mathbb{R}^n)$ satisfying*
$u\left(t,x\right) - \Psi\left(t,x\right) \ge 0 = u\left(\tau(\omega),\xi(\omega)\right) - \Psi\left(\tau(\omega),\xi(\omega)\right)$
for all (t,x) in a neighborhood of (ξ,τ), $\mathbb{P}$-a.e. on the set $\{0<\tau<T\}$, it holds that

(a) on the event $\{0<\tau<T\}$,

$$\mathrm{A}_{f,g}\left(\Psi\left(\tau,\xi\right)\right)-D_y\Psi\left(\tau,\xi\right)D_t\varphi\left(\tau,\xi\right)\geq 0,\ \mathbb{P}\text{-}a.e.;$$

(b) on the event $\{0<\tau<T\}\cap\{\xi\in\partial\Theta\}$,

$$\max\left\{\mathrm{A}_{f,g}\left(\Psi\left(\tau,\xi\right)\right)-D_y\Psi\left(\tau,\xi\right)D_t\varphi\left(\tau,\xi\right),\ -\frac{\partial\Psi}{\partial n}\left(\tau,\xi\right)-\phi\left(\tau,\xi,\Psi\left(\tau,\xi\right)\right)\right\}\geq 0,\mathbb{P}\text{-}a.e. \tag{2.3}$$

(3) A random field $u\in\mathcal{C}\left(\mathbf{F}^B,[0,T]\times\overline{\Theta}\right)$ *is called a stochastic viscosity solution of SPDIE* (f,g) (1.1) *if it is both a stochastic viscosity subsolution and a a stochastic viscosity supersolution.*

Remark 2.2 *We remark that if* f, ϕ *are deterministic and* $g\equiv 0$, *the flow* η *becomes* $\eta(t,x,y)=y$ *and* $\Psi(t,x)=\varphi(t,x)$, $\forall\ (t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}$. *Thus, definition* 2.1 *coincides with the definition of (deterministic) viscosity solution of PDIE* $(f,0,\phi)$ *given in [15].*

Next, the following notion of a random viscosity solution will be a bridge linking the stochastic viscosity solution and its deterministic counterpart.

Definition 2.3 *A random field* $u\in C(\mathbf{F}^B,[0,T]\times\overline{\Theta})$ *is called an* ω*-wise viscosity solution if for* $\mathbb{P}$*-almost all* $\omega\in\Omega$, $u(\omega,\cdot,\cdot)$ *is a deterministic viscosity solution of SPDIE* $(f,0,\phi)$.

22 Doss-Sussmann transformation

In this subsection, we study the Doss-Sussmann transformation. It enables us to convert SPDIE (f,g,ϕ) to an SPDIE $(\widetilde{f},0,\widetilde{\phi})$, where $\widetilde{f}$ and $\widetilde{\phi}$ are well-defined random field depending on f, g and ϕ respectively. We get the following important result.

Proposition 2.4 *Assume* $(\mathbf{A1})$–$(\mathbf{A5})$ *hold. A random field* u *is a stochastic viscosity sub- (resp. super)-solution to SPDIE* (f,g,ϕ) (1.1) *if and only if* $v(\cdot,\cdot)=\varepsilon(\cdot,\cdot,u(\cdot,\cdot))$ *is a stochastic viscosity sub-(resp. super)-solution to SPDIE* $(\widetilde{f},0,\widetilde{\phi})$, *with*

$$\begin{aligned}
&\widetilde{f}(t,x,y,(z^{(k)})_{k=1}^{\infty})\\
=\ &\frac{1}{D_y\eta(t,x,y)}\Big[f\Big(t,x,\eta(t,x,y),\Big(D_y\eta(t,x,y)z^{(k)}+\sigma(x)D_x\eta(t,x,y)\mathbf{1}_{\{k=1\}}\\
&+\int_{\mathbb{R}}\theta^k(t,x,y,u)\nu(du)\Big)_{k=1}^{\infty}\Big)\\
&-\frac{1}{2}gD_yg(t,x,\eta(t,x,y))+L_x\eta(t,x,y)+\sigma(x)D_{xy}\eta(t,x,y)\left(z^{(1)}+\int_{\mathbb{R}}\theta^1(t,x,y,u)\nu(du)\right)\\
&+\frac{1}{2}D_{yy}\eta(t,x,y)\sum_{k=1}^{\infty}\left|z^{(k)}+\frac{1}{D_y\eta(t,x,y)}\int_{\mathbb{R}}\theta^k(t,x,y,u)\nu(du)\right|^2\Big]
\end{aligned} \tag{2.4}$$

and

$$\widetilde{\phi}(t,x,y) = \frac{1}{D_y\eta(t,x,y)}\left[h(t,x,\eta(t,x,y)) + D_x\eta(t,x,y)\nabla\psi(x)\right]. \tag{2.5}$$

The process θ is defined by

$$\theta^k(t,x,y,u) = [\eta(t,x+\sigma(x)u,y) - \eta(t,x,y) - D_x\eta(t,x,y)u]p_k(u). \tag{2.6}$$

Remark 2.5 *Let us recall that under the assumption* (**A5**) *the random field η belongs to $C^{0,2,2}(F^B,[0,T]\times\mathbb{R}^n\times\mathbb{R})$, and hence that the same is true for ε. Then, considering the transformation $\Psi(t,x) = \eta(t,x,\varphi(t,x))$, we obtain*

$$\begin{aligned}
D_x\Psi &= D_x\eta + D_y\eta D_x\varphi,\\
D_{xx}\Psi &= D_{xx}\eta + 2(D_{xy}\eta)(D_x\varphi)^* + (D_{yy}\eta)(D_x\varphi)(D_x\varphi)^* + (D_y\eta)(D_{xx}\varphi).
\end{aligned}$$

