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# On the incoherence of noiselet and Haar bases 

Tomas Tuma, Paul Hurley<br>IBM Research, Zurich Laboratory 8803 Rüschlikon, Switzerland<br>E-mail: \{uma,pah\}@zurich.ibm.com


#### Abstract

: Noiselets are a family of functions completely uncompressible using Haar wavelet analysis. The resultant perfect incoherence to the Haar transform, coupled with the existence of a fast transform has resulted in their interest and use as a sampling basis in compressive sampling. We derive a recursive construction of noiselet matrices and give a short matrix-based proof of the incoherence.


## 1. Introduction

The noiselet basis, originally described in [2], has garnered interest recently because noiselets (1) are maximally incoherent to the Haar basis and (2) have a fast algorithm for their implementation. Thus, they have been employed in compressive sampling to sample signals that are sparse in the Haar domain [1].
The work presented here was motivated by the observation that it had not been previously shown in a straightforward way that the discrete Haar transform is maximally incoherent to a discretized version of the noiselet transform. Additionally, the exact form of a noiselet matrix needed to be inferred from the original work.
The main contributions are the derivation of a recursive, tensor product-based, construction of noiselet matrices, the unitary matrices that result from the noiselet transform for discrete input, and an intuitive proof showing its incoherence to the corresponding Haar matrix.

## 2. Preliminaries

### 2.1 General definitions

Definition 1. Let $A$ be an $m \times n$ matrix, and $B$ be a matrix of an arbitrary size. The Kronecker product of $A$ and $B$ is

$$
A \otimes B=\left[\begin{array}{ccc}
a_{11} B & \cdots & a_{1 n} B \\
\vdots & \ddots & \vdots \\
a_{m 1} B & \cdots & a_{m n} B
\end{array}\right]
$$

The Kronecker product (see e.g. [4]) is a bilinear and associative operator which is not generally commutative. It can be combined with a standard maxtrix multiplication as follows:

$$
(A \otimes B)(C \otimes D)=A C \otimes B D
$$

whenever the products $A C, B D$ exist. This property is sometimes called the mixed product property.

Definition 2. Let $A$ be a $m \times n$ matrix. $A(k, *)$ denotes the (row) vector $(A(k, 1) \quad A(k, 2) \quad \ldots \quad A(k, n))$ while, $A(*, l)$ similarly denotes the (column) vector $(A(1, l) \quad A(2, l) \quad \ldots \quad A(m, l))^{T}$.

### 2.2 Noiselets

Noiselets [2] are functions that are completely uncompressible under the Haar transform. The family of noiselets is constructed on the interval $[0,1)$ as follows:

$$
\begin{aligned}
f_{1}(x) & =\chi_{[0,1)}(x), \\
f_{2 n}(x) & =(1-i) f_{n}(2 x)+(1+i) f_{n}(2 x-1) \\
f_{2 n+1}(x) & =(1+i) f_{n}(2 x)+(1-i) f_{n}(2 x-1)
\end{aligned}
$$

Here, $\chi_{[0,1)}(x)=1$ on the definition interval $[0,1)$ and 0 otherwise. It is shown in [2] that $\left\{f_{j}\right\}$ is a basis:

Theorem 1. The set $\left\{f_{j} \mid j=2^{N}, \ldots, 2^{N+1}-1\right\}$ is an orthogonal basis of the vector space $V_{2^{N}}$, which is the space of all possible approximations at the resolution $2^{N}$ of functions in $L^{2}[0,1)$.

### 2.3 Haar Transform

Haar wavelet transform can be described by a real square matrix. For our purposes, it is advantageous to recursively build the Haar matrix using the Kronecker product [3]:

$$
H_{n}=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
H_{n / 2} \otimes(1
\end{array}\right)
$$

The iteration starts with $H_{1}=[1]$. The normalization constant $\frac{1}{\sqrt{2}}$ ensures that $H_{n}^{T} H_{n}=I$. Haar wavelets are the rows of $H_{n}$.

