Asymmetric Multi-channel Sampling in Shift Invariant Spaces

Sinuk Kang (1) and K.H. Kwon (1)

(1) KAIST, 335 Gwahangro, Yuseong-gu, Daejeon 305-701, S. Korea.
sukang@kaist.ac.kr, khkwon@kaist.edu

Abstract:

We consider a multi-channel sampling with asymmetric sampling rates in shift invariant spaces, while related previous works have supposed that each channel has a symmetric(uniform) sampling rate. Motivated by the fact that shift invariant spaces are isomorphic images of $L^2[0, 2\pi]$, we obtain a sampling expansion in shift invariant spaces by using frame or Riesz basis expansion in $L^2[0, 2\pi]$. The samples in the expansion are expressed in terms of frame coefficients of an appropriate function with respect to a certain frame in $L^2[0, 2\pi]$. The involved reconstruction functions are given explicitly by using the frame operator. We also present relation between asymmetric multi-channel sampling and symmetric one.

1. Introduction

Reconstructing a band-limited signal f from samples which are taken from several channeled versions of f is called multi-channel sampling. The multi-channel sampling method goes back to the work of Shannon [6] and Fogel [2], where the reconstruction of a band-limited signal from samples of the signal and of its derivatives was suggested. Generalized sampling expansion for arbitrary multi-channel sampling was introduced first by Papoulis [5].

Papoulis' result has been extended to a general shift-invariant space [1, 7, 8]. Here, a shift invariant space $V(\phi)$ with a generator $\phi \in L^2(\mathbb{R})$ is defined by the closed subspace of $L^2(\mathbb{R})$ spanned by integer translates $\{\phi(t-n) : n \in \mathbb{Z}\}$ of ϕ. Recently García and Pérez-Villalón [3] derived stable generalized sampling in a shift-invariant space by using some special dual frames in $L^2[0, 1]$.

The previous works related to the multi-channel sampling have assumed that numbers of samples from each channel are uniform, namely, sampling rates of channels are same. In this paper we consider a multi-channel sampling with asymmetric sampling rates in shift invariant spaces. We find an expression for the samples as frame coefficients of an appropriate function in $L^2[0, 2\pi]$ with respect to some particular frame in $L^2[0, 2\pi]$ and present the sufficient and necessary condition under which a sequence of functions of particular form becomes a frame or a Riesz basis for $L^2[0, 2\pi]$. Using isomorphism between a shift invariant space $V(\phi)$ and $L^2[0, 2\pi]$, we derive sampling theory in $V(\phi)$ with some Riesz generator ϕ and find a formula of reconstruction functions by means of the frame operator. The theory contains both a frame and Riesz basis expansion as sampling formulas.

2. Asymmetric multi-channel sampling

Assume that $\phi(t)$ is everywhere well defined complex valued square integrable function on $\mathbb{R}$ throughout the paper. Moreover, let $\phi(t)$ be a Riesz generator with $C_\phi(t) < \infty$ for any $t \in \mathbb{R}$ so that $V(\phi)$ is an RKHS(see Proposition 2.4 in [4]). We now are given a LTI system $\{L_j[\cdot]\}_{j=1}^N$ whose impulse response is $\{l_j(t) : l_j \in L^2(\mathbb{R}),\ j = 1, 2, \cdots, N\}$. The aim of this paper is to recover any $f(t) \in V(\phi)$ via discrete samples from $\{L_j[f]\}_{j=1}^N$ as

$$f(t) = \sum_{j=1}^{N} \sum_{n\in\mathbb{Z}} L_j[f](\sigma_j + r_j n)s_{j,n}(t), \qquad (1)$$

where $\{s_{j,n}(t) : j = 1, \cdots, N \text{ and } n \in \mathbb{Z}\}$ is a frame or a Riesz bases of $V(\phi)$ and $0 \le \sigma_j < r_j$ with a positive integer r_j for $j \in \{1, 2, \cdots, N\}$.