Moreover, since for all $(t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}$ the equality $\varepsilon(t,x,\eta(t,x,y)) = y$ holds $\mathbb{P}$-almost surely, we also have

$$\begin{aligned}
D_x\varepsilon + D_y\varepsilon D_x\eta &= 0,\\
D_y\varepsilon D_y\eta &= 1,\\
D_{xx}\varepsilon + 2(D_{xy}\varepsilon(D_x\eta)^* + (D_{yy}\varepsilon)(D_x\eta)(D_x\eta)^* + (D_y\varepsilon)(D_{xx}\eta) &= 0,\\
(D_{xy}\varepsilon)(D_y\eta) + (D_{yy}\varepsilon)(D_x\eta)(D_y\eta) + (D_y\varepsilon)(D_{xy}\eta) &= 0,\\
(D_{yy}\varepsilon)(D_y\eta)^2 + (D_y\varepsilon)(D_{yy}\eta) &= 0,
\end{aligned}$$

where all the derivatives of the random field $\varepsilon(\cdot,\cdot,\cdot)$ are evaluated at $(t,x,\eta(t,x,y))$, and all those of $\eta(\cdot,\cdot,\cdot)$ are evaluated at (t,x,y).

Proof of Proposition 2.4. We shall only prove that if $u\in C(\mathbf{F}^B;[0,T]\times\mathbb{R}^n)$ is a stochastic viscosity subsolution to SPDIEs (f,g,ϕ), then $v(\cdot,\cdot) = \varepsilon(\cdot,\cdot,u(\cdot,\cdot))\in C(\mathbf{F}^B,[0,T]\times\mathbb{R}^n)$ is a stochastic viscosity subsolution to SPDIE$(\widetilde{f},0,\widetilde{\phi})$. The remaining part can be proved without enough difficulties in the similar way.

To this end, let $u\in C(\mathbf{F}^B;[0,T]\times\mathbb{R}^n)$ be a stochastic viscosity subsolution to SPDIEs (f,g,ϕ) and let $v(t,x) = \varepsilon(t,x,u(t,x))$. Let us take $\tau\in\mathcal{M}^B_{0,T},\xi\in L^2(\mathcal{F}^B_\tau,\mathbb{R}^n)$ arbitrarily, and let $\varphi\in C^{1,2}(\mathcal{F}^B_\tau,\mathbb{R}^n)$ be such that

$$v(\omega,t,x) - \varphi(\omega,t,x) \le 0 = v(\omega,\tau(\omega),\xi(\omega)) - \varphi(\omega,\tau(\omega),\xi(\omega))$$

for all (t,x) in a neighborhood of (ξ,τ), $\mathbb{P}$-a.e. on the set $\{0<\tau<T\}$.

Setting $\Psi(t,x) = \eta(t,x,\varphi(t,x))$ and since mapping $y\mapsto\eta(t,x,\varphi(t,x,y))$ is strictly increasing, we have

$$\begin{aligned}
u(t,x) - \Psi(t,x) &= \eta(t,x,v(t,x)) - \eta(t,x,\varphi(t,x))\\
&\le 0 = \eta(\tau,\xi,v(\tau,\xi)) - \eta(\tau,\xi,\varphi(\tau,\xi)) = u(\tau,\xi) - \Psi(\tau,\xi),
\end{aligned}$$

for all (t,x) in a neighborhood of (ξ,τ), $\mathbb{P}$-a.e. on the set $\{0<\tau<T\}$. Therefore, since u is a stochastic viscosity subsolution to SPDIE(f,g,ϕ), it follows that $\mathbb{P}$-a.e. on $\{0<\tau<T\}$,

$$\mathrm{A}_{f,g}\left(\Psi\left(\tau,\xi\right)\right)-D_{y}\Psi\left(\tau,\xi\right)D_{t}\varphi\left(\tau,\xi\right)\geq 0. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned} L\Psi(t,x) &= L_x\eta(t,x,\varphi(t,x))+D_y\eta(t,x,\varphi(t,x))L\varphi(t,x) \\ &\quad +\sigma(x)D_{xy}\eta(t,x,\varphi(t,x))(D_x\varphi(t,x)) \\ &\quad +\frac{1}{2}D_{yy}\eta(t,x,\varphi(t,x))(D_x\varphi(t,x))^2, \end{aligned}$$

where L_x is the same as the operator L, with all the derivatives taken with respect to the second variable x from which together with (2.4), we obtain

$$D_y\varepsilon(t,x,\Psi(t,x))A_{f,g}(\Psi(t,x))=A_{\widetilde{f},0}(\varphi(t,x)).$$

Finally, in virtue of (2.7), we get

$$A_{\widetilde{f},0}(\varphi(\tau,\xi))\geq D_t\varepsilon(\tau,\xi).$$

That is, part (a) of Definition 2.1. is established. To derive part (b), noting that for all $(t,x)\in[0,T]\times\partial\Theta$, we have

$$\begin{aligned} \frac{\partial\Psi}{\partial n}(t,x) &= D_x\Psi(t,x)\cdot\nabla\psi(x) \\ &= D_x\eta(t,x,\varphi(t,x))\cdot\nabla\psi(x)+D_y\eta(t,x,\varphi(t,x))D_x\varphi(t,x)\cdot\nabla\psi(x) \\ &= D_x\eta(t,x,\varphi(t,x))\cdot\nabla\psi(x)+D_y\eta(t,x,\varphi(t,x))\frac{\partial\varphi}{\partial n}(t,x). \end{aligned}$$

This shows that

$$\frac{\partial\Psi}{\partial n}(\tau,\xi)+\phi(\tau,\xi,\Psi(\tau,\xi))=D_x\eta(\tau,\xi,\varphi(t,x))\left(\frac{\partial\varphi}{\partial n}(\tau,\xi)+\widetilde{\phi}(\tau,\xi,\varphi(\tau,\xi))\right)$$

where $\widetilde{\phi}$ is defined by (2.5). Because $D_y\eta(t,x,y)$ is strictly positive, we have $\mathbb{P}$-a.s. on $\{0<\tau<T\}\cap\{\xi\in\partial\Theta\}$

$$\min\left\{A_{\widetilde{f},0}(\varphi(\tau,\xi))-D_t\varepsilon(\tau,\xi),-\frac{\partial\varphi}{\partial n}(\tau,\xi)-\widetilde{\phi}(\tau,\xi,\varphi(\tau,\xi))\right\}\geq 0.$$