## 3. Matrix construction of noiselets

First we extend and discretize the noiselet functions.
Definition 3. The extensions of noiselets to the interval $\left[0,2^{m}-1\right]$ sampled at points $0, \ldots, 2^{m}-1$ is the series


Figure 1: Noiselet matrix: graphical view. In figures (a) and (b), the black and white colors denote values of -0.25 and 0.25 respectively. In figures (c) and (d), the black, gray and white colors denote values of $-0.125,0$ and 0.125 respectively. .
of functions $f_{m}(k, l)$

$$
\begin{aligned}
f_{m}(1, l) & = \begin{cases}1 & l=0, \ldots, 2^{m}-1 \\
0 & \text { otherwise }\end{cases} \\
f_{m}(2 k, l) & =(1-i) f_{m}(k, 2 l)+(1+i) f_{m}\left(k, 2 l-2^{m}\right) \\
f_{m}(2 k+1, l) & =(1+i) f_{m}(k, 2 l)+(1-i) f_{m}\left(k, 2 l-2^{m}\right)
\end{aligned}
$$

where $m$ denotes the range of extension, $k=1, \ldots, 2^{m+1}$ is the function index and $l=0, \ldots, 2^{m}-1$ is the sample index.

Starting with a $1 \times 1$ matrix $N_{1}$, a sequence of noiselet matrices $N_{1}, N_{2}, N_{4}, \ldots, N_{2^{m}}$ of sizes $1 \times 1,2 \times 2,4 \times 4$, $\ldots, 2^{m} \times 2^{m}$, respectively, is generated. The rows of the $N_{n}$ matrix are noiselets which form an orthonormal basis for the space $\mathbb{C}^{n}$.

Definition 4. For $n=1, N_{1}=[1]$. Then the $n \times n$ noiselet matrix $N_{n}$ is built up recursively according to:

$$
N_{n}(k, *)=\frac{1}{2}(1-i \quad 1+i) \otimes N_{n / 2}\left(\frac{k}{2}, *\right)
$$

when $k=0,2,4, \ldots, n-2$ and

$$
N_{n}(k, *)=\frac{1}{2}(1+i \quad 1-i) \otimes N_{n / 2}\left(\frac{k-1}{2}, *\right)
$$

when $k=1,3, \ldots, n-1$.

Lemma 1. Let $m>0$. The noiselet matrices $N_{1}, N_{2}, N_{4}, \ldots, N_{2^{m}}$ are built up from a series of discretised and extended noiselets $f_{m}$ :

$$
N_{n}(k, l)=f_{m}\left(n+k, \frac{2^{m}}{n} l\right), \quad k, l=0, \ldots, n-1
$$

Proof. Let $m>0$ be fixed. For $n=1$

$$
N_{1}(0,0)=f_{m}(1,0)=1
$$

By induction, for a matrix of size $n=2^{p}, p=1, \ldots, m$, its basis vector $k=0,2,4, \ldots, n-2$ and vector indices $l=0, \ldots, \frac{n}{2}-1$

$$
\begin{aligned}
N_{n}(k, l) & =(1-i) N_{n / 2}\left(\frac{k}{2}, l\right) \\
& =(1-i) f_{m}\left(\frac{n}{2}+\frac{k}{2}, \frac{2^{m}}{\frac{n}{2}} l\right)=f_{m}\left(n+k, \frac{2^{m}}{n} l\right) .
\end{aligned}
$$

For the same $n, k$ and $l=\frac{n}{2}, \ldots, n-1$,

$$
\begin{aligned}
N_{n}(k, l) & =(1+i) N_{n / 2}\left(\frac{k}{2}, l-\frac{n}{2}\right) \\
& =(1+i) f_{m}\left(\frac{n}{2}+\frac{k}{2}, 2 \frac{2^{m} l}{n}-2^{m}\right)=f_{m}\left(n+k, \frac{2^{m}}{n} l\right)
\end{aligned}
$$

To see this, observe that $f_{m}$ is zero outside of $\left[0,2^{m}-1\right]$ and therefore, the first half of samples of $f_{m}(k, l)$ are defined exclusively by the expression $(1 \pm i) f_{m}(k, 2 l)$
whereas the second half of the samples are defined exclusively by $(1 \pm i) f_{m}\left(k, 2 l-2^{m}\right)$.
For $k$ odd ( $k=1,3, \ldots, n-1$ ) the proof is similar.
Specially, the noiselet matrix $N_{n}$ for $n=2^{m}$ can be found as the "tail" of the function series $f_{m}$. Indeed, the expression in Theorem 1 becomes $N(k, l)=f_{m}(n+k, l)$ for $n=2^{m}$.