2.1 An expression for the samples

Define an isomorphism J from $L^2[0, 2\pi]$ onto $V(\phi)$ by

$$JF(t) = \frac{1}{2\pi} \sum_{n\in\pi} \langle F(\xi), e^{-in\xi}\rangle \phi(t-n),\ F(\xi) \in L^2[0, 2\pi].$$

By the isomorphism $J : L^2[0, 2\pi] \to V(\phi)$, the reconstruction formula (1) is equivalent to the following one:

$$F(\xi) = \sum_{j=1}^{N} \sum_{n\in\mathbb{Z}} L_j[f](\sigma_j+r_j n)S_{j,n}(\xi),\ F(\xi) \in L^2[0, 2\pi], \qquad (2)$$

where $f(t) = JF(t)$ and $s_{j,n}(t) = JS_{j,n}(t)$. Notice further that $L_j f(\sigma_j + r_j n)$ is represented by an inner product of $F(\xi)$ and some function in $L^2[0, 2\pi]$.

Lemma 2.1.1 *Let $L[\cdot]$ be a LTI system with an impulse response $l(t) \in L^2(\mathbb{R})$ and $\psi(t) = L[\phi](t) = (\phi * l)(t)$.*

(a) L is a bounded operator from $L^2(\mathbb{R})$ into $L^\infty(\mathbb{R})$, $\|f * l\|_\infty \le \|f\|_2 \|l\|_2$ and $Lf(t) \in C^o_\infty(\mathbb{R})$,

(b) $\sup_{\mathbb{R}} C_\psi(t) < \infty$,

(c) (cf. Lemma 2 in [3]) for any $f(t) = (\mathbf{c} * \phi)(t)$ with $\mathbf{c} \in \ell^2$ in $V(\phi)$, $L[f](t) = (\mathbf{c}*\psi)(t)$ converges absolutely and uniformly on $\mathbb{R}$. For any $f(t) = JF(t) \in V(\phi)$ with $F(\xi) \in L^2[0, 2\pi]$,

$$L[f](t) = \langle F(\xi), \frac{1}{2\pi}\overline{Z_\psi(t, \xi)}\rangle_{L^2[0,2\pi]}.$$

In particular,

$$L[f](\sigma_j+r_jn) = \langle F(\xi), \frac{1}{2\pi}\overline{Z_\psi(\sigma_j, \xi)e^{-ir_jn\xi}}\rangle_{L^2[0,2\pi]}. \quad (3)$$

2.2 The sampling theorem

For a given LTI system $\{L_j[\cdot]\}_{j=1}^N$, let $L_j\phi(t) = \psi_j(t)$, $1 \le j \le N$. Using equation (3), the expansion (2) is equivalent to

$$F(\xi) = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} \langle F(\xi), \frac{1}{2\pi}\overline{Z_{\psi_j}(\sigma_j, \xi)e^{-ir_jn\xi}}\rangle_{L^2[0,2\pi]} \cdot S_{j,n}(\xi),\ F(\xi) \in L^2[0, 2\pi],$$

where $f(t) = JF(t)$ and $s_{j,n}(t) = JS_{j,n}(t)$.
For convenience, we introduce a few more notations. Let $g_j(\xi) \in L^2[0, 2\pi]$ for $1 \le j \le N$, $g_{j,m_j}(\xi) := g_j(\xi)e^{ir_j(m_j-1)\xi}$ for $1 \le m_j \le \frac{r}{r_j}$ and

$$G(\xi) = [Dg_{1,1}(\xi), Dg_{1,2}(\xi), \cdots, Dg_{1,\frac{r}{r_1}}(\xi), Dg_{2,1}(\xi), \cdots, Dg_{N,\frac{r}{r_N}}(\xi)]^T,$$

where D is an unitary operator from $L^2[0, 2\pi]$ onto $L^2(I)^r$ defined by $(DF)(\xi) = [F(\xi), F(\xi + \frac{2\pi}{r}), \cdots, F(\xi + (r-1)\frac{2\pi}{r})]^T$, $F(\xi) \in L^2[0, 2\pi]$. Note that $G(\xi)$ is the $\sum_{j=1}^N \frac{r}{r_j} \times r$ matrix whose entries are in $L^2[0, \frac{2\pi}{r}]$. And define $\lambda_M(\xi)$(resp. $\lambda_m(\xi)$) as the largest(resp. the smallest) eigenvalue of $r \times r$ matrix $G(\xi)^*G(\xi)$, β_G as $\|\lambda_M(\xi)\|_\infty$ and α_G as $\|\lambda_m\|_0$.