That is, v is a stochastic viscosity subsolution of SPDIE$(\widetilde{f},0,\widetilde{\phi})$. ■

3 Generalized BDSDELs and SPDIEs with Neumann boundary condition

The main object of this section is to show how a semi-linear SPDIE (f, g, ϕ) (1.1) is related to the so-called generalized BDSDELs (GBDSDELs, for short) initiated by Hu and Ren[10], in the Markovian case. To begin with, let us introduce another complete probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ on which we define a Lévy process L characterized by the following famous Lévy-Khintchine formula

$$\mathbb{E}(\mathrm{e}^{iuL_t}) = \mathrm{e}^{-t\Phi(u)} \text{ with } \Phi(u) = -ibu + \frac{\sigma^2}{2}u^2 - \int_{\mathbb{R}} \left(\mathrm{e}^{iuy} - 1 - iuy\mathbf{1}_{\{|y|\leq 1\}}\right)\nu(dy).$$

Thus L is characterized by its Lévy triplet (b, σ, ν) where $b \in \mathbb{R}, \sigma^2 \geq 0$ and ν is a measure defined in $\mathbb{R}\backslash\{0\}$ which satisfies that
(i) $\int_{\mathbb{R}}(1 \wedge y^2)\nu(dy) < +\infty$,
(ii) $\exists\ \varepsilon > 0$ and $\lambda > 0$ such that $\int_{(-\varepsilon,\varepsilon)^c} \mathrm{e}^{\lambda|y|}\nu(dy) < +\infty$.
This implies that the random variable L_t have moment of all orders, i.e $m_1 = \mathbb{E}(L_1) = b + \int_{|y|\geq 1} y\nu(dy)$ and $m_i = \int_{-\infty}^{+\infty} y^i\nu(dy) < \infty,\ \ \forall\, i \geq 2$. For the background on Lévy processes, we refer the reader to [2, 17].

We define the following family of σ-fields:

$$\mathcal{F}_t^L = \sigma(L_r - L_s, s \leq r \leq t) \vee \mathcal{N}',$$

where $\mathcal{N}'$ denotes all the $\mathbb{P}'$-null sets in $\mathcal{F}'$. Denote $\mathbf{F}^L = (\mathcal{F}_t^L)_{0\leq t\leq T}$.

Next, we consider the product space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ where

$$\bar{\Omega} = \Omega \otimes \Omega'; \quad \bar{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}', \ ; \quad \bar{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}',$$

and define $\mathcal{F}_t = \mathcal{F}_{t,T}^B \otimes \mathcal{F}_t^L$ for all $t \in [0, T]$. We remark that $\mathbf{F} = \{\mathcal{F}_t,\ t \in [0, T]\}$ is neither increasing nor decreasing so that it does not a filtration. Further, we assume that random variables $\xi(\omega),\ \omega \in \Omega$ and $\zeta(\omega'),\ \omega' \in \Omega'$ are considered as random variables on $\bar{\Omega}$ via the following identifications:

$$\xi(\omega, \omega') = \xi(\omega); \quad \zeta(\omega, \omega') = \zeta(\omega').$$

We denote by $(H^{(i)})_{i\geq 1}$ the Teugels Martingale associated with the Lévy process $\{L_t : t \in [0, T]\}$. More precisely

$$H^{(i)} = c_{i,i}Y^{(i)} + c_{i,i-1}Y^{(i-1)} + \cdots + c_{i,1}Y^{(1)}$$

where $Y_t^{(i)} = L_t^i - m_i$ for all $i \geq 1$ with L_t^i a power-jump process. That is $L_t^1 = L_t$ and $L_t^i = \sum_{0<s<t}(\Delta L_s)^i$ for all $i \geq 2$, where $X_{t^-} = \lim_{s\nearrow t} X_s$ and $\Delta X_t = X_t - X_{t^-}$. It was shown in Nualart and Schoutens [14] that the coefficients $c_{i,k}$ correspond to

the orthonormalization of the polynomials $1, x, x^2, ...$ with respect to the measure $\mu(dx) = x^2 d\nu(x) + \sigma^2\delta_0(dx)$:

$$q_{i-1}(x) = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}.$$

We set

$$p_i(x) = xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \cdots + c_{i,1}x^1.$$

The martingale $(H^{(i)})_{i=1}^{\infty}$ can be chosen to be pairwise strongly orthonormal martingale.

We consider the following spaces of processes:

1. $\mathcal{M}^2(\ell^2)$ denotes the space of ℓ^2-valued, square integrable and $\mathcal{F}_t$ -predictable processes $\varphi = \{\varphi_t : \ t \in [0,T]\}$ such that

$$\|\varphi\|^2_{\mathcal{M}^2} = \mathbb{E}\int_0^T \|\varphi_t\|^2 dt < \infty.$$

2. $\mathcal{S}^2(\mathbb{R})$ is the subspace of $\mathcal{M}^2(\mathbb{R})$ formed by the $\mathcal{F}_t$-adapted, right continuous with left limit (rcll) processes $\varphi = \{\varphi_t : \ t \in [0,T]\}$ such that

$$\|\varphi\|^2_{\mathcal{S}^2} = \mathbb{E}\left(\sup_{0\le t\le T} |\varphi_t|^2\right) < \infty.$$

Finally, let $\mathcal{E}^2 = \mathcal{S}^2(\mathbb{R}) \times \mathcal{M}^2(\ell^2)$ be endowed with the norm

$$\|(Y,Z)\|^2_{\mathcal{E}^2} = \mathbb{E}\left(\sup_{0\le t\le T} |Y_t|^2 + \int_0^T \|Z_t\|^2 dt\right).$$