## 4. Incoherence of noiselets and Haar

In what follows, we adhere to the terminology of basis coherence which is common in the field of compressive sampling. See for example [1] for details on these definitions and related literature.
Mutual coherence of two bases is defined as the maximum scalar product of any pair of their basis vectors:

Definition 5. Mutual coherence between two orthonormal bases $\Psi$, $\Phi$ is

$$
\mu(\Psi, \Phi)=\max _{k, j}\left|\left\langle\psi_{k}, \phi_{j}\right\rangle\right| .
$$

The minimal coherence is usually termed maximal or perfect incoherence, which means that $\mu(\Psi, \Phi)=O(1)$. In other words, the matrix of scalar products $\Psi \Phi^{*}$ is "flat". As Candès and Romberg suggest [1], we will show the perfect incoherence of Haar and noiselets in the following setting. Given an orthonormal $n \times n$ Haar matrix $H$, we compute the matrix of scalar products for a corresponding noiselet matrix $N$ normalized such that $N^{*} N=n I$. By doing so, the product will be flat with all values having the magnitude of 1 .
For clarity of the main proof, it saves some technical work to define a "twisted" noiselet basis.

Definition 6. The twisted noiselet matrix $\hat{N}_{1}=[1]$.
Then the $n \times n$ twisted noiselet matrix $\hat{N}_{n}$ is built up recursively by

$$
\hat{N}_{n}(k, *)=\frac{1}{2} \hat{N}_{n / 2}\left(\frac{k}{2}, *\right) \otimes(1-i \quad 1+i)
$$

when $k=0,2,4, \ldots, n-2$ and

$$
\hat{N}_{n}(k, *)=\frac{1}{2} \hat{N}_{n / 2}\left(\frac{k-1}{2}, *\right) \otimes(1+i \quad 1-i)
$$

when $k=1,3, \ldots, n-1$.
The difference between this and the definition of the noiselet matrix $N$ (Definition 4) is that the order of operands in the Kronecker product is changed. In fact, each one is just a permutation of the other.

Lemma 2. For $n=2^{m}$, the bases $N_{n}, \hat{N}_{n}$ consist of the same set of basis vectors.

Proof. Indeed, we can write $\hat{N}_{n}=P_{n} N_{n}$ where $P$ is the permutation matrix:
$P(k, *)= \begin{cases}(1 & 0) \otimes P_{n / 2}\left(\frac{k}{2}, *\right) \\ \left(\begin{array}{ll}0 & 1) \otimes P_{n / 2}\left(\frac{k-1}{2}, *\right)\end{array}\right. & k=0,2,4, \ldots, n-2 \\ , \ldots, n-1\end{cases}$
starting with $P=[1]$.

The claim holds for $n=1$. For $n=2,4,8, \ldots, 2^{m}$,

$$
P_{n} N_{n}(k, l)=P_{n}(k, *) N_{n}(l, *)^{T}
$$

as it can easily be shown that $N_{n}$ is symmetric. Using the recurrent equations for $P_{n}$ and $N_{n}$ and applying the mixed product rule, we get, for $k=0,2,4, \ldots, n-2$,

$$
P_{n} N_{n}(k, l)=\frac{1}{2}(1-i) P_{n / 2}\left(\frac{k}{2}, *\right) N_{n / 2}\left(*, \frac{l}{2}\right)
$$

when $l=0,2,4, \ldots, n-2$ and

$$
P_{n} N_{n}(k, l)=\frac{1}{2}(1+i) P_{n / 2}\left(\frac{k-1}{2}, *\right) N_{n / 2}\left(*, \frac{l}{2}\right)
$$