Lemma 2.2.1 *Let $g_j \in L^2[0, 2\pi]$ and r_j be a positive integer for $1 \le j \le N$. Define r as the least common multiplier of $\{r_j\}_{j=1}^N$. Then $\{\overline{g_j(\xi)}e^{-ir_jn\xi} : 1 \le j \le N,\ n \in \mathbb{Z}\}$ is a*

(a) Bessel sequence in $L^2[0, 2\pi]$ if and only if $\|\lambda_M(\xi)\|_\infty < \infty$ if and only if $g_j \in L^\infty[0, 2\pi]$ for $1 \le j \le N$. In this case, optimal bound is $\frac{2\pi}{r}\|\lambda_M(\xi)\|_\infty$;

(b) frame of $L^2[0, 2\pi]$ if and only if $0 < \|\lambda_m(\xi)\|_0 \le \|\lambda_M(\xi)\|_\infty < \infty$ so that $r \le \sum_{j=1}^N \frac{r}{r_j}$ and optimal bounds are $\frac{2\pi}{r}\|\lambda_m(\xi)\|_0 \le \frac{2\pi}{r}\|\lambda_M(\xi)\|_\infty$;

(c) Riesz basis of $L^2[0, 2\pi]$ if and only if frame of $L^2[0, 2\pi]$ and $r = \sum_{j=1}^N \frac{r}{r_j}$, i.e., $1 = \sum_{j=1}^N \frac{1}{r_j}$ if and only if $g_j(\xi) \in L^\infty[0, 2\pi]$ for $1 \le j \le N$, $1 = \sum_{j=1}^N \frac{1}{r_j}$ and $|\det G(\xi)| \ge \exists\alpha > 0$ a.e..

Appealing to the setting $g_j(\xi) = \frac{1}{2\pi}Z_{\psi_j}(\sigma_j, \xi)$ for $1 \le j \le N$, we have

Theorem 2.2.2 *Let $\phi(t)$ be a Riesz generator with $C_\phi(t) < \infty$, $t \in \mathbb{R}$ and $\{L_j[\cdot]\}_{j=1}^N$ be LTI systems with an impulse response $\{l_j(t)\}_{j=1}^N \in L^2(\mathbb{R})$. Let $\{\psi_j(t) = (\phi * l_j)(t)\}_{j=1}^N$, $r_j \ge 1$ an integer and $0 \le \sigma_j < r_j$.*

(a) If $0 < \alpha_G \le \beta_G < \infty$, i.e., $0 < \alpha_G$ and $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$, $1 \le j \le N$, then there is a frame $\{s_{j,n}(t) : 1 \le j \le N,\ n \in \mathbb{Z}\}$ of $V(\phi)$ for which

$$f(t) = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}} L_jf(\sigma_j+r_jn)s_{j,n}(t),\ f(t) \in V(\phi). \quad (4)$$

(b) Assume that $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$, $1 \le j \le N$. Then there is a frame $\{s_{j,n}(t) : 1 \le j \le N,\ n \in \mathbb{Z}\}$ of $V(\phi)$ for which (4) holds if and only if $0 < \alpha_G$.

(c) Assume that $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$, $1 \le j \le N$. Then there is a Riesz basis $\{s_{j,n}(t) : 1 \le j \le N,\ n \in \mathbb{Z}\}$ of $V(\phi)$ for which (4) holds if and only if $0 < \alpha_G$ and $1 = \sum_{j=1}^N \frac{1}{r_j}$.