31 A class of reflected diffusion process and GBDSDELs

We now introduce a class of reflected diffusion process. Let Θ be a regular convex and bounded subsect of $\mathbb{R}^n$, which is such that for a function $\psi \in C_b^2(\mathbb{R}^n)$, $\Theta = \{x \in \mathbb{R}^n : \ \psi(x) > 0\}$, $\partial\Theta = \{x \in \mathbb{R}^n : \ \psi(x) = 0\}$ and for all $x \in \partial\Theta$, $\nabla\psi(x)$ coincides with the unit normal pointing towards the interior of Θ (see [11]). Under assumption (**A1**), we know from [11] that for every $(t,x) \in [0,T] \times \overline{\Theta}$ there exists a unique pair of progressively measurable process $(X_s^{t,x}, A_s^{t,x})_{t\le s\le T}$, which is a solution to the following reflected SDE:

$$\begin{cases} \mathbb{P}(X_s^{t,x} \in \overline{\Theta},\ s \ge t) = 1 \\ X_s^{t,x} = x + \displaystyle\int_t^s \sigma(X_{r^-}^{t,x}) dL_r + \int_t^s \nabla\psi(X_s^{t,x}) dA_s^{t,x},\ s \ge t, \end{cases} \tag{3.1}$$

where $A_s^{t,x} = \int_t^s \mathbf{1}_{\{X_r^{t,x}\in\partial\Theta\}} dA_r^{t,x}$, $A^{t,x}$ is an increasing process with bounded variation on $[0,T]$, $0 < T < \infty$, $A_0 = 0$. Furthermore, we have the following proposition.

Proposition 3.1 *There exists a constant $C > 0$ such that for all x, $x' \in \overline{\Theta}$,*

$$\mathbb{E}\left[\sup_{0\leq s\leq T}|X_s^x - X_s^{x'}|^4\right] \leq C|x-x'|^4$$

and

$$\mathbb{E}\left[\sup_{0\leq s\leq T}|A_s^x - A_s^{x'}|^4\right] \leq C|x-x'|^4.$$

The main subject in this section is the following GBDSDELs, for $(t,x) \in [0;T]\times\mathbb{R}^n$,

$$\begin{cases} \text{(i) } \mathbb{E}\left[\sup_{t\leq s\leq T}|Y_s^{t,x}|^2 + \int_t^T \|Z_s^{t,x}\|^2 ds\right] < \infty; \\ \text{(ii) } Y_s^{t,x} = u_0(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^T \phi(r, X_r^{t,x}, Y_r^{t,x})dA_r^{t,x} \\ \qquad + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x})\, dB_r - \sum_{i=1}^{\infty}\int_s^T (Z_r^{t,x})^{(i)} dH_r^{(i)}, \;\; t \leq s \leq T. \end{cases} \tag{3.2}$$

Remark 3.2 *In what follows, we will assume $n = 1$. The multidimensional case can be completed without major difficulties.*

Let us recall an existence and uniqueness result appear in [10] and a generalized version of the Itô-Ventzell formula whose proof is analogous to the corresponding one in Buckdahn-Ma [4] replacing the Brownian motion W by the Teugels martingale $(H^{(i)})_{i\geq 1}$.

Theorem 3.3 *Assume that* (**A1**)–(**A5**) *hold. For each $(t,x) \in [0,T]\times\mathbb{R}$, GBDSDEL* (3.2) *has a unique solution $(Y^{t,x}, Z^{t,x}) \in \mathcal{E}^2$.*

Theorem 3.4 *Suppose that $M \in C^{0,2}(\mathbf{F}, [0,T]\times\mathbb{R})$ is a semimartingale in the sense that for every spatial parameter $x \in \mathbb{R}$ the process $t \mapsto M(t,x)$, $t \in [0,T]$, is of the form:*

$$M(t,x) = M(0,x) + \int_0^t G(s,x)ds + \int_0^t \langle N(s,x), dB_s\rangle + \sum_{i=1}^{\infty}\int_0^t K^{(i)}(s,x)dH_s^{(i)},$$

where $G \in C^{0,2}(\mathbf{F}^B, [0,T]\times\mathbb{R})$, $N \in C^{0,2}(\mathbf{F}^B, [0,T]\times\mathbb{R};\mathbb{R}^d)$, and the process K belongs to $C^{0,2}(\mathbf{F}^L, [0,T]\times\mathbb{R};\ell^2)$. We also consider the process $\alpha \in C(\mathbf{F}, [0,T])$ of the form

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \theta_s dA_s + \int_0^t \gamma_s dB_s + \sum_{i=1}^{\infty}\int_0^t \delta_s^{(i)} dH_s^{(i)}$$

where $\beta, \theta \in \mathcal{S}^2(\mathbb{R})$, $\gamma \in \mathcal{M}^2(\mathbb{R}^d)$, and $\delta \in \mathcal{M}^2(\ell^2)$. Then the following equality holds $\mathbb{P}$-almost surely for all $0 \leq t \leq T$:

$$\begin{aligned}
M(t,\alpha_t) &= M(0,\alpha_0) + \int_0^t G(s,\alpha_s)ds + \int_0^t \langle N(s,\alpha_s), dB_s\rangle + \sum_{i=1}^{\infty}\int_0^t K^{(i)}(s,\alpha_s)dH_s^{(i)} \\
&+ \int_0^t D_xM(s,\alpha_s)\beta_s ds + \int_0^t D_xM(s,\alpha_s)\theta_s dA_s + \int_0^t \langle D_xM(s,\alpha_s), \gamma_s dB_s\rangle \\
&+ \sum_{i=1}^{\infty}\int_0^t D_xM(s,\alpha_s)\delta_s^{(i)}dH_s^{(i)} - \frac{1}{2}\sum_{i=1}^{d}\int_0^t D_{xx}M(s,\alpha_s)|\gamma_s^i|^2 ds \\
&+ \frac{1}{2}\sum_{i=1}^{\infty}\int_0^t D_{xx}M(s,\alpha_s)|\delta_s^{(i)}|^2 ds + \sum_{i=1}^{\infty}\int_0^t D_xK^{(i)}(s,\alpha_s)\delta_s^{(i)}ds \\
&- \sum_{i=1}^{d}\int_0^t D_xN^i(s,\alpha_s)\gamma_s^i ds.
\end{aligned}$$