when $l=1,3, \ldots, n-1$. By induction,

$$
P_{n} N_{n}(k, *)=\frac{1}{2} \hat{N}_{n / 2}\left(\frac{k}{2}, *\right) \otimes(1-i \quad 1+i)
$$

for even $k$ indices. This situation for odd $k$ is similar.
Now the main result can be shown.
Theorem 2. Let $n=2^{m}$ where $m$ is a non-negative integer. Let $N_{n}$ be the noiselet matrix of size $n \times n$ and let $H_{n}$ be the Haar matrix of size $n \times n$. Then $H_{n}$ and $N_{n}$ are maximally incoherent.
Proof. Without loss of generality, assume the bases are normalized such that $H_{n}^{T} H_{n}=I$ and $N_{n}^{*} N_{n}=n I$. For the case of $n=1$,

$$
H_{1} N_{1}^{*}=[1] \cdot[1]=[1]
$$

For $n=2^{m}, m>1$, the incoherence is shown by induction. Suppose we know maximal incoherence holds for $\frac{n}{2}$ and we want to show it for $n$. In the induction step, we use the iterative construction of the Haar matrix by means of Kronecker product. By computing the product

$$
H_{n} \hat{N}_{n}^{*}=H\left(N_{n}^{*} P_{n}^{*}\right)=\left(H_{n} N_{n}^{*}\right) P_{n}^{T}
$$

we will still be able to conclude on magnitude of the elements of $\left(H_{n} N_{n}^{*}\right)$, since permutation matrices do not change magnitudes.
The product $H_{n} \hat{N}_{n}^{*}$ can be computed per-column; we take the $j$-th column of $\hat{N}_{n}^{*}, j=0,2,4, \ldots, n-2$ and transform it by $H_{n}$, getting

$$
\left.\left.\begin{array}{rl}
H_{n} \hat{N}_{n}^{*}(*, j)= & \frac{1}{\sqrt{2}}\left[\begin{array}{cc}
H_{n / 2} \otimes(1 & 1
\end{array}\right) \\
I_{n / 2} \otimes(1 & -1
\end{array}\right]\right] \text {. } \frac{1}{\sqrt{2}} \hat{N}_{n / 2}^{*}\left(*, \frac{j}{2}\right) \otimes\left(\begin{array}{ll}
1-i & 1+i)^{*}
\end{array}\right.
$$

Note the altered normalization factor of noiselets. Now the mixed product property can be applied to get

$$
\begin{aligned}
& \frac{1}{2}\left[\begin{array}{l}
H_{n / 2} \hat{N}_{n / 2}^{*}\left(*, \frac{j}{2}\right) \otimes\left(\begin{array}{ll}
1 & 1
\end{array}\right)\left[\begin{array}{l}
1+i \\
1-i
\end{array}\right] \\
I_{n / 2} \hat{N}_{n / 2}^{*}\left(*, \frac{j}{2}\right) \otimes\left(\begin{array}{ll}
1 & -1
\end{array}\right)\left[\begin{array}{l}
1+i \\
1-i
\end{array}\right]
\end{array}\right]= \\
& \frac{1}{2}\left[\begin{array}{l}
H_{n / 2} \hat{N}_{n / 2}^{*}\left(*, \frac{j}{2}\right) * 2 \\
I_{n / 2} \hat{N}_{n / 2}^{*}\left(*, \frac{j}{2}\right) * 2 i
\end{array}\right] .
\end{aligned}
$$

By induction, it follows that $\left|H_{n / 2} \hat{N}_{n / 2}^{*}\left(i, \frac{j}{2}\right)\right|=1$ and $\left|I_{n / 2} \hat{N}_{n / 2}^{*}\left(i, \frac{j}{2}\right)\right|=1$ for $i=1, \ldots, \frac{n}{2}$. The Kronecker multiplication is only by entries with magnitude 2 , thus the resulting magnitudes are $\frac{1}{2} * 2=1$. The proof is equivalent for $j=1,3, \ldots, n-1$.

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