In all cases, sampling series (4) converges in $L^2(\mathbb{R})$, absolutely on $\mathbb{R}$ and uniformly on any subset of $\mathbb{R}$ on which $C_\phi(t)$ is bounded.

Remark 2.2.3 *Asymmetric multi-channel sampling series with LTI system $\{L_j[\cdot]\}_{j=1}^N$ whose impulse response is $\{l_j(t)\}_{j=1}^N$ can be considered as symmetric multi-channel sampling series with LTI system $\{\tilde{L}_{j,m_j}[\cdot]\}_{j=1,m_j=1}^{N,\frac{r}{r_j}}$ with impulse response $\{\tilde{l}_{j,m_j}(t)\}_{j=1,m_j=1}^{N,\frac{r}{r_j}}$, where $\tilde{l}_{j,m_j}(t) = l_j(r_j(m_j-1)+t)$.*

2.3 Reconstruction functions

Let S be a frame operator with frame $\{\overline{g_j(\xi)}e^{-ir_jn\xi}\}_{j,n}$. For any $F(\xi) \in L^2[0, 2\pi]$,

$$SF(\xi) = \sum_{j=1}^{N}\sum_{m_j=1}^{\frac{r}{r_j}} \overline{g_j(\xi)}e^{-ir_j(m_j-1)\xi} \cdot\frac{2\pi}{r}g_{j,m}(\xi)^TDF(\xi)$$

so that

$$DSF(\xi) = \frac{2\pi}{r}G^*G(\xi)DF(\xi).$$

Then, from Lemma 2.2.1 (b), there exists $(G^*G)^{-1}(\xi)$ a.e. such that

$$D(S^{-1}(\overline{g_j(\xi)}e^{-ir_jn\xi})) = \frac{r}{2\pi}(G^*G)^{-1}(\xi)D(\overline{g_j(\xi)}e^{-ir_jn\xi})$$

for $1 \le j \le N$ and $n \in \mathbb{Z}$. Hence,

$$\{s_{j,n}\}_{j,n} = \{\frac{r}{2\pi}JD^{-1}[(G^*G)^{-1}(\xi)D(\overline{g_j(\xi)}e^{-ir_jn\xi})]\}_{j,n}.$$

Remark 2.3.1 *One sufficient condition under which $\{s_{j,n}\}_{j,n}$ is translates of a single function in $L^2[0, 2\pi]$ is that r divides r_j for all $1 \le j \le N$. Since r is the least common multiplier of $\{r_j\}_{j=1}^N$, the condition holds if and only if $r = r_j$ for all $1 \le j \le N$.*

References:

[1] I. Djokovic, P. P. Vaidyanathan, Generalized sampling theorems in multiresolution subspaces, *IEEE Trans. Signal Process.*, 45:583-599, 1997.

[2] L. J. Fogel, A note on the sampling theorem, *IRE Tran. Infor. Theory IT-1*:47-48, 1995.

[3] A. G. García and G. Pérez-Villarón, Dual frames in $L^2(0,1)$ connected with generalized sampling in shift-invariant spaces, *Appl. Comput. Harmon. Anal.*, 20:422-433, 2006.

[4] J. M. Kim, K. H. Kwon, Sampling expansion in shift invariant spaces, *Intern. J Wavelets, Multiresolution and Inform. Processing*, 6(2):223-248, 2008.

[5] A. Papoulis, Generalized sampling expansion, *IEEE Trans. Circuits Systems*, 24(11), 652-654, 1977.

[6] C. E. Shannon, Communication in the presence of noise, *Proc. IRE*, 37:10-21, 1949.

[7] M. Unser, J. Zerubia, Generalized sampling: Stability and performance analysis, *IEEE trans. Signal Process.*, 45(12):2941-2950, 1997.

[8] M. Unser, J. Zerubia, A generalized sampling theory without band-limiting constraints, *IEEE Trans. Circuits Syst. 2*, 45(8):959-969, 1998.