32 Existence of stochastic viscosity solution

In this section we prove the existence of the stochastic viscosity solution to the SPDIEs (f, g, ϕ). Our main idea is to apply the Doss transformation to the GBDSDEL (3.2) to obtain resulting GBDSDEL without the stochastic integral against dB, which naturally become a GBSDEL with new generators being exactly $\widetilde{f}$ and $\widetilde{\phi}$. For this, for each $(t,x) \in [0,T] \times \mathbb{R}, t \leq s \leq T$, let us define the following processes,

$$\begin{aligned}
U_s^{t,x} &= \varepsilon(s, X_s^{t,x}, Y_s^{t,x}), \\
(V^{(1)})_s^{t,x} &= D_y\varepsilon(s, X_s^{t,x}, Y_s^{t,x})(Z^{(1)})_s^{t,x} + \sigma(X_s^{t,x})D_x\varepsilon(s, X_s^{t,x}, Y_s^{t,x}) \\
&+ \int_{\mathbb{R}} [\varepsilon(s, X_s^{t,x} + \sigma(X_s^{t,x})u, Y_s^{t,x}) - \varepsilon(s, X_s^{t,x}, Y_s^{t,x}) \\
&- D_x\varepsilon(s, X_s^{t,x}, Y_s^{t,x})\sigma(X_s^{t,x})u]p_1(u)\nu(du), \\
(V^{(k)})_s^{t,x} &= D_y\varepsilon(s, X_s^{t,x}, Y_s^{t,x})(Z^{(k)})_s^{t,x} \\
&+ \int_{\mathbb{R}} [\varepsilon(s, X_s^{t,x} + \sigma(X_s^{t,x})u, Y_s^{t,x}) - \varepsilon(s, X_s^{t,x}, Y_s^{t,x}) \\
&- D_x\varepsilon(s, X_s^{t,x}, Y_s^{t,x})\sigma(X_s^{t,x})u]p_k(u)\nu(du), \\
& k \in \{2, \cdots\}.
\end{aligned} \tag{3.3}$$

From Proposition 3.4 appear in [4], the process $\{(U_s^{t,x}, V_s^{t,x}), \quad s \in [t,T]\}$ belongs to $\mathcal{E}$ for each $(t,x) \in [0,T] \times \overline{\Theta}$.

Now we are ready to give the following result.

Theorem 3.5 *For each $(t,x) \in [0,T] \times \overline{\Theta}$, the pair $(U^{t,x}, V^{t,x})$ is the unique solution* *of the following GBSDEL:*

$$\begin{aligned} U_s^{t,x} &= u_0(X_T^{t,x}) + \int_t^T \widetilde{f}(r, X_r^{t,x}, U_r^{t,x}, V_r^{t,x})dr + \int_s^T \widetilde{\phi}(r, X_r^{t,x}, U_r^{t,x})dA_r^{t,x} \\ &\quad - \sum_{k=1}^{\infty} \int_s^T (V_r^{t,x})^{(k)} dH_r^{(k)}, \;\; t \le s \le T, \end{aligned} \tag{3.4}$$

where $\widetilde{f}$ *and* $\widetilde{\phi}$ *are given by* (2.4) *and* (2.5) *respectively.*

Proof For clarity, $(X^{t,x}, Y^{t,x}, Z^{t,x}, U^{t,x}, V^{t,x})$ will be replaced by (X, Y, Z, U, V) throughout this proof. As it is shown in [4], the mapping $(X, Y, Z) \mapsto (X, U, V)$ is one-to-one, with the inverse transformation:

$$\begin{aligned} Y_s &= \eta(s, X_s, U_s), \\ Z_s^{(1)} &= D_y\eta(s, X_s, U_s)V_s^{(1)} + \sigma(X_s)D_x\eta(s, X_s, U_s) \\ &\quad + \int_{\mathbb{R}} [\eta(s, X_s + \sigma(X_s)u, U_s) - \eta(s, X_s, U_s) - D_x\eta(s, X_s, U_s)\sigma(X_s)u]p_1(u)\nu(du), \\ Z_s^{(k)} &= D_y\eta(s, X_s, U_s)V_s^{(k)} \\ &\quad + \int_{\mathbb{R}} [\eta(s, X_s + \sigma(X_s)u, U_s) - \eta(s, X_s, U_s) - D_x\eta(s, X_s, U_s)\sigma(X_s)u]p_k(u)\nu(du), \\ &\quad k \in \{2, \cdots\}. \end{aligned} \tag{3.5}$$

Thanks to (3.3) and (3.5), the uniqueness of GBSDEL (3.4) follows from GBDSDEL (3.2). Thus, the proof reduces to show that (U, V) is a solution of the GBSDEL (3.4). To this end, let us remark that $U_T = Y_T = u_0(X_T)$. Moreover, applying the generalized Itô-Ventzell formula (see Theorem 4.2) to $\varepsilon(s, X_s, Y_s)$, and after a little calculation we obtain

$$
\begin{aligned}
U_t &= u_0(X_T) + \int_t^T D_y\varepsilon(s,X_s,Y_s)f(s,X_s,Y_s,Z_s)ds + \int_t^T D_y\varepsilon(s,X_s,Y_s)\phi(s,X_s,Y_s)dA_s \\
&\quad - \sum_{k=1}^{\infty}\int_t^T D_y\varepsilon(s,X_s,Y_s)Z_s^{(k)}dH_s^{(k)} - m_1\int_t^T D_x\varepsilon(s,X_s,Y_s)\sigma(X_s)ds \\
&\quad - \int_t^T D_x\varepsilon(s,X_s,Y_s)\sigma(X_s)dH_s^{(1)} \\
&\quad - \int_t^T D_x\varepsilon(s,X_s,Y_s)\nabla\psi(X_s)dA_s - \frac{1}{2}\int_t^T \sigma(X_s)^* D_{xx}\varepsilon(s,X_s,Y_s)\sigma(X_s)ds \\
&\quad - \sum_{k=1}^{\infty}\int_t^T\int_{\mathbb{R}}[\varepsilon(s,X_s+\sigma(X_s)u,Y_s) - \varepsilon(s,X_s,Y_s) - D_x\varepsilon(s,X_s,Y_s)\sigma(X_s)u]p_k(u)\nu(du)dH_s^{(k)} \\
&\quad + \int_t^T\int_{\mathbb{R}}[\varepsilon(s,X_s+\sigma(X_s)u,Y_s) - \varepsilon(s,X_s,Y_s) - D_x\varepsilon(s,X_s,Y_s)\sigma(X_s)u]\nu(du)ds \\
&\quad + \frac{1}{2}\sum_{k=1}^{\infty}\int_t^T D_{yy}\varepsilon(s,X_s,Y_s)|Z_s^{(k)}|^2ds - \int_t^T \sigma^*(X_s)D_{xy}\varepsilon(s,X_s,Y_s)Z_s^{(1)}ds \\
&\quad - \frac{1}{2}\int_t^T D_y\varepsilon(s,X_s,Y_s)\langle g, D_y g\rangle(s,X_s,Y_s)ds. \qquad (3.6) \\
U_t &= u_0(X_T) + \int_t^T F(s,X_s,Y_s,Z_s)ds + \int_t^T \Phi(s,X_s,Y_s)dA_s - \sum_{k=1}^{\infty}\int_t^T V^{(k)}dH_s^{(k)},
\end{aligned}
$$

where

$$
\begin{aligned}
F(s,x,y,z) &= D_y\varepsilon f(s,x,y,z) - m_1 D_x\varepsilon\sigma(x) + \frac{1}{2}D_{yy}\varepsilon\|z\|^2 - \sigma^*(x)D_{xy}\varepsilon z^{(1)} \\
&\quad - \frac{1}{2}\sigma^*(x)D_{xx}\varepsilon\sigma(x) - \frac{1}{2}D_y\varepsilon\langle g, D_y g\rangle(s,x,y) \\
&\quad + \int_{\mathbb{R}}[\varepsilon(s,x+\sigma(x)u,y) - \varepsilon(s,x,y) - D_x\varepsilon\sigma(x)u]\nu(du) \qquad (3.7)
\end{aligned}
$$

and

$$\Phi(s,x,y) = D_y\varepsilon\phi(s,x_s,y,z) - D_x\varepsilon\nabla\psi(x), \qquad (3.8)$$

replaced $\varepsilon(s,x,y)$ by ε. Comparing (3.6) with (3.4), it suffice to show that

$$F(s,X_s,Y_s,Z_s) = \widetilde{f}(s,X_s,U_s,V_s), \quad \forall\, s\in[0,T],\ \mathbb{P}\text{-a.s}, \qquad (3.9)$$

and

$$\Phi(s,X_s,Y_s) = \widetilde{\phi}(s,X_s,U_s), \ \ \forall\, s\in[0,T],\ \mathbb{P}\text{-a.s.} \qquad (3.10)$$

To this end, if we write $\sigma(X_s) = \sigma_s$ and recall (2.6) together with Remark 2.5 we obtain the following equalities:

$$D_x\varepsilon(s,X_s,Y_s)\sigma(X_s) = -D_y\varepsilon(s,X_s,Y_s)\sigma(X_s)D_x\eta(s,X_s,U_s) \tag{3.11}$$

$$\begin{aligned}(D_y\varepsilon)f\left(s,X_s,Y_s,(Z_s^{(k)})_{k=0}^{\infty}\right) &= (D_y\varepsilon)f\Big(s,X_s,\eta(s,X_s,U_s),\Big(D_y\eta V_s^{(k)}+\sigma_s^*(D_x\eta)\mathbf{1}_{\{k=1\}}\\ &\quad +\int_{\mathbb{R}}\theta^k(s,X_s,U_s,u)\nu(du)\Big)_{k=1}^{\infty}\Big)\\ \sigma_s^*(D_{xy}\varepsilon)Z_s^{(1)} &= \sigma_s^*(D_{xy}\varepsilon)D_y\eta(s,X_s,U_s)V^{(1)}+\sigma_s^*(D_{xy}\varepsilon)\sigma_s^*D_x\eta\\ &\quad +\sigma_s^*(D_{xy}\varepsilon)\int_{\mathbb{R}}\theta^1(s,X_s,U_s,u)\nu(du) \end{aligned}\tag{3.12}$$

$$\begin{aligned}-\frac{1}{2}(D_{yy}\varepsilon)\sum_{k=1}^{\infty}|Z_s^{(k)}|^2 &= \frac{1}{2}(D_y\varepsilon)(D_{yy}\eta)\sum_{k=1}^{\infty}|V_s^{(k)}|^2+(D_y\varepsilon)^2(D_{yy}\eta)V_s^{(1)}\sigma_s(D_x\eta)\\ &\quad +\frac{1}{2}(D_y\varepsilon)(D_{yy}\eta)|\sigma_s(D_x\eta)(D_y\varepsilon)|^2\\ &\quad +(D_y\varepsilon)^2D_{yy}\eta\sum_{k=1}^{\infty}V_s^{(k)}\int_{\mathbb{R}}\theta^k(s,X_s,U_s,u)\nu(du)\\ &\quad +\frac{1}{2}(D_y\varepsilon)(D_{yy}\eta)\sum_{k=1}^{\infty}\left|D_y\varepsilon\int_{\mathbb{R}}\theta^k(s,X_s,U_s,u)\nu(du)\right|^2\\ &\quad +(D_y\varepsilon)(D_{yy}\eta)(D_y\varepsilon)^2\sigma_sD_x\eta\int_{\mathbb{R}}\theta^1(s,X_s,U_s,u)\nu(du).\end{aligned}\tag{3.13}$$

Hence plugging (3.11)-(3.13) in (3.7), we get

$$\begin{aligned}F(s,X_s,Y_s,Z_s) &= D_y\varepsilon\Big[f\Big(s,X_s,\eta,\Big(D_y\eta V_s^{(k)}+\sigma_s^*(D_x\eta)\mathbf{1}_{\{k=1\}}+\int_{\mathbb{R}}\theta^k(s,X_s,U_s,u)\nu(du)\Big)_{k=0}^{\infty}\Big)\\ &\quad +m_1\sigma_sD_x\eta+\frac{1}{2}(D_{yy}\eta)\sum_{k=1}^{\infty}\left|V_s^{(k)}+D_y\varepsilon\int_{\mathbb{R}}\theta^k(s,X_s,U_s,u)\nu(du)\right|^2\\ &\quad -\frac{1}{2}\langle g,D_yg\rangle(s,X_s,\eta)\Big]\\ &\quad +V^{(1)}\sigma_s^*\Big[(D_x\eta)(D_y\varepsilon)^2(D_{yy}\eta)-D_y\eta(D_{xy}\varepsilon)\Big]\\ &\quad +\left(\int_{\mathbb{R}}\theta^1(s,X_s,U_s,u)\nu(du)\right)\sigma_s^*\Big[(D_x\eta)(D_y\varepsilon)^2(D_{yy}\eta)-D_y\eta(D_{xy}\varepsilon)\Big]\\ &\quad \Big[\frac{1}{2}(D_{yy}\eta)(D_y\varepsilon|\sigma_s(D_x\eta)(D_y\varepsilon)|^2-\sigma_s^*(D_{xy}\varepsilon)\sigma_s^*D_x\eta \qquad (3.14)\\ &\quad -\frac{1}{2}\sigma^*(x)D_{xx}\varepsilon\sigma_s+\int_{\mathbb{R}}[\varepsilon(s,X_s+\sigma_su,Y_s)-\varepsilon(s,X_s,Y_s)-(D_x\varepsilon)\sigma_su]\nu(du)\Big],\end{aligned}$$

where all the derivatives of the random field $\varepsilon(\cdot,\cdot,\cdot)$ are to be evaluated at the point $(s,x,\eta(s,x,y))$, and all those of $\eta(\cdot,\cdot,\cdot)$ at (s,x,y). On other hand, using again Remark 2.5, we have

$$\begin{aligned}-\frac{1}{2}\sigma^*(x)(D_{xx}\varepsilon)\sigma_s &= (\sigma_s)^2 D_{xy}\varepsilon D_x\eta - \frac{1}{2}(D_y\varepsilon)D_{yy}\eta|\sigma_s D_x\eta D_y\varepsilon|^2 \\ &\quad +\frac{1}{2}(D_y\varepsilon)(\sigma_s)^2(D_{xx}\eta) \qquad (3.15)\end{aligned}$$

and

$$D_x\eta(D_y\varepsilon)^2(D_{yy}\eta) - D_{xy}\varepsilon D_y\eta = D_y\varepsilon D_{xy}\eta. \qquad (3.16)$$

The equalities in (3.15) and (3.16)), together with $D_y\varepsilon(s,X_s,Y_s) = (D_y\eta)^{-1}(s,X_s,U_s)$, imply that

$$\begin{aligned}F(s,X_s,Y_s,Z_s) &= D_y\varepsilon\Big[f\left(s,X_s,\eta,\left(D_y\eta V_s^{(k)} + \sigma_s^*(D_x\eta)\mathbf{1}_{\{k=1\}} + \int_{\mathbb{R}}\theta^k(s,X_s,U_s,u)\nu(du)\right)_{k=0}^{\infty}\right) \\ &\quad +m_1\sigma_s D_x\eta + \frac{1}{2}(D_{yy}\eta)\sum_{k=1}^{\infty}\left|V_s^{(k)} + D_y\varepsilon\int_{\mathbb{R}}\theta^k(s,X_s,U_s,u)\nu(du)\right|^2 \\ &\quad -\frac{1}{2}\langle g, D_y g\rangle(s,X_s,\eta)\Big] \\ &\quad +\frac{1}{2}(D_y\varepsilon)\sigma_s^2(D_{xx}\eta) + (D_y\varepsilon)\sigma_s D_{xy}\eta\left(V_s^{(1)} + \int_{\mathbb{R}}\theta^1(s,X_s,U_s,u)\nu(du)\right) \\ &\quad +\int_{\mathbb{R}}[\varepsilon(s,X_s+\sigma_s u,Y_s) - \varepsilon(s,X_s,Y_s) - (D_x\varepsilon)\sigma_s u]\nu(du).\end{aligned}$$

Next, using again Remark 2.5 together with change variable $(t=-u)$ we have

$$\begin{aligned}&\int_{\mathbb{R}}[\varepsilon(s,X_s+\sigma_s u,Y_s) - \varepsilon(s,X_s,Y_s) - (D_x\varepsilon)\sigma_s u]\nu(du) \\ &= -D_y\varepsilon\int_{\mathbb{R}}[\eta(s,X_s+\sigma_s u,U_s) - \eta(s,X_s,U_s) - (D_x\eta)\sigma_s u]\nu(du) \\ &= D_y\varepsilon\int_{\mathbb{R}}[\eta(s,X_s+\sigma_s u,U_s) - \eta(s,X_s,U_s) - (D_x\eta)\sigma_s u]\nu(du).\end{aligned}$$

Finally we obtain

$$\begin{aligned} F(s,X_s,Y_s,Z_s) &= D_y\varepsilon\Big[f\left(s,X_s,\eta,\left(D_y\eta V_s^{(k)}+\sigma_s^*(D_x\eta)\mathbf{1}_{\{k=1\}}+\int_{\mathbb{R}}\theta^k(s,X_s,U_s,u)\nu(du)\right)_{k=0}^{\infty}\right) \\ &\quad +m_1\sigma_s D_x\eta+\frac{1}{2}(D_{yy}\eta)\sum_{k=1}^{\infty}\left|V_s^{(k)}+D_y\varepsilon\int_{\mathbb{R}}\theta^k(s,X_s,U_s,u)\nu(du)\right|^2 \\ &\quad -\frac{1}{2}\langle g,D_yg\rangle(s,X_s,\eta)\Big] \\ &\quad +\frac{1}{2}(D_y\varepsilon)\sigma_s^2(D_{xx}\eta)+(D_y\varepsilon)\sigma_s D_{xy}\eta\left(V_s^{(1)}+\int_{\mathbb{R}}\theta^1(s,X_s,U_s,u)\nu(du)\right) \\ &\quad +D_y\varepsilon\int_{\mathbb{R}}[\eta(s,X_s+\sigma_s u,U_s)-\eta(s,X_s,U_s)-(D_x\eta)\sigma_s u]\nu(du). \end{aligned} \tag{3.17}$$

Since the expressions in (3.14) and (3.17) are equal, this shows the equality in (3.9).

Next, we show the equality in (3.10). In fact,

$$\begin{aligned} \Phi(s,X_s,Y_s) &= D_y\varepsilon\phi(s,X_s,Y_s)-D_x\varepsilon\nabla\psi(X_s) \\ &= D_y\varepsilon(s,X_s,Y_s)[D_x\eta(s,X_s,U_s)\nabla\psi(X_s)+\phi(s,X_s,\eta(s,X_s,U_s)] \\ &= \frac{1}{D_y\eta(s,X_s,U_s)}[D_x\eta(s,X_s,U_s)\nabla\psi(X_s)+\phi(s,X_s,\eta(s,X_s,U_s)] \\ &= \widetilde{\phi}(s,X_s,U_s). \end{aligned} \tag{3.18}$$

This ends the proof of theorem. ■

We are now ready to prove the existence of the stochastic viscosity solutions of SPDIE (f,g,ϕ). Let us define for each $(t,x)\in[0;T]\times\overline{\Theta}$ two random fields

$$u(t,x)=Y_t^{t,x},\ \ v(t,x)=U_t^{t,x}. \tag{3.19}$$

Theorem 3.6 *Assume that* (**A1**)–(**A5**) *hold. Then, the random field v is a stochastic viscosity solution of SPDIE $(\widetilde{f},0,\widetilde{\phi})$ and hence u is a stochastic viscosity solution to SPDIE (f,g,ϕ).*

Proof Let define $u(t,x)=Y_t^{t,x}$ and $v(t,x)=U_t^{t,x}$ where Y and U are given as above. We have

$$u(t,\omega,x)=\eta(t,\omega,v(t,\omega,x))\ \text{ and }\ v(t,\omega,x)=\varepsilon(t,\omega,v(t,\omega,x)). \tag{3.20}$$

Since $Y_s^{x,t}$ is $\mathcal{F}_{t,s}^L\vee\mathcal{F}_{s,T}^B$-measurable, it follows that $Y_t^{x,t}$ is $\mathcal{F}_{t,T}^B$-measurable. Therefore, $u(t,x)$ is $\mathcal{F}_{t,T}^B$- measurable and so it is independent of $\omega'\in\Omega'$. Consequently, according to proposition 1.7 in [10], we have $u\in C(\mathbf{F}^B,[0,T]\times\overline{\Theta})$. Moreover, (3.20) implies that v belongs to $C(\mathbf{F}^B,[0,T]\times\overline{\Theta})$. We emphasis that as an $\mathbf{F}^B$-progressively measurable ω-wise viscosity solution is automatically a stochastic viscosity solution (see Definition

2.3), it suffice to show that v is an ω-wise viscosity solution to SPDIE$(\widetilde{f}, 0, \widetilde{\phi})$. To do it, let us denote, for a fixed $\omega \in \Omega$,

$$\overline{U}^{\omega}(\omega') = U(\omega, \omega'), \;\; \overline{V}^{\omega}(\omega') = V(\omega, \omega').$$

Then $(\overline{U}^{\omega}, \overline{V}^{\omega})$ is the unique solution of the GBDSDELs with coefficient $(\widetilde{f}(\omega, \cdot, \cdot, \cdot), \;\; \widetilde{\phi}(\omega, \cdot, \cdot))$, and as it is shown by Ren and Otmani in [15], $\bar{v}(\omega, t, x) = \overline{U}^{\omega}_t$ is a viscosity solution to SPDIE$(\widetilde{f}(\omega, \cdot, \cdot, \cdot), \widetilde{\phi}(\omega, \cdot, \cdot))$ with nonlinear Neumann boundary condition. By Blumenthal's 0-1 law it folows that $\mathbb{P}'(\overline{U}^{\omega}_t(\omega') = U_t(\omega, \omega')) = 1$. Hence we get $\bar{v}(t, x) = v(t, x)$ $\mathbb{P}$-almost surely for all $(t, x) \in [0, T]\Theta$. Therefore, for every ω fixed the function $v \in C(\mathbf{F}^B, [0, T] \times \Theta)$ is a viscosity solution to the SPDE $(\widetilde{f}(\omega, \cdot, \cdot, \cdot), \widetilde{\phi}(\omega, \cdot, \cdot))$. Hence, by definition it is an ω-wise viscosity solution and hence a stochastic viscosity solution to SPDIE$(\widetilde{f}, 0, \widetilde{\phi})$. The conclusion of the theorem now follows from Theorem 3.5. ■